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# 1 **Wave propagation in random 2D turbulence: a** 2 **multi-scale approach**

3 **Valentin Resseguier<sup>1,2</sup>†, Erwan Hascoët<sup>3</sup> and Bertrand Chapron<sup>4</sup>**

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9 To study two-dimensional dispersive waves propagating over turbulent flows, a new and less  
10 restrictive fast waves approximation is proposed using a multiscale setting. In this ansatz,  
11 correlation lengths of the random small scale turbulence components can be considered  
12 negligible in the wave packet propagating frame. Still, the large-scale flow can be relatively  
13 strong, to significantly impact wavenumbers along the propagating rays. New theoretical  
14 results, numerical tools and proxies are derived to describe ray and wave action distributions.  
15 All model parameters can robustly be calibrated from the large-scale flow component only.  
16 We illustrate our purpose with ocean surface gravity waves propagating in different types  
17 of surface currents. The multiscale solution is demonstrated to efficiently document wave  
18 trapping effects by intense jets.

19 **Key words:**

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20 **MSC Codes** 76B15, 76M35, 82B31, 82C31, 37H10, 37L55

## 21 **1. Introduction**

22 This paper aims to revisit the ray-path concept for fast waves propagating over heterogeneous  
23 turbulent flows. Considering ocean surface wave propagation, many authors have already  
24 discussed the random changes of rays subject to a random current (Voronovich 1991; White  
25 & Fornberg 1998; Smit & Janssen 2019), and consequences on wave action distributions.  
26 Closures have been derived in the Eulerian setting (Bal & Chou 2002; Klyatskin & Koshel  
27 2015; Borcea *et al.* 2019; Kafiabad *et al.* 2019; Bôas & Young 2020; Garnier *et al.* 2020).  
28 Some of these approaches can be traced back to wave-wave interactions models, e.g.  
29 McComas & Bretherton (1977) (see also Kafiabad *et al.* 2019, and reference therein). In  
30 most cases, the central assumption is either delta-correlated turbulence (Voronovich 1976;  
31 Klyatskin 2005; Klyatskin & Koshel 2015) and/or fast waves in comparison to fluid flow

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32 velocities (White & Fornberg 1998; Dysthe 2001; Bal & Chou 2002; Borcea *et al.* 2019;  
 33 Kafiabad *et al.* 2019; Smit & Janssen 2019; Bôas & Young 2020; Garnier *et al.* 2020; Boury  
 34 *et al.* 2023; Wang *et al.* 2023). Medium variations may be slow and delta-correlations are  
 35 hardly justifiable in a fixed frame. Though, attached to a fast propagating wave group, the  
 36 medium may seem to vary rapidly, and the delta-correlation assumption makes more sense.  
 37 Another common assumption is frozen turbulence. In such a case, weak currents also implies  
 38 conservation along ray of intrinsic frequency, wavenumber, and group velocity magnitude in  
 39 two dimension (Boury *et al.* 2023). Subsequently, most of wave dynamics closures neglect  
 40 variations and diffusion of frequency or wavenumber.

41 The diffusion of the wave action at large distance with a multiscale decomposition of the  
 42 current has already been reported Bal & Chou (2002). However, an explicit formulation for  
 43 the diffusivity has solely been derived for a zero large-scale current. More generally, fast  
 44 wave models mostly rely either on zero or constant current components at larger scales. West  
 45 (1978), for instance, discussed acoustic waves in a two-component random media, but no  
 46 velocity was involved.

47 Hereafter, the proposed two-scale velocity decomposition falls into the family of stochastic  
 48 transport models (Kunita 1997; Mikulevicius & Rozovskii 2004; Resseguier *et al.* 2020a;  
 49 Zhen *et al.* 2023), including dynamics under Location Uncertainty (LU) (Mémmin 2014;  
 50 Resseguier *et al.* 2017a) and Stochastic Advection by Lie Transport (SALT) (Holm 2015).  
 51 Under this framework, resulting conservative stochastic closures build on time delta-  
 52 correlation for the small-scale turbulence component (Cotter *et al.* 2017). Nonlinear wave  
 53 Hamiltonian dynamics and wave influence on currents (e.g. stokes drift) have then been  
 54 derived (e.g. Crisan & Holm 2018; Bauer *et al.* 2020; Holm 2021; Holm & Luesink 2021;  
 55 Dinvey & Mémmin 2022; Holm *et al.* 2023). Considering a single-wavevector current, solutions  
 56 for a monochromatic shallow water wave were developed Mémmin *et al.* (2022). In the present  
 57 study, our objective is restricted to the influence of turbulent flows on linear waves.

58 After first recalling the principle of ray tracing method, we present the multiscale  
 59 framework for fast waves dynamics, its physical ground and a calibration method for the  
 60 closure. Simplified stochastic equations are then derived for ray dynamics and wave action  
 61 spectrum, in both Lagrangian and Eulerian settings. For illustrative examples, numerical  
 62 tools, analytic models and proxies are applied to ocean surface gravity waves propagating  
 63 through two types of 2D turbulent flows: a typical slow homogeneous turbulence and a jet  
 64 case.

## 65 2. Characteristics of wave-packet rays

66 Isolating a single progressive group of quasi-regular wave train, it follows a form  
 67  $h(\mathbf{x}, t)e^{i\phi(\mathbf{x}, t)} + \text{c.c.}$ , for most properties. If a packet is to be followed, the phase,  $\phi(\mathbf{x}, t)$ ,  
 68 must smoothly vary along the propagation, i.e.  $\phi(\mathbf{x}, t)$  is differentiable. The relative frequency  
 69 is then  $\omega = -\partial_t \phi(\mathbf{x}, t)$ , and the wave number vector  $\mathbf{k} = \nabla \phi(\mathbf{x}, t)$ , with wavenumber  $k = \|\mathbf{k}\|$   
 70 and direction given by the normalized wave-vector,  $\tilde{\mathbf{k}} = \mathbf{k}/k = \begin{pmatrix} \cos \theta_k \\ \sin \theta_k \end{pmatrix}$ . To first order, such  
 71 a train of waves is dispersive and the intrinsic frequency reads

$$72 \quad \omega - \mathbf{k} \cdot \mathbf{v} = \omega_0 = \begin{cases} \text{cst. } \frac{1}{\alpha} k^\alpha, & \alpha \neq 0 \\ \text{cst. } \log(k), & \alpha = 0 \end{cases} \quad (2.1)$$

73 and propagates with its group velocity  $\mathbf{v}_g = \nabla_{\mathbf{k}}\omega$ , constantly modified by the local velocity  
74 of the currents  $\mathbf{v}$ ,

$$75 \quad \frac{d\mathbf{x}_r}{dt} = \mathbf{v}_g = \mathbf{v}_g^0 + \mathbf{v}, \quad (2.2)$$

76 where  $\mathbf{x}_r$  is the centroid of a wave group,  $\mathbf{v}_g^0 = \frac{\partial\omega_0(k)}{\partial k}\tilde{\mathbf{k}}$  is the group velocity without  
77 currents, i.e. solely depending on the wave vector. For  $\alpha = 1$ , the medium is non-dispersive  
78 (e.g. acoustic waves).  $\alpha = 1/2$  corresponds to gravity waves over deep ocean ( $\omega_0 = \sqrt{gk}$ ).  
79 The dominant wave-vector  $\mathbf{k}$  within the group evolves according to

$$80 \quad \frac{d\mathbf{k}}{dt} = -\nabla_{\mathbf{v}}\mathbf{k}. \quad (2.3)$$

81 Equations (2.2)-(2.3) are the Hamilton's eikonal equations. Along the propagating ray,  
82 velocity gradients induce linear variations. Decelerating currents will, for instance, shorten  
83 waves, and reduce the group velocity. Traveling over fields of random velocities  $\mathbf{v}$ , the wave-  
84 vector  $\mathbf{k}$  will also become randomly distributed. Scattering of ocean surface wave packets  
85 by random currents can generally be assumed to be weak, with  $\|\mathbf{v}\|$  of order  $0.5 \text{ m}\cdot\text{s}^{-1}$ ,  
86 much smaller than  $v_g^0 = \|\mathbf{v}_g^0\|$  of order  $10 \text{ m}\cdot\text{s}^{-1}$ . Yet, cumulative effects of these random  
87 surface currents can lead to strong convergence or divergence between initially nearby ray  
88 trajectories.

89 To complete the wave field description, the wave action,  $A(\mathbf{x}, t) = h^2(\mathbf{x}, t)/\omega_0(k(\mathbf{x}, t))$ ,  
90 is considered to be an adiabatic invariant in absence of source terms. Wave action is then  
91 crucial to anticipate wave transformations by currents (White 1999). Unlike wave energy,  
92 wave action is conserved, in the absence of wave generation or dissipation. This action is  
93 the integral over wave-vectors of the action spectrum,  $N$ , also related to the wave energy  
94 spectrum,  $E$ :

$$95 \quad A(\mathbf{x}, t) = \int d\mathbf{k} N(\mathbf{x}, \mathbf{k}, t) = \int d\mathbf{k} \frac{E(\mathbf{x}, \mathbf{k}, t)}{\omega_0(k, t)}. \quad (2.4)$$

96 Action and energy spectrum quantify action and energy by unit of surface (unit of  $\mathbf{x}$ ) and by  
97 unit of wave-vector surface (unit of  $\mathbf{k}$ ). Consider the  $(\mathbf{x}, \mathbf{k})$  variable change between different  
98 times  $t_i$  and  $t_f$  integrating the characteristic eikonal equations (2.2)-(2.3)

$$99 \quad \begin{pmatrix} \mathbf{x}_r(t_i) \\ \mathbf{k}(t_i) \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{x}_r(t_f) \\ \mathbf{k}(t_f) \end{pmatrix} \quad (2.5)$$

100 By the Liouville theorem and since  $\nabla_{\mathbf{x}} \cdot \frac{d\mathbf{x}_r}{dt} + \nabla_{\mathbf{k}} \cdot \frac{d\mathbf{k}}{dt} = 0$ , the state-space of the "packet-  
101 by-packet" approach (the  $(\mathbf{x}, \mathbf{k})$  space) does not contract nor dilates along time. Therefore,  
102 if wave dissipation is neglected, the wave action spectrum  $N$  is conserved Lavrenov (2013),  
103 i.e.

$$104 \quad N(\mathbf{x}_r(t_i), \mathbf{k}(t_i), t_i) = N(\mathbf{x}_r(t_f), \mathbf{k}(t_f), t_f). \quad (2.6)$$

105 This result is extremely useful because it only involves quantities of the characteristics, i.e.  
106 each Fourier mode can be modified independently of the others. The wave energy spectrum  
107 can be computed from the characteristics

$$108 \quad E(\mathbf{x}_r(t_f), \mathbf{k}(t_f), t_f) = \frac{\omega_0(\mathbf{k}(t_f))}{\omega_0(\mathbf{k}(t_i))} E(\mathbf{x}_r(t_i), \mathbf{k}(t_i), t_i). \quad (2.7)$$

109 starting with an initial incoming wave spectrum  $E(\mathbf{x}_r(t_i), \mathbf{k}(t_i), t_i)$  for every wave-vectors  
110  $\mathbf{k}(t_i)$ , starting from a small set of spatial points  $\mathbf{x}_r(t_i)$ .

