A practical global existence and uniqueness result for stochastic differential equations on Riemannian manifolds of bounded geometry

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Abstract

In this paper, we establish a result for existence and uniqueness of stochastic differential equations on Riemannian manifolds, for regular inhomogeneous tensor coefficients with stochastic drift, under geometrical-only hypothesis on the manifold, so-called manifolds of bounded geometry, this hypothesis is consistent with the maximal regularity result for parabolic equations obtained by Herbert Amann. Furthermore, we provide a stochastic flow estimate for the solutions.

Contents

1 Introduction

In this paper, we establish an existence and uniqueness result of stochastic differential equations on Riemannian manifolds, for regular inhomogeneous tensor coefficients with stochastic drift, under the hypothesis that the manifold is of bounded geometry, the definition is given in Section 2. There is a vast literature on this class of manifold since it seems to be a reasonable setting when studying function spaces on (possibly) non-compact manifolds (any compact manifold is of bounded geometry, and a large class of model Riemannian manifolds are elements of this family). This hypothesis allows one to construct an atlas associated with a regular partitioning of unity. The main theorem proved here is consistent with the maximal regularity result for parabolic equations obtained in [1], since it has been proven in [2] that the *uniformly regular manifolds* hypothesis used in the former is equivalent to the definition of manifold of bounded geometry, the definition of *uniformly regular manifolds* is also given in Section 2. This result is mainly motivated from a modelization point of view; when a geometrical structure on the state space is of great interest, one wants a ready-to-use theorem for the well-posedness of the model under geometrical hypothesis.

From the pioneering work of K. Ito [3], there has been a wide development of the literature on stochastic differential equations on manifolds. Hsu [4] proved an existence result, for the process associated to the Laplace-Beltrami operator, up to an explosion time controlled by the curvature of the underlying manifold. For a general deterministic homogeneous operator, under the hypothesis that the manifold is compact, we have the existence and uniqueness result in *e.g* [5]. As highlighted by Elworthy in [6], there is a conceptual difficulty when introducing stochastic differential equations on manifold; it can be tackled with *orthonormal frame bundle* and Stratonovich calculus leading to the

definition of *rolling without slipping* of *moving frame* solutions, introduced by Eells and Elworthy in [7], we will explicit this conceptual difficulty and introduce the tools needed in the next subsection. We also refer to [8], [9] for the study of semimartingale valued on a manifold.

Notations : In this paper, we use the following notations, $|\cdot|$ is the absolute value, $\|\cdot\|$ is the Euclidean norm. If ϕ is a local chart, we set $\phi^{-1} = \psi$, so that ψ^α is the inverse of ϕ^α, where α is the index of the chart. We use capital letters for tensors, lowercases for their associated coefficients in local charts and the Einstein summation convention. The symbol $\doteq$ indicates that the right part of the equation is written in coordinates. The manifold considered here are supposed to be Riemanian, we write g_{ij} (respectively g^{ij}), the components of the metric tensor (resp. its inverse) in a local coordinate system. We suppose that they are equipped with their Levi-Civita connection, and we write $\nabla_{\partial_i}\partial_j \doteq \Gamma^k_{ij}$, the Christoffel symbol, given, in the case of the Levi-Civita connection by $\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$. Let $T^n_m M$ be the (n,m)-tensor bundle over M consisting of tensors being contravariant of order n and covariant of order m, then $C^k(T^n_m M)$ denote the space of k-differentiable sections of $T^n_m M$. $C^k_b(T^n_m M)$ indicates that the section is bounded for the tensor norm $\|\cdot\|_{k,\infty}$, defined as $\|A\|_{k,\infty} = \max_{i=0}^k \sup_M |\nabla^i A|_g$, where ∇^i is the i-th covariate derivative of the tensor; if k is not specified then one should assume $k = 0$. The notation $A \cdot B$ indicates the scalar product of the vector fields A with B, every time we will consider an operation on tensors, we will specify it in local coordinates. The notation $\circ$ denotes Stratonovich integral.

We begin with a brief overview of the conceptual technicality encountered when introducing stochastic systems on manifolds, we recommend the article of Elworthy [6], which specifically addresses this matter. Formally we want to push a process with white noise on a manifold, and define stochastic characteristics associated with some second-order operator. We need to specify how to map the noise onto the tangent space, *a priori* there is no canonical way to define this notion, here are three main approaches to give sense to the desired notion,

- *embedded*: given an isometrical embedding of the manifold as a submanifold on $\mathbb{R}^n$, Nash theorem ensures the existence of such embedding, we can use the n-dimensional Brownian motion and project it onto the d-dimensional tangent space at each point, with $n \leq d$.
- *chart covering dependent*: if we fix an atlas and a partition of unity of manifold, this defines a noise-pushing map from the basis of the tangent space in charts.
- *rolling without slipping*: the equation is now set on OM the manifold of ordered orthonormal basis of the tangent space TM, thus defining an equation on a moving basis of the tangent space along a curve on M, that follows the desired dynamic.

The first two notions are *extrinsic*, in the sense that they need some structure outside of the Manifold. The third notion, introduced by Eells and Elwothy in the case of the Laplace-Beltrami Operator, is *intrinsic*, it seems more natural to stick to the third notion that is *per se* more geometrical than the others.

1.1 Orthonormal frame bundle and horizontal lift

We now precise the third notion that we will use in the rest of this paper. First, we need to introduce a few geometrical definitions and results. We refer to [4, chp 2, p35] and [10, sec 7.2, p.127] for a detailed presentation of the following concepts.

Definition-Proposition 1.1. Let (M, g) be a d-dimesional Riemannian manifold, let OM denote the set of all orthonormal basis of the tangent space at each point of M,

$$OM = \{(x, E_1, \cdots, E_d) | x \in M, (E_1, \cdots, E_d) \text{ is an orthonormal basis of } T_x M\},$$

and denote by $\pi : OM \to M$, the canonical projection. Then, there exists a manifold structure on OM, that makes (OM, π) a principal bundle over M, called the *orthonormal frame bundle*. An element $u \in OM$, is identifiable as an isometry $u : \mathbb{R}^d \to T_{\pi(u)}M$, for an element $\lambda \in \mathbb{R}^d$, we will note $u\lambda = \sum_{i=1}^d E_i \lambda^i \in T_{\pi(u)}M$.

It is rather tempting to find a straightforward way to map the noise onto the tangent space by fixing a global section of OM, however, the existence of such a section implies that OM is globally trivializable; as a counter-example, $\mathbb{S}^2$ the 2-sphere doesn't satisfy this property, we thus need to specify how the frame is transported along the path of the process. To this purpose, we need to introduce the *horizontal lift* of a vector field.

Definition-Proposition 1.2 (Horizontal Lift)**.** Let (M, g) be a d-dimesional Riemannian manifold, and OM denote its orthonormal frame bundle, with projection $\pi : OM \to M$. A smooth curve (u_t) taking values in OM, is said to be horizontal if for each $e \in \mathbb{R}^d$ the vector field $(u_t e)$ is parallel along the M-valued curve $(\pi(u_t))$. A tangent vector $Y \in T_u OM$ is said to be horizontal if it is the tangent vector of a horizontal curve at u. The space of horizontal vectors at u is denoted by $H_u OM$, we have the decomposition

$$T_u OM = V_u OM \oplus H_u OM,$$

where $V_u OM$ is the subspace of vertical vectors, that are tangent to the fibre $T_u OM$. It follows that the canonical projection π, induces an isomorphism $\pi_{\mathsf{h}} : H_u OM \to T_{\pi(u)} M$, and for each $B \in T_x M$ and a frame u at x, there is a unique horizontal vector B^{h}, the horizontal lift of B to u, such that $\pi_{\mathsf{h}}(B^{\mathsf{h}}) = B$. Thus if B is a vector field on M, then B^{h} is a vector field on OM.
In coordinates $\{x^i, \zeta_k^j\}$, the lifted vector field writes,

$$B^{\mathsf{h}} \doteq b^i(x) \frac{\partial}{\partial x^i} - \Gamma_{ij}^k(x) b^i(x) \zeta_m^j \frac{\partial}{\partial \zeta_m^k}.$$

The differential structure of a manifold is built upon local trivializations, thus a continuous adapted process on a manifold is a solution to the desired stochastic differential equation if it is a solution in charts. To this end, we need to introduce a *local regularisation* of the dynamical system, that is an atlas, where the integrals involved in coordinates are well-defined.

Definition 1.3 (Regular Localization)**.** Let (M, g) be a d-dimensional Riemannian Manifold. Given $((\Omega, \mathscr{F}, \mathbb{P}), W.)$, a filtered probability space with the usual hypothesis, equipped with a d-dimensional Brownian motion and $A : [0,T] \to C_b^2(T_1^1 M), B : [0,T] \times \Omega \to C_b(T^1 M)$, such that B is progressively measurable. We say that an atlas $\{(O^\alpha, \phi^\alpha) =: \Lambda^\alpha, \alpha \in \mathfrak{K}\}$, is a *local regularization* of the stochastic differential system associated to (A, B), if in any chart $\alpha \in \mathfrak{K}, \exists\, C > 0$, such that

$$\sup_{t \in [0,T], x \in \phi^\alpha(O^\alpha)} \max_{i=1}^{d} |b^i(t, \omega, x)| < C, \mathbb{P} - a.s,$$

$$\sup_{t \in [0,T], x \in \phi^\alpha(O^\alpha)} \max_{i,j,k=1}^{d} |\partial_k^n a_l^i(t, x)| < C, \text{ for } n = 0, 1,$$

$$\sup_{x \in \phi^\alpha(O^\alpha)} \max_{k,i,j=1}^{d} |\Gamma_{ij}^k(x))| < C, \sup_{x \in \phi^\alpha(O^\alpha)} \max_{i,j=1}^{d} |g_{ij}(x)| < C,$$

where $a_k^l(t, x)$, $b^i(t, \omega, x)$, $g_{ij}(x)$, $\Gamma_{ij}^k(x)$ are respectively the coefficients of the tensors A, B, the metric tensor g and the Christoffel symbol in the coordinate chart α.

