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► **To cite this version:**

François Boulier, François Lemaire. On Formal Power Series Solutions of Regular Differential Chains. 2024. hal-04547418v2

HAL Id: hal-04547418

<https://hal.science/hal-04547418v2>

Preprint submitted on 29 Apr 2024 (v2), last revised 11 Sep 2024 (v3)

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On Formal Power Series Solutions of Regular Differential Chains

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Abstract. In their celebrated 1984 article, Denef and Lipshitz provide an algorithm for deciding whether a finite system of ordinary differential polynomials has formal power series solutions. This algorithm has never been implemented. In this paper, we describe some key algorithmic tools for transforming this decision result into practical software.

1 Introduction

We are concerned with [10, Theorem 3.1] which provides an algorithm for deciding whether a finite system of ordinary differential polynomials, with coefficients in $\mathbb{Q}[x]$ has formal power series solutions in $\mathcal{F}[[x]]$ where \mathcal{F} stands for \mathbb{C} , \mathbb{R} or \mathbb{Q}_p .

The problem belongs to differential algebra, which is an algebraic theory for systems of ordinary or partial polynomial differential equations, founded by Joseph Fels Ritt in the first half of the former century [26, 27] and developed by Ellis Robert Kolchin [18].

The addressed problem is more difficult than deciding whether 1 belongs to the differential ideal generated by the input differential polynomials, which is solved by differential elimination methods in both the ordinary and the partial case [6]. Indeed differential elimination methods solve the existence problem of formal power series solutions in $\mathcal{F}[[x - x_0]]$ where x_0 stands for some unspecified expansion point (or initial value) while Denef and Lipshitz Theorem addresses the existence problem of formal power series solutions, in the ordinary case, for a given expansion point x_0 . In the partial case, the same problem turns out to be algorithmically undecidable [10, Theorem 4.11].

The proof of [10, Theorem 3.1] is difficult. It is remarkable that it solves the more general (and interesting!) problem of deciding whether a finite system of ordinary differential polynomials has formal power series solutions the coefficients of which satisfy polynomial constraints.

This second problem is important since it contains as a particular case the existence problem of formal power series solutions for given initial values. Consider any Runge-Kutta scheme [14, 15]: the entries of the Butcher tableaux are obtained by identifying the coefficients of the formal power series solution of the initial value problem under consideration (which is thus assumed to exist and

be unique) with the ones of the Taylor expansion of the Runge-Kutta scheme. The same observation holds for all classical numerical integration schemes.

In this paper, we provide basic algorithmic tools, based on the theories of regular chains and regular differential chains, for solving key issues arising in the proof of [10, Theorem 3.1] and describe a partial implementation available in the recent Python `DifferentialAlgebra` package. By lack of time, it was not possible to finish a general algorithm solving the addressed problem. However, readers who would compare our text to the original one will guess that the structure of the proof of this general algorithm is very likely to be quite different from the one of Denef and Lipshitz (though all essential theoretical tools are given in [10]). In particular, we address the uniqueness problem of formal power series solutions (see subcase 3.1 of Section 5).

In Section 2, we provide some mathematical background including an original interesting example (Example 2) and a related undecidability result. Section 3 is almost completely borrowed from [10] with a slight generalization of [10, Lemma 2.3] (our Lemma 2). Section 4 is a synthesis of known results on regular chains and does not contain any new result. Section 5 shows how to apply the results given in Section 3 to the context of formal power series the coefficients of which are taken modulo squarefree regular chains. The content of this whole section is new. Section 6 is new also: it explains how to easily solve an algorithmic problem raised in Section 3. The fact that the content of this section is easy is good news. Indeed, the corresponding step in [10, page 227], does not look that simple at first sight (it is actually related to the step which does not generalize well to the partial case). Let us quote it (here, V denotes a general algebraic variety and I_i a family of multivariate polynomials): *compute (by elimination theory over an algebraically closed field)*

