



HAL
open science

The spatial average of solutions to SPDEs is asymptotically independent of the solution

Ciprian A Tudor, Jérémy Zurcher

► **To cite this version:**

Ciprian A Tudor, Jérémy Zurcher. The spatial average of solutions to SPDEs is asymptotically independent of the solution. 2024. ⟨hal-04549075⟩

HAL Id: hal-04549075

<https://hal.science/hal-04549075v1>

Preprint submitted on 17 Apr 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

The spatial average of solutions to SPDEs is asymptotically independent of the solution

Ciprian A. Tudor and Jérémy Zurcher

CNRS, Université de Lille
Laboratoire Paul Painlevé UMR 8524
F-59655 Villeneuve d'Ascq, France.
ciprian.tudor@univ-lille.fr
jeremy.zurcher@univ-lille.fr

April 17, 2024

Abstract

Let $(u(t, x), t \geq 0, x \in \mathbb{R}^d)$ be the solution to the stochastic heat or wave equation driven by a Gaussian noise which is white in time and white or correlated with respect to the spatial variable. We consider the spatial average of the solution $F_R(t) = \frac{1}{\sigma_R} \int_{|x| \leq R} (u(t, x) - 1) dx$, where $\sigma_R^2 = \mathbf{E} \left(\int_{|x| \leq R} (u(t, x) - 1) dx \right)^2$. It is known that, when R goes to infinity, $F_R(t)$ converges in law to a standard Gaussian random variable Z . We show that the spatial average $F_R(t)$ is actually asymptotic independent by the solution itself, at any time and at any point in space, meaning that the random vector $(F_R(t), u(t, x_0))$ converges in distribution, as $R \rightarrow \infty$, to $(Z, u(t, x_0))$, where Z is a standard normal random variable independent of $u(t, x_0)$. By using the Stein-Malliavin calculus, we also obtain the rate of convergence, under the Wasserstein distance, for this limit theorem.

2010 AMS Classification Numbers: 60H15, 60H07, 60G15, 60F05.

Key Words and Phrases: stochastic heat equation, stochastic wave equation; Malliavin derivative, Stein's method; Wasserstein distance.

1 Introduction

In the last decade, a large number of research papers analyzed the asymptotic behavior of the solution to various stochastic partial differential equations (SPDEs in the sequel). The context is as follows. Consider a general SPDE of the form

$$\mathcal{L}u(t, x) = \sigma(u(t, x))\dot{W}(t, x), \quad t \geq 0, x \in \mathbb{R}^d, \quad (1.1)$$

with initial condition $u(0, x) = 1$ for every $x \in \mathbb{R}^d$. In (1.1), \mathcal{L} is first or second order differential operator with constant coefficients, $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a globally Lipschitz function and W is a space-time white noise (presented in the next section). The mild solution to (1.1) is expressed via the integral equation

$$u(t, x) = 1 + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) \sigma(u(s, y)) W(ds, dy), \quad (1.2)$$

where G is the Green kernel associated with the operator \mathcal{L} (i.e. the solution of $\mathcal{L}G = \delta_0$ in the sense of distributions) and the stochastic integral with respect to W is the Dalang-Walsh stochastic integral (see Section 2.1).

Assume that there exists a unique solution to (1.2) (which happens for the situations considered in our work). The purpose is to study the asymptotic behavior, as $R \rightarrow \infty$, of the quantity

$$F_R(t) = \frac{1}{\sigma_R} \int_{|x| \leq R} (u(t, x) - 1) dx, \quad (1.3)$$

where $\sigma_R^2 = \mathbf{E} \left(\int_{|x| \leq R} (u(t, x) - 1) dx \right)^2$ and $|\cdot|$ denotes the Euclidean norm in \mathbb{R}^d . The functional $F_R(t)$ is called *the spatial average* of the solution to the SPDE (1.1) at time t . In many situations, it has been proven that, for fixed $t > 0$ (in the sequel, we denote by $\rightarrow^{(d)}$ the convergence in distribution),

$$F_R(t) \rightarrow^{(d)} Z \sim \mathcal{N}(0, 1),$$

and estimates for the total variation distance between the laws of $F_R(t)$ and $Z \sim \mathcal{N}(0, 1)$ has been obtained. For the case of the stochastic heat equation (i.e. $\mathcal{L} = \frac{\partial}{\partial t} - \frac{1}{2}\Delta$ in (1.1), with Δ the Laplacian on \mathbb{R}^d), we refer to the papers [13], [14] or [16], which treated the situations when the noise is white or colored in space. The case of the stochastic wave equation (i.e. one takes $\mathcal{L} = \frac{\partial^2}{\partial t^2} - \Delta$ in (1.1) and we assume in addition $\frac{\partial u}{\partial t}(0, x) = 0$ for every $x \in \mathbb{R}^d$) with white noise in time, and white or correlated spatial covariance, has been studied in [6], [11], [17]. More precise statements of the results obtained for the heat and wave equations with white noise in time are presented in Theorem 1 and Theorem 2. A large variety of other situations has been considered in the literature: the case of the fractional noise in time in [2], [19], [16] or [18], the case of the fractional heat equation (i.e. the standard Laplacian is replaced by the fractional Laplacian) in [1], the case of time-independent noise in [3], [4] and [5] or the case of delta-initial condition in [7], [15].

In this work, we aim at studying the asymptotic independence between the spatial average $F_R(t)$ given by (1.3) and the solution (1.2) itself at time t and at a fixed spatial point $x_0 \in \mathbb{R}^d$. In other words, we show that, for every $t > 0$ and $x \in \mathbb{R}^d$, the family of random vectors

$$(F_R(t), u(t, x_0), R > 0)$$

converges in distribution, as $R \rightarrow \infty$, to the random vector

$$(Z, u(t, x_0)),$$

where $Z \sim \mathcal{N}(0, 1)$ is independent of $u(t, x_0)$. This implies that the spatial average $F_R(t)$ is asymptotically independent of the solution at any time and at any point in space. Intuitively, that means that large values of the solution are unlikely to be accompanied by large values of the spatial average, the extreme values of u have no effect on the behavior of the spatial average when the integration domain in (1.3) becomes larger and larger. For a recent work on the concept of asymptotic independence, see [10].

We treat several situations: the case of the (nonlinear) stochastic heat and wave equations with space-time white noise and with white noise in time and correlated spatial covariance given by the Riesz kernel. In all these situations, we prove that the functional $F_R(t)$ given by (1.3) is asymptotically independent of $u(t, x_0)$ and in addition, we give estimates for the Wasserstein distance between the probability distributions of $(F_R(t), u(t, x_0))$ and $(Z, u(t, x_0))$ (where $Z \sim \mathcal{N}(0, 1)$ is independent of $u(t, x_0)$), when R is large enough.

To prove these results, we use the variant of the Stein's method recently developed in [20] and [21]. This method allows to give bounds, in terms of the Malliavin operators, for the Wasserstein distance between the law of a random vector (X, Y) (with components differentiable in the Malliavin sense)

and the law of the vector (Z, Y) , with independent components and with Z a standard normal random variable. We actually slightly adapt the method from [20] and [21], by considering the situation when X is a Skorohod integral.