1.2 Moving frame strong solution

We are now ready to state the definition of the *rolling without slipping* or *moving frame* solution of a stochastic differential system on a manifold, introduced by Eells and Elworthy in [7]. We recommend [4], [5], [11], for a presentation of this notion.

Definition 1.4. Let (M, g) be a d-dimensional Riemannian Manifold, Given $((\Omega, \mathscr{F}, \mathbb{P}), W.)$ be a filtered probability space with the usual hypothesis, equipped with a d-dimensional Brownian motion, and $\{e_i\}$ are the coordinate unit vectors of $\mathbb{R}^d$. Let $A : [0,T] \to C_b^2(T_1^1 M)$, and $B : [0,T] \times \Omega \to C_b(T^1 M)$ a progressively measurable tensor valued process. For $u \in OM$ define, $A_i^{\mathsf{h}}(t, u) := (A(t, \pi(u)) u e_i)^{\mathsf{h}} := (A_i(t, u))^{\mathsf{h}}, B^{\mathsf{h}}(t, \omega, u) = (B(t, \omega, \pi(u)))^{\mathsf{h}}$. A progressively measurable adapted and continuous process $(U.)_{[0,T]}$ taking values in OM, is said to be a moving frame strong solution of

$$dU_t = A_i^{\mathsf{h}}(t, U_t) \circ dW_t^i + B^{\mathsf{h}}(t, U_t) dt \text{ on } T_{U_t} OM, \qquad (E_M^{\mathsf{h}})$$

with $\mathscr{F}_0$-measurable initial value $U_0 \in OM$, if there exists, a *regular localization*, that *affirms* the equation, that is, if for any $t \in [0,T], \alpha \in \mathfrak{K}$, defining, $\Omega_t^\alpha := \{\omega \in \Omega, \pi(U_t(\omega)) \in O^\alpha\}$, $\tau_t^\alpha : \Omega_t^\alpha \to [t,T]$ the exit time from O^α, and $(\xi^i(t), \zeta_k^j(t)) = (\phi^\alpha(\pi(U_t)), d\phi^\alpha(\pi(U_t))U_t\partial/\partial x^k)$ the local trivialization of the process in the chart, with $\partial/\partial x^k$ the associated basis. We have that, $(\xi^i(t), \zeta_k^j(t))$ is $\mathbb{P}$-as on $\Omega_t^{\Lambda^\alpha}$ up to τ_t^α solution of,

$$\begin{cases} d\xi^i(t) = (a_l^i(t,\xi(t))\zeta_m^l(t)) \circ dW_t^m + b^i(t,\xi(t))dt, \\ d\zeta_m^k(t) = (-\Gamma_{ij}^k(\xi(t))\zeta_m^i(t)a_l^j(t,\xi(t))\zeta_n^l(t)) \circ dW_t^n + (-\Gamma_{ij}^k(\xi(t))b^i(t,\xi(t))\zeta_m^j(t))dt, \end{cases} \tag{E_Λ}$$

where a_l^i, b^i and Γ_{ij}^k are respectively the coefficients associated to the tensors A, B and the Christoffel symbol, in the chart Λ^α.

All the hypotheses on the regular localization ensure that equation (E_Λ) is well-defined. Indeed all the coefficients involved are supposed to be bounded in the chart. And the C^2 regularity of the tensor A allows us to define the Stratonovich integral. Finally, since U_t is an orthonormal basis of $T_{\pi(U_t)}M$, we have in coordinate that $\zeta_k^i g_{ij}(\xi)\zeta_k^j = 1$, which implies, since the coefficients of the metric tensor are bounded, that $|\zeta_k^i| < K$ for some K possibly depending on the chart.

Remark 1.5. Since any element of OM can be identified as a point x in M and an associated orthonormal basis of T_xM, equation (E_M^{h}) describes a moving frame along a process $X_t = \pi(U_t)$ taking values in M, itself solution of :

$$dX_t = A_i(t,U_t) \circ dW_t^i + B(t,X_t)dt \text{ on } T_{X_t}M.$$

The fact that the equation on X is not autonomous, translates the need to keep track of how the noise should push the process on the manifold.

Remark 1.6. We need Stratonovich integral to give a chart-independent definition of a solution. Indeed, without the chain rule given by Stratonovich integral, we would get a second-order term when changing chart, corresponding to Ito's corrective term, that wouldn't correspond to the change of coordinate of a tensor field, and thus the process wouldn't be a solution of the same equation by change of coordinate.

The notion of solution considered here is similar to [5], but one can also define Semimartingale solution [8], [11], where the characterization of solution is against smooth function $C^\infty([0,T] \times M, \mathbb{R})$. It is classical [5], [11] to check that we have an analog to Ito's formula for moving frame solutions. For the sake of completeness, we give here a proof that is adapted to the general differential equations that we consider in this paper.

Proposition 1.7. *Let $(U_t)_t$ be a moving frame strong solution, then $(X_t := \pi(U_t))_t$ is a semimartingale solution, i.e, it is a continuous adapted process taking values in M such that, $\forall \varphi \in C^{1,2}([0,T] \times M, \mathbb{R}), \forall t \geq s \in [0,T]$, $(M_t^s)_{t \geq s}$ defined by,*

$$M_t^s := \varphi(t,X_t) - \varphi(s,X_s) - \int_s^t (\partial_t\varphi + (B + \frac{1}{2}\nabla A \cdot A) \cdot \nabla\varphi + \frac{1}{2}\Sigma \cdot \nabla^2\varphi)(u,X_u)du,$$

is a martingale, and it holds that,

$$\varphi(t,X_t) - \varphi(s,X_s) = \int_s^t (\partial_t\varphi + B \cdot \nabla\varphi)(u,X_u)du + \int_s^t (A_i(u,U_u) \cdot \nabla\varphi) \circ dW_u^i.$$

or in Ito's form,

$$\varphi(t,X_t) - \varphi(s,X_s) = \int_s^t (\partial_t\varphi + (B + \frac{1}{2}\nabla A \cdot A) \cdot \nabla\varphi + \frac{1}{2}\Sigma \cdot \nabla^2\varphi)(u,X_u)du + \int_s^t (A_i(u,U_u) \cdot \nabla\varphi)dW_u^i,$$

where Σ is a $(2,0)$-tensor, defined as $\Sigma = A \cdot A^*$, in coordinates it writes, $\sigma^{ij} \doteq a_k^i g^{kl} a_l^j$.

Proof. Let $\varphi \in C^{1,2}([0,T] \times M, \mathbb{R})$, since $(U_\cdot)_{[0,T]}$ is a moving frame solution, there exists a *regular localization* $\{\Lambda^\alpha =: (O^\alpha, \phi^\alpha), \alpha \in \mathfrak{K}\}$, that affirms the equation. Since M is second countable, we can assume that $\mathfrak{K}$ is at most countable. For $\alpha \in \mathfrak{K}$, define the following partition of unity,

$$\tilde{O}^0 = O^0, \tilde{O}^i = O^n / (\cup_{j=0}^{i-1} \tilde{O}^i).$$

Let $s \in [0,T]$, and define $\Omega_s^\alpha = \{\omega \in \Omega, \pi(U_s(\omega)) \in \tilde{O}^\alpha\}$, then up to a $\mathbb{P}$-nullset, $\Omega = \sqcup_{\alpha\in\mathfrak{K}}\Omega_s^\alpha$. And, since U is a solution, on the event Ω_s^α, up to the exit time $\tau_s^\alpha : \Omega_s^\alpha \to [s,T]$ of O^α, we have that the process in a local chart is a solution of (E_Λ). Thus using Stratonovich chain rule on φ in coordinates, we obtain on the event Ω_s^α, for any $t > s$,

$$\varphi(\tau_s^\alpha \wedge t, \xi_{\tau_s^\alpha\wedge t}) - \varphi(s,\xi_s) = \int_s^{\tau_s^\alpha\wedge t} (\partial_t\varphi + b^j\partial_j\varphi)(v,\xi_v)dv + \int_s^{\tau_s^\alpha\wedge t} (a_i^j(t,\xi_v)\zeta_k^i \cdot \partial_j\varphi) \circ dW_v^k. \quad (1)$$

Define, the following $(\mathscr{F}_{[s,T]})$-stopping time $\tau_s : \Omega \to [s,T]$, as $\tau_s = \tau_s^\alpha$ on Ω_s^α. Since we have the disjoint union of the events Ω_s^α, and rewriting in a coordinate independent manner (1), we obtain up to a $\mathbb{P}$-nullset on Ω, using the notation $X_t := \pi(U_t)$,

$$\varphi(\tau_s\wedge t, X_{\tau_s\wedge t}) - \varphi(s,X_s) = \int_s^{\tau_s\wedge t} (\partial_t\varphi + B\cdot\nabla\varphi)(v,X_v)dv + \int_s^{\tau_s\wedge t} (A_k(t,U_v)\cdot\nabla\varphi(v,X_v)) \circ dW_v^k. \quad (2)$$

Now let $s < t \in [0,T]$, define $t_k^n = \frac{(t-s)k}{n} + s$, from (2), $\mathbb{P}$-a.s for any $n \in \mathbb{N}^*$,

$$\begin{aligned}\varphi(t,X_t) - \varphi(s,X_s) - &\int_s^t (\partial_t\varphi + B\cdot\nabla\varphi)(v,X_v)dv - \int_s^t (A_k(t,U_v)\cdot\nabla\varphi(v,X_v)) \circ dW_v^k \\ &= \sum_{k=0}^n \Big(\varphi(t_{k+1}^n, X_{t_{k+1}^n}) - \varphi(t_{k+1}^n \wedge \tau_{t_{k+1}^n}, X_{t_{k+1}^n\wedge\tau_{t_{k+1}^n}}) \\ &+ \int_{t_k^n}^{t_{k+1}^n\wedge\tau_{t_{k+1}^n}} (\partial_t\varphi + B\cdot\nabla\varphi)(v,X_v)dv + \int_{t_k^n}^{t_{k+1}^n\wedge\tau_{t_{k+1}^n}} (A_k(t,U_v)\cdot\nabla\varphi(v,X_v)) \circ dW_v^k \Big).\end{aligned}$$

Since the process is $\mathbb{P}$-a.s continuous, and from the density of the sequences (t_k^n), we can show that the right term tends to 0 $\mathbb{P}$-a.s, and get the desired Stratonovich development.