### 111 3. A new fast wave assumption

112 Commonly, the Eulerian current  $\mathbf{v}$  is decomposed into a low-frequency large-scale component  
113  $\bar{\mathbf{v}}$  and a transient small-scale unresolved component  $\mathbf{v}'$ :

$$114 \quad \mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}'. \quad (3.1)$$

115 From now, we shall consider divergence-free two-dimensional currents only.

#### 116 3.1. The ray Lagrangian correlation time

117 To better characterize the wave dynamics in such a random environment, the covariance of the  
118 fluid velocity can be evaluated in the wave group frame. To take into account the small-scale  
119 unresolved component  $\mathbf{v}'$ , its Eulerian spatio-temporal covariance is considered, assuming  
120 statistical homogeneity and stationarity for the Eulerian velocity  $\mathbf{v}'_E(t, \mathbf{x}) = \mathbf{v}'(t, \mathbf{x})$

$$121 \quad C_{ij}^{\mathbf{v}'_E}(\delta t, \delta \mathbf{x}) = \mathbb{E} \left( v'_i(t, \mathbf{x}) v'_j(t + \delta t, \mathbf{x} + \delta \mathbf{x}) \right) = \mathbb{E} \left( v'_i(t, \mathbf{x}_r(t)) v'_j(t + \delta t, \mathbf{x}_r(t) + \delta \mathbf{x}) \right) \quad (3.2)$$

122 where  $\mathbf{x}_r$  is solution of (2.2) with an arbitrary initial position  $\mathbf{x}_r^0$ . Then, we define,  $\mathbf{v}'_R(t) =$   
123  $\mathbf{v}'(t, \mathbf{x}_r(t))$ , the Lagrangian velocity along the ray  $\mathbf{x}_r(t)$ . The temporal covariance of the  
124 small-scale component  $\mathbf{v}'$  – in the wave group frame – is the covariance of that Lagrangian  
velocity:

$$125 \quad C_{ij}^{\mathbf{v}'_R}(\delta t) = \mathbb{E} \left( v'_i(t, \mathbf{x}_r(t)) v'_j(t + \delta t, \mathbf{x}_r(t + \delta t)) \right) = C_{ij}^{\mathbf{v}'_E}(\delta t, \mathbf{x}_r(t + \delta t) - \mathbf{x}_r(t)), \quad (3.3)$$

126 Assume a typical isotropic form for this covariance:

$$127 \quad C^{\mathbf{v}'_E}(\delta t, \delta \mathbf{x}) = C \left( \frac{|\delta t|}{\tau_{\mathbf{v}'}} + \frac{\|\delta \mathbf{x}\|}{l_{\mathbf{v}'}} \right), \quad (3.4)$$

128 the covariance can be evaluated in the wave group frame for small time increment  $\delta t$ :

$$129 \quad C^{\mathbf{v}'_R}(\delta t) = C \left( \frac{|\delta t|}{\tau_{\mathbf{v}'}} + \frac{\|\mathbf{x}_r(t' + t) - \mathbf{x}_r(t')\|}{l_{\mathbf{v}'}} \right) = C \left( \left( \frac{1}{\tau_{\mathbf{v}'}} + \frac{\|\mathbf{v}_g\|}{l_{\mathbf{v}'}} \right) |\delta t| + O(\delta t^2) \right). \quad (3.5)$$

130 Therefore,  $\left( \frac{1}{\tau_{\mathbf{v}'}} + \frac{\|\mathbf{v}_g\|}{l_{\mathbf{v}'}} \right)^{-1}$  is the correlation time of  $\mathbf{v}'(t, \mathbf{x}_r(t))$ . The same derivation is valid  
131 for  $\nabla(\mathbf{v}')^\top(t, \mathbf{x}_r(t))$ . For fast waves, the along-ray correlation time of the small-scale velocity  
132 can be approximated by  $l_{\mathbf{v}'}/v_g^0$ . The ratio  $\epsilon$ , between this along-ray correlation time and  
133 the characteristic time of the wave group properties evolution, will then control the time  
134 decorrelation assumption of  $\mathbf{v}'$ :

$$135 \quad \epsilon = \frac{l_{\mathbf{v}'}}{v_g^0} \|\nabla \mathbf{v}'^\top\| \sim \frac{l_{\mathbf{v}'}}{l_v} \frac{\|\mathbf{v}\|}{v_g^0}. \quad (3.6)$$

136 Note the Eulerian small-scale velocity  $\mathbf{v}'$  is not necessarily time uncorrelated, as assumed  
137 in Voronovich (1976); Klyatskin & Koshel (2015). Yet, for small enough  $\epsilon$ , the Lagrangian  
138 small-scale velocity along the ray can be considered time uncorrelated. From the expression  
139 of  $\epsilon$ , such a condition depends upon:

- 140 •  $v_g^0$ , the fast wave group velocity
- 141 •  $\|\mathbf{v}\|$ , often slow but not always negligible compared to the intrinsic wave group,  $v_g^0$ .
- 142 •  $l_{\mathbf{v}'}/l_v$ , related to the separation between large scales  $\bar{\mathbf{v}}$  and small scales  $\mathbf{v}'$ , e.g. the  
143 spatial filtering cutoff of the large-scale velocity  $\bar{\mathbf{v}}$ , but also related to its kinetic energy (KE)  
144 distribution over spatial scales, typically the spectrum slope.

145 This along-ray partially time-decorrelation assumption is less restrictive than the usual  
 146 fast wave approximation (White & Fornberg 1998; Dysthe 2001; Bal & Chou 2002; Borcea  
 147 *et al.* 2019; Kafiabad *et al.* 2019; Smit & Janssen 2019; Bôas & Young 2020; Garnier *et al.*  
 148 2020; Boury *et al.* 2023; Wang *et al.* 2023) – say  $\frac{\|\mathbf{v}\|}{v_g^0} \ll 1$  – and than the SALT-LU time-  
 149 decorrelation used for turbulence dynamics (Mémín 2014; Holm 2015; Cotter *et al.* 2017;  
 150 Resseguier *et al.* 2020a) – say  $\frac{l_{v'}}{l_v} \ll 1$ . Similarly, this last validity criterion can be obtained  
 151 replacing in (3.2)-(3.6)  $\mathbf{x}_r$  by the fluid particle Lagrangian path  $\mathbf{x}$  (solution of  $\frac{d\mathbf{x}}{dt} = \mathbf{v}$ )  
 152 and thus  $\mathbf{v}_g^0$  by  $\mathbf{v}$ . These asymptotic models often rely on averaging or homogenization  
 153 techniques (Papanicolaou & Kohler 1974; White & Fornberg 1998) to derive Markovian  
 154 dynamics involving various types of diffusivity.

### 155 3.2. Ray absolute diffusivity and turbulence statistics: calibration

156 Diffusivity is a natural tool to specify statistics of uncorrelated random media. For waves in  
 157 random media, we shall specify multi-point statistics, and the Fourier space is convenient for  
 158 this purpose. We will first present scalar diffusivity and then distribute it over spatial scales  
 159 to fully calibrate the random velocity  $\mathbf{v}'$ .

160 The absolute diffusivity (or Kubo-type formula) usually corresponds, in the so-called  
 161 diffusive regime, to the variance per unit of time of a fluid particle Lagrangian path  $\frac{d\mathbf{x}(t)}{dt} =$   
 162  $\mathbf{v}_L(t) = \mathbf{v}(t, \mathbf{x}(t))$ . It is approximately equal to the velocity variance times its correlation time.  
 163 The Eulerian velocity covariance (3.4) will thus induce an absolute diffusivity (Piterbarg &  
 164 Ostrovskii 1997; Klyatskin 2005)

$$165 \quad \frac{1}{2}a^L = \int_0^\infty d\delta t C^{v'_L}(\delta t) = \int_0^\infty d\delta t C^{v'_E}(\delta t, \mathbf{x}(t+\delta t) - \mathbf{x}(t)) \approx \frac{1}{2}\tau_{v'} C(0). \quad (3.7)$$

166 Along a propagating wave group, a ray absolute diffusivity occurs and slightly differs from  
 167 the usual absolute diffusivity to become

$$168 \quad \frac{1}{2}a^R = \int_0^\infty d\delta t C^{v'_R}(\delta t) \approx \frac{1}{2} \left( \frac{1}{\tau_{v'}} + \frac{\|\mathbf{v}_g\|}{l_{v'}} \right)^{-1} C(0) \approx \frac{1}{2} \frac{l_{v'}}{v_g^0} C(0). \quad (3.8)$$

169 The absolute diffusivity sets the amplitude of the small scale velocity  $\mathbf{v}'$ . To calibrate  
 170 its spatial correlations, we may focus on its Fourier transform,  $\widehat{\mathbf{v}}'(\boldsymbol{\kappa}, t)$ , denoting by  $\boldsymbol{\kappa} =$   
 171  $\boldsymbol{\kappa} \begin{pmatrix} \cos \theta_\kappa \\ \sin \theta_\kappa \end{pmatrix}$ , the surface current wave-vector. By analogy with the current kinetic energy  
 172 spectra  $E_\kappa = \frac{1}{2} \oint_0^{2\pi} d\theta_\kappa \kappa \frac{\|\widehat{\mathbf{v}}(\boldsymbol{\kappa}, t)\|^2}{(2\pi)^2}$ , Resseguier *et al.* (2017b, 2020b) decompose the absolute  
 173 diffusivity scale by scale. Referring it to Absolute Diffusivity Spectral Density (ADSD), they  
 174 defined it by

$$175 \quad \frac{1}{2}A^R(\boldsymbol{\kappa}) = \frac{1}{2} \frac{1/\boldsymbol{\kappa}}{v_g^0(k)} E_k(\boldsymbol{\kappa}), \quad (3.9)$$

176 where  $k$  denotes the wave wavenumber and  $\boldsymbol{\kappa}$  the current wavenumber. To calibrate an  
 177 equivalent noise, we model  $\mathbf{v}'$  by  $\sigma dB_t/dt$ , where  $dB_t/dt$  is a spatio-temporal white noise  
 178 and  $\sigma$  denotes a spatial filtering operator which encodes spatial correlations through  $A_{v'}^R$  and  
 179 the horizontal incompressibility condition ( $\nabla \cdot \sigma = 0$ ). Specifically, we assume an ADSD  
 180 power law,  $A^R(\boldsymbol{\kappa}) \approx A_0^R \boldsymbol{\kappa}^{-\mu}$ , and we choose a divergence-free isotropic spatial filter  $\nabla^\perp \check{\psi}_\sigma$   
 181 such that

$$182 \quad \mathbf{v}' = \sigma dB_t/dt = \nabla^\perp \check{\psi}_\sigma \star dB_t/dt, \quad (3.10)$$

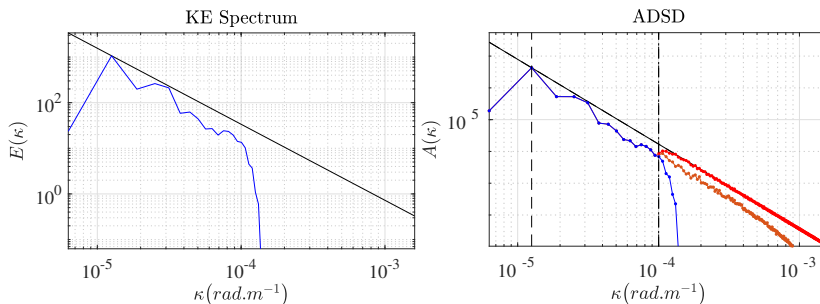


Figure 1: KE spectrum ( $m^2 \cdot s^{-2} / (\text{rad} \cdot m^{-1})$ ) (left) and ADSD ( $m^2 \cdot s^{-1} / (\text{rad} \cdot m^{-1})$ ) (right) of the resolved high-resolution velocity,  $A^R$ , in brown, low-resolution velocity,  $A_{\bar{v}}^R$ , in blue, and modeled stochastic velocity,  $A_{v'}^R(\kappa) = A_0^R \kappa^{-\mu} - A_{\bar{v}}^R(\kappa)$ , in red. For the ADSD power law,  $A^R(\kappa) \approx A_0^R \kappa^{-\mu}$ , we impose the theoretical KE spectrum slope  $-\frac{5}{3}$  (black solid line), coherently with homogeneous SQG dynamics (see "Numerical results" section). The residual ADSD (red line) is set to extrapolate that power law at small scales.