We organized our paper as follows. Section 2 constitutes a preliminary section where we describe the main objects that appear in our work and we present our new results. In Section 3 we develop our mathematical tools: we prove a new Stein-Malliavin bound to evaluate the Wasserstein distance between the law of a random vector (X, \mathbb{Y}) , where X is a Skorohod integral and Y is an arbitrary d -dimensional vector with components differentiable in the Malliavin sense, and the vector (Z, Y) , where $Z \sim \mathcal{N}(0, \sigma^2)$ and Z, \mathbb{Y} are independent. Section 4 contains the proofs of our main results while Section 5 is the Appendix where we included the basic elements of Malliavin calculus.

2 Preliminaries and goals of the paper

In this preliminary part, we present the stochastic heat and wave equations and some properties of their solutions. We then recall some results obtained in the literature regarding the asymptotic behavior of the spatial average for these equations and we also describe our new findings.

2.1 The Gaussian noise and the Dalang-Walsh stochastic integral

Let $(\Omega, \mathcal{F}, \mathbf{P})$ a probability space. We consider an integer $d \geq 1$ and let $\mathcal{B}_b(\mathbb{R}^d)$ be the set of bounded Borel subsets of \mathbb{R}^d . Consider the centered Gaussian field $W = (W(t, A), t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d))$ which is white in time and colored in space by the Riesz kernel. That is, for every $s, t \geq 0$ and $A, B \in \mathcal{B}_b(\mathbb{R}^d)$, we have the following expression for the covariance of W

$$\mathbf{E}[W(t, A)W(s, B)] = (s \wedge t) \int_A \int_B f_\alpha(x - y) \, dx \, dy, \quad (2.4)$$

where the function $f_\alpha : \mathbb{R}^d \rightarrow \mathbb{R}$ is given by

$$f_\alpha(x) = \begin{cases} \frac{1}{|x|^{d-\alpha}} & \text{if } 0 < \alpha < d \\ \delta_0(x) & \text{if } \alpha = 0. \end{cases}$$

When $\alpha = 0$, we say that the noise is white in space, and then the covariance (2.4) reduces to

$$\mathbf{E}[W(t, A)W(s, A)] = (s \wedge t) \text{Leb}(A \cap B), \quad (2.5)$$

where Leb stands for the Lebesgue measure on \mathbb{R}^d .

We define the Hilbert space H as the completion of the set $\{\mathbf{1}_{[0,t]} \otimes \mathbf{1}_A, t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$ under the scalar product, for all $t, s \geq 0$ and $A, B \in \mathcal{B}_b(\mathbb{R}^d)$:

$$\langle \mathbf{1}_{[0,t]} \otimes \mathbf{1}_A, \mathbf{1}_{[0,s]} \otimes \mathbf{1}_B \rangle_H := (t \wedge s) \int_A \int_B f_\alpha(x - y) \, dx \, dy.$$

Note that if $\alpha = 0$ (the white-noise case), then H is simply $L^2(\mathbb{R}_+ \times \mathbb{R}^d)$. We have, for $f, g \in H$,

$$\langle f, g \rangle_H = \int_0^\infty ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dy' f(u, y) g(v, y') f_\alpha(y - y'). \quad (2.6)$$

If $f = \mathbf{1}_{[0,t]} \otimes \mathbf{1}_A$, we set $W(f) := W(t, A)$ for $t \geq 0$ and $A \in \mathcal{B}_b(\mathbb{R}^d)$. Then we extend this relation to H , by linearity and density. In this way, the family $(W(g), g \in H)$ becomes an isonormal process (see also [22]). We denote by D and δ the Malliavin derivative and the Skorohod integral with respect to W , see Section 5.

For $t > 0$, denote by \mathcal{F}_t the σ -algebra generated by the random variables $(W(s, A), 0 \leq s \leq t, A \in \mathcal{B}_b(\mathbb{R}^d))$. A stochastic field $(X(t, y), t \geq 0, y \in \mathbb{R}^d)$ is said to be $(\mathcal{F}_t)_{t \geq 0}$ -adapted if $X(t, x)$ is \mathcal{F}_t -measurable for every $t \geq 0$ and for every $x \in \mathbb{R}^d$. Consider a random field $(X(t, y), t \geq 0, y \in \mathbb{R}^d)$, jointly measurable and $(\mathcal{F}_t)_{t \geq 0}$ -adapted such that

$$\mathbf{E}\|X\|_H^2 = \mathbf{E} \int_0^\infty du \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dy' X(s, y) X(s, y') f_\alpha(y - y') < \infty. \quad (2.7)$$

For such a stochastic field, we can define the Dalang-Walsh integral (see [8] or [23])

$$\int_0^\infty \int_{\mathbb{R}^d} X(s, y) W(ds, dy).$$

This stochastic integral satisfies the Itô-type isometry

$$\mathbf{E} \left(\int_0^\infty \int_{\mathbb{R}^d} X(s, y) W(ds, dy) \right)^2 = \mathbf{E}\|X\|_H^2,$$

where $\|\cdot\|_H = \langle \cdot, \cdot \rangle_H$ is the norm in H .

2.2 The stochastic heat and wave equations: Convergence of the spatial average

This part contains some basic facts concerning the heat and wave equations. We also recall the results for the normal convergence of the spatial average, which constitute the starting of our investigation.

2.2.1 The heat equation

We consider the following heat equation: for all $t \geq 0$ and $x \in \mathbb{R}^d$:

$$\frac{\partial u}{\partial t}(t, x) = \frac{1}{2} \Delta u(t, x) + \sigma(u(t, x)) \dot{W}(t, x), \quad (2.8)$$

with initial condition $u(0, x) = 1$ for all $x \in \mathbb{R}^d$, where W is the noise having the correlation (2.4), and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is supposed to be globally Lipschitz. We understand the solution in the *mild sense*, meaning that

$$u(t, x) = 1 + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) \sigma(u(s, y)) W(ds, dy), \quad (2.9)$$

where the integral must be understood in the Dalang-Walsh sense (see Section 2.1), and G is the Green kernel given by, for all $t > 0$ and $x \in \mathbb{R}^d$,

$$G(t, x) := \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{\frac{d}{2}}}. \quad (2.10)$$

This mild solution (2.9) exists if and only if $d < 2 + \alpha$ if $\alpha \in (0, d)$ i.e. W is a white-colored noise and if and only if $d = 1$ when W is a space-time white noise (see for instance [8]). Note that G satisfies the semi-group property

$$\int_{\mathbb{R}^d} G(t, x' - y) G(s, y - x) dy = G(t+s, x' - x), \quad (2.11)$$

for every $s, t \geq 0$ and $x, x' \in \mathbb{R}^d$. We have the following bound for all $p \geq 1$ and $T > 0$ (see e.g. [8] or [22])

$$\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^d} \mathbf{E} [|u(t, x)|^p] < \infty. \quad (2.12)$$

The object of interest of this paper is the normalized *spatial average of the solution*, defined by (1.3). In the case of the stochastic heat equation, we use the following notation: for every $R > 0$ and $t \geq 0$, let

$$F_R^H(t) := \frac{1}{\sigma_R} \int_{|x| \leq R} (u(t, x) - 1) dx, \quad (2.13)$$

where $\sigma_R^2 := \text{Var} \left(\int_{|x| \leq R} (u(t, x) - 1) dx \right)$. By a Fubini argument, and since the Dalang-Walsh integral coincides with the Skorohod integral for adapted integrands (see (5.39)), we have

$$F_R^H(t) = \delta(v_{R,t}),$$

where $v_{R,t} \in L^2(\mathbb{R}_+ \times \mathbb{R})$ is defined as

$$v_{R,t}(s, y) := \frac{1}{\sigma_R} \left(\int_{|x| \leq R} G(t-s, x-y) dx \right) \sigma(u(s, y)) \mathbf{1}_{[0,t]}(s). \quad (2.14)$$

The following theorem has been proved in [13] and [14]. For now, and in the rest of a paper, C is a strictly positive constant that could change from every line to an other.