For the second Ito-like identity, it is the same reasoning, we just need to explicit the Ito-Stratonovich equivalence. Omitting for the sake of conciseness t, ξ in the differential equation, the equation in a local chart is,

$$\begin{aligned} d\xi^i =& (a_l^i\zeta_m^l)dW_t^m + \frac{1}{2}(\partial_j a_l^i a_k^j \zeta_m^k \zeta_m^l - a_l^i \zeta_m^n \zeta_m^j a_n^k \Gamma_{kj}^l)dt + b^i dt, \\ d\zeta_m^k =& (-\Gamma_{ij}^k \zeta_m^i \zeta_q^l a_l^j)dW_t^q - (\Gamma_{ij}^k b^i \zeta_m^j)dt \\ &- \frac{1}{2}(\Gamma_{ij}^k \zeta_m^i \partial_n a_l^j \zeta_q^l a_p^n \zeta_q^p + \partial_n \Gamma_{ij}^k \zeta_m^i a_l^j \zeta_q^j a_p^n \zeta_q^p - \Gamma_{ij}^k \Gamma_{np}^i \zeta_m^n a_{l'}^p \zeta_q^{l'} a_l^j \zeta_q^l - \Gamma_{ij}^k \zeta_m^i a_l^j \Gamma_{np}^l \zeta_q^n \zeta_q^{l'} a_{l'}^p)dt. \end{aligned}$$

From Ito's lemma in chart, on the event Ω_t^α, we obtain that,

$$d\varphi(t,\xi_t) = \Big(\partial_t\varphi + b^i\partial_i\varphi + \frac{1}{2}\sum_{m=1}^d (\partial_j a_l^i a_k^j \zeta_m^k \zeta_m^l - a_l^i \zeta_m^n \zeta_m^j a_n^k \Gamma_{kj}^l)\partial_l\varphi + \frac{1}{2}\sum_{m=1}^d a_l^i \zeta_m^l \zeta_m^k a_k^j \partial_{ij}\varphi\Big)dt + a_l^i\zeta_m^l\partial_i\varphi dW_t^m. \quad (3)$$

Now since by definition, ζ_k^j is the j-coordinate in chart associated to the k^{th} vector of an orthonormal basis of $T_{\pi(U_t)}M$, $\zeta_k^j g_{ij} \zeta_m^i = \delta_{mk}$. And this implies that $\sum_{m=1}^d \zeta_m^i \zeta_m^j = g^{ij}$, i.e the i,j components of the inverse of the metric tensor, thus equation (3) rewrites,

$$d\varphi(t,\xi_t) = \Big(\partial_t\varphi + b^i\partial_i\varphi + \frac{1}{2}(\partial_j a_l^i a_k^j g^{kl} - a_l^i g^{nj} a_n^k \Gamma_{kj}^l)\partial_l\varphi + \frac{1}{2}a_l^i g^{kl} a_k^j \partial_{ij}\varphi\Big)dt + a_l^i\zeta_m^l\partial_i\varphi dW_t^m. \quad (4)$$

Introduicing the tensor $\Sigma = A\cdot A^*$, defined in coordinate by $\sigma^{ij} = a_l^i g^{kl} a_k^j$, and noting that in coordinate $\nabla A \doteq \partial_j a_k^i - a_k^l \Gamma_{jl}^i)$, we can rewrite equation (4) in a coordinate independent manner as,

$$d\varphi(t,X_t) = \Big(\partial_t\varphi + (B + \frac{1}{2}\Sigma + \frac{1}{2}\nabla A\cdot A^*)\cdot\nabla\varphi\Big)dt + A_m(t,U_t)\cdot\nabla\varphi(t,X_t)dW_t^m.$$

This concludes the proof. □

2 Main restults.

2.1 Geometric hypothesis

In order to formulate our main theorem, we introduce the following hypotheses on the geometry of the manifold.

Definition 2.1. Let $i : M \to \mathbb{R}_+$ be the injectivity radius function, defined by $i(x)$ is the largest radius for which the exponential map $\exp_x$ is a diffeomorphism on $\{y \in M, d(x,y) < i(x)\}$. Note $i(M) = \inf_{m\in M} i(x)$. A manifold is said to have a positive injectivity radius if $i(M) > 0$.

Definition 2.2 (Manifold of bounded Geometry)**.** A Riemannian manifold (M, g) equipped with its Levi-Civita connection is said to be *of bounded geometry* if: it is a *complete* metric space, it has *positive injectivity radius*, and if all covariant derivatives of the Riemann curvature tensor are bounded,

$$\||\nabla^k R|_g\|_\infty \leq C(k), \forall k \in \mathbb{N}_0.$$

Definition 2.3 (Uniformly Regular Atlas)**.** Let (M, g) be a Riemannian manifold of dimension d endowed with a Riemannian metric g, such that the underlying topological space is separable. An Atlas $\mathfrak{A} = (O_\kappa, \phi_\kappa)_{\kappa\in\mathfrak{K}}$ for M is said to be normalized, if $\phi_\kappa(O_\kappa) = \mathbb{B}^d$ unit centered ball in $\mathbb{R}^d$. The atlas $\mathfrak{A}$ is said to have *finite multiplicity* if there exists $N \in \mathbb{N}$ such that any intersection of more that N coordinate patches is empty. An atlas $\mathfrak{A}$ is said to fulfill the uniformly shrinkable condition: if it is normalized and there exists $r \in (0, 1)$ such that $\{\psi_\kappa(r\mathbb{B}), \kappa \in \mathfrak{K}\}$ is a cover of M. We say that $\mathfrak{A}$ is a *Uniformly Regular Atlas* on (M, g) if,

- $\mathfrak{A}$ is uniformly shrinkable and has finite multiplicity.
- $\|\phi_\eta \circ \psi_\kappa\|_{k,\infty} \leq c(k), \forall \eta, \kappa \in \mathfrak{K}, \forall k \geq 0$ s.t $O_\eta \cap O_\kappa \neq \emptyset$.
- $\|\psi_\kappa^* g\|_{k,\infty} \leq c(k), \kappa \in \mathfrak{K}$ and $k \in \mathbb{N}_0$, where $\psi_\kappa^* g$ is the pullback metric in the chart κ.
- $\exists K \geq 1$, such that, $\|\zeta\|^2/K \leq \psi_\kappa^* g(x)(\zeta, \zeta) \leq K\|\zeta\|^2, \quad \forall \zeta \in \mathbb{R}^d, \forall x \in \mathbb{B}^d, \forall \kappa \in \mathfrak{K}$.

Such an atlas is always countable. We will denote by $(r, K, c, \mathfrak{A})$ a uniformly regular atlas on M.

The notion of uniformly regular manifold was introduced by Herbert Amann, to prove a maximal regularity result for parabolic equations. A notable result proven by M. Disconzi, Y. Shao, and G. Simonett in [2], is the equivalence between the geometrical definition of Manifold of bounded geometry and the existence of a uniformly regular atlas. A few examples of Manifolds of bounded geometry, are any Euclidean space, any isometric images of uniformly regular Riemannian manifolds, any compact Manifold, the d-dimensional hyperbolic space or the d-dimensional sphere.

Lemma 2.4 ([12], Theorem 4.1, p.22)**.** Let (M, g) be a manifold of bounded geometry. Then, it admits a *Uniformly Regular Atlas* $(r, K, c, \mathfrak{A} = \{(O^\alpha, \phi^\alpha) = \Lambda^\alpha\}_{\alpha\in\mathfrak{K}})$.

The construction of the atlas relies on the positivity of the injectivity radius and the existence of normal coordinate systems around each point. And the estimates of the metric tensor and its derivatives in normal coordinates rely on Jacobi fields theory,(T. Aubin [13][Ch 1, sec. 8, Lemma 2.26, p20] and J. Eichorn [14]).

2.2 Existence and uniqueness result

We are now able to state our main theorem.

Theorem 2.5 (Existence and Uniqueness)**.** *Given a Riemannian Manifold of bounded geometry (M, g), and $((\Omega, \mathscr{F}, \mathbb{P}), W.)$ a filtered probability space with the usual hypothesis, equipped with a d-dimensional Brownian motion. Let A be a deterministic $(1,1)$-tensor valued function $A : [0, T] \to C_b^2(T_1^1 M)$, and B be a tensor valued adapted processes, $B : [0, T] \times \Omega \to C_b^1(T^1 M)$, suppose that $\exists C > 0$, such that,*

$$\|\nabla B\|_\infty, \|\nabla A\|_\infty, \|\nabla^2 A\|_\infty, \|B\|_\infty, \|A\|_\infty < C \qquad \forall t \in [0, T], \mathbb{P}\text{-}a.s,$$

where $|\cdot|_\infty$ designe the tensor norm induced by g, and ∇^k the k^{th}-covariant derivative operator.