183 and  $\frac{\partial i \kappa^3}{(2\pi)^2} |\widehat{\psi}_{\sigma}(\kappa)|^2 = \frac{1}{2} \oint_0^{2\pi} d\theta_{\kappa} \kappa \frac{\|\widehat{\sigma dB_t}(\kappa)\|^2}{(2\pi)^2 dt} = A_{v'}^R(\kappa)$ . The power law assumption enables  
 184 automatic closure calibration:  $A_{v'}^R(\kappa) = A_0^R \kappa^{-\mu} - A_{\bar{v}}^R(\kappa)$ , from instantaneous large-scale  
 185 current statistics  $A_{\bar{v}}^R$  only (Resseguier *et al.* 2020b) as illustrated in figure 1.

#### 186 4. Statistical wave dynamics

187 In a stochastic framework, the Stratonovich or Itô notations can both be used (Kunita  
 188 1997; Oksendal 1998). Under Stratonovich calculus rules, expressions become similar  
 189 to deterministic ones. Specifically, stochastic versions of linearized dynamical equations  
 190 are obtained by replacing  $\mathbf{v}$  by  $\bar{\mathbf{v}} + \sigma \circ dB_t/dt$ . Then, the stochastic transport of phase,  
 191  $\frac{d}{dt} \phi = \omega_0(\|\nabla \phi\|)$ , i.e. – up to that velocity replacement – the Stratonovich dispersion  
 192 relation is exactly (2.1). The method of characteristics also applies. Note, one can switch  
 193 from Stratonovich to Itô notations, where  $\mathbf{v}'$  corresponds to  $\sigma dB_t/dt$ . The characteristics  
 194 equations (2.2)-(2.3) also remain unchanged for homogeneous and isotropic  $\mathbf{v}'$ :

$$195 \begin{cases} d\mathbf{x}_r = (\mathbf{v}_g^0 + \bar{\mathbf{v}})dt + \sigma dB_t, \\ d\mathbf{k} = -\nabla(\bar{\mathbf{v}}dt + \sigma dB_t)^{\top} \mathbf{k}. \end{cases} \quad (4.1)$$

##### 196 4.1. Single-ray stochastic differential equations

197 When studying a single ray in an homogeneous and isotropic turbulence (3.10), the wave-  
 198 vector dynamics simplifies. In the local crest-oriented frame, the influence of small scale  
 199 currents can be solely represented by four one-dimensional white noise forcings.

200 Notably, dynamics of wave-vectors (2.3) are similar to tracer gradient dynamics (Bühler  
 201 2009; Plougonven & Zhang 2014). Only the coupled ray path dynamics (2.2) differs.  
 202 Accordingly, we follow the notations and derivations of the mixing analysis from Lapeyre  
 203 *et al.* (1999) and references therein. Without loss of generality, the large-scale velocity can  
 204 be parameterized as

$$205 \bar{\mathbf{v}} = \bar{v} \begin{pmatrix} \cos \bar{\theta} \\ \sin \bar{\theta} \end{pmatrix} \quad \text{and} \quad \nabla \bar{\mathbf{v}}^{\top} = \frac{1}{2} \begin{bmatrix} \bar{\sigma} \sin 2\bar{\phi} & \bar{\omega} + \bar{\sigma} \cos 2\bar{\phi} \\ -\bar{\omega} + \bar{\sigma} \cos 2\bar{\phi} & -\bar{\sigma} \sin 2\bar{\phi} \end{bmatrix}. \quad (4.2)$$

206 The influence of the large-scale currents on the ray dynamics (4.1), expressed in the local  
 207 crest-oriented frame  $(\bar{\mathbf{k}}, \bar{\mathbf{k}}^{\perp})$ , is straightforward (Lapeyre *et al.* 1999). The small-scale

208 currents force the ray dynamics through a stochastic noise. For a single ray  $(\mathbf{x}_r, \mathbf{k}) =$   
 209  $(x_r, y_r, k \cos \theta_k, k \sin \theta_k)$ , this noise can be rigorously described by four independent one-  
 210 dimensional white noises only (see Appendix),  $\dot{B}_t^{(1)}$ ,  $\dot{B}_t^{(2)}$ ,  $\dot{B}_t^{(3)}$ , and  $\dot{B}_t^{(4)}$ , and:

$$211 \quad \frac{d}{dt}x_r = v_g^0 \cos \theta_k + \bar{v} \cos \bar{\theta} + \sqrt{a_0} \dot{B}_t^{(1)}, \quad (4.3)$$

$$212 \quad \frac{d}{dt}y_r = v_g^0 \sin \theta_k + \bar{v} \sin \bar{\theta} + \sqrt{a_0} \dot{B}_t^{(2)}, \quad (4.4)$$

$$213 \quad \frac{d}{dt} \log k = -\bar{\sigma} \sin(\zeta) + \gamma_0 + \sqrt{\gamma_0} \dot{B}_t^{(3)}, \quad (4.5)$$

$$214 \quad \frac{d}{dt} \theta_k = \frac{1}{2}(\bar{\omega} - \bar{\sigma} \cos(\zeta)) + \sqrt{3\gamma_0} \dot{B}_t^{(4)}, \quad (4.6)$$

215 where  $\zeta = 2(\theta_k + \bar{\phi})$  and

$$216 \quad a_0 = \frac{1}{2dt} \mathbb{E} \|\sigma dB_t\|^2 = \int_0^{+\infty} A_{v'}^R(k) dk, \quad (4.7)$$

$$217 \quad \gamma_0 = \frac{1}{8dt} \mathbb{E} \|\nabla_{\mathbf{x}}(\sigma dB_t)\|^2 = \frac{1}{4} \int_0^{+\infty} k^2 A_{v'}^R(k) dk \quad (4.8)$$

218 Diffusivity constants depend through (3.9) on both the correlation length and the spectrum  
 219 slope of the small-scale velocity. Contrasting the classical fast wave approximation, the  
 220 wavenumber does vary. This is due to (i) the finite large-scale strain rate,  $\bar{\sigma}$ , and (ii) the  
 221 small-scale isotropic velocity model (3.10). This isotropy assumption and its implication are  
 222 discussed in Appendix. Note that neither large-scale nor small-scale component is assumed  
 223 to be steady, even though the small-scale time variations are mainly driven by the wave  
 224 speed and not by their Eulerian correlation time. The current unsteadiness can also lead  
 225 to wavenumber variations (Dong *et al.* 2020; Cox *et al.* 2023; Boury *et al.* 2023). Given a  
 226 known wavevector angle, it leads to a wavenumber evolution

$$227 \quad k(t) = k(0) \exp\left(-\int_0^t \bar{\sigma} \sin(2(\theta_k + \bar{\phi}))\right) \exp\left(\gamma_0 t + \sqrt{\gamma_0} B_t^{(3)}\right), \quad (4.9)$$

228 and hence the complete wavevector distribution, i.e. the wave spectrum. The second  
 229 exponential factor in (4.9) is a geometric Brownian motion. Its mean diverges in time  
 230 exponentially rapidly. Physically, shear and strain of  $\mathbf{v}'$  tends to shorten the wavelength  
 231 (Voronovich 1991; Boury *et al.* 2023) leading to this divergence. This factor has a log-normal  
 232 distribution, suggesting possible extreme transient wavenumber events. This generalizes  
 233 previous results Voronovich (1991); Klyatskin & Koshel (2015), obtained with neglecting  
 234 the time-correlated current component,  $\bar{\mathbf{v}}$ .

235 For completeness, the action distribution over space and wave vector can be derived. Some  
 236 approaches consider finite-size wave trains either through additional equations (Jonsson  
 237 1990; White & Fornberg 1998) or re-meshing (Hell *et al.* in preparation). Otherwise,  
 238 each ray transports its action spectrum (2.6) and we need to numerically combine many  
 239 rays (Lavrenov 2013), or rely on analytic approximations. Typically, we solve (4.3)-(4.5)  
 240 exhibiting,  $p(\mathbf{x}, \mathbf{k} | \mathbf{x}_r^0, \mathbf{k}_r^0, t)$ , the distribution of the ray  $(\mathbf{x}, \mathbf{k})$  at time  $t$  given initial conditions  
 241  $(\mathbf{x}_r^0, \mathbf{k}^0)$ . Then, by analogy with tracers in incompressible turbulence (Piterbarg & Ostrovskii  
 242 1997, equation (1.31)) we can evaluate the wave action spectrum mean – or any point-wise  
 243 statistics – as follows

$$244 \quad \mathbb{E}N(\mathbf{x}, \mathbf{k}, t) = \iint d\mathbf{x}_r^0 d\mathbf{k}^0 N^0(\mathbf{x}_r^0, \mathbf{k}_r^0) p(\mathbf{x}, \mathbf{k} | \mathbf{x}_r^0, \mathbf{k}^0, t) \quad (4.10)$$

245 where  $N^0$  is the initial wave action spectrum. Integrating this expression over wavevectors,



246 we note that the wave action mean solely depends of group positions distribution

$$247 \quad \mathbb{E}A(\mathbf{x}, t) = \iint d\mathbf{x}_r^0 d\mathbf{k}^0 N^0(\mathbf{x}_r^0, \mathbf{k}_r^0) p(\mathbf{x}|\mathbf{x}_r^0, \mathbf{k}^0, t) \quad (4.11)$$

248 Multi-point action statistics – e.g. focusing  $\mathbb{E}\|\nabla_x A\|^2$  – rely on multi-ray correlations,  
 249 encoded in the stochastic characteristic equations (4.1), but not the simplified model (4.3)-  
 250 (4.6). Alternatively, Eulerian descriptions of wave action dynamics directly provide action  
 251 distribution over space and wave vector.

#### 252 4.2. Eulerian dynamics and action diffusion

253 Wave action spectrum is transported along a 4-dimensional volume-preserving stochastic  
 254 flow (4.1). Again by analogy with incompressible turbulence (Resseguier *et al.* 2017a), the  
 255 stochastic transport of wave action spectrum in Itô notations reads

$$256 \quad \begin{aligned} \partial_t N + (\mathbf{v}_g^0 + \bar{\mathbf{v}} + \sigma \frac{dB_t}{dt}) \cdot \nabla_x N + \left( -\nabla_x (\bar{\mathbf{v}} + \sigma \frac{dB_t}{dt})^\top \mathbf{k} \right) \cdot \nabla_k N \\ 257 \quad = \begin{bmatrix} \nabla_x \\ \nabla_k \end{bmatrix} \cdot \left( \mathbf{D} \begin{bmatrix} \nabla_x \\ \nabla_k \end{bmatrix} N \right) = \frac{1}{2} a_0 \Delta_x N + \frac{1}{2} \gamma_0 \frac{1}{k} \partial_k \left( k^3 \partial_k N \right) + \frac{3}{2} \gamma_0 \partial_{\theta_k}^2 N. \end{aligned} \quad (4.12)$$

258 The RHS is reminiscent to Eq. (3.16) in Bôas & Young (2020) and Eq. (36) in Smit &  
 259 Janssen (2019), and more generally to rapid wave models. Nevertheless, equation (4.12) is  
 260 not averaged and explicitly involves large-scale currents and noise terms (terms with factor  
 261  $\frac{dB_t}{dt}$ ). Differences with Smit & Janssen (2019); Bôas & Young (2020) for the diffusivity  
 262 estimates and the detailed computation of the  $4 \times 4$  diffusion matrix  $\mathbf{D}$  can be found in  
 263 Appendix. Itô notations of (4.12) explicitly separate mean terms (e.g. diffusion terms) and  
 264 zero-mean noise terms. Here, the Eulerian Itô notations reveal that coefficients  $\frac{1}{2}a_0$ ,  $\frac{1}{2}\gamma_0$ , and  
 265  $\frac{3}{2}\gamma_0$  act to diffuse wave action in space, wavenumber and wave-vector angle, respectively.