Theorem 1. *Let $\beta := d - \alpha \in (0, d]$, $T > 0$ and let v be defined on (2.14). Then for all $t > 0$ and $R > 0$, $F_R^H(t) \in \mathbb{D}^{1,2}$. Moreover,*

1. *when $\beta = d = 1$ (i.e. $\alpha = 0$, the white-noise case), there exists a constant $C > 0$ depending on T such that for all $R \geq 0$ and $t \in (0, T]$*

$$d_{\text{TV}} \left(F_R^H(t), \mathcal{N}(0, 1) \right) \leq \sqrt{\text{Var} \left[\langle DF_R^H(t), v_{R,t} \rangle_{L^2(\mathbb{R}_+ \times \mathbb{R})} \right]} \leq \frac{C}{\sqrt{R}}, \quad (2.15)$$

where d_{TV} is the total variation distance. Moreover, as R goes to ∞ , $\sigma_R \sim C\sqrt{R}$, where C depends on T and α .

2. *for $0 < \alpha < d$ (the white-colored noise case), we have for some $C > 0$ depending on α and T :*

$$d_{\text{TV}} \left(F_R^H(t), \mathcal{N}(0, 1) \right) \leq \sqrt{\text{Var} \left[\langle DF_R^H(t), v_{R,t} \rangle_H \right]} \leq \frac{C}{R^{\frac{\beta}{2}}}. \quad (2.16)$$

Moreover, as R goes to ∞ , $\sigma_R \sim CR^{1-\frac{\beta}{2}}$.

For the definition of the total variation distance d_{TV} , we refer to e.g. [13]. However, we do not need it in our work. One last very useful result is the following lemma allowing to estimate the Malliavin derivative of $u(t, x)$ in terms of the corresponding Green kernel. Those results are also derived in [13] and [14].

Lemma 1. *Let $T > 0$, $t \in (0, T]$ and $x \in \mathbb{R}^d$. Then for all $p \geq 2$, $u(t, x) \in \mathbb{D}^{1,p}$ and for all $s \in (0, T]$ and $y \in \mathbb{R}^d$:*

$$\mathbf{E} \left[|D_{s,y} u(t, x)|^p \right]^{\frac{1}{p}} \leq C_{p,T} G(t-s, x-y). \quad (2.17)$$

2.2.2 The wave equation in dimension 1

We consider the following one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) + \sigma(u(t, x)) \dot{W}(t, x), \quad t \geq 0, x \in \mathbb{R}, \quad (2.18)$$

with initial condition $u(0, x) = 1$ and $\frac{\partial u}{\partial t}(0, x) = 0$ for all $x \in \mathbb{R}$, where W is the noise having the covariance (2.4), and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is supposed to be globally Lipschitz. We understand again the solution in the *mild sense*, meaning that

$$u(t, x) = 1 + \int_0^t \int_{\mathbb{R}^d} G_1(t-s, x-y) \sigma(u(s, y)) W(ds, dy), \quad (2.19)$$

where the integral $W(ds, dy)$ is the Dalang-Walsh integral introduced in Section 2.1 and G_1 is the Green kernel given by for all $t > 0$ and $x \in \mathbb{R}$

$$G_1(t, x) := \frac{1}{2} \mathbf{1}_{\{|x| \leq t\}}. \quad (2.20)$$

Then we have the following bound for all $p \geq 1$ and $T > 0$ (see [8] or [22])

$$\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} \mathbf{E} \left[|u(t, x)|^p \right] < \infty. \quad (2.21)$$

For every $R > 0$ and $t \geq 0$, we consider the spatial average of the solution

$$F_R^W(t) := \frac{1}{\sigma_R} \int_{-R}^R (u(t, x) - 1) dx, \quad (2.22)$$

where $\sigma_R^2 := \text{Var} \left(\int_{-R}^R (u(t, x) - 1) dx \right)$. We can write

$$F_R^W(t) = \delta(v_{R,t}),$$

where

$$v_{R,t}(s, y) := \frac{1}{\sigma_R} \left(\int_{-R}^R G_1(t-s, x-y) dx \right) \sigma(u(s, y)) \mathbf{1}_{[0,t]}(s). \quad (2.23)$$

The following theorem has been proved in the reference [11].

Theorem 2. *Let $\beta := 1 - \alpha \in [0, 1)$, and v defined on (2.23).*

1. *When $\beta = 1$ (the white-noise case), we have for all $t > 0$ and R large enough*

$$d_{\text{TV}} \left(F_R^W(t), \mathcal{N}(0, 1) \right) \leq \sqrt{\text{Var} \left[\langle DF_R^W(t), v_{R,t} \rangle_{L^2(\mathbb{R}_+ \times \mathbb{R})} \right]} \leq \frac{C}{\sqrt{R}}, \quad (2.24)$$

Moreover, as R goes to ∞ , $\sigma_R \sim C\sqrt{R}$.

2. *For $0 < \alpha < 1$ (the white-colored noise case), we have for all $t > 0$ and R large enough*

$$d_{\text{TV}} \left(F_R^W(t), \mathcal{N}(0, 1) \right) \leq \sqrt{\text{Var} \left[\langle DF_R^W(t), v_{R,t} \rangle_H \right]} \leq \frac{C}{R^{\frac{\beta}{2}}}. \quad (2.25)$$

Moreover, as R goes to ∞ , $\sigma_R \sim CR^{1-\frac{\beta}{2}}$.

As for the heat equation we have a lemma giving an estimate of the Malliavin derivative of the solution (it has been proven in [11], Lemma 2.2).

Lemma 2. *Let $T > 0$, $t \in (0, T]$ and $x \in \mathbb{R}^d$. Then for all $p \geq 2$, $u(t, x) \in \mathbb{D}^{1,p}$ and for all $s \in (0, T]$ and $y \in \mathbb{R}^d$:*

$$\mathbf{E} \left[|D_{s,y} u(t, x)|^p \right]^{\frac{1}{p}} \leq C_{p,T} G_1(t-s, x-y). \quad (2.26)$$

2.3 Results

We define first the distance we will work with. For every random variable X , real or vector valued, we will denote by \mathbf{P}_X or $\mathcal{L}(X)$ its law. We define $\text{Lip}(1)$ the set of functions $h : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ which are Lipschitz continuous, with a Lipschitz constant less or equal to 1. In other words,

$$\text{Lip}(1) := \left\{ h : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \left| \sup_{\substack{\mathbf{x}, \mathbf{x}' \in \mathbb{R}^{d+1} \\ \mathbf{x} \neq \mathbf{x}'}} \frac{|h(\mathbf{x}) - h(\mathbf{x}')|}{|\mathbf{x} - \mathbf{x}'|} \leq 1 \right. \right\}.$$

We define the *Wasserstein distance* for every random variable \mathbb{X}, \mathbb{Y} valued in $\mathbb{R} \times \mathbb{R}^d$ such that $h(\mathbb{X}) \in L^1(\mathbf{P})$ and $h(\mathbb{Y}) \in L^1(\mathbf{P})$ as :

$$d_W(\mathcal{L}(\mathbb{X}), \mathcal{L}(\mathbb{Y})) := \sup_{h \in \text{Lip}(1)} \left| \mathbf{E}[h(\mathbb{X})] - \mathbf{E}[h(\mathbb{Y})] \right|.$$

We state the first main result which shows that the spatial average of the solution of the heat equation is asymptotically independent of the solution itself.