Then, for any $\mathscr{F}_0$-measurable initial condition $U_0 \in OM$, there exists a unique solution to the lifted equation (E_M^{h}),

$$dU_t = A_i^{\mathsf{h}}(t, U_t) \circ dW_t^i + B^{\mathsf{h}}(t, U_t)dt \text{ on } T_{U_t}OM,$$

in the sens of Definition 1.4, where $B^{\mathsf{h}}(t, \omega, u) = (B(t, \omega, \pi(u))^{\mathsf{h}}$, and $A_i^h(t, u) = (A(t, \pi(u))ue_i)^{\mathsf{h}}$, with e_i the i^{th} coordinate unit vector of $\mathbb{R}^d$.

Remark 2.6. There is no integrability assumption on the initial condition for this kind of solution. However, A is required to be deterministic, and smooth, this comes from the imposed Stratonovich formalism, as mentioned in the previous section, to define the stochastic differential equation in a chart-independent manner. Extending this result to stochastic fields A could be possible, but one should use Ito-Wenzel lemma.

We will first prove the following lemma, which gives an existence and uniqueness result in local charts of the process.

Lemma 2.7. Under the assumptions and notations of Theorem 2.5, $\forall s \in [0, T]$, for any chart Λ^β from a uniformly regular atlas $(r, K, c, \mathfrak{A} = \{(O^\alpha, \phi^\alpha) = \Lambda^\alpha\}_{\alpha \in \mathfrak{K}})$, for any $\mathscr{F}_s$-measurable $(\xi(s), \zeta(s)) \in B_d(0, 1) \times B_d(0, K)^d$. There exists an unique adapted continuous process $(\xi^i(t), \zeta_k^j(t))_{t\in[s,T]}$, with initial condition $(\xi(s), \zeta(s))$, solution of

$$\begin{cases} d\xi^i(t) = (\lambda(\xi)a_l^i(t, \xi)\zeta_m^l) \circ dW_t^m + \lambda(\xi)b^i(t, \xi)dt, \\ d\zeta_m^k(t) = (-\lambda(\xi)\Gamma_{ij}^k\zeta_m^i a_l^j\zeta_n^l) \circ dW_t^n + (-\lambda(\xi)\Gamma_{ij}^k b^i\zeta_m^j)dt, \end{cases} \tag{λE_Λ}$$

where λ is a bump function to be specified, confining the process.
Furthermore, the process stays in $B_d(0, 1) \times B_d(0, K^2)^d$,

$$(\xi(t), \zeta(t)) \in B_d(0, 1) \times B_d(0, K^2)^d \quad \forall t \in [s, T],$$

with the following property

$$\zeta_k^i(t)g_{ij}(\xi_t)\zeta_m^j(t) \equiv \zeta_k^i(s)g_{ij}(\xi_s)\zeta_m^j(s) \quad \forall t \in [s, T], \tag{Z_Λ}$$

with g_{ij} is the metric tensor in the chart Λ^β.
Moreover, we have a uniform in any chart estimate,

$$\mathbb{P}\Big(\sup_{[s,t]} \|\xi_u - \xi_s\| > \rho | \mathscr{F}_s\Big) \leq C|t - s|^2, \tag{$\mathbb{P}_{\text{EXIT}}$}$$

where C is a constant depending on $\rho, K, c(0), c(1), \||A|_g\|_\infty, \||\nabla A|_g\|_\infty, \||B|_g\|_\infty$,.

Proof. Let $\lambda : \mathbb{R}^d \to [0, 1]$ be the following $C_b^2(\mathbb{R}^d) \cap W^{3,\infty}(\mathbb{R}^d)$ bump function,

$$\lambda(x) = \begin{cases} 1 \text{ if } \|x\| < \frac{1+2r}{3}, \\ -6(\frac{3\|x\|-2r-1}{1-r})^5 + 15(\frac{3\|x\|-2r-1}{1-r})^4 - 10(\frac{3\|x\|-2r-1}{1-r})^3 + 1 \text{ if } \|x\| \in \big[\frac{1+2r}{3}, \frac{2+r}{3}\big], \\ 0 \text{ otherwise.} \end{cases}$$

Introduce, the Ito Stochastic differential equation associated to the Stratonovich equation (λE_Λ). For the sake of conciseness, we note $a_j^i = \lambda(\xi_t)a_j^i(t, \xi)$, $b^i = \lambda(\xi)b^i(t, \xi)$. We omit t in the differential equation at time t, and ξ_t in the Christoffel symbol, so when we note $\partial_n a_l^j$, we mean $\partial_n\lambda(\xi_t)a_j^i(t, \xi) + \lambda(\xi_t)\partial_n a_j^i(t, \xi)$. The Ito's counterpart of equation (λE_Λ) writes as

$$\begin{aligned} d\xi^i =& (a_l^i\zeta_m^l)dW_t^m + \frac{1}{2}(\partial_j a_l^i a_k^j\zeta_m^k\zeta_m^l - a_l^i\zeta_m^n\zeta_m^j a_n^k\Gamma_{kj}^l)dt + b^i dt, \\ d\zeta_m^k =& (-\Gamma_{ij}^k\zeta_m^i\zeta_q^l a_l^j)dW_t^q - (\Gamma_{ij}^k b^i\zeta_m^j)dt \\ & - \frac{1}{2}(\Gamma_{ij}^k\zeta_m^i\partial_n a_l^j\zeta_q^l a_p^n\zeta_q^p + \partial_n\Gamma_{ij}^k\zeta_m^i a_l^j\zeta_q^j a_p^n\zeta_q^p - \Gamma_{ij}^k\Gamma_{np}^i\zeta_m^n a_{l'}^p\zeta_q^{l'} a_l^j\zeta_q^l - \Gamma_{ij}^k\zeta_m^i a_l^j\Gamma_{np}^l\zeta_q^n\zeta_q^{l'} a_{l'}^p)dt. \end{aligned}$$

Since the coefficients and the Christoffel symbol are regular in the chart, if we introduce the following stopping time,

$$\tau = \inf\{t > s, \max_{m=1}^d \|\zeta_m(t)\| > 2K^2\},$$

using classical results, we obtain existence and uniqueness of the localized stochastic equation, since all the coefficients involved are $\mathbb{P}$-a.s Lipschitz and bounded up to the killing time. Now by proving the first property (Z_Λ) of the lemma, for the localized system, this will imply that the localized system stays in a ball inside the localization, since from the uniform equivalence of the norm in the chart, we obtain that,

$$|\zeta_k(t)|^2 \leq K\zeta_k^i g^{ij}\zeta_k^j(t) = K\zeta_k^i g^{ij}\zeta_k^j(s) \leq K^2|\zeta_k(s)|^2 \leq K^4.$$

Thus the localized solution is a solution of the non-localized equation. Proving (Z_Λ), is the straight forward following computation. We can use Ito Stratonovich equivalence [15, Prop.2.21, p.295], and obtain that (ξ, ζ) is a solution of the localized version of the Stratonovich equation (λE_Λ). Again we omit for conciseness $t, \xi(t)$ when evaluating a matrix value function or a coordinate in time, and the bump function λ.

$$\begin{aligned}
d(\zeta_m^i(t)g_{ij}(\xi)\zeta_n^j(t)) =& g_{ij}\zeta_n^j \circ d\zeta_m^i + g_{ij}\zeta_m^i \circ d\zeta_n^j + \zeta_n^j\zeta_m^i \circ dg_{ij} \\
=& (\partial_q g_{ij}\zeta_m^i\zeta_n^j a_l^q\zeta_k^l - g_{ij}\zeta_n^j\Gamma_{pq}^i\zeta_m^p a_l^q\zeta_k^l - g_{ij}\zeta_m^i\Gamma_{qp}^j\zeta_n^p a_l^q\zeta_k^l) \circ dW_t^k \\
&+ (\partial_q g_{ij}\zeta_m^i\zeta_n^j b^q - \Gamma_{pq}^i b^q\zeta_m^i\zeta_n^p g_{ij} - \Gamma_{pq}^i b^q\zeta_m^p\zeta_n^j g_{ij})dt \\
=& (\zeta_m^i\zeta_n^j a_l^q\zeta_k^l(\partial_q g_{ij} - g_{il}\Gamma_{jq}^l - g_{lj}\Gamma_{iq}^l)) \circ dW_t^k \\
&+ (\zeta_m^i\zeta_n^j b^q(\partial_q g_{ij} - g_{il}\Gamma_{jq}^l - g_{lj}\Gamma_{iq}^l))dt
\end{aligned}$$

Now by the following identity, $\partial_k g_{ij} - g_{il}\Gamma_{jk}^l - g_{lj}\Gamma_{ik}^l \doteq \nabla g = 0$, e.g [16], we obtain that, $d(\zeta_m^i(t)g_{ij}(\xi)\zeta_n^j(t)) = 0$. Thus,

$$\zeta_m^i(t)g_{ij}(\xi(t))\zeta_n^j(t) \equiv \zeta_m^i(s)g_{ij}(\xi(s))\zeta_n^j(s) \quad \forall t \in [s,T].$$

We now prove estimate ($\mathbb{P}_{\text{EXIT}}$). First, recall that,

$$\xi^i(u) - \xi^i(s) = M^i(s,u) + \frac{1}{2}\int_s^u (\partial_j a_l^i a_k^j\zeta_m^k\zeta_m^l - a_l^i\zeta_m^n\zeta_m^j a_n^k\Gamma_{kj}^l)dv + \int_s^u b^i dv,$$

where $M^i(s,u) := \int_s^u a_l^i\zeta_m^l dW_v^m$ is a martingale.