### 266 5. Numerical experiments

267 To illustrate these developments, we consider ocean surface gravity waves propagating over  
 268 a dynamical flow region. Ray tracing through synthetic surface currents will provide a  
 269 benchmark. It will be shown that a broad range of the current scales can be replaced by  
 270 the stochastic parametrization (3.10) without affecting ray scattering and action distribution.  
 271 Theoretical results (4.3)-(4.12) will suggest approximate analytic solutions.

#### 272 5.1. Surface current dynamics

273 Simplified upper ocean dynamics are considered to follow:

$$274 \quad (\partial_t + \mathbf{v} \cdot \nabla) \Theta = 0 \quad \text{with} \quad \mathbf{v} = -\nabla^\perp (-\Delta)^{-\xi} \Theta. \quad (5.1)$$

275 Two extreme cases: the Surface Quasi-Geostrophic dynamics ( $\xi = \frac{1}{2}$ ) (Pierrehumbert  
 276 1994; Lapeyre 2017), abbreviated SQG, and the two-dimensional Euler dynamics ( $\xi = 1$ ),  
 277 abbreviated 2D Euler. SQG is a dynamics with an extreme locality (KE spectrum slope  $-5/3$ )  
 278 whereas 2D Euler has an extreme non-locality (KE spectrum slope  $-3$ ). The objective is to test  
 279 how the proposed closures apply for both dynamics – and possibly beyond validity of rapid  
 280 wave models – to also be useful for any more realistic upper ocean dynamics. Additionally,  
 281 test cases are developed to assess the multiscale stochastic closure in both homogeneous and  
 282 heterogeneous propagating medium. First, surface fast waves travel in a homogeneous and  
 283 isotropic SQG turbulence, then waves propagate in a spatially localized 2D Euler turbulence,  
 284 mimicking an oceanic jet. For both SQG and 2D Euler dynamics, a reference simulation is

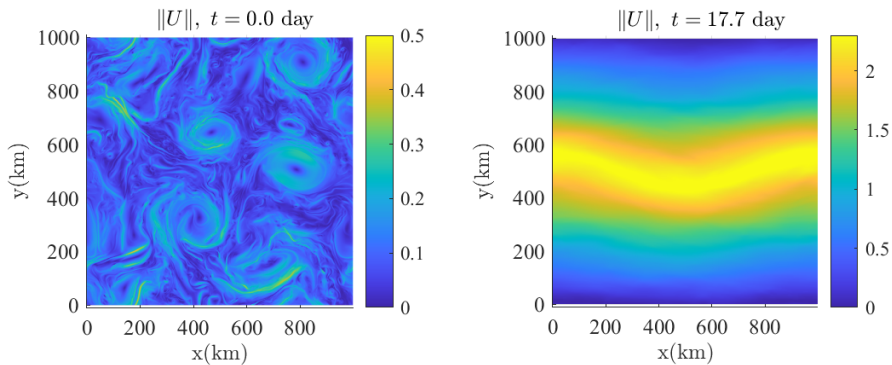


Figure 2: Current velocity norm of the SQG homogeneous turbulence (left) and of the 2D Euler jet current at high-resolution ( $512 \times 512$ ).

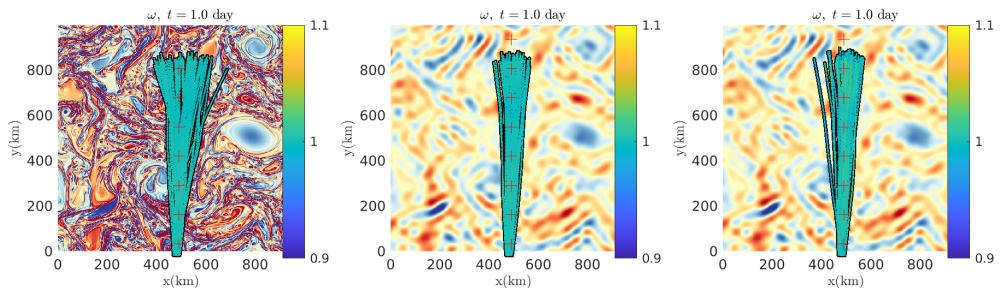


Figure 3: Swell interacting with a high-resolution ( $512 \times 512$ ) deterministic SQG current (left), a low-resolution ( $32 \times 32$ ) deterministic SQG current (middle) and a low-resolution ( $32 \times 32$ ) deterministic SQG current plus (one realization of) the time-uncorrelated stochastic current (right) – colored by the corresponding wave amplitude (right-hand side colorbar) – computed by forward advection and superimposed on the current vorticity  $\omega = \nabla^\perp \cdot \mathbf{v}$ . The red cross indicate where the bidirectional wave spectra of figure 4 are computed.

285 obtained at a resolution  $512 \times 512$  for a 1000-km squared domain, through a pseudo-spectral  
 286 code (Resseguier *et al.* 2017b, 2020b). Once initialized, the current velocity  $\mathbf{v}$  is about  $0.1$   
 287  $\text{m}\cdot\text{s}^{-1}$  for the homogeneous turbulence and  $1 \text{ m}\cdot\text{s}^{-1}$  for the jet (see figure 2).

288

### 5.2. Rays scattering in homogeneous SQG turbulence

289 A wave system enters the bottom boundary, propagating to the top. The carrier incident wave  
 290 has an intrinsic wave group velocity of  $10 \text{ m/s}$ , i.e. wavelength  $\lambda = 250 \text{ m}$ . Its envelope  
 291 is Gaussian with an isotropic spatial extension of  $30\lambda$ . The left panels of figures 3 and 4  
 292 illustrates the resulting branched regime, spreading the wavevectors (figure 4) of the incoming  
 293 waves. From bottom to top, spectral diffusion occurs (figure 4), in the direction orthogonal  
 294 (here  $k_x$ ) to the propagation (here  $k_y$ ), in line with equation (4.6). This accelerates – along  
 295 the propagation – the wave position spread (figure 3). This acceleration is explained by the  
 296 ray equation (4.3) dominated by the intrinsic wave group velocity.

297 To mimic a badly resolved  $\bar{\mathbf{v}}$  field,  $\mathbf{v}$  is smoothed at a resolution  $32 \times 32$ . Using this coarse-  
 298 scale current, middle panels of figures 3 and 4, the branched regime is strongly depleting,  
 299 i.e. the spectral small-scale turbulence diffusion is missing. A stochastic current  $\mathbf{v}'$  is then

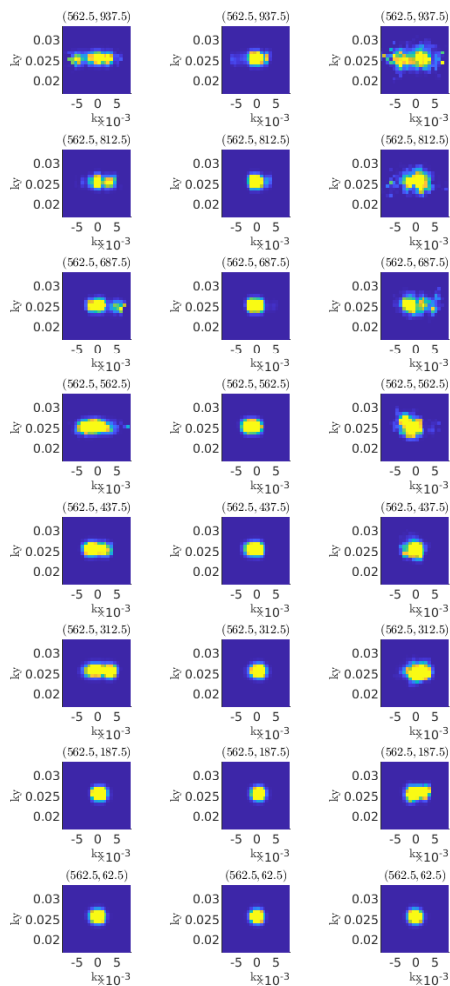


Figure 4: Bidirectional wave spectra, computed by backward advection, at 8 locations along a vertical axis (the mean wave propagation direction) resulting from a swell interacting with a high-resolution ( $512 \times 512$ ) deterministic SQG current (left), a low-resolution ( $32 \times 32$ ) deterministic SQG current (middle) and a low-resolution ( $32 \times 32$ ) deterministic SQG current plus (one realization of) the stochastic model (3.10) (right). The spatial locations where the spectra are calculated are specified on top of each image. We point out these locations on figure 3 by the red crosses.

300 added for ray tracing (4.1). This stochastic component is divergence-free and has a self-  
 301 similar distribution of energy across spatial scales (3.10) (see figure 1). The resulting spatial  
 302 and spectral spreads are now comparable to simulations with high-resolution currents. For  
 303 this setting, the stochastic closure provides satisfying results for a sufficiently well-resolved  
 304 large-scale current. The key decorrelation ratio  $\epsilon = \frac{l_{v'}}{l_v} \frac{\|\bar{v}\|}{v_g^0}$  indeed depends on the resolution  
 305 through  $l_{v'}$ . The large-scale current  $\bar{v}$  is resolved on a  $32 \times 32$  grid, i.e. with a resolution

306  $l_{v'} = \frac{\|\nabla \mathbf{v}'\|}{\|\nabla \mathbf{v}'^T\|} l_v = 0.33 l_v$ . As such  $\epsilon = 4.1 \times 10^{-3}$ , computed with  $v_g^0 \simeq 10 \text{ m.s}^{-1}$  and  
 307  $\|\mathbf{v}'\| \simeq 0.12 \text{ m.s}^{-1}$ , so  $\frac{\|\mathbf{v}'\|}{v_g^0} \simeq 1.2 \times 10^{-2}$ , which is sufficiently small to make the proposed  
 308 model applicable.

309 From the ADSD estimate (equation (3.9) illustrated by figure 1) and (4.7)-(4.8), evaluations  
 310 of the diffusivity coefficients  $a_0$  and  $\gamma_0$  are straightforward. Previously discussed Smit &  
 311 Janssen (2019), the spatial diffusivity is extremely weak:  $a_0 = 6.4 \times 10^{-1} \text{ m}^2.\text{s}^{-1}$  (spatial  
 312 variations in ray equations (4.3)-(4.4) of about  $\sqrt{a_0 t} = 230 \text{ m}$  during 1 day). In contrast, the  
 313 spectral angle diffusivity is large:  $3\gamma_0 = 3.0 \times 10^{-8} \text{ rad}^2.\text{s}^{-1}$ . Along our 1-day simulation,  
 314 neglecting large-scale velocity influence, (4.6) leads to a Brownian wave vector angle  
 315 variations  $\delta\theta_k = \theta_k - \theta_k(0) = \sqrt{3\gamma_0} B_t^{(4)}$  with a standard deviation  $\sigma_{\delta\theta_k} = \sqrt{3\gamma_0 t} =$   
 316  $5.2 \times 10^{-2} \text{ rad} \approx 3.0^\circ$ , eventually increasing the wave group spectral maximal extension from  
 317  $\pm 2\sigma_{k_x} = \pm 2 \frac{2\pi}{30\lambda} = \pm 1.7 \times 10^{-3} \text{ rad.m}^{-1}$  to  $\pm 2\sigma_{k_x} \approx \pm 2 \sqrt{\left(\frac{2\pi}{30\lambda}\right)^2 + (k\sigma_{\delta\theta_k})^2} = \pm 3.1 \times 10^{-3}$   
 318  $\text{rad.m}^{-1}$ , confirmed by figure 4. This figure also illustrates the wave action diffusion induced  
 319 by diffusivity  $\gamma_0$ , well predicted by the Eulerian wave action model (4.12). In the branched  
 320 regime, the increased angle variability leads, by advection, to a spatial spread. The simplified  
 321 ray equation (4.3) gives  $\delta x \approx \int_0^t v_g^0 \cos \theta_k \approx v_g^0 \int_0^t \delta\theta_k \approx v_g^0 \sqrt{3\gamma_0} \int_0^t B_t^{(4)}$  with maximal  
 322 extension  $\pm 2\sigma_x \approx \pm 2v_g^0 \sqrt{\gamma_0 t^3} \approx \pm 52 \text{ km}$ , in agreement with figure 3. Finally, we estimate a  
 323 first-order delay along the propagation  $\delta t = t - (y - y(0))/v_g^0 \approx \int_0^t (1 - \sin \theta_k) \approx \frac{1}{2} \int_0^t \delta\theta_k^2 \approx$   
 324  $\frac{3}{2} \gamma_0 \int_0^t (B_t^{(4)})^2$ , with mean value  $\mathbb{E}\delta t = \frac{3}{4} \gamma_0 t^2$ .