$$E = \{q \in \mathbb{N} \mid \exists i \forall t \in V, I_i(q, t) = 0\}.$$

Related work. On the symbolic computing side, the results of [10] we are dealing with are known by the differential algebra community. In particular, they have been used for deciding whether expressions depending on power series are zero [24, 30]. They have recently gained a renewal of interest because of their relationship with tropical differential algebraic geometry [13, 11, 1, 12]. On the numerical side, they should be related to the computation of the numerical index of DAE [15] and to the problem of numerically integrating differential-algebraic equations (DAE) by computing Taylor series coefficients, which is solved by the DAETS software starting from a structural analysis of the DAE [25, 23]. Note that the recent [32] has pointed out weaknesses in the structural analysis stage of DAETS, for specific values of the DAE parameters. However, all these works do not mention the results of Denef and Lipshitz, probably because of the very algebraic nature of [10]. A precise compared analysis of all these works still remains to be done.

2 Mathematical Background

On the one hand, [10, Theorem 3.1] is concerned with formal power series solutions of systems of non autonomous ordinary differential polynomial equations i.e. differential polynomials which explicitly depend on the unique independent variable x , assuming that differentiations are performed with respect to x . On the other hand, classical Ritt-Kolchin differential algebra only considers autonomous differential polynomials with coefficients in a field. There is no fundamental difference between these two approaches as long as non autonomous equations have polynomial coefficients in x since it is always possible to reduce the framework of [10] to the one of Ritt and Kolchin by performing a change of derivation $d/dx = d/d\xi$, viewing the former independent variable x occurring in the coefficients as a new differential indeterminate subject to $\dot{x} = 1$ and transforming the expansion point x_0 of the sought formal power series into an initial value for $x(\xi)$. This being understood, we will often consider non autonomous equations within Ritt-Kolchin theory.

Let \mathcal{F} be an algebraically closed ordinary differential field of characteristic zero (we do not consider the case of \mathcal{F} being \mathbb{R} or \mathbb{Q}_p but we admit the case of \mathcal{F} being the algebraic closure of \mathbb{Q}), $\mathcal{F}\{y_1, \dots, y_\nu\}$ be an ordinary differential polynomial ring in ν differential indeterminates over \mathcal{F} and $\Sigma \subset \mathcal{F}\{y_1, \dots, y_\nu\}$ be a system of differential polynomials. We are concerned by the existence and uniqueness problem of formal power series solutions $\bar{y}_1, \dots, \bar{y}_\nu$ of Σ in $\mathcal{F}[[x-x_0]]$ for some prescribed initial values, where x_0 denotes a prescribed expansion point. To simplify notations, we assume that x_0 is the origin and seek solutions in $\mathcal{F}[[x]]$.

In principle, the computation of a formal power series solution of some differential polynomial $f \in \mathcal{F}\{y\}$ can be achieved by the following process. Assign unknown coefficients \bar{y}_i to the sought formal power series

$$\bar{y} = \bar{y}_0 + \bar{y}_1 x + \frac{\bar{y}_2}{2} x^2 + \dots \quad (1)$$

Assume f has order n in y so that $f = f(x, y, \dot{y}, \dots, y^{(n)})$. Given any non-negative integer p , beware to distinguish $f^{(p)}(x, \bar{y}, \bar{y}', \dots, \bar{y}^{(n+p)})$ which is the formal power series obtained by evaluating the p th derivative of f at $y = \bar{y}$ from $f^{(p)}(0, \bar{y}_0, \bar{y}_1, \dots, \bar{y}_{n+p})$ which is a polynomial of $\mathcal{F}[\bar{y}_0, \bar{y}_1, \dots, \bar{y}_{n+p}]$ obtained by renaming each derivative $y^{(i)}$ as \bar{y}_i and replacing x by 0 (the expansion point) in $f^{(p)}$. The next formula is well-known (see [28, Lemma, page 160])

$$f(x, \bar{y}, \bar{y}', \dots, \bar{y}^{(n)}) = \sum_{p=0}^{\infty} \frac{1}{p!} f^{(p)}(0, \bar{y}_0, \bar{y}_1, \dots, \bar{y}_{n+p}) x^p.$$

The unknown coefficients \bar{y}_i must then be chosen so that they annihilate the infinite polynomial system $f^{(p)}(0, \bar{y}_0, \bar{y}_1, \dots, \bar{y}_{n+p}) = 0$ for $p \geq 0$.