Noting that, $\partial_j a_l^i - a_l^i\Gamma_{kj}^l \doteq \nabla A$, from the uniform equivalence of the norm, Cauchy-Schwratz inequality and the estimate on $\max_{k=1}^d \|\zeta_k\|$, there exists a constant C depending on K only such that,

$$\|\partial_j a_l^i a_k^j\zeta_m^k\zeta_m^l - a_l^i\zeta_m^n\zeta_m^j a_n^k\Gamma_{kj}^l\|_{\mathbb{R}^d} \leq C\||\nabla A|_g\|_\infty\||A|_g\|_\infty.$$

Similarly, $\|b\|_{\mathbb{R}^d} \leq K\||B|_g\|_\infty$. From the convexity of the fourth power, we have the following bound,

$$\sup_{[s,t]}\|\xi_u - \xi_s\|^4 \leq 8(\sup_{[s,t]}\|M_u^s\|^4 + (t-s)^4 C(\|\nabla A\|\|A\| + \|B\|)^4),$$

where C depends on K only. Now, note that the quadratic variation of the i-th component of the martingale M is expressed as $\langle M^s\rangle_t^i = \int_s^t a_l^i\zeta_m^l a_k^i\zeta_m^k ds$, which implies that there exists a constant C depending on K only, such that $|\langle M^s\rangle_t^i| \leq |t-s|C\||A|_g\|_\infty$. Now, summing up, and using successively, Markov Inequality on the quartic moment and Burkholder-Davis-Gundis Inequality,

$$\begin{aligned}
\mathbb{P}\Big(\sup_{[s,t]}\|\xi_u - \xi_s\| > \rho|\mathscr{F}_s\Big) \leq& \frac{1}{\rho^4}\mathbb{E}\Big[\sup_{[s,t]}\|\xi_u - \xi_s\|^4|\mathscr{F}_s\Big] \\
\leq& \frac{1}{\rho^4}\Big(\mathbb{E}\Big[8\sup_{[s,t]}\|M_u^s\|^4|\mathscr{F}_s\Big] + C(t-s)^4\Big) \\
\leq& C\Big(\mathbb{E}\Big[\|(\langle M^s\rangle_t^i)_{i=1,\cdots,d}\|^2|\mathscr{F}_s\Big] + (t-s)^4\Big) \\
\leq& C|t-s|^2.
\end{aligned}$$

We obtain the desired estimate, for a constant C, only depending on K, ρ, the tensor norm of A, ∇A and B, uniformly in any local coordinate. □

We now can prove the main theorem.

Proof of Theorem 2.5. We will use Ito's random switching construction [3], which we adapt to the moving frame solution.

Step 1: Contruction of the process.

Since (M,g) is of bounded geometry, from Lemma 2.4, there exists a uniformly regular atlas $(r,K,c,\mathfrak{A}=\{(O^\alpha,\phi^\alpha)=\Lambda^\alpha\}_{\alpha\in\mathfrak{K}})$. Define $O_0^\alpha=\psi^\alpha(r\mathbb{B}^d)$. Since $\mathfrak{A}$ is uniformly regular $(O_0^\alpha)_{\alpha\in\mathfrak{K}}$ covers M. Introduce the following partition of unity,

$$\tilde{O}^i=O_0^i/(\bigcup_{k=0}^{i-1}\tilde{O}^k).$$

Fix $m\in\mathbb{N}^*$, let $t_k^m=\frac{kT}{m}$, for $k=0,\cdots,m$, and for any chart Λ^α in $\mathfrak{A}$, define the $\mathscr{F}_0$-measurable random variables,

$$(\xi_0,\zeta_0)^{(\alpha,U_0)}=\begin{cases}(\phi^\alpha(\pi(U_0)),d\phi^\alpha(\pi(U_0))U_0\partial/\partial x^j)\text{ if }U_0\in\tilde{O}^\alpha\\(0_{\mathbb{R}^d},0_{\mathbb{R}d^2})\text{ otherwise.}\end{cases}$$

From lemma 2.7, for any chart $\alpha\in\mathfrak{K}$ there exists a unique solution to (λE_Λ), starting from t_0^m, with initial condition $(\xi_0,\zeta_0)^{(\alpha,U_0)}$, we call this process $(\xi_\cdot,\zeta_\cdot)_{[0,t_1^m]}^{(\alpha,t_0^m,U_0)}$. And from property (Z_Λ), on the event $\{\pi(U_0)\in\tilde{O}^\alpha\}$,

$$\zeta_m^i(t)g_{ij}(\xi(t))\zeta_k^j(t)\equiv\zeta_l^i(t_0)g_{ij}(\xi(t_0))\zeta_k^j(t_0)=\delta_{lk}.$$

The last equality holds from the construction of ζ_0. This implies that ζ remains an orthonormal basis of $T_{\psi^\alpha(\xi)}M$, and that $(\xi_\cdot,\zeta_\cdot)_{[0,t_1^m]}^{(\alpha,t_0^m,U_0)}$ stays identifiable as an element of OM. Now set,

$$\overset{(m)}{U}_\cdot=(\psi^\alpha(\xi_\cdot^{(\alpha,t_0^m,U_0)}),d_\xi\psi^\alpha\zeta_\cdot^{(\alpha,t_0^m,U_0)})\in OM\text{ on }[t_0^m,t_1^m]\text{ if }\pi(U_0)\in\tilde{O}^\alpha.$$

Since $\mathfrak{K}$ is countable, $(\overset{(m)}{U}_\cdot)_{[t_0^m,t_1^m]}$ is an adapted process defined up to a $\mathbb{P}$-nullset. By continuing this procedure, define at step $k=1,\cdots,m-1$, for any chart α the $\mathscr{F}_{t_k^m}$-measurable random variable,

$$(\xi_{t_k^m},\zeta_{t_k^m})^{(\alpha,U_{t_k^m}^{(m)})}=\begin{cases}(\phi^\alpha(\pi(U_{t_k^m}^{(m)})),d\phi^\alpha(\pi(U_{t_k^m}^{(m)}))U_{t_k^m}^{(m)}\partial/\partial x^j)\text{ if }U_{t_k^m}^{(m)}\in\tilde{O}^\alpha\\(0,0)\text{ otherwise.}\end{cases}$$

And from lemma 2.7, for any chart α, note by $(\xi_\cdot,\zeta_\cdot)_{[t_k,t_{k+1}^m]}^{(\alpha,U_{t_k^m}^{(m)})}$ the unique solution of (λE_Λ) starting at t_k^m with initial condition $(\xi_{t_k^m},\zeta_{t_k^m})^{(\alpha,U_{t_k^m}^{(m)})}$. And extend the process $\overset{(m)}{U}$ as,

$$\overset{(m)}{U}_\cdot=(\psi^\alpha(\xi_\cdot^{(\alpha,t_k^m,U_{t_k^m}^{(m)})}),d_\xi\psi^\alpha\zeta_\cdot^{(\alpha,t_k^m,U_{t_k^m}^{(m)})})\in OM\text{ on }[t_k^m,t_{k+1}^m]\text{ if }\pi(U_{t_k^m})\in\tilde{O}^\alpha.$$

For $m\in\mathbb{N}^*,k=0,\cdots,m-1$, define the following events,

$$\Omega_{\alpha\beta}^{(k,m)}=\left\{\omega\in\Omega,\pi(U_{(t_k^m,\omega)}^{(m)})\in\tilde{O}^\alpha,\pi(U_{(t_{k+1}^m,\omega)}^{(m)})\in\tilde{O}^\beta,\xi_{([t_k^m,t_{k+1}^m])}^{(\alpha,t_k^m,U_{t_k^m}^{(m)})}\in B_{\mathbb{R}^d}\left(0,\frac{1+2r}{3}\right)\right\}$$

$$\overset{(m)}{\Omega}=\bigsqcup_{n_0,\cdots,n_m\in\mathfrak{K}^m}\bigcap_{k=0,\cdots,m-1}\Omega_{n_k,n_{k+1}}^{(k,m)}$$

And introduce the adapted process $(U_\cdot)_{[0,T]}$, defined up to a $\mathbb{P}$-nullset on the event $\bigcup_{m\in\mathbb{N}}\overset{(m)}{\Omega}$ as,

$$U_\cdot=\begin{cases}\overset{(1)}{U}_\cdot\text{ on }\overset{(1)}{\Omega},\\\overset{(m)}{U}_\cdot\text{ on }\overset{(m)}{\Omega}/(\bigcup_{n=1}^{m-1}\overset{(n)}{\Omega}).\end{cases}$$