### 325 5.3. Wave groups trapped in a 2D Euler turbulent jet

326 Tests are now performed for rays traveling in fast and strongly heterogeneous 2D Euler flows.  
 327 Classical fast wave models – assuming flows of weak amplitude and often uniform statistics –  
 328 are expected to fail here. Jets exhibit strong current gradients (e.g. Kudryavtsev *et al.* 2017)  
 329 creating strong ray focusing and possibly rogue events. Passing through localized spatial  
 330 structures, caustics can appear but solely from unrealistically collimated wave trains (White  
 331 & Fornberg 1998; Heller *et al.* 2008; Wang *et al.* 2023). Occurrences strongly reduce for  
 332 finite directional spread (Slunyaev & Shrira 2023). Here, wave groups are trapped in a jet, but  
 333 nonlinear wave interactions are neglected. The high-resolution numerical simulations (see  
 334 figure 5) reveal that even linear wave trains are well trapped in adversarial currents. Freund  
 335 & Fleischman (2002) observed a similar behavior for acoustic waves in a 3D turbulent jet.  
 336 Note that during our simulation, rays cross the domain several times (because of the doubly  
 337 periodic boundary conditions, see appendix for technical details). Top (resp. bottom) of the  
 338 jet, the vorticity and thus – at first order – rays curvatures (Dysthe 2001) are negative (resp.  
 339 positive). Therefore, rays oscillate around the jet. A toy model can explain this behavior.  
 340 Following the multiscale stochastic approach (4.3)-(4.6), wave scattering is also taken into  
 341 account.

342 For very-coarsened-grained ( $4 \times 4$ ) current  $\bar{\mathbf{v}}$ , oscillation remains, but most of the scattering  
 343 vanishes, as illustrated by figure 6. Moreover, the curvature of the jet creates artificial wave  
 344 focusing at  $t = 8$  and 10 days. Introducing a time-uncorrelated model (3.10) corrects the  
 345 resolution issue on figure 7. Figure 8 plots the current ADSD. The current is strong ( $\|\mathbf{v}'\| \simeq 1.4$   
 346  $\text{m.s}^{-1}$ ), and the usual fast wave approximation cannot be applied ( $\frac{\|\mathbf{v}'\|}{v_g^0} \simeq 1.2 \times 10^{-1}$ ). However,  
 347 the proposed modified fast wave model is valid, even at the very coarse  $4 \times 4$  resolution.  
 348 Indeed, 2D Euler spectra are steeper than for SQG dynamics, and the length scale ratio is

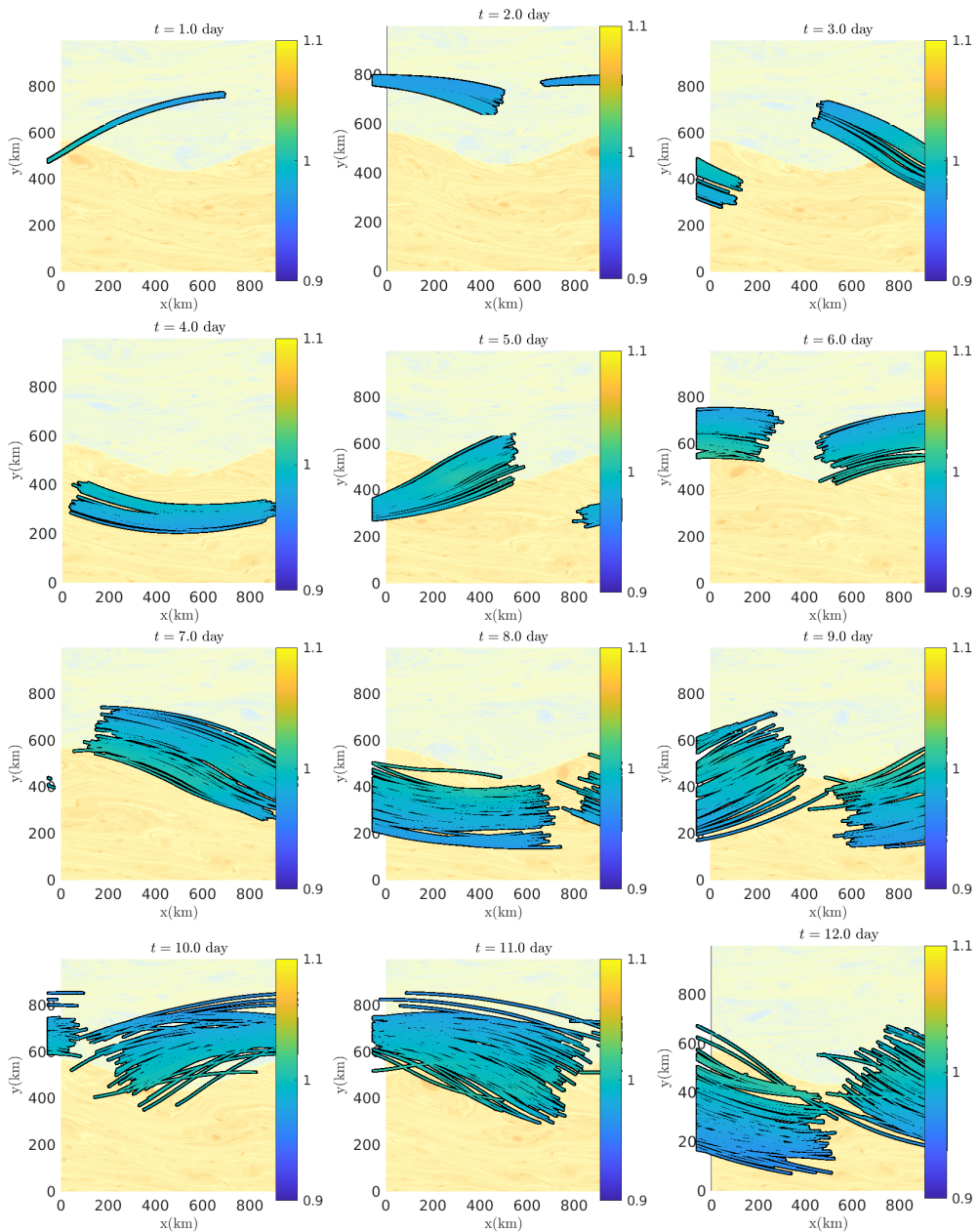


Figure 5: Rays facing a high-resolution ( $512 \times 512$ ) deterministic 2D Euler jet current – colored by the corresponding wave amplitude (right-hand side colorbar) – computed by forward advection and superimposed on the current vorticity  $\omega = \nabla^\perp \cdot \mathbf{v}$ .

349 already significant at this resolution,  $\frac{l_{v'}}{l_v} = 0.14$ , and the derived time-decorrelation ratio is  
 350 small:  $\epsilon = \frac{l_{v'}}{l_v} \frac{\|\mathbf{v}\|}{v_g} = 1.6 \times 10^{-2}$ .

351 Furthermore, by approximating the under-resolved current  $\bar{\mathbf{v}}$ , an analytic stochastic solution  
 352 can be obtained for a ray traveling against the current. The large-scale pattern of the jet takes

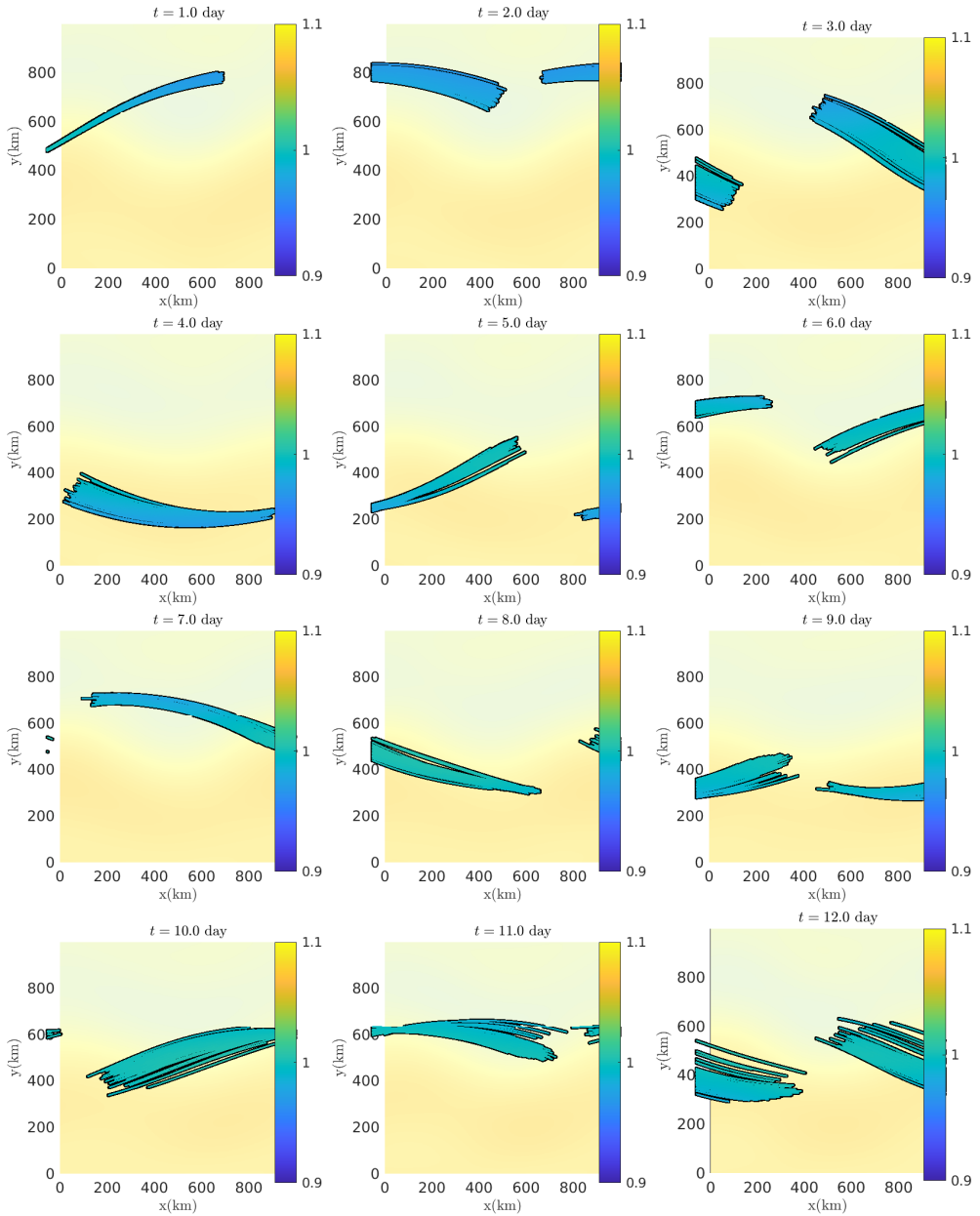


Figure 6: Rays facing a low-resolution ( $4 \times 4$ ) deterministic 2D Euler jet current – colored by the corresponding wave amplitude (right-hand side colorbar) – computed by forward advection and superimposed on the current vorticity  $\omega = \nabla^\perp \cdot \mathbf{v}$ .