Let us now give two examples illustrating the issues raised by our problem.

Example 1. Consider the following system:

$$\begin{aligned} x \dot{y} &= a y, \\ \dot{a} &= 0. \end{aligned} \tag{2}$$

The second equation tells us that a is a constant. Concerning y , one of the formal power series solutions is $y = 0$ (the zero function). However, if $a \in \mathbb{N}$, another formal power series solution is the monomial $y = x^a$ which has its a first coefficients in common with the zero function. Developing this example and using Matiassevitch negative answer [20] to Hilbert's Tenth Problem, Singer [29, pages 89-90] (see also [10, Proposition 3.3]) proved that the following problem is algorithmically undecidable: *determine whether a finite system Σ with coefficients in $\mathbb{Q}[x]$ has a nonzero formal power series solution.*

Example 2. Consider the following equation with coefficients in $\mathbb{Q}[x]$:

$$f(x, y, \dot{y}, \ddot{y}) = x y \ddot{y} + \dot{y} + y^2 + 1 = 0. \tag{3}$$

It has a formal power series solution in $\mathcal{F}[[x]]$ (which is then uniquely defined) if and only if its initial value (we denote y_i rather than \bar{y}_i for legibility) $y_0 \neq -1/a$ for all $a \in \mathbb{N}^*$. Moreover, if $y_0 = -1/a$ then $f(x, y, \dot{y}, \ddot{y}) = 0 \pmod{x^a}$ admits a solution. This can easily be seen by considering the polynomial system to be solved in order to have a formal power series solution:

$$\begin{aligned} f(0, y_0, y_1, y_2) &= y_1 + y_0^2 + 1 \\ f'(0, y_0, \dots, y_3) &= (y_0 + 1) y_2 + 2 y_0 y_1 \\ f''(0, y_0, \dots, y_4) &= (2 y_0 + 1) y_3 + (2 y_1 + 2 y_0) y_2 + 2 y_1^2 \\ f^{(3)}(0, y_0, \dots, y_5) &= (3 y_0 + 1) y_4 + (6 y_1 + 2 y_0) y_3 + 6 y_1 y_2 + 3 y_2^2 \\ &\vdots \end{aligned}$$

Given any y_0 , there exists a unique value for y_1 which annihilates the first equation. If $y_0 \neq -1/a$ for all $a \in \mathbb{N}^*$ then the leading coefficients of the equations do not vanish and the polynomial system has a unique solution. If $y_0 = -1/a$ for some $a \in \mathbb{N}^*$ then the leading coefficient of $f_a = f^{(a)}(0, y_0, \dots, y_{a+2})$ vanishes. By considering the signs of the monomials, it is easy to see that the first a equations force y_1, \dots, y_a to have negative values while f_a forces y_a to have a positive value: the system is inconsistent.

An undecidability result. In the differential polynomial ring in ν differential indeterminates y_1, \dots, y_ν , introduce ν copies $f_i(x, y_i, \dot{y}_i, \ddot{y}_i)$ of the differential polynomial f introduced in the above example, for $i = 1, \dots, \nu$. Consider the differential equation (the product of the f_i):

$$f_1 f_2 \cdots f_\nu = 0, \tag{4}$$

in the differential indeterminates y_1, \dots, y_ν subject to the following polynomial system on the initial values $y_{1,0}, \dots, y_{\nu,0}$ and ν other indeterminates a_1, \dots, a_ν ,

where $p \in \mathbb{Q}[x_1, \dots, x_\nu]$:

$$\Omega \begin{cases} y_{i,0} (a_i + 1) + 1 = 0, & (i = 1, \dots, \nu), \\ p(a_1, \dots, a_\nu) = 0. \end{cases}$$

Assume $(a_1, \dots, a_\nu) \in \mathbb{N}^\nu$ is a zero of p . Then the differential equation (4) has no formal power series solutions. Conversely, assume (a_1, \dots, a_ν) is a zero of p such that (say) $a_1 \notin \mathbb{N}$. Then there exists a formal power series \bar{y}_1 with initial value $y_{1,0}$ which annihilates the first factor f_1 of (4). Thus the tuple of series $(\bar{y}_1, \bar{y}_2, \dots, \bar{y}_\nu)$ where \bar{y}_k is any formal power series with initial value $y_{k,0}$ for $k = 2, \dots, \nu$, is a formal power series solution of the differential equation (4).