Theorem 3. *Let $T > 0$, $F_R^H = (F_R^H(t), t \in [0, T])$ given by (2.13) and let $x_0 \in \mathbb{R}$ be an arbitrary point. $\beta := d - \alpha$.*

1. *If $\beta = d = 1$, we have for every $t \in (0, T]$ and for R large,*

$$d_W(\mathcal{L}(F_R^H(t), u(t, x_0)), \mathcal{N}(0, 1) \otimes \mathcal{L}(u(t, x_0))) \leq \frac{C}{\sqrt{R}}, \quad (2.27)$$

where $C > 0$ depends on T .

2. *If $0 < \alpha < d$, then for R large enough,*

$$d_W(\mathcal{L}(F_R^H(t), u(t, x_0)), \mathcal{N}(0, 1) \otimes \mathcal{L}(u(t, x_0))) \leq \frac{C}{R^{\frac{\beta}{2}}}, \quad (2.28)$$

where $C > 0$ depends on T and α .

A similar phenomenon happens in the case of the stochastic wave equation in spatial dimension $d = 1$.

Theorem 4. *Let $T > 0$, $F_R^W = (F_R^W(t), t \geq 0)$ given by (2.22) and $x_0 \in \mathbb{R}$. Let $\beta := 1 - \alpha$*

1. *If $\beta = 1$, i.e. in the space-time white noise case, there exists a constant $C > 0$ depending on T and x_0 such that for every $t \in (0, T]$ and for all $R > x_0 + 2T$:*

$$d_W(\mathcal{L}(F_R^W(t), u(t, x_0)), \mathcal{N}(0, 1) \otimes \mathcal{L}(u(t, x_0))) \leq \frac{C}{\sqrt{R}}; \quad (2.29)$$

2. *If $0 < \alpha < 1$ (the white colored noise case), then there exists a constant $C > 0$ depending on T , α and x_0 such that for every $t \in (0, T]$ and for all $R > x_0 + 2T$:*

$$d_W(\mathcal{L}(F_R^W(t), u(t, x_0)), \mathcal{N}(0, 1) \otimes \mathcal{L}(u(t, x_0))) \leq \frac{C}{R^{\frac{\beta}{2}}}. \quad (2.30)$$

3 A fundamental Stein's bound

We begin our proofs by introducing a new Stein's bound for a couple of random variables, by slightly adapting a result from [21]. The main result of this section is Proposition 1, where we give a bound for the Wasserstein distance between the law of (X, \mathbb{Y}) , where X is a Skorohod integral, and the law of (Z, Y) with $Z \sim \mathcal{N}(0, \sigma^2)$ independent of \mathbb{Y} . The idea to prove it is to use the following observation (see for instance Lemmas 1 and 2 in [21], see also [20]): if X is a real random variable and \mathbb{Y} a random vector, then the couple (X, \mathbb{Y}) is independent and X follows $\mathcal{N}(0, \sigma^2)$ if and only if for every function $f : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ belonging to a large class of function, we have

$$\sigma^2 \mathbf{E} \left[\frac{\partial f}{\partial x}(X, \mathbb{Y}) \right] = \mathbf{E}[X f(X, \mathbb{Y})].$$

To the above identity, one can associate a Stein's equation as follows: let $h : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\mathbf{E}[|h(Z, \mathbb{Y})|] < \infty$, where $Z \sim \mathcal{N}(0, \sigma^2)$ is independent of \mathbb{Y} . We set, for all $x \in \mathbb{R}$ and $\mathbf{y} \in \mathbb{R}^d$:

$$\sigma^2 \frac{\partial f}{\partial x}(x, \mathbf{y}) - x f(x, \mathbf{y}) = h(x, \mathbf{y}) - \mathbf{E}[h(Z, \mathbf{y})]. \quad (3.31)$$

We will use the following lemma, proved in Proposition 1 in [21].

Lemma 3. *Let $h : \mathbb{R} \rightarrow \mathbb{R}^d$ be of C^1 with partial derivatives bounded by 1, the equation (3.31) admits a unique bounded solution. Moreover, there exists a constant $C > 0$ which does not depend of h such that*

$$\max \left\{ \left\| \frac{\partial f_h}{\partial x} \right\|_{\infty}, \max_{1 \leq j \leq d} \left\| \frac{\partial f_h}{\partial y_j} \right\|_{\infty} \right\} \leq C.$$

We can state and prove the following proposition, which constitutes a generalization of Proposition 2.2 in [13].

Proposition 1. *Let $X = \delta(v)$, where $v \in \text{Dom}(\delta) \subset L^2(\Omega \times H)$ and let $\mathbb{Y} = (Y_1, \dots, Y_d)$ be a vector-valued random variable. Suppose that $\mathbf{E}[X^2] = \sigma^2$, $X \in \mathbb{D}^{1,2}$ and $\mathbb{Y} = (Y_1, \dots, Y_d)$ with $Y_j \in \mathbb{D}^{1,2}$ for each $j = 1, \dots, d$. Then we have*

$$d_W \left(\mathcal{L}(X, \mathbb{Y}), \mathcal{N} \left(0, \sigma^2 \right) \otimes \mathcal{L}(\mathbb{Y}) \right) \leq C \left\{ \sqrt{\text{Var} [\langle DX, v \rangle_H]} + \sum_{j=1}^d \sqrt{\mathbf{E} [\langle v, DY_j \rangle_H^2]} \right\}. \quad (3.32)$$

Proof. Suppose first that h is also C^1 on \mathbb{R}^{d+1} and it has partial derivatives bounded by 1. Let $Z \sim \mathcal{N}(0, \sigma^2)$ be a random variable independent of \mathbb{Y} . Then, by the Stein's equation (3.31)

$$\begin{aligned} \mathbf{E}[h(X, \mathbb{Y})] - \mathbf{E}[h(Z, \mathbb{Y})] &= \mathbf{E} [h(X, \mathbb{Y}) - \mathbf{E}[h(Z, \mathbb{Y})]] \\ &= \mathbf{E} \left[\sigma^2 \frac{\partial f_h}{\partial x}(X, \mathbb{Y}) - X f_h(X, \mathbb{Y}) \right] \\ &= \mathbf{E} \left[\sigma^2 \frac{\partial f_h}{\partial x}(X, \mathbb{Y}) \right] - \mathbf{E} [\delta(v) f_h(X, \mathbb{Y})]. \end{aligned}$$