353 a quadratic form

$$354 \quad \bar{u} \approx \bar{U}_0 - \frac{1}{2}\bar{\beta} \left( y - \frac{L_y}{2} \right)^2 \quad \text{and} \quad \bar{v} \approx 0, \quad \text{with} \quad \bar{U}_0, \bar{\beta} < 0 \quad (5.2)$$

355 Note, the toy model (5.2) simply considers a straight jet, neglecting its curvature. For weak  
 356 subgrid currents and a ray,  $(x_r, y'_r + \frac{L_y}{2}, k, \theta_k)$ , propagating mainly to the right,  $\theta_k$  is small

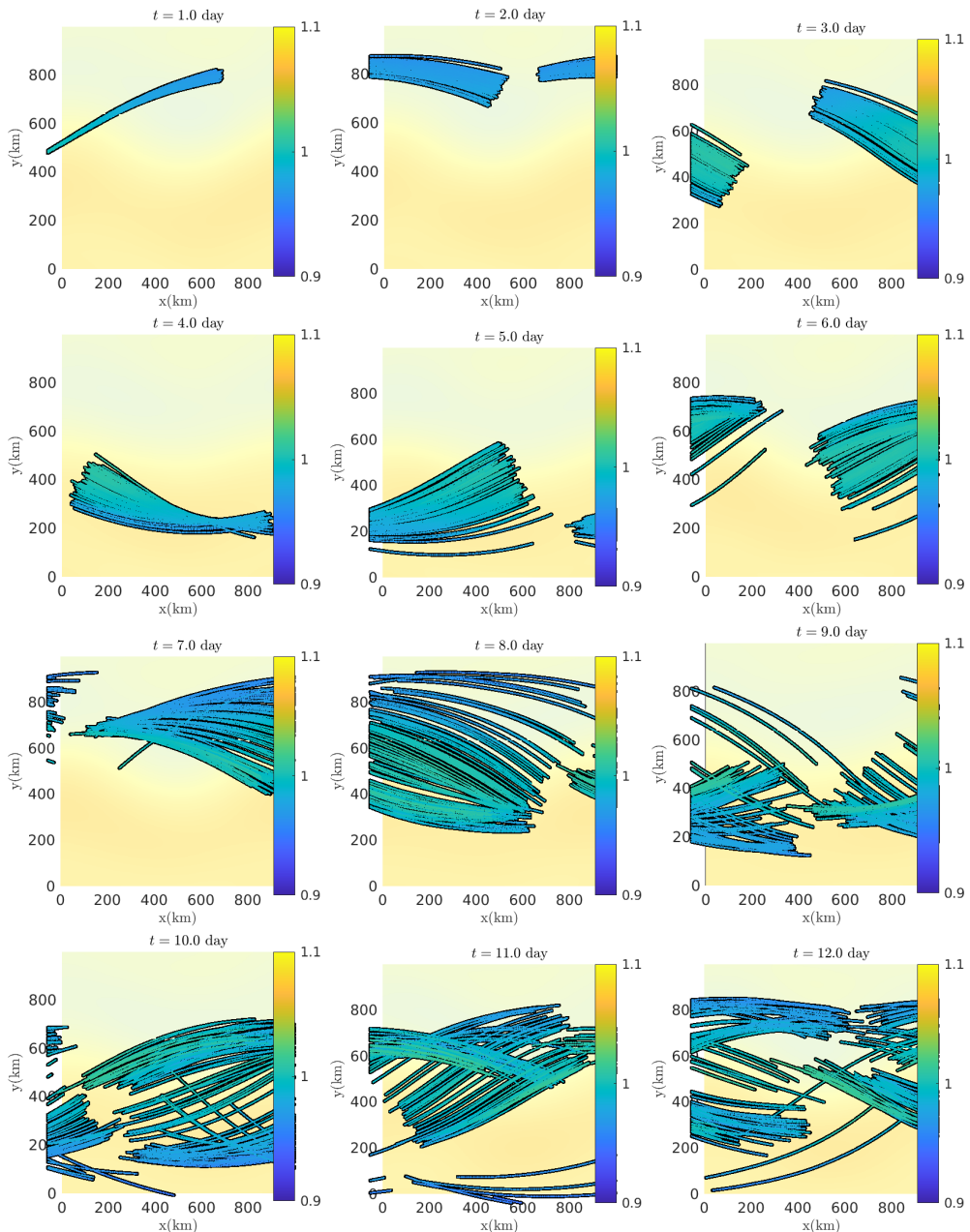


Figure 7: Rays facing a low-resolution ( $4 \times 4$ ) deterministic 2D Euler jet current plus (one realization of) the time-uncorrelated stochastic model – colored by the corresponding wave amplitude (right-hand side colorbar) – computed by forward advection and superimposed on the current vorticity  $\omega = \nabla^{\perp} \cdot \mathbf{v}$ .

357 and the simplified ray equation (4.4) determines the group position with respect to the jet  $y'_r$

358 
$$\frac{d}{dt} y'_r \approx v_g^0 \sin(\theta_k) = v_g^0 \theta_k + O(\theta_k^2). \quad (5.3)$$

359 For frozen turbulence, the wave-number and hence  $v_g^0$  will not significantly vary. The other

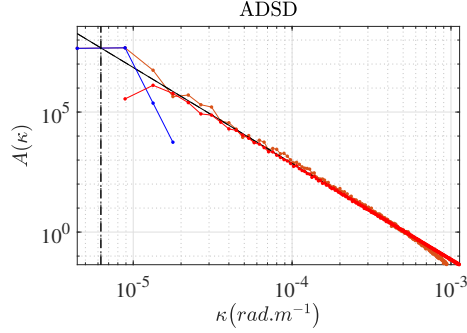


Figure 8: ADSD ( $m^2 \cdot s^{-1} / (\text{rad} \cdot m^{-1})$ ) of the resolved high-resolution jet velocity in brown, low-resolution jet velocity in blue, and modeled stochastic velocity, in red. The theoretical spectrum slope  $-3$  (black solid line) is imposed, consistent with homogeneous 2D Euler dynamics. The residual ADSD (red line) is set to extrapolate that power law at small scales.

360 ray equation (4.3) localizes the group along the jet,  $x_r \approx x_r(0) + (v_g^0 - \bar{u})t$ , dropping the  
 361  $O(\theta_k^2)$  from now on. Moreover,  $\tilde{\mathbf{k}}^\perp \cdot \nabla \bar{\mathbf{v}}^\top \tilde{\mathbf{k}} \approx -\partial_y \bar{u}$  and the dynamics of wave vector angle  
 362 (4.6) simplifies to a stochastic oscillator equation:

$$363 \quad \frac{d^2}{dt^2} y'_r = v_g^0 \frac{d}{dt} \theta_k = -\partial_y (v_g^0 \bar{u}) + v_g^0 \sqrt{3\gamma_0} \dot{B}_t^{(4)} = -\bar{\omega}_r^2 y'_r + v_g^0 \sqrt{3\gamma_0} \dot{B}_t^{(4)}. \quad (5.4)$$

364 with  $\bar{\omega}_r = \sqrt{|v_g^0 \bar{\beta}|}$ . Here  $v_g^0 \bar{u}$  plays the role of a potential, trapping the rays in the jet vicinity,  
 365 whereas the noise accounts for wave scattering. Solution of this linear equation is known  
 366 (e.g. Resseguier *et al.* 2017a, Eq.(51)-(55)):

$$367 \quad y_r(t) = \underbrace{\frac{L_y}{2} + y'_r(0) \cos(\bar{\omega}_r t) + \frac{v_g^0}{\bar{\omega}_r} \theta_k(0) \sin(\bar{\omega}_r t)}_{=\mathbb{E}(y_r(t))} + \underbrace{Y_{\gamma_0} \sqrt{\bar{\omega}_r} \int_0^t \sin(\bar{\omega}_r(t-r)) dB_r^{(4)}}_{=y_r''(t)}, \quad (5.5)$$

368 with  $Y_{\gamma_0} = v_g^0 \sqrt{3\gamma_0 / \bar{\omega}_r^3}$ . The wavevector angle solution is similar. The solution ensemble  
 369 mean,  $\mathbb{E}y_r$ , is a simple coherent deterministic oscillator. This mean solution describes well  
 370 the interaction between the group and the under-resolved current from figure 6. From the  
 371 coarse-scale vorticity shear plotted in figure 9 in the vicinity of the jet, we can estimate  
 372  $\bar{\beta} = -2.7 \times 10^{-11} \text{m}^{-1} \cdot \text{s}^{-1}$ . It yields an oscillation frequency  $\bar{\omega}_r = 1.3 \times 10^{-5} \text{rad} \cdot \text{s}^{-1}$  i.e. a  
 373 period of  $2\pi / \bar{\omega}_r = 5.7$  days, in agreement with the ray tracing simulations. Note that the  
 374 high-resolution vorticity shear, left panel of figure 9, does not suggest any relevant values to  
 375 explain the ray oscillations. Only the proposed multiscale current decomposition provides a  
 376 quantitative explanation for these oscillations, and by extension for trapping rays inside the  
 377 jet. Added to the mean solution, the random parts,  $y_r''(t)$ , are continuous summations of zero-  
 378 mean incoherent wave fluctuations. At each time  $r$ , the additive random forcing introduces  
 379 an oscillation. But, the influence of the past excitations is weighed by sine wave due to the  
 380 phase change. The group position and wavevector angle are Gaussian random variables (as  
 381 linear combinations of independent Gaussian variables). Therefore, their finite dimensional  
 382 law (*i.e.* the multi-time probability density function) are entirely defined by their mean and  
 383 covariance functions. Specifically,

$$384 \mathbb{E}(y_r''(t) y_r''(t + \tau)) = \frac{1}{4} Y_{\gamma_0}^2 (\cos(\bar{\omega}_r \tau) (2\bar{\omega}_r t - \sin(2\bar{\omega}_r t)) + \sin(\bar{\omega}_r \tau) (1 - \cos(2\bar{\omega}_r t))). \quad (5.6)$$



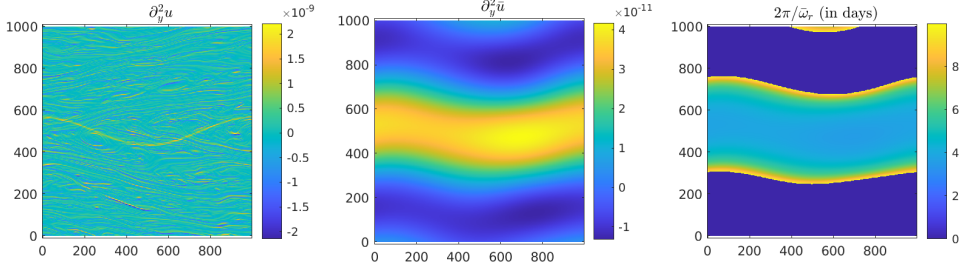


Figure 9: Vorticity shear  $\partial_y^2 u$  of the deterministic 2D Euler jet current at high-resolution ( $512 \times 512$ ) (left), at low-resolution ( $4 \times 4$ ) (middle), and the corresponding swell system period  $2\pi/\bar{\omega}_r$ . Far from the jet ( $\pm 200$  km away), the vorticity shear becomes zero or even positive, so period larger than 10 days are cropped.