Summarizing, the differential equation (4) has formal power series solutions for all solutions of the polynomial system Ω if and only if p has no zero in \mathbb{N}^ν . By Matiassevitch's Theorem, the following problem is thus undecidable: *determine if an ordinary differential polynomial system Σ with coefficients in $\mathbb{Q}[x]$ whose initial values are subject to a polynomial system Ω with coefficients in \mathbb{Q} has formal power series solutions for all solutions of Ω .*

A variant. Consider the following system Σ of the differential polynomial ring $\mathcal{R} = \mathbb{Q}\{y_1; y_{i,0}; z_i; a_i\}$ (for $i = 1, \dots, \nu$) where all symbols are differential indeterminates (including x and the $y_{i,0}$). As above, the f_i are ν copies of (3) and $p \in \mathbb{Q}[x_1, \dots, x_\nu]$.

$$\Sigma \begin{cases} f_1 f_2 \cdots f_\nu = 0, \\ \dot{x} = 1, \\ y_{i,0} + x z_i = y_i, & i = 1, \dots, \nu, \\ \dot{y}_{i,0} = 0, & i = 1, \dots, \nu, \\ y_{i,0} (a_i + 1) + 1 = 0, & i = 1, \dots, \nu, \\ p(a_1, \dots, a_\nu) = 0. \end{cases}$$

Since we are looking for solutions as formal power series, the third equation states that the differential indeterminates $y_{i,0}$ encode the initial values of y_i for $i = 1, \dots, \nu$. System Σ is a polynomial DAE hence a particular case of a polynomial differential system. Let us define precisely what we mean by "initial values compatible with Σ ": they are the constant solutions of the ideal $\{\Sigma\} \cap \mathcal{R}_0$ where $\{\Sigma\}$ denotes the perfect differential ideal generated by Σ (see [18, III, page 122]) and \mathcal{R}_0 the ring of the order zero differential polynomials of \mathcal{R} . We see that the initial values compatible with Σ can be prolonged into formal power series if and only if there exists some $1 \leq i \leq \nu$ such that $a_i \notin \mathbb{N}$. The following problem is thus seen to be undecidable: *given an ordinary differential polynomial system Σ with coefficients in \mathbb{Q} , determine whether all initial values compatible with Σ can be prolonged into formal power series solutions of Σ .*

3 The Prolongation Bounds

This section aims at presenting and slightly generalizing the part of Denef and Liphitz paper which is important for our problem. This part is a subset of [10, Lemma 2.2 to Theorem 3.1].

The key result is the following Lemma, due to Adolf Hurwitz [17, pages 328-329]. It is restated in [10, Lemma 2.2]. For a better relationship with these articles, we keep in this section, the somewhat unusual notation f_n for the separant.

Lemma 1. *Let f be a differential polynomial of order n in $\mathcal{F}\{y\}$. Let $k \in \mathbb{N}$. Then*

$$f^{(2k+2)} = y^{(n+2k+2)} f_n + y^{(n+2k+1)} f_{n+1} + y^{(n+2k)} f_{n+2} + \cdots + y^{(n+k+2)} f_{n+k} + f_{n+k+1}, \quad (5)$$

where the f_j are differential polynomials of order at most j in y for all indices $j = n, n+1, \dots, n+k+1$ and $f_n = \partial f / \partial y^{(n)}$. The differential polynomials $f_{n+1}, f_{n+2}, \dots, f_{n+k+1}$ depend on k but f_n does not.