By duality formula (5.38) for the divergence operator and by chain rule of Malliavin derivative

$$\begin{aligned} \mathbf{E}[h(X, \mathbb{Y})] - \mathbf{E}[h(Z, \mathbb{Y})] &= \mathbf{E} \left[\sigma^2 \frac{\partial f_h}{\partial x}(X, \mathbb{Y}) \right] - \mathbf{E} [\langle v, Df_h(X, \mathbb{Y}) \rangle_H] \\ &= \mathbf{E} \left[\frac{\partial f_h}{\partial x}(X, \mathbb{Y}) \left(\sigma^2 - \langle v, DX \rangle \right) \right] - \sum_{j=1}^d \mathbf{E} \left[\frac{\partial f_h}{\partial y_j}(X, \mathbb{Y}) \langle v, DY_j \rangle \right]. \end{aligned}$$

Using again the duality relationship (5.38), we have

$$\mathbf{E}[\langle v, DX \rangle] = \mathbf{E}[\delta(v)^2] = \mathbf{E}[X^2] = \sigma^2,$$

we conclude by Cauchy–Schwarz inequality and by Lemma 3 that

$$\left| \mathbf{E}[h(X, \mathbb{Y})] - \mathbf{E}[h(Z, \mathbb{Y})] \right| \leq C \left\{ \sqrt{\text{Var}[\langle DX, v \rangle_H]} + \sum_{j=1}^d \sqrt{\mathbf{E}[\langle v, DY_j \rangle_H^2]} \right\}, \quad (3.33)$$

where the constant C does not depend on h . This inequality is true for every $h \in \text{Lip}(1)$ being C^1 . For a general $h \in \text{Lip}(1)$, we approximate it for every $\varepsilon > 0$ by

$$h_\varepsilon(x, y) := \mathbf{E} [h(x + \sqrt{\varepsilon}N, \mathbf{y} + \sqrt{\varepsilon}\mathbf{N})],$$

where $N \sim \mathcal{N}(0, 1)$ and $\mathbf{N} \sim \mathcal{N}_d(0, I_d)$ are independent. Then $h_\varepsilon \in \text{Lip}(1)$ is C^1 and uniformly converges to h on $\mathbb{R} \times \mathbb{R}^d$ as ε goes to zero. Consequently, we get

$$\left| \mathbf{E}[h(X, \mathbb{Y})] - \mathbf{E}[h(Z, \mathbb{Y})] \right| \leq C \left\{ \sup_{\mathbb{R} \times \mathbb{R}^d} |h - h_\varepsilon| + \sqrt{\text{Var}[\langle DX, v \rangle_H]} + \sum_{j=1}^d \sqrt{\mathbf{E}[\langle v, DY_j \rangle_H^2]} \right\}.$$

Since this holds for every $\varepsilon > 0$, we conclude on (3.33) for every $h \in \text{Lip}(1)$ by using the uniform convergence of h_ε to h , and so in (3.32). \square

4 Proofs

This section is consecrated to the proofs of our main results, Theorem 3 and Theorem 4. The two proofs follow the same main line but the calculations are pretty different since they strongly depend on the expression of the Green kernel.

4.1 Proof of Theorem 3

We separate the proofs of the cases when the noise is white or correlated with respect to the space variable.

4.1.1 White noise case

We suppose that $\alpha = 0$ and so $d = 1$. Let $t \in [0, T]$ and $R > 0$. Recall that $F_R^H(t) = \delta(v_{R,t})$ where $v_{R,t} \in L^2(\mathbb{R}_+ \times \mathbb{R})$ is defined by (2.14). Then, since $F_R^H(t) \in \mathbb{D}^{1,2}$, we apply Proposition 1 to get

$$\begin{aligned} & d_W \left(\mathcal{L} \left(F_R^H(t), u(x_0, t) \right), \mathcal{N}(0, 1) \otimes \mathcal{L}(u(x_0, t)) \right) \\ & \leq C \left\{ \sqrt{\text{Var}[\langle DF_R^H(t), v_{R,t} \rangle_H]} + \sqrt{\mathbf{E}[\langle v_{R,t}, Du(x_0, t) \rangle_H^2]} \right\}. \end{aligned} \quad (4.34)$$

By Theorem 1, the first term satisfies

$$\sqrt{\text{Var}[\langle DF_R^H(t), v_{R,t} \rangle_H]} \leq \frac{C}{\sqrt{R}}.$$

All we need to do is to estimate the second term. By definition, we have

$$\left\langle v_{R,t}, Du(t, x_0) \right\rangle_H = \frac{1}{\sigma_R} \int_{-R}^R \left[\int_0^t \int_{\mathbb{R}} \sigma(u(s, y)) G(t-s, x-y) D_{s,y} u(t, x_0) dy ds \right] dx.$$

Hence, the expectation of the square of this term yields to, by using Cauchy–Schwarz inequality :

$$\begin{aligned}
\mathbf{E} \left[\left\langle v_{R,t}, Du(t, x_0) \right\rangle_H^2 \right] &= \frac{1}{\sigma_R^2} \int_{-R}^R \int_{-R}^R \int_0^t \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} G(t-s, x-y) G(t-s', x'-y') \\
&\quad \cdot \mathbf{E} \left[\sigma(u(s, y)) D_{s,y} u(t, x_0) \sigma(u(s', y')) D_{s',y'} u(t, x_0) \right] dy ds dy' ds' dx dx' \\
&\leq \frac{1}{\sigma_R^2} \int_{-R}^R \int_{-R}^R \int_0^t \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} G(t-s, x-y) G(t-s', x'-y') \\
&\quad \cdot \mathbf{E} \left[\sigma(u(s, y))^4 \right]^{\frac{1}{4}} \mathbf{E} \left[\sigma(u(s', y'))^4 \right]^{\frac{1}{4}} \mathbf{E} \left[D_{s,y} u(t, x_0)^4 \right]^{\frac{1}{4}} \mathbf{E} \left[D_{s',y'} u(t, x_0)^4 \right]^{\frac{1}{4}} \\
&\quad dy ds dy' ds' dx dx'.
\end{aligned}$$

Since σ is globally Lipschitz (we denote by $\|\sigma\|_{\text{Lip}}$ its Lipschitz constant), we have by using (2.21) that for all $s \in [0, t]$ and $y \in \mathbb{R}$:

$$\sup_{0 \leq t \leq T} \sup_{y \in \mathbb{R}} \mathbf{E} \left[\sigma(u(s, y))^4 \right] \leq 8|\sigma(0)|^4 + 8\|\sigma\|_{\text{Lip}}^4 \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} \mathbf{E} \left[u(t, x)^4 \right] < \infty. \quad (4.35)$$

Moreover, by Lemma 1, we have for all $s \in [0, t]$ and $y \in \mathbb{R}$:

$$\mathbf{E} \left[D_{s,y} u(t, x_0)^4 \right]^{\frac{1}{4}} \leq CG(t-s, x_0-y).$$

Consequently, we have

$$\mathbf{E} \left[\left\langle v_{R,t}, Du(t, x_0) \right\rangle_H^2 \right] \leq \frac{C}{\sigma_R^2} \int_{-R}^R \int_{-R}^R \int_0^t \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} G(t-s, x-y) G(t-s', x'-y') \\
G(t-s, x_0-y) G(t-s', x_0-y') dy ds dy' ds' dx dx'.$$

By semi-group property (2.11):

$$\mathbf{E} \left[\left\langle v_{R,t}, Du(t, x_0) \right\rangle_H^2 \right] \leq \frac{C}{\sigma_R^2} \int_{-R}^R \int_{-R}^R \int_0^t \int_0^t G(2(t-s), x-x_0) G(2(t-s'), x'-x_0) ds ds' dx dx'.$$

To conclude, we do the majoration $\int_{-R}^R \leq \int_{\mathbb{R}}$, we compute $\int_{\mathbb{R}} G(2(t-s), x) dx = 2(t-s)$ and we find

$$\mathbf{E} \left[\left\langle v_{R,t}, Du(t, x_0) \right\rangle_H^2 \right] \leq \frac{C}{\sigma_R^2} \leq \frac{C}{R},$$

by Theorem 1. Then the estimate (2.27) is obtained.