385 In particular, the variance of the vertical positions reads  $\sigma_y^2(t) = \frac{1}{4} Y_{\gamma_0}^2 (2\bar{\omega}_r t - \sin(2\bar{\omega}_r t))$ . At  
 386  $t = 2\pi/\bar{\omega}_r$ , the group has oscillated once around the jet and the maximal position extension  
 387 reaches  $\pm 2\sigma_y = \pm 2\sqrt{\pi} Y_{\gamma_0} = \pm 42$  km, well confirmed by ray simulations. In contrast,  
 388 usual fast wave models (e.g. Smit & Janssen 2019) do not consider the interplay between  
 389 smooth and rough currents, and hence solely predict a classical branched regime with a much  
 390 faster vertical location spreading:  $\pm 2\sigma_y = \pm 2\sqrt{(2\pi)^3/3} Y_{\gamma_0} = \pm 217$  km. For large time, our  
 391 multiscale approach predicts a scaling in  $t$ , much slower than the usual  $t^3$  scaling of the  
 392 branched regime.

393 From the group vertical location and wavevector angle, we can also solve (4.5) analytically  
 394 to estimate the group wavenumber variations. For small wavevector angles,  $-\int_0^t \bar{\sigma} \sin(\zeta) \approx$   
 395  $2 \int_0^t \bar{\omega} \theta_k = 2\bar{\beta} \int_0^t y'_r \theta_k$  and (4.9) together with the analytic solutions for  $y'_r$  and  $\theta_k$   
 396 give a closed stochastic expression for the group wavenumber. Thus, the wavenumber  
 397 factor  $\exp(2\bar{\beta} \int_0^t y'_r \theta_k)$  oscillates at frequency  $2\bar{\omega}_r$  and the oscillations modulate the wave  
 398 amplitude:  $h = \sqrt{E} = \sqrt{\omega_0 N} = \text{cst.} k^{\frac{1}{4}}$ . The modulations are associated with wave-current  
 399 energy exchanges Boury *et al.* (2023), visible in the colored rays, figures 5, 6 and 7, when  
 400 the groups enter and exit the jet.

401 Finally, the conditional ray distribution,  $p(\mathbf{x}, \mathbf{k} | \mathbf{x}_r^0, \mathbf{k}^0, t)$ , the action spectrum mean from  
 402 (4.10) and the action mean from (4.11) can all be derived. For a system initially localized  
 403 in  $(0, \frac{L_y}{2})$  with action  $A^0$ , wavenumber  $k^0$  and a  $\sigma_{\delta\theta_k}^0$ -width Gaussian angular spreading,  
 404 propagating to the right, the action mean reads

$$405 \quad \mathbb{E}A(x, y, t) = A^0 \delta\left(x - (v_g^0(k^0) - \bar{u}(y))t\right) \mathcal{N}\left(y - \frac{L_y}{2} \middle| \tilde{\sigma}_y^2(t)\right), \quad (5.7)$$

406 with  $\mathcal{N}\left(\bullet \middle| \tilde{\sigma}_y^2(t)\right)$ , a Gaussian function with variance  $\tilde{\sigma}_y^2(t) = \sigma_y^2(t) + \left(\frac{v_g^0}{\bar{\omega}_r} \sin(\bar{\omega}_r t) \sigma_{\delta\theta_k}^0\right)^2$ .

407 The action is advected in the horizontal direction, and slowly diffuses along the vertical  
 408 direction.

## 409 6. Conclusion

410 Developed to generalize the ray-path concept for waves propagating over an heterogeneous  
 411 turbulence, a practical stochastic framework is derived. For fast waves, the smallest scales  
 412 of a turbulent flow decorrelate along the wave propagation. Flows with steeper spectra

413 decorrelate faster, leading to a broader validity range of fast wave approximations. The  
 414 proposed framework encodes both large-scale refraction and random scattering effects on  
 415 wave statistical properties. The mean wave-action statistics are directly linked to resolved  
 416 strain-rate and vorticity, but also to unresolved KE spectral properties. Both Eulerian and  
 417 Lagrangian views are presented. A convenient calibration method is also proposed for the  
 418 subgrid parametrization.

419 Anticipated, random horizontal currents delay wave arrival and augment the initial  
 420 radiative transport equation with a directional diffusive term. These phenomena are illustrated  
 421 with numerical simulations, analytical solutions, and quantitative proxies describing weak  
 422 homogeneous turbulence. Using these proxies, measured delays in ray arrivals, estimated  
 423 wave energy spectral characteristics and decays, and/or varying directional spread shall then  
 424 be used to more quantitatively interpreted the upper the turbulent underlying flow properties.

425 The generalized fast wave approximation does takes into account wavenumber variation  
 426 and handles strong heterogeneous flows, like localized jets with strong current gradients. As  
 427 compared to numerical simulations, numerical and theoretical results explain and quantify  
 428 ray trapping effects by jets, unlike usual fast wave approaches.

429 Among fast wave literature, isotropic diffusion and, hence, wavenumber diffusion may  
 430 (e.g. Voronovich 1991) or may not (e.g. Bôas & Young 2020) comes into play (see last  
 431 Appendix for details). Future works could adapt our convenient stochastic calculus framework  
 432 to the second models family. Besides, further analytical developments could consider finite-  
 433 size wave groups, their dynamics (Jonsson 1990; White & Fornberg 1998) and statistical  
 434 distributions, or alternatively the Eulerian action dynamics (4.12) with all its multi-point  
 435 stochastic structure. When achieved, this next theoretical development could provide new  
 436 means to analyze wave dynamics with subsequent fast simulations of ensembles. Beside  
 437 comprehension and analysis, our stochastic simulation tools aim at eventually facilitate  
 438 future ensemble-based data assimilation algorithms (Smit *et al.* 2021).

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447 **Author contributions.** VR developed the theory. VR and EH wrote the code and performed numerical  
 448 experiments. VR and BC wrote the paper.

## 449 Appendix A. Stochastic forcing covariance

450 In this appendix, we will compute the conditionnal covariance of the stochastic forcing of  
 451 our eikonal characteristic equations (4.1), that is:

$$452 \quad 2\mathbf{D} \triangleq \frac{1}{dt} \mathbb{E}_t \left\{ \left( \sigma dB_t \right) \left( \sigma dB_t \right)^\top \right\} = \begin{bmatrix} \mathbf{a} & \Sigma_{\eta, \sigma} \\ \Sigma_{\eta, \sigma}^\top & \Sigma_\eta \end{bmatrix} \quad (\text{A } 1)$$

453 where,  $d\eta_t = -\nabla(\sigma dB_t)^\top \mathbf{k}$ , denotes the wave-vector stochastic forcing and,  $\Sigma_\eta dt$ , its  
 454 covariance, and  $\mathbb{E}_t\{\bullet\} \triangleq \mathbb{E}\{\bullet | \mathbf{x}_r(t), \mathbf{k}(t)\}$  stands for the conditional expectation evaluated  
 455 with given characteristics  $(\mathbf{x}_r(t), \mathbf{k}(t))$  at the current time  $t$ . Note that in the appendix we  
 456 use  $\text{It}\bar{o}$  notations only.

457 The subgrid velocity,  $\mathbf{v}' = \sigma d\mathbf{B}_t/dt$  is constructed in Fourier space with a divergence-free  
458 isotropic spatial filter  $\nabla^\perp \psi_\sigma$  (see (3.10)).

$$459 \quad \widehat{\mathbf{v}}'(\boldsymbol{\kappa}, t) = \int d\mathbf{x} \mathbf{v}'(\mathbf{x}, t) e^{-i\mathbf{k} \cdot \mathbf{x}} = \overline{\sigma d\mathbf{B}_t/dt}(\boldsymbol{\kappa}, t) = i\boldsymbol{\kappa}^\perp \widehat{\psi}_\sigma(\boldsymbol{\kappa}) \overline{d\mathbf{B}_t}(\boldsymbol{\kappa})/dt, \quad (\text{A } 2)$$

460 The computation of the variance tensor  $\mathbf{a}$  is classical and straightforward from the definition  
461 of the inverse Fourier transform and the identity  $\mathbb{E} \left\{ \overline{d\mathbf{B}_t}(\boldsymbol{\kappa}_1) \overline{d\mathbf{B}_t}^\top(\boldsymbol{\kappa}_2) \right\} = (2\pi)^2 \delta(\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2) dt$ .  
462 We simply need to split the integral of the stochastic forcing spectrum over the current  
463 wavevector  $\boldsymbol{\kappa} = \kappa(\cos \theta_\kappa, \sin \theta_\kappa)$ :

$$464 \quad \mathbf{a} = \frac{1}{(2\pi)^4 dt} \iint d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_1 \mathbb{E}_t \left\{ \overline{\sigma d\mathbf{B}_t}(\boldsymbol{\kappa}_1, \mathbf{k}) \overline{\sigma d\mathbf{B}_t}^\top(\boldsymbol{\kappa}_2, \mathbf{k}) \right\} e^{i(\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2) \cdot \mathbf{x}}, \quad (\text{A } 3)$$

$$465 \quad = \frac{1}{(2\pi)^2} \int d\boldsymbol{\kappa} \kappa^2 |\widehat{\psi}_\sigma(\boldsymbol{\kappa})|^2 \begin{pmatrix} -\sin \theta_\kappa \\ \cos \theta_\kappa \end{pmatrix} \begin{pmatrix} -\sin \theta_\kappa \\ \cos \theta_\kappa \end{pmatrix}^\top, \quad (\text{A } 4)$$

$$466 \quad = \frac{1}{(2\pi)^2} \int_0^\infty \oint_0^{2\pi} d\kappa d\theta_\kappa \kappa^3 |\widehat{\psi}_\sigma(\boldsymbol{\kappa})|^2 \begin{pmatrix} \sin^2 \theta_\kappa & -\sin \theta_\kappa \cos \theta_\kappa \\ -\sin \theta_\kappa \cos \theta_\kappa & \cos^2 \theta_\kappa \end{pmatrix}, \quad (\text{A } 5)$$

$$467 \quad = \frac{2}{2\pi} a_0 \oint_0^{2\pi} d\theta_\kappa \begin{pmatrix} \sin^2 \theta_\kappa & -\sin \theta_\kappa \cos \theta_\kappa \\ -\sin \theta_\kappa \cos \theta_\kappa & \cos^2 \theta_\kappa \end{pmatrix}, \quad (\text{A } 6)$$

$$468 \quad = a_0 \mathbb{I}_d. \quad (\text{A } 7)$$

469 Now, the Fourier transform of the wave-vector stochastic forcing is

$$470 \quad d\widehat{\boldsymbol{\eta}}_t = -\nabla(\overline{\sigma d\mathbf{B}_t})^\top \mathbf{k} = -i\boldsymbol{\kappa}(i\boldsymbol{\kappa}^\perp \widehat{\psi}_\sigma \overline{d\mathbf{B}_t}) \cdot \mathbf{k} = -\boldsymbol{\kappa}(\mathbf{k}^\perp \cdot \boldsymbol{\kappa}) \widehat{\psi}_\sigma \overline{d\mathbf{B}_t}. \quad (\text{A } 8)$$