Let now $q \in \mathbb{N}$. Differentiating q more times formula (5) we get

$$\begin{aligned} f^{(2k+2+q)} &= y^{(n+2k+2+q)} f_n + y^{(n+2k+2+q-1)} [f_{n+1} + q f'_n] \\ &\quad + \cdots + y^{(n+2k+2+q-r)} \left[f_{n+r} + q f'_{n+r-1} + \cdots + \binom{q}{r} f_n^{(r)} \right] \\ &\quad + \cdots + y^{(n+2k+2+q-k)} \left[f_{n+k} + q f'_{n+k-1} + \cdots + \binom{q}{k} f_n^{(k)} \right] \\ &\quad + h_{n+k+q+1}, \end{aligned} \quad (6)$$

where $h_{n+k+q+1}$ has order at most $n+k+q+1$ in y .

We will apply Lemma 1 in the differential polynomial ring $\mathcal{F}\{y_1, \dots, y_\nu\}$ equipped with a *ranking* [18] which is a total ordering over the infinite set of the derivatives of the differential indeterminates y_1, \dots, y_ν . In this setting, the derivative denoted $y^{(n)}$ in Lemma 1 is the *leading derivative* of f i.e. the highest derivative occurring in f with respect to the ranking and the differential polynomial f_n is the *separant* of f . Assume the leading derivative of f is a derivative of (say) y_1 . Lemma 1 will be applied with $y = y_1$ and the differential polynomial ring $\mathcal{F}\{y_2, \dots, y_\nu\}$ (more accurately, its field of fractions) in place of the base field \mathcal{F} .

In the sequel, we will apply the Lemma for the following value of k . The two next definitions and Lemma 2 form a slight generalization of [10, Lemma 2.3].

Definition 1. *(definition of k and c_0)*

Let $f(x, y, \dots, y^{(n)})$ be a differential polynomial of order n and $\bar{y} \in \mathcal{F}[[x]]$ be a formal power series which does not annihilate the separant f_n of f . Then one defines k as the valuation of $f_n(\bar{y})$ i.e. the nonnegative integer such that

$$f_n(x, \bar{y}, \bar{y}', \dots, \bar{y}^{(n)}) = c_0 x^k + c_1 x^{k+1} + \cdots \quad (c_0 \neq 0)$$

Denote \bar{y}_i the coefficients of the ‘‘initial value’’ encoding series \bar{y} .

$$\bar{y} = \bar{y}_0 + \bar{y}_1 x + \frac{\bar{y}_2}{2} x^2 + \cdots \quad (7)$$

Evaluate (6) at $(x, y, \dot{y}, \dots) = (0, \bar{y}, \bar{y}', \dots)$. Then $f_n^{(k)}(0, \bar{y}_0, \bar{y}_1, \dots, \bar{y}_{n+k}) = c_0 \neq 0$. Thus the last term of (6) evaluates to a nonzero value. This remark allows us to state the next definition.

Definition 2. (definition of r and $A(q)$)

Let f, \bar{y}, k be as in Definition 1. One defines r ($0 \leq r \leq k$) as the smallest integer such that

$$\left[f_{n+r} + q f'_{n+r-1} + \dots + \binom{q}{r} f_n^{(r)} \right] (0, \bar{y}_0, \bar{y}_1, \dots) \neq 0. \quad (8)$$

The left hand side of (8) is a nonzero polynomial of $\mathcal{F}[q]$, denoted $A(q)$.

The next Lemma is more precise than [10, Lemma 2.3] which does not state that the series \bar{y} is unique.

Lemma 2. Let f, \bar{y}, k be as in Definition 1. Let r and $A(q)$ be as in Definition 2. Let $\gamma \in \mathbb{N}$ be bigger than any integer root of $A(q)$. Denote $\beta = 2k + 2 + \gamma + r$ and $\delta = n + 2k + 2 + \gamma$. Suppose

$$f(x, \bar{y}, \bar{y}', \dots, \bar{y}^{(n)}) = 0 \pmod{x^\beta}. \quad (9)$$

Then there exists a unique formal power series $\bar{\bar{y}} \in \mathcal{F}[[x]]$ such that

$$\bar{\bar{y}} = \bar{y} \pmod{x^\delta}, \quad (10)$$

and

$$f(x, \bar{\bar{y}}, \bar{\bar{y}}', \dots, \bar{\bar{y}}^{(n)}) = 0. \quad (11)$$