Let us prove that U is defined $\mathbb{P}$-a.s, by proving that, $\lim_m \mathbb{P}\Big(\overset{(m)}{\Omega}\Big) = 1$. To this end, define $\tilde{\Omega}^{(k,m)}_{n_k} := \Big\{X_{t^m_k} \in \tilde{O}^{n_k}, \sup_{[t^m_k, t^m_{k+1}]} \|\xi^{(\alpha,k,m)}_{(u)}\| < \frac{1+2r}{3}\Big\}$, where $X_t := \pi(U_t)$. And note that,

$$\Big\{X_{t^m_k} \in \tilde{O}^{n_k}, \sup_{[t^m_k, t^m_{k+1}]} \|\xi^{(\alpha,k,m)}_{(u)} - \xi^{(\alpha,k,m)}_{(t^m_k)}\| < \frac{1-r}{3}\Big\} \subset \tilde{\Omega}^{(k,m)}_{n_k},$$

so that, for any event $A \in \mathscr{F}_{t^m_k}$,

$$\begin{aligned}
\mathbb{P}\Big(A \cap \tilde{\Omega}^{(k,m)}_{n_k}\Big) \geq & \mathbb{P}\Big(A \cap \Big\{X_{t^m_k} \in \tilde{O}^{n_k}, \sup_{[t^m_k, t^m_{k+1}]} \|\xi^{(\alpha,k,m)}_{(u)} - \xi^{(\alpha,k,m)}_{(t^m_k)}\| < \frac{1-r}{3}\Big\}\Big) \\
= & \int_{A \cap \{X_{t^m_k} \in \tilde{O}^{n_k}\}} \mathbb{P}\Big(\sup_{[t^m_k, t^m_{k+1}]} \|\xi^{(\alpha,k,m)}_{(u)} - \xi^{(\alpha,k,m)}_{(t^m_k)}\| < \frac{1-r}{3} \Big| \mathscr{F}_{t^m_k}\Big)\mathbb{P}(d\omega) \\
\geq & \int_{A \cap \{X_{t^m_k} \in \tilde{O}^{n_k}\}} \Big(1 - \frac{C}{m^2}\Big)\mathbb{P}(d\omega) = \mathbb{P}\Big(A \cap \{X_{t^m_k} \in \tilde{O}^{n_k}\}\Big)\Big(1 - \frac{C}{m^2}\Big),
\end{aligned}$$

where C is a constant depending on the tensor norms of A and B, some bounds on the geometry and $\frac{1-r}{3}$, uniformly in any chart. Thus we can conclude that,

$$\mathbb{P}\Big(\Omega^{(m)}\Big) = (1 - \frac{C}{m^2}) \sum_{n_0, \cdots n_{m-1}} \mathbb{P}\Big(\Omega^{(1,m)}_{(n_1,n_2)} \cap \cdots \cap \Omega^{(m-2,m)}_{(n_{m-2}, n_{m-1})}\Big) \geq \Big(1 - \frac{C}{m^2}\Big)^m \xrightarrow{m \to +\infty} 1. \tag{5}$$

Step 2: Verify that the process is a solution.

We constructed a continuous adapted process, and we now verify that it is indeed a solution for the regular localization given by the restriction of the uniformly regular atlas used in the construction, specifically $\{(O^\alpha_0 =: \psi^\alpha(B_{\mathbb{R}^d}(0,r), \phi^\alpha), \alpha \in \mathfrak{K}\}$. Let $t \in [0,T], \alpha \in \mathfrak{K}$, and introduce as in the definition (1.4), $\Omega^\alpha_t := \{\omega \in \Omega, \pi(U_t(\omega)) \in O^\alpha_0 =: \psi^\alpha(B_{\mathbb{R}^d}(0,r))\} \in \mathscr{F}_t$, and the exit time $\tau^\alpha_t : \Omega^\alpha_t \to [t,T]$ of $O^\alpha_0 \subset O^\alpha$. From the convergence of $\Omega^{(m)}$ in (5), we obtain that,

$$\Omega^\alpha_t = \Big(\bigcup_{m \in \mathbb{N}} \Omega^\alpha_t \cap \overset{(m)}{\Omega}\Big) \cup \Omega_0,$$

where Ω_0 is a $\mathbb{P}$-nullset. For any $m \in \mathbb{N}^*$, recall that $\overset{(m)}{\Omega} = \bigsqcup_{(n_0, \cdots, n_m) \in \mathfrak{K}^m} \bigcap_{k=0,\cdots,m-1} \Omega^{(k,m)}_{n_k, n_{k+1}}$. Take a multi-index $\vec{n} \in \mathfrak{K}^m$ and define

$$\Omega^{(m)}_{\vec{n}} = \bigcap_{k=0,\cdots,m-1} \Omega^{(k,m)}_{n_k, n_{k+1}}.$$

Then, either $\Omega^\alpha_t \bigcap \Omega^{(m)}_{\vec{n}} = \varnothing$ and the property is verified, or $\Omega^\alpha_t \bigcap \Omega^{(m)}_{\vec{n}} \neq \varnothing$ and there exists $k = 0, \cdots, m-1$, such that $\frac{kT}{m} \leq t < \frac{(k+1)T}{m}$, and $O^\alpha_0 \cap \psi^{n_k}(B(0, \frac{1+2r}{3})) \neq \varnothing$, and if we denote $(\bar{\xi}^i, \bar{\zeta}^j_k)_{[t,\tau^\alpha_t)}$ the process $U_\cdot$ in coordinate Λ^α, from the change of coordinate rule and by construction of $U_\cdot$ on the event, $\forall s \in [t, \tau^\alpha_t)$, if $\frac{lT}{m} \leq s < \frac{(l+1)T}{m}$ then,

$$(\bar{\xi}(s), \bar{\zeta}_k(s)) = ((\varphi^\alpha \circ \psi^{n_l})(\xi^{(n_l,t_l,U^{(m)}_{t_k})}_{(s)}), \partial_j(\varphi^\alpha \circ \psi^{n_l})\big(\zeta^{(n_l,t_l,U^{(m)}_{t_k})}_{k(s)}\big)^j)$$

For the sake of clarity, we introduce the following notations for the up coming computations. First $\bar{x}(y) := (\varphi^\alpha \circ \psi^\beta)(y)$, $\frac{\partial \bar{x}^i}{\partial x^k}(y) = \partial_k(\varphi^\alpha \circ \psi^\beta(y))^i$, respectively, $x(\bar{y}) := (\varphi^\beta \circ \psi^\alpha)(\bar{y})$, $\frac{\partial x^i}{\partial \bar{x}^k}(\bar{y}) = \partial_k(\varphi^\beta \circ \psi^\alpha(\bar{y}))^i$, then the change of coordinate rule writes, $(\bar{\xi}^i, \bar{\zeta}^j_k) = (\bar{x}^i(\xi), \frac{\partial \bar{x}^j}{\partial x^l}(\xi)\zeta^l_k)$. From the construction of the process U, on the considered event, ξ stays where the bump function is equal to 1. From

the Stratonovich chain rule, we obtain the following equation on $(\bar{\xi}^i, \bar{\zeta}^j_k)$, again, we omit t, ξ for the matrix-valued functions.

$$\begin{cases} d\bar{\xi}^i &= (\frac{\partial \bar{x}^i}{\partial x^l} a^l_m \zeta^m_k) \circ dW^k_t + \frac{\partial \bar{x}^i}{\partial x^l} b^l dt, \\ d\bar{\zeta}^j_k &= (\frac{\partial^2 \bar{x}^j}{\partial x^l \partial x^i} a^i_m \zeta^m_n \zeta^l_k - \frac{\partial \bar{x}^j}{\partial x^p} \Gamma^p_{il} a^i_m \zeta^m_n \zeta^l_k) \circ dW^n_t + (\frac{\partial^2 \bar{x}^j}{\partial x^l \partial x^i} b^i \zeta^l_k - \frac{\partial \bar{x}^j}{\partial x^p} \Gamma^p_{il} b^i \zeta^l_k) dt. \end{cases}$$

Every time there is a repeating index, we introduce $\frac{\partial \bar{x}^j}{\partial x^l} \frac{\partial x^l}{\partial \bar{x}^p} = \partial_p (\varphi^\alpha \circ \psi^\beta (\varphi^\beta \circ \psi^\alpha))^j = \delta^j_p$, and thus the change of coordinate of the concerned tensors, e.g $\bar{a}^i_j = \frac{\partial \bar{x}^i}{\partial x^l} \frac{\partial x^k}{\partial \bar{x}^j} a^l_k$, where $\bar{a}$(resp. a) is the coefficients of A the $(1,1)$-tensor in the chart Λ^α(resp. Λ^β). We then obtain,

$$\begin{cases} d\bar{\xi}^i &= (\bar{a}^i_m \bar{\zeta}^m_k) \circ dW^k_t + \bar{b}^i dt, \\ d\bar{\zeta}^j_k &= (\frac{\partial^2 \bar{x}^j}{\partial x^l \partial x^i} \frac{\partial x^i}{\partial \bar{x}^q} \bar{a}^q_m \bar{\zeta}^m_n \frac{\partial x^l}{\partial \bar{x}^p} \bar{\zeta}^p_k - \frac{\partial \bar{x}^j}{\partial x^m} \Gamma^m_{il} \frac{\partial x^i}{\partial \bar{x}^q} \bar{a}^q_{m'} \bar{\zeta}^{m'}_n \frac{\partial x^l}{\partial \bar{x}^p} \bar{\zeta}^p_k) \circ dW^n_t \\ &\quad + (\frac{\partial^2 \bar{x}^j}{\partial x^l \partial x^i} \frac{\partial x^i}{\partial \bar{x}^q} \bar{b}^q \frac{\partial x^l}{\partial \bar{x}^p} \bar{\zeta}^p_k - \frac{\partial \bar{x}^j}{\partial x^m} \Gamma^m_{il} \frac{\partial x^i}{\partial \bar{x}^q} \bar{b}^q \frac{\partial x^l}{\partial \bar{x}^p} \bar{\zeta}^p_k) dt. \end{cases}$$