4.1.2 White-colored noise case

We suppose now that $d \geq 1$ and $0 < \alpha < d$. We still have $F_R^{\text{H}}(t) = \delta(v_{R,t})$, with v defined in (2.14), and the estimation (4.34). By Theorem 1, we have

$$\sqrt{\text{Var} [\langle DF_R^{\text{H}}(t), v_{R,t} \rangle_H]} \leq \frac{C}{R^{\frac{\beta}{2}}}.$$

So we focus on the second term. By the definition of the scalar product on H (see (2.6)),

$$\left\langle v_{R,t}, Du(t, x_0) \right\rangle_H = \frac{1}{\sigma_R} \int_{|x| \leq R} \left[\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma(u(s, y)) G(t-s, x-y) D_{s,z} u(t, x_0) \frac{dy dz}{|y-z|^\beta} ds \right] dx.$$

By using Cauchy–Schwarz inequality, (4.35) and Lemma 1, we have

$$\begin{aligned}
& \mathbf{E} \left[\left\langle v_{R,t}, Du(t, x_0) \right\rangle_H^2 \right] \\
& \leq \frac{C}{\sigma_R^2} \int_{|x| \leq R} \int_{|x'| \leq R} \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t-s, x-y) G(t-s', x'-y') \\
& \quad G(t-s, x_0-z) G(t-s', x_0-z') \frac{dy \, dz}{|y-z|^\beta} \, ds \frac{dy' \, dz'}{|y'-z'|^\beta} \, ds' \, dx' \, dx \\
& = \frac{C}{\sigma_R^2} \left(\int_{|x| \leq R} \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t-s, y) G(t-s, z) \frac{dy \, dz}{|z-y+(x-x_0)|^\beta} \, ds \, dx \right)^2 \\
& = \frac{C}{\sigma_R^2} \left(\int_{|x| \leq R} \int_0^t \mathbf{E} \left[\frac{1}{|x-x_0-\sqrt{t-s}(Z_1-Z_2)|^\beta} \right] \, ds \, dx \right)^2,
\end{aligned}$$

where Z_1 and Z_2 are independent and follows the standard d -dimensional Gaussian law $\mathcal{N}_d(0, I_d)$. By doing the change of variables $x \leftarrow \frac{x}{R}$ and using the fact that

$$\sup_{y \in \mathbb{R}^d} \int_{|x| \leq 1} \frac{dx}{|x+y|^\beta} < \infty,$$

we conclude that

$$\begin{aligned}
\mathbf{E} \left[\left\langle v_{R,t}, Du(t, x_0) \right\rangle_H^2 \right] & \leq \frac{CR^{2d-2\beta}}{\sigma_R^2} \left(\int_0^t \mathbf{E} \left[\int_{|x| \leq 1} \frac{dx}{\left| x - \frac{x_0 + \sqrt{t-s}(Z_1-Z_2)}{R} \right|^\beta} \right] \, ds \right)^2 \\
& \leq \frac{CR^{2d-2\beta}}{\sigma_R^2}.
\end{aligned}$$

By Theorem 1, σ_R^2 is equivalent to $CR^{2d-\beta}$ for R large, so we conclude that

$$\mathbf{E} \left[\left\langle v_{R,t}, Du(t, x_0) \right\rangle_H^2 \right] \leq \frac{C}{R^\beta}.$$

This leads to (2.28) and concludes the proof.

4.2 Proof of Theorem 4

The idea of the proof is the same as for the previous theorem, except that the semi-group property does not hold for the Green kernel associated to the wave equation. Instead, we will compute explicitly the integrals appearing in the proof.

4.2.1 White noise case

Suppose that $\alpha = 0$ and $d = 1$. Then we write $F_R^W(t) = \delta(v_{R,t})$, where $v_{R,t}$ is given by (2.23). Then, by doing the same steps as in the proof of Theorem 3, by using Theorem 2 and Proposition 1, we only need to focus on estimating $\mathbf{E} \left[\left\langle v_{R,t}, Du(t, x_0) \right\rangle^2 \right]$. With the use of Cauchy–Schwarz inequality, inequality (4.35) that still holds for the wave equation and Lemma 2, we conclude that

$$\begin{aligned}
\mathbf{E} \left[\left\langle v_{R,t}, Du(t, x_0) \right\rangle^2 \right] & \leq \frac{C}{\sigma_R^2} \int_{-R}^R \int_{-R}^R \int_0^t \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} G_1(t-s, x-y) G_1(t-s', x'-y') \\
& \quad G_1(t-s, x_0-y) G_1(t-s', x_0-y') \, dy \, ds \, dy' \, ds' \, dx \, dx' \\
& = \frac{C}{\sigma_R^2} \left(\int_{-R}^R \int_0^t \int_{\mathbb{R}} G_1(t-s, x-y) G_1(t-s, x_0-y) \, dy \, ds \, dx \right)^2.
\end{aligned}$$

We go on by computing this integral, by using the explicit expression of G_1 given in (2.20). Then, we have for $s \in [0, t]$ and $x \in \mathbb{R}$ that

$$\begin{aligned} & \int_{\mathbb{R}} G_1(t-s, x-y) G_1(t-s, x_0-y) dy \\ = & \left(x_0 - x - 2(t-s) \right) \mathbf{1}_{[x_0-2(t-s), x_0]}(x) + \left(x - x_0 - 2(t-s) \right) \mathbf{1}_{[x_0, x_0+2(t-s)]}(x), \end{aligned}$$

where, by convention, $[a, b] = \emptyset$ if $a > b$. Hence, we have for $R > x_0 + 2t$:

$$\begin{aligned} \mathbf{E} \left[\left\langle v_{R,t}, Du(t, x_0) \right\rangle^2 \right] & \leq \frac{C}{\sigma_R^2} \left[\int_0^t \int_{\max\{-R, x_0-2(t-s)\}}^{\min\{x_0, R\}} (x_0 - x - 2(t-s)) dx ds \right. \\ & \quad \left. + \int_0^t \int_{\max\{-R, x_0\}}^{\min\{x_0+2(t-s), R\}} (x - x_0 - 2(t-s)) dx ds \right] \\ & = \frac{C}{\sigma_R^2} \left[\int_0^t \int_{x_0-2(t-s)}^{x_0} (x_0 - x - 2(t-s)) dx ds \right. \\ & \quad \left. + \int_0^t \int_{x_0}^{x_0+2(t-s)} (x - x_0 - 2(t-s)) dx ds \right] \\ & = \frac{C}{\sigma_R^2}. \end{aligned}$$

By Theorem 2, σ_R^2 is equivalent to CR when R goes to infinity, so we conclude that

$$\mathbf{E} \left[\left\langle v_{R,t}, Du(t, x_0) \right\rangle^2 \right] \leq \frac{C}{\sqrt{R}},$$

which proves the inequality (2.29).