471 Then, applying the crest-oriented rotation matrix,  $\mathbf{M}_k = \begin{bmatrix} \tilde{\mathbf{k}} & \tilde{\mathbf{k}}^\perp \end{bmatrix}$ , leads to

$$472 \quad d\widehat{\mathbf{Z}}_t = \mathbf{M}_k^\top d\widehat{\boldsymbol{\eta}}_t = -\begin{pmatrix} \tilde{\mathbf{k}} \cdot \boldsymbol{\kappa} \\ \tilde{\mathbf{k}}^\perp \cdot \boldsymbol{\kappa} \end{pmatrix} (\mathbf{k}^\perp \cdot \boldsymbol{\kappa}) \widehat{\psi}_\sigma \overline{d\mathbf{B}_t} = -\begin{pmatrix} \cos \delta\theta \sin \delta\theta \\ \sin^2 \delta\theta \end{pmatrix} \kappa^2 k \widehat{\psi}_\sigma \overline{d\mathbf{B}_t}, \quad (\text{A } 9)$$

473 with  $\delta\theta = \theta_\kappa - \theta_k$ . From there, we can evaluate the conditional covariance matrix  $\Sigma_Z =$   
474  $\frac{1}{dt} \mathbb{E}_t \{ d\mathbf{Z}_t d\mathbf{Z}_t^\top \}$  of  $d\mathbf{Z}_t$  as before:

$$475 \quad \Sigma_Z = \frac{1}{(2\pi)^4 dt} \iint d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_1 \mathbb{E}_t \left\{ d\widehat{\mathbf{Z}}_t(\boldsymbol{\kappa}_1, \mathbf{k}) \overline{d\widehat{\mathbf{Z}}_t}^\top(\boldsymbol{\kappa}_2, \mathbf{k}) \right\} e^{i(\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2) \cdot \mathbf{x}}, \quad (\text{A } 10)$$

$$476 \quad = \frac{1}{(2\pi)^2} \int_0^\infty \oint_0^{2\pi} d\kappa d\delta\theta \kappa^5 k^2 |\widehat{\psi}_\sigma(\boldsymbol{\kappa})|^2 \begin{pmatrix} \cos^2 \delta\theta \sin^2 \delta\theta & \cos \delta\theta \sin^3 \delta\theta \\ \cos \delta\theta \sin^3 \delta\theta & \sin^4 \delta\theta \end{pmatrix}, \quad (\text{A } 11)$$

$$477 \quad = \gamma_0 k^2 \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}. \quad (\text{A } 12)$$

478 Finally, we come back to the canonical frame to get  $\Sigma_\eta$

$$479 \quad \Sigma_\eta = \mathbb{E}_t \{ d\boldsymbol{\eta}_t d\boldsymbol{\eta}_t^\top \} = \mathbf{M}_k \Sigma_Z \mathbf{M}_k^\top = \gamma_0 k^2 \left[ \tilde{\mathbf{k}} \tilde{\mathbf{k}}^\top + 3\tilde{\mathbf{k}}^\perp (\tilde{\mathbf{k}}^\perp)^\top \right] \quad (\text{A } 13)$$

480 For noises cross-correlations, by isotropy, it is also straightforward to show that

$$481 \quad \Sigma_{\eta, \sigma} = 0. \quad (\text{A } 14)$$

482 The stochastic forcings of  $\mathbf{x}_r$  and  $\mathbf{k}$  are hence (conditionally) independent from one another.

## 483 Appendix B. Single ray dynamics

484 The Itô noise  $\begin{pmatrix} \sigma dB_t \\ d\eta_t \end{pmatrix}$  is white in time and conditionally Gaussian. Its conditional single-  
 485 point distribution is fully determined by its zero mean and its local covariance matrix (given  
 486 by equations (A 1), (A 7), (A 13) and (A 14)). In particular, we can replace this noise by  
 487 another zero-mean Gaussian vector with the same covariance without changing the single-  
 488 ray dynamics – typically replacing  $\sigma dB_t$  by  $\sqrt{a_0} \begin{pmatrix} dB_t^{(1)} \\ dB_t^{(2)} \end{pmatrix}$  and  $dZ_t$  by  $-\sqrt{\gamma_0}k \begin{pmatrix} dB_t^{(3)} \\ \sqrt{3}dB_t^{(4)} \end{pmatrix}$ . It  
 489 yields the simplified ray equations (4.3)-(4.4).

490 Then note that from Itô lemma (Oksendal 1998)  $d\tilde{\mathbf{k}} = d \begin{pmatrix} \cos \theta_k \\ \sin \theta_k \end{pmatrix} = \tilde{\mathbf{k}}^\perp d\theta_k - \frac{1}{2}\tilde{\mathbf{k}}d < \theta_k, \theta_k >_t$  where  $< \bullet, \bullet >_t$  denotes the quadratic covariation. Thus,

$$492 \quad d\mathbf{k} = dk\tilde{\mathbf{k}} + kd\tilde{\mathbf{k}} + d < k, \tilde{\mathbf{k}} > = (dk - \frac{1}{2}kd < \theta_k, \theta_k >_t)\tilde{\mathbf{k}} + (kd\theta_k + d < k, \theta_k >_t)\tilde{\mathbf{k}}^\perp \quad (\text{B } 1)$$

493 Projecting this equation and  $d\mathbf{k} = -\nabla\bar{\mathbf{v}}^\top \mathbf{k}dt + d\eta_t$  on  $\tilde{\mathbf{k}}$  and  $\tilde{\mathbf{k}}^\perp$ , we have

$$494 \quad \begin{cases} dk = -\tilde{\mathbf{k}} \cdot \nabla\bar{\mathbf{v}}^\top \mathbf{k}dt + (dZ_t)_1 + \frac{1}{2}kd < \theta_k, \theta_k >_t \\ kd\theta_k = -\tilde{\mathbf{k}}^\perp \cdot \nabla\bar{\mathbf{v}}^\top \mathbf{k}dt + (dZ_t)_2 - d < k, \theta_k >_t \end{cases}, \quad (\text{B } 2)$$

$$495 \quad \begin{cases} dk = -\tilde{\mathbf{k}} \cdot \nabla\bar{\mathbf{v}}^\top \mathbf{k}dt + (dZ_t)_1 + \frac{1}{2}k^{-1}d < Z_2, Z_2 >_t \\ d\theta_k = -\tilde{\mathbf{k}}^\perp \cdot \nabla\bar{\mathbf{v}}^\top \tilde{\mathbf{k}}dt + k^{-1}(dZ_t)_2 + \frac{1}{2}k^{-2}d < Z_1, Z_2 >_t \end{cases}. \quad (\text{B } 3)$$

496 The treatment of the large-scale terms  $\tilde{\mathbf{k}} \cdot \nabla\bar{\mathbf{v}}^\top \tilde{\mathbf{k}}$  and  $\tilde{\mathbf{k}}^\perp \cdot \nabla\bar{\mathbf{v}}^\top \tilde{\mathbf{k}}$  is classical. Interested  
 497 readers can refer to Lapeyre *et al.* (1999) for details. From Ito lemma again,  $d \log k =$   
 498  $dk/k - \frac{1}{2}d < k, k >_t/k^2$  leading to the simplified wave-vector dynamics (4.5)-(4.6).

## 499 Appendix C. Subgrid flow anisotropy and comparison with other works

500 Throughout this paper, we have considered an isotropic model for the stochastic subgrid  
 501 velocity (3.10). The isotropic diffusivity matrix  $\mathbf{a} = a_0\mathbb{I}_d$  is a good illustration of this. In  
 502 contrast, many authors (e.g. White & Fornberg 1998; Bôas & Young 2020; Smit & Janssen  
 503 2019) assume isotropic and homogeneous turbulence and obtain anisotropic stochastic  
 504 subgrid models for  $\frac{\|\bar{\mathbf{v}}\|}{v_g^0} \rightarrow 0$ . In these approaches, the integral over  $\delta\theta$  in diffusivity matrix  
 505 computations (A 4) and (A 12) involve singular integrations over the direction  $\mathbf{v}_g^0 = v_g^0\tilde{\mathbf{k}}$ . It  
 506 makes appear a Dirac delta function,  $2\pi\delta(\boldsymbol{\kappa} \cdot \mathbf{v}_g^0) = \frac{2\pi}{\kappa v_g^0} (\delta(\theta_\kappa - \theta_k - \frac{\pi}{2}) + \delta(\theta_\kappa - \theta_k + \frac{\pi}{2}))$   
 507 (see Appendix in Bôas & Young 2020). This precision imposes a statistical anisotropy for  
 508  $\sigma dB_t$  (oriented along  $\mathbf{k}$ ) and  $d\eta_t$  (oriented along  $\mathbf{k}^\perp$ ), eventually leading to a covariance  
 509  $\Sigma_Z = \gamma_0 k^2 \begin{bmatrix} 0 & 0 \\ 0 & 16 \end{bmatrix}$  (Eq. (3.17) in Bôas & Young (2020) and Eq. (24) in Smit & Janssen  
 510 (2019)), no noise  $dZ_1$ , and no Brownian motion  $B_t^{(3)}$ . Moreover, because of scaling  
 511 assumption, Bôas & Young (2020) neglect the spatial diffusivity matrix,  $\mathbf{a}$ , while Smit  
 512 & Janssen (2019) find  $\mathbf{a} = 4a_0 \left( \mathbb{I}_d + \frac{5}{4}\tilde{\mathbf{k}}\tilde{\mathbf{k}}^\top \right)$  (Eq. (22)-(23)). In this anisotropic framework,  
 513 the Stratonovich wavevector equation (2.3),  $d\mathbf{k} = -\nabla(\bar{\mathbf{v}}dt + \sigma \circ dB_t)^\top \mathbf{k}$ , would involve an  
 514 additional drift term in Itô notations.

515 Further developing this anisotropic stochastic closure is an interesting avenue. Neverthe-

516 less, in the present study, we adopt the isotropic model for  $\sigma dB_t$ , which is much more  
517 convenient for multi-ray numerical simulations.

## 518 **Appendix D. Jet simulation**

519 Again, currents are simulated at a resolution  $512 \times 512$  on a 1000-km-width squared domain  
520 through the same code. A backward velocity  $v_{Bk}$  forces a leftward jet structure.

$$521 \quad \partial_t \omega + \mathbf{v} \cdot \nabla \omega = S_\omega \quad \text{with} \quad \mathbf{v} = \nabla^\perp \Delta^{-1} (\omega + \omega_{Bk}). \quad (\text{D } 1)$$

522  $S_\omega$  encompasses the linear drag and the hyperviscosity, and the background vorticity,  $\omega_{Bk}$ ,  
523 is a smooth step function with a wavy interface at  $y = Y_{Bk}(x)$ :

$$524 \quad \omega_{Bk}(x, y) = \Omega_{Bk} \left( \frac{1}{2} - \text{erf} \left( \frac{y - Y_{Bk}(x)}{L_y^\omega} \right) \right) \quad \text{with} \quad Y_{Bk}(x) = L_y \left( \frac{1}{2} + \frac{1}{30} \cos \left( \frac{2\pi}{L_x} x \right) \right) \quad (\text{D } 2)$$

525 To better highlight the interplay between ray oscillations and scattering, we consider  
526 very-collimated swells, with a spatial extension of  $100\lambda = 25$  km.

527 Besides, the curvature of the simulated jet can force an additional faster oscillation around  
528 the jet for small enough wavevector angle. Indeed, a wave group traveling exactly rightward  
529 would cross an alternation of positive and negative vorticity regions with a period  $L_x / (v_g^0 -$   
530  $\bar{U}_0) \approx 1$  day  $< 2\pi / \bar{\omega}_r$ . Here, we set an initial wavevector angle large enough to prevent the  
531 additional harmonics.

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