Rewriting the second equation as,

$$\begin{aligned} d\bar{\zeta}^j_k =& \big((\frac{\partial^2 \bar{x}^j}{\partial x^l \partial x^i} \frac{\partial x^i}{\partial \bar{x}^q} \frac{\partial x^l}{\partial \bar{x}^p} - \frac{\partial \bar{x}^j}{\partial x^m} \Gamma^m_{il} \frac{\partial x^i}{\partial \bar{x}^q} \frac{\partial x^l}{\partial \bar{x}^p}) \bar{a}^q_{m'} \bar{\zeta}^{m'}_n \bar{\zeta}^p_k \big) \circ dW^n_t \\ &+ \big((\frac{\partial^2 \bar{x}^j}{\partial x^l \partial x^i} \frac{\partial x^i}{\partial \bar{x}^q} \frac{\partial x^l}{\partial \bar{x}^p} - \frac{\partial \bar{x}^j}{\partial x^m} \Gamma^m_{il} \frac{\partial x^i}{\partial \bar{x}^q} \frac{\partial x^l}{\partial \bar{x}^p}) \bar{b}^q \bar{\zeta}^p_k \big) dt. \end{aligned}$$

And by combining, $\frac{\partial^2 \bar{x}^j}{\partial x^l \partial x^i} \frac{\partial x^i}{\partial \bar{x}^q} \frac{\partial x^l}{\partial \bar{x}^p} + \frac{\partial^2 x^l}{\partial \bar{x}^p \partial \bar{x}^q} \frac{\partial \bar{x}^j}{\partial x^l} = \partial_{pq} (\varphi^\alpha \circ \psi^\beta (\varphi^\beta \circ \psi^\alpha))^j = 0$, with the change of coordinate of the Christoffel symbol identity,

$$\bar{\Gamma}^j_{pq} = \frac{\partial \bar{x}^j}{\partial x^m} \Gamma^m_{il} \frac{\partial x^i}{\partial \bar{x}^q} \frac{\partial x^l}{\partial \bar{x}^p} + \frac{\partial^2 x^l}{\partial \bar{x}^p \partial \bar{x}^q} \frac{\partial \bar{x}^j}{\partial x^l},$$

we conclude that,

$$-\bar{\Gamma}^j_{pq} = (\frac{\partial^2 \bar{x}^j}{\partial x^l \partial x^i} \frac{\partial x^i}{\partial \bar{x}^q} \frac{\partial x^l}{\partial \bar{x}^p} - \frac{\partial \bar{x}^j}{\partial x^m} \Gamma^m_{il} \frac{\partial x^i}{\partial \bar{x}^q} \frac{\partial x^l}{\partial \bar{x}^p}).$$

And thus, $(U_\cdot)$ is a solution of (E^{h}_M) in the sense of Definition 1.4.

Step 3: Proof of uniqueness.

Let $(U^1_\cdot), (U^2_\cdot)$ be two solutions of (E^{h}_M), with initial condition equal $\mathbb{P}$-a.s, we can suppose that they are solutions on the same uniformly regular atlas. Indeed, if U^1 is a solution in atlas $\mathfrak{A}^1$, it is a solution in atlas $\mathfrak{A}^2$. Choose a countable dense subset $\mathscr{D}$ of $[0,T]$ with $0 \in \mathscr{D}$. For $t_0 \in \mathscr{D}$, define the events $E^\alpha_{t_0} = \{U^1_{t_0} = U^2_{t_0} \in O^\alpha_0\}$ and $\Omega^\alpha_{[t_0,t_1]} = \{E^\alpha_{t_0}, t_1 < \tau^\alpha_{t_0}\}$, where $\tau^\alpha_{t_0} : E^\alpha_{t_0} \to [t,T]$, is the minimum of the exit time of O^α_0 for each of the processes. Then by hypothesis, since they are both solutions of (E^{h}_M), up to a $\mathbb{P}$-nullset, that we note $Z^\alpha_{[t_0,t_1]}$, they are solutions in the chart Λ^α of the same Euclidean stochastic differential equation, and from the uniqueness result of Lemma 2.7, both processes are equal on $\Omega^\alpha_{[t_0,t_1]} / Z^\alpha_{[t_0,t_1]}$.

Taking $\omega \in \Omega$, both process are continuous a.s, let $s \in [0,T]$ and assume that the processes agree on $(\omega, [0,s))$, then there exists $\alpha \in \mathfrak{K}$, with $\omega \in \Omega^\alpha_{[t_0,t_1]}$ for some $t_0, t_1 \in \mathscr{D}$ satisfying $0 \le t_0 \le s < t_1 \le T$. Then if $\omega \notin Z^\alpha_{[t_0,t_1]}$ we have that the processes agree on $(\omega, [0,t_1])$. Noting that t_1 is strictly greater than s, using the sample continuity and the fact that $\bigcup_{t_0,t_1 \in \mathscr{D}, \alpha \in \mathfrak{K}} Z^\alpha_{[t_0,t_1]}$ is a $\mathbb{P}$-nullset, we obtain the desired result. □

2.3 Stochastic flow estimate and integrability

As mentioned earlier, the type of solution considered here does not require an integrability assumption, in this section, we will prove a Lipschitz in time integrability estimate on the flow of the solutions associated with the drifted Laplace-Beltrami operator, *i.e* associated to $B \cdot \nabla_g + \frac{1}{2}\Delta_g$, where Δ_g is the Trace of the Hessian, in local coordinates it writes $\Delta_g f = g^{ij} \partial_{ij} f - g^{ij} \Gamma^k_{ij} \partial_k f$. In the Euclidean setting, this is straightforward from the triangular inequality of the norm, Jensen's inequality and

Burkholder-Davis-Gundy inequality. On a Riemannian manifold, one needs to use Ito's formula on the distance function, however, outside the injectivity radius the distance function fails to be differentiable. For example, on the 1-dimensional Torus, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, then the distance function $d(0,\cdot)$ is not differentiable at $1/2$. Nevertheless, on a connected manifold of bounded geometry, it is possible to construct a substitute of the distance function with natural smoothness properties, this is the subject of the following result due to Yu.A. Kordyukov.

Lemma 2.8. Suppose that (M,g) is a *connected* manifold of bounded geometry. There exists a function $\tilde{d} : M \times M \to [0,\infty)$ satisfying the following conditions,

- there exists $\rho > 0$ such that:
$$|\tilde{d}(x,y) - d(x,y)| < \rho, \quad \forall x, y \in M.$$
- $\forall x \in M$, $\tilde{r}_x : y \mapsto \tilde{d}(x,y)$ is $C^\infty(M)$, and its derivatives are uniformly bounded in x, in other words, for $n > 0$,
$$\|\nabla^n \tilde{r}_x\|_\infty \leq C(n) \quad \forall x \in M$$

This relies on an appropriate partitioning of the unity to regularise the distance function. There exists a proof in English, in [17][Lemma 2.1 p70], where the statement gives a uniform bound in coordinates of $|\partial_y^\alpha \tilde{d}(x,y)|$, to conclude, one uses the uniform equivalence of the euclidean norm in a local chart with the pullback metric.

We also need to control the Laplacian of the distance function inside the *cut locus*, that is, the set of points sufficiently close, so that they are uniquely connected from the starting point with a geodesic.

Proposition 2.9 (Laplacian Comparison Theorem, [18] Thm 3.4.2, p.90)**.** *Let $x_0 \in M$ and note $r(x) = d(x_0,x)$, suppose that the sectional curvature is bounded from above by K_1^2 and the Ricci curvature is bounded from below by $-(n-1)K_2^2$. Then inside the cut locus of x_0,*

$$(n-1)K_1 cot(K_1 r(x)) \leq \Delta_g r(x) \leq (n-1)K_2 coth(K_2 r(x)).$$

In particular, $r\Delta_M r$ is uniformly bounded on any compact subset of M within the cut locus.

We now can state the stochastic flow estimate.

Theorem 2.10. *Let (M,g) a connected Riemannian Manifold of bounded goemetry. Let $((\Omega,\mathscr{F},\mathbb{P}),W.)$ be a filtered probability space and suppose that $(U.)_{[0,T]}$ is a moving frame solution of*

$$dU_t = H_i^{\mathsf{h}}(U_t) \circ dW_t^i + B^{\mathsf{h}}(t,U_t)dt$$

where $B \in L^\infty([0,T]\times\Omega, C_b(T^1M))$, and $H_i^{\mathsf{h}}(u) = (ue_i)^{\mathsf{h}}$, we note $X_t = \pi(U_t)$.

Then, we have the flow estimate,

$$\mathbb{E}[d^p(X_s,X_t)] < C(|t-s|^p + |t-s|^{p/2}) \quad \forall p \geq 1, \tag{6}$$

where C depends on $T, \|B\|_\infty$ and geometrical bounds.

If there exists $x_* \in M$ such that, $\mathbb{E}[d^p(x^*,X_0)] < +\infty$, for some random variable, we note $X_0 \in L^p(\Omega, d(x_*,\cdot)d\mathbb{P})$. Then (6) ensures that the solution stays in the same space, if the initial condition lives in this space.

Proof. Let us introduce the following notations, $r_y(x) = d(x,y)$ and $R_{s,t} = d(X_s,X_t) = r_{X_s}(X_t)$ for any $s < t \in [0,T]$. Since M is of bounded geometry, according to Lemma 2.8 there exists a C^∞ regularisation, in the second variable, of the distance function, that we denote by $\tilde{d} : M \times M \to [0,\infty[$. Similarly we note by $\tilde{r}_x(y) = \tilde{d}(x,y)$, and $\tilde{R}_{s,t} = \tilde{d}(X_s,X_t)$.