4.2.2 White-colored noise case

Suppose that $0 < \alpha < 1$. By repeating the same steps as before, we have

$$\mathbf{E} \left[\left\langle v_{R,t}, Du(t, x_0) \right\rangle_H^2 \right] \leq \frac{C}{\sigma_R^2} \left(\int_{-R}^R \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{G_1(t-s, x-y) G_1(t-s, x_0-z)}{|y-z|^\beta} dy dz ds dx \right)^2. \quad (4.36)$$

We need to compute the double integral with respect to y and z .

Lemma 4. *Let $x \in \mathbb{R}$ and $s \in [0, t]$. Then*

$$\begin{aligned} & (1-\beta)(2-\beta) \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{G_1(t-s, x-y) G_1(t-s, x_0-z)}{|y-z|^\beta} dy dz \\ = & \left((x_0 - x + 2(t-s))^{2-\beta} + (x_0 - x - 2(t-s))^{2-\beta} - 2(x_0 - x)^{2-\beta} \right) \mathbf{1}_{(-\infty, x_0-2(t-s))}(x) \\ & + \left((x_0 - x + 2(t-s))^{2-\beta} + (x - x_0 + 2(t-s))^{2-\beta} - 2(x_0 - x)^{2-\beta} \right) \mathbf{1}_{[x_0-2(t-s), x_0]}(x) \\ & + \left((x - x_0 + 2(t-s))^{2-\beta} + (x_0 - x + 2(t-s))^{2-\beta} - 2(x - x_0)^{2-\beta} \right) \mathbf{1}_{(x_0, x_0+2(t-s)]}(x) \\ & + \left((x - x_0 + 2(t-s))^{2-\beta} + (x - x_0 - 2(t-s))^{2-\beta} - 2(x - x_0)^{2-\beta} \right) \mathbf{1}_{(x_0+2(t-s), +\infty)}(x). \end{aligned}$$

Going back on (4.36), we actually have with the previous lemma, with $R > x_0 + 2t$:

$$\begin{aligned}
& \mathbf{E} \left[\left\langle v_{R,t}, Du(t, x_0) \right\rangle_H^2 \right] \\
\leq & \frac{C}{\sigma_R^2} \left(C' + \int_0^t \int_{-R}^{x_0 - 2(t-s)} \left((x_0 - x + 2(t-s))^{2-\beta} + (x_0 - x - 2(t-s))^{2-\beta} \right. \right. \\
& \left. \left. - 2(x_0 - x)^{2-\beta} \right) dx ds \right. \\
& \left. + \int_0^t \int_{x_0 + 2(t-s)}^R \left((x - x_0 + 2(t-s))^{2-\beta} + (x - x_0 - 2(t-s))^{2-\beta} - 2(x - x_0)^{2-\beta} \right) dx ds \right)^2,
\end{aligned}$$

where C' is another constant (with respect to R) that may depend on α, T, x_0 . Then

$$\begin{aligned}
& \mathbf{E} \left[\left\langle v_{R,t}, Du(t, x_0) \right\rangle_H^2 \right] \\
\leq & \frac{C}{\sigma_R^2} \left(C' + \int_0^t \left((x_0 + R + 2(t-s))^{3-\beta} + (x_0 + R - 2(t-s))^{3-\beta} - 2(x_0 + R)^{3-\beta} \right) ds \right. \\
& \left. + \int_0^t \left((R - x_0 + 2(t-s))^{3-\beta} + (R - x_0 - 2(t-s))^{3-\beta} - 2(R - x_0)^{3-\beta} \right) ds \right)^2 \\
= & \frac{C}{\sigma_R^2} \left\{ C' + \frac{1}{2(4-\beta)} \left((x_0 + R + 2t)^{4-\beta} - (x_0 + R)^{4-\beta} - (x_0 + R - 2t)^{4-\beta} + (x_0 + R)^{4-\beta} \right) \right. \\
& + \frac{1}{2(4-\beta)} \left((R - x_0 + 2t)^{4-\beta} - (R - x_0)^{4-\beta} - (R - x_0 - 2t)^{4-\beta} + (R - x_0)^{4-\beta} \right) \\
& \left. + 2t(x_0 + R)^{3-\beta} - 2t(R - x_0)^{3-\beta} \right\}^2 \\
= & \frac{C}{\sigma_R^2} \left\{ C' + \frac{1}{2(4-\beta)} \left((R + x_0 + 2t)^{4-\beta} - (R - (x_0 + 2t))^{4-\beta} \right. \right. \\
& \left. \left. + (R + 2t - x_0)^{4-\beta} - (R - (2t - x_0))^{4-\beta} \right) - 2t(x_0 + R)^{3-\beta} - 2t(R - x_0)^{3-\beta} \right\}^2.
\end{aligned}$$

By expanding $(R + u)^{4-\beta} = R^{4-\beta} + u(4-\beta)R^{3-\beta} + \frac{u^2(4-\beta)(3-\beta)}{2}R^{2-\beta} + O(R^{1-\beta})$ when R grows to infinity, and expanding $(u + R)^{3-\beta}$ too, we conclude that

$$\mathbf{E} \left[\left\langle v_{R,t}, Du(t, x_0) \right\rangle_H^2 \right] \leq \frac{C}{\sigma_R^2} \left(C' + C'' R^{1-\beta} \right) \leq \frac{C R^{2-2\beta}}{\sigma_R^2}.$$

By Theorem 2, σ_R^2 is equivalent to $C R^{2-\beta}$ for R large. Consequently, we have

$$\mathbf{E} \left[\left\langle v_{R,t}, Du(t, x_0) \right\rangle_H^2 \right] \leq \frac{C}{\sigma_R^2} \left(C' + C'' R^{1-\beta} \right) \leq \frac{C}{R^\beta}.$$

This concludes in (2.30), and the proof.

Let us end this section with some comments:

Remark 1. • *In Theorems 3 and 4, the distance between the probability laws of $(F_R(t), u(t, x_0))$ and the random vector $(Z, u(t, x_0))$, where $Z \sim \mathcal{N}(0, 1)$ is independent of $u(t, x_0)$, is the sum of*

two terms: the first quantifies Wasserstein distance between $F_R(t)$ and Z and the second somehow measures the correlation between the spatial average $F_R(t)$ and $u(t, x_0)$ when R is larger and larger. It can be noticed from the proofs of Theorems 3 and 4 that these two terms are of the same order when the integration domain becomes larger and larger (i.e. $R \rightarrow \infty$).

- In the case of the stochastic wave equation, we restricted to the situation when $d = 1$. This is because our calculations strongly rely on the expression of the Green kernel G_1 . On the other hand, it seems that the main idea used in this work could be extended to the case when the spatial dimension is bigger than one.