Now, start by noting that

$$\begin{aligned}
d^p(X_s,X_t) &= d^p(X_s,X_t)\mathbb{1}_{\{\sup_{u\in[s,t]} d(X_s,X_u)<\epsilon/2\}} + d^p(X_s,X_t)\mathbb{1}_{\{\sup_{u\in[s,t]} d(X_s,X_u)\geq\epsilon/2\}} \\
&\leq d^p(X_s,X_t)\mathbb{1}_{\{\sup_{u\in[s,t]} d(X_s,X_u)<\epsilon/2\}} + (\tilde{d}(X_s,X_t)+\rho)^p\mathbb{1}_{\{\sup_{u\in[s,t]} d(X_s,X_u)\geq\epsilon/2\}} \\
&\leq d^p(X_s,X_t)\mathbb{1}_{\{\sup_{u\in[s,t]} d(X_s,X_u)<\epsilon/2\}} + C_p(\tilde{d}^p(X_s,X_t)+\rho^p)\mathbb{1}_{\{\sup_{u\in[s,t]} d(X_s,X_u)\geq\epsilon/2\}},
\end{aligned}$$

where ρ is a constant given in Lemma 2.8. The last inequality holds from the convexity of the p-th power on $\mathbb{R}_+$.

We will first show that $\mathbb{E}[\tilde{R}^p_{s,t}]$ is finite. Since $\tilde{r}_{X_s}$ is C^∞, from Ito's formula, we obtain that,

$$\tilde{R}_{s,t} = \int_s^t (\nabla \tilde{r}_{X_s} \cdot B + \frac{1}{2}\Delta_g \tilde{r}_{X_s})(u, X_u)du + \int_s^t (\nabla \tilde{r}_{X_{\tau^n}})(X_u) \cdot U_u e_i dW^i_u.$$

From Lemma 2.8, and the bound on B, there exists C depending on $\|B\|_\infty$ and some constant depending of the geometry, such that, $\sup_{u\in[0,T]} \sup_{x\in M} |(\nabla \tilde{r}_{X_s} \cdot B + \frac{1}{2}\Delta_g \tilde{r}_{X_s})(u,x)| < C \quad \omega\mathbb{P}-a.s.$ Denote by $\tilde{M}_{s,u} = \int_s^u (\nabla \tilde{r}_{X_s})(X_v) \cdot U_v e_i dW^i_v$. Then,

$$\mathbb{E}[\tilde{d}^p(X_s, X_t)] \leq C(|t-s|^p + \mathbb{E}[\sup_{u\in[s,t]} \tilde{M}^p_{s,u}]) \leq C(|t-s|^p + \mathbb{E}[\langle \tilde{M}_s \rangle^{p/2}_t]),$$

from Burkholder-Davis-Gundis inequality. Since $|\nabla \tilde{r}_{X_s}(X_t) \cdot U_t e_i| \leq |\nabla \tilde{r}_{X_s}(X_t)|_g |U_t e_i|_g = |\nabla \tilde{r}_{X_s}(X_t)|_g \leq \||\nabla \tilde{r}|_g\|_\infty$, where $\||\nabla \tilde{r}|_g\|_\infty = \sup_{x,y\in M} |\nabla \tilde{r}_x(y)|_g$, we obtain that,

$$\mathbb{E}[\tilde{d}^p(X_s, X_t)] \leq C(|t-s|^p + |t-s|^{p/2}). \tag{7}$$

Now, to tackle the distance function, recall that $r^2_x : M \to [0, \infty[$, defined as $r^2_x(y) = d^2(x,y)$ is in $C^\infty(B_M(x,\epsilon))$, thus on the event $\{d(X_s, X_t) < \epsilon/2\}$, from Ito's formula, we obtain that,

$$dR^2_{s,t} = (2R_{s,t}(\nabla r_{X_s}(X_t) \cdot B(t, X_t) + \frac{1}{2}\Delta_g r_{X_s}(X_t)) + |\nabla r_{X_s}(X_t)|^2)dt + 2R_{s,t}(\nabla r_{X_s})(X_t) \cdot U_t e_i dW^i_t$$

Now observe that from the comparison proposition 2.9, we have that, $r\Delta_g r \leq (d-1)coth(Lr)Lr$, where L is a bound on the Ricci curvature. From the property of the *coth* function, we deduce that $r\Delta_g r \leq Cr + 1$, for some C. Furthermore, the distance has the property that $|\nabla r|_g = 1$ at any point of differentiability, and by definition of U_t, we have that $|U_t e_i|_g = 1$ for any t, $\mathbb{P}$-a.s. In a similar way as in (7), this gives,

$$\mathbb{E}[d^p(X_s, X_t)\mathbb{1}_{\{\sup_{u\in[s,t]} d(X_s, X_u) < \epsilon/2\}}] \leq C(|t-s|^p + |t-s|^{p/2}). \tag{8}$$

Finally, introducing the exit time $\tau = \inf\{u > s, d(X_s, X_u) > \epsilon\} \bigwedge T$.

$$\begin{aligned}\mathbb{P}\big(\sup_{u\in[s,t]} d(X_s, X_u) \geq \epsilon/2\big) &= \mathbb{P}\big(\sup_{u\in[s,t\wedge\tau]} d(X_s, X_u) \geq \epsilon/2\big) \\ &\leq C_{p,\epsilon}\mathbb{E}\big[\sup_{u\in[s,t\wedge\tau]} d^{2p}(X_s, X_u)\big] \leq C|t-s|^p \wedge 1,\end{aligned} \tag{9}$$

Summing-up the estimates (7), (8) and (9), in

$$\mathbb{E}[d^p(X_s, X_t)] \leq \mathbb{E}[d^p(X_s, X_t)\mathbb{1}_{\{\sup_{u\in[s,t]} d(X_s, X_u) < \epsilon/2\}}] + \sqrt{\mathbb{E}[(\tilde{d}(X_s, X_t) + \rho)^{p/2}]}\sqrt{\mathbb{P}\big(\sup_{u\in[s,t]} d(X_s, X_u) \geq \epsilon/2\big)},$$

concludes the proof. □

Remark 2.11. Note that by controlling a sequence of exit time, only a bound from below on the Ricci curvature and on the injectivity radius are needed to prove this estimate. It is also possible to prove this statement for a general tensor A as in Theorem 2.5.

3 Appendix

3.1 Riemannian Geometry

We recall here a few definitions linked to curvature in Riemannian geometry needed for the definition of a manifold of bounded geometry. In the following, (M, g) is a Riemannian manifold equipped with its Levi-Civita connection. We recommend J. Lee's book [16], for the notions of *vector bundle*, *Riemannian metric*, *connection*, *tensor fields* and for a detailed presentation of curvature, with the natural derivation of the Riemann curvature tensor. In this paper, we refer to the following definition of a Manifold.

Definition 3.1 (Smooth Manifold)**.** A d-dimensional manifold M is second countable topological space, such that there exists an atlas $\{O^\alpha, \phi^\alpha\}_{\alpha\in\mathfrak{K}}$, where the O^α forms an open cover of M and ϕ^α is a diffeomorphism from M to $\mathbb{R}^n$. A couple (O, ϕ) is called a local chart.

At the first order, a Riemannian manifold locally resembles an Euclidean space. At the second order, the Riemann curvature tensor characterizes how the manifold deviates from being locally isometric to Euclidean space.

Definition 3.2 (Riemann Curvature Tensor)**.** Let $X, Y, Z \in C^2(TM)$ be three vector fields on M. We define the *Riemann curvature tensor* with the following formula,

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

where $[X, Y]$ is the Lie bracket of the vector fields. In coordinates it writes,

$$R^l_{ijk} = \partial_j\Gamma^l_{ik} - \partial_k\Gamma^l_{ij} + \Gamma^l_{jm}\Gamma^m_{ik} - \Gamma^l_{km}\Gamma^m_{ij}.$$

From the Riemann curvature tensor we can introduce, the *Ricci curvature* tensor, it is defined as the contraction of the Riemann tensor. In coordinate it writes,

$$\mathrm{Ric}_{ij} = R^k_{ikj}.$$

We say that the Ricci curvature tensor is bounded from below, if there exists a scalar $\lambda > 0$, such that $\forall X \in C(TM)$,

$$-\lambda g(X,X) \leq \mathrm{Ric}(X,X) \text{ (resp. } \mathrm{Ric}(X,X) \leq \lambda g(X,X)),$$

or equivalently in coordinates, $-\lambda g_{ij} \leq \mathrm{Ric}_{ij}$ in the sens of positive definite matrices.

We also introduce, for $u, v \in T_xM$ two linearly independent tangent vectors in the tangent space at a point x, the sectional curvature as,

$$\frac{g_x(R_x(u,v),u)}{g_x(u,u)g_x(v,v) - g_x(u,v)^2}.$$

Finally, recall that the covariant derivative of a tensor is induced from the connection on the tangent space, it is defined as follows, for a tensor field $A \in C^1(T^n_mM)$, the covariate derivative is a $(n, m+1)$-tensor, defined in coordinate as,

$$(\nabla A) \doteq \partial_i a^{k_1\cdots k_n}_{j_1\cdots j_m} + \Gamma^{k_1}_{il} a^{lk_2\cdots k_n}_{j_1\cdots j_m} + \ldots + \Gamma^{k_n}_{il} a^{k_1\cdots k_{n-1}l}_{j_1\cdots j_m} - \Gamma^l_{j_1 i} a^{k_1\cdots k_n}_{lj_2\cdots j_m} - \ldots - \Gamma^l_{j_n i} a^{k_1\cdots k_n}_{j_1\cdots j_{m-1}l}.$$

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