5 Appendix: Malliavin calculus

Let us next describe the basic tools from Malliavin calculus needed in this work. We refer to [12] for a detailed presentation. Let $(W(h), h \in H)$ be an isonormal process, i.e. a centered Gaussian family such that for every $h, g \in H$,

$$\mathbf{E}W(h)W(g) = \langle h, g \rangle_H,$$

We introduce $C_p^\infty(\mathbb{R}^n)$ as the space of smooth functions with all their partial derivatives having at most polynomial growth at infinity, and \mathcal{S} as the space of simple random variables of the form

$$F = f(W(h_1), \dots, W(h_n)),$$

where $f \in C_p^\infty(\mathbb{R}^n)$ and $h_i \in H$, $1 \leq i \leq n$. Then the Malliavin derivative DF is defined as H -valued random variable

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), \dots, W(h_n))h_i. \quad (5.37)$$

For any $p \geq 1$, the operator D is closable as an operator from $L^p(\Omega)$ into $L^p(\Omega; H)$. Then $\mathbb{D}^{1,p}$ is defined as the completion of \mathcal{S} with respect to the norm

$$\|F\|_{1,p} = (\mathbf{E}[|F|^p] + \mathbf{E}(\|DF\|_H^p))^{1/p}.$$

The adjoint operator δ of the derivative is defined through the duality formula

$$\mathbf{E}(\delta(u)F) = \mathbf{E}(\langle u, DF \rangle_H), \quad (5.38)$$

valid for any $F \in \mathbb{D}^{1,2}$ and any $u \in \text{Dom } \delta \subset L^2(\Omega; H)$. The operator δ is also called the Skorokhod integral since, in the case of the standard Brownian motion, it coincides with an extension of the Itô integral introduced by Skorokhod (see e.g. [12]).

In our context, the Hilbert space H coincides with $L^2(\mathbb{R}_+ \times \mathbb{R})$ when W is space-time white noise and it is characterized by the scalar product (2.6) when W is a white noise in time with spatial covariance given by the Riesz kernel. In particular, for any adapted random field X which is jointly measurable and satisfies (2.7) belongs to the domain of δ , and $\delta(X)$ coincides with the Walsh integral (introduced in Section 2).

$$\delta(X) = \int_0^\infty \int_{\mathbb{R}} X(s, y)W(ds, dy). \quad (5.39)$$

Funding: Both authors acknowledge partial support from Labex CEMPI (ANR-11-LABX-007-01). C. Tudor also acknowledges support from the ANR project SDAIM 22-CE40-0015, ECOS SUD (project C2107), Japan Science and Technology Agency CREST (grant JPMJCR2115) and by the Ministry of Research, Innovation and Digitalization (Romania), grant CF-194-PNRR-III-C9-2023.

References

- [1] O. Assaad, D. Nualart, C.A. Tudor, L. Viitasaari (2021): *Quantitative normal approximations for the stochastic fractional heat equation*. Stoch. Partial Differ. Equ. Anal. Comput. 10 (2022), no. 1, 223–254.
- [2] R. M. Balan, D. Nualart, L. Quer-Sardanyons, G. Zheng (2022): *The hyperbolic Anderson model: Moment estimates of the Malliavin derivatives and applications*. Stoch. PDEs: Anal. Comput. 10(3), 757-827.
- [3] R. M. Balan, W. Yuan (2022): *Spatial integral of the solution to hyperbolic Anderson model with time-independent noise*. Stoch. Process. Appl. 152, 177–207.
- [4] R. M. Balan, W. Yuan (2023): *Central limit theorems for heat equation with time-independent noise: The regular and rough cases*. Inf. Dim. Anal. Quantum Probab. Rel. Top. 26 (02), 2250029.
- [5] R. M. Balan, W. Yuan (2023): *Hyperbolic Anderson model with time-independent rough noise: Gaussian fluctuations*. Preprint arXiv:2305.05043.
- [6] R. Bolanos-Guerrero, D. Nualart, G. Zheng (2021): *Averaging 2d stochastic wave equation*. Electr. J. Probab. 26, Paper No. 102.
- [7] L. Chen, D. Khoshnevisan, D. Nualart F. Pu (2022): *Spatial ergodicity and central limit theorems for parabolic Anderson model with delta initial condition*. J. Funct. Anal. 282, 109290.
- [8] R. Dalang (1999): *Extending the martingale measure stochastic integral with applications to spatially homogeneous SPDE's*. Electr. J. Probab. 4, 1-29. pp. Erratum in Electr. J. Probab. 6 (2001), 5 pp.
- [9] R. Dalang, L. Quer-Sardanyons (2011): *Stochastic integrals for SPDE's : a comparison*. Expo. Math. 29 (1), 67–109.
- [10] Yu. A. Davydov, S. Novikov (2021): *Remarks on asymptotic independence*. eoriya Veroyatnostei i ee Primeneniya, 66 (1), 55–72.
- [11] F. Delgado-Vences, D. Nualart, G. Zheng (2020): *A central limit theorem for the stochastic wave equation with fractionnal noise*, Ann. Inst. Henri Poincaré Probab. Stat. 56 (4), 3020–3042.
- [12] D. Nualart (2006): *The Malliavin Calculus and Related Topics*, Second edition.
- [13] J. Huang, D. Nualart, L. Viitasaari (2020): *A Central Limit Theorem for the stochastic heat equation*. Stochastic Process. Appl. 130 (12), 7170-7184.
- [14] J. Huang, D. Nualart, L. Viitasaari, G. Zheng (2020) : *Gaussian Fluctuations for the Stochastic Heat Equation with colored noise*. Stoch. Partial Differ. Equ. Anal. Comput. 8 (2), 402-421.
- [15] D. Khoshnevisan, D. Nualart, F. Pu (2021): *Spatial stationarity, ergodicity and CLT for the parabolic Anderson model with delta initial condition in dimension $d \geq 1$* . SIAM J. Math. Anal. 53, 2084-2133.
- [16] D. Nualart, P. Xia, G. Zheng (2022): *Quantitative central limit theorems for the parabolic Anderson model driven by colored noises*. Electr. J. Probab. 27, Paper No. 120, 43.
- [17] D. Nualart, G. Zheng (2022): *Central limit theorems for stochastic wave equations in dimensions one and two*. Stoch. PDE: Anal. Comput. 10(2), 392–418.

- [18] D. Nualart, G. Zheng (2020): *Averaging Gaussian functionals*. Electr. J. Probab. 25, Paper No. 48, 54.
- [19] D. Nualart, X. Song, G. Zheng (2020): *Spatial averages for the Parabolic Anderson model driven by rough noise*. ALEA Lat. Am. J. Probab. Math. Stat. 18 (1), 907–943.
- [20] L. Pimentel (2022): *Integration by parts and the KPZ two-point function*. Ann. Probab. 50, 1755–1780.
- [21] C. A. Tudor (2023) : *Multidimensional Stein method and quantitative asymptotic independence*. Preprint.
- [22] C. A. Tudor (2022): *Stochastic Partial Differential Equations with Additive Gaussian noise: Analysis and Inference*. World Scientific.
- [23] Walsh, J. B. (1986): *An introduction to stochastic partial differential equations*. In: *Ecole d'été de Probabilités de St. Flour XIV*, pp. 266-439, Lect. Notes in Math. 1180, Springer-Verlag, Berlin.