Proof. The existence of $\bar{\bar{y}}$ is proved by [10, Lemma 2.3]. Let us address the uniqueness. In the proof of [10, Lemma 2.3] it is written that $\bar{\bar{y}}$ is determined as follows for $q \geq \gamma + r$ (we use our notations introduced in (1) and (7))

$$\bar{\bar{y}}_i = \bar{y}_i \quad (i = 0, \dots, \delta - 1), \quad (12)$$

$$A(q) \bar{\bar{y}}_{n+2k+2+q-r} = B_q \quad (13)$$

where the polynomial B_q depends on \bar{y}_j for $j < n + 2k + 2 + q - r$ only. Observe that $q \geq \gamma + r$ implies that $n + 2k + 2 + q - r \geq \delta$. From (10) and (12), the coefficients $\bar{\bar{y}}_i$ of $\bar{\bar{y}}$ for $i = 0, \dots, \delta - 1$ are uniquely defined. The other coefficients are uniquely defined by (13).

Remark. The series \bar{y} satisfies condition (9) if and only if its coefficients form a zero of the following system of β polynomial (prolongation) equations:

$$f(0, \bar{y}_0, \dots, \bar{y}_n) = \dots = f^{(\beta-1)}(0, \bar{y}_0, \dots, \bar{y}_{\delta-1}) = 0. \quad (14)$$

3.1 An approximation Lemma

The following Lemma 3 is a version of [10, Lemma 2.9] slightly simplified to fit our needs.

Lemma 3. *Let \mathcal{F}_0 be a finitely generated subfield of \mathcal{F} . Let \mathfrak{p} be a prime ideal of $\mathcal{F}_0[x_1, \dots, x_r]$ and $W_j(x_1, \dots, x_r)$, where $j \in \mathbb{N}$, be a collection of polynomials over \mathcal{F}_0 of bounded degree. Let V be the algebraic variety of \mathfrak{p} with coordinates in \mathcal{F} . Suppose that, for every $j \in \mathbb{N}$ there exists $(x_1, \dots, x_r) \in V$ such that $W_j(x_1, \dots, x_r) \neq 0$ and that there exists a nonsingular point $(a_1, \dots, a_r) \in V$.*

Then there exists a point $(b_1, \dots, b_r) \in V$ such that $W_j(x_1, \dots, x_r) \neq 0$ for all $j \in \mathbb{N}$.

3.2 Examples

Example 3. We consider the following ODE arising from the brachistochrone and the tautochrone problems. The general solution is a cycloid generated by a circle of diameter 1. For $\bar{y}_0 = 0$ it has no formal power series (but a formal Puiseux series) solution. For $\bar{y}_0 = 1$, it has two formal power series solutions (one for the cycloid and one for the singular solution $y(x) = 1$ which touches the cycloid at this point).

$$f = y \dot{y}^2 + y - 1.$$

We have $n = 1$ and $f_n(x, y, \dot{y}) = 2y\dot{y}$. First consider the case $\bar{y}_0 = 0$. In order to secure k , assume $\bar{y}_1 \neq 0$. Then $f_n(x, y, \dot{y}) = 2\bar{y}_1^2 x + \dots$ and $k = 1$. Since the equation is autonomous, we have $r = k = 1$ and $A(q) = 2\bar{y}_1^2 q + 9\bar{y}_1^2 + 1$. The polynomial $A(q)$ has no nonnegative integer root. We can thus choose $\gamma = 0$ which gives us $\beta = \delta = 5$. The polynomial $f(0, \bar{y}_0, \bar{y}_1)$ is equal to 1. Thus (14) is inconsistent, proving that no formal power series solution exists. This example shows that the bounds β and δ may be pessimistic.

Consider the case $\bar{y}_0 = 1$. Looking at the polynomials $f^{(i)}(0, \bar{y}_0, \dots)$ for $i = 0, \dots, 4$, we see that only two choices are possible for the 5 first coefficients $\bar{y}_0, \dots, \bar{y}_4$ which are $(1, 0, 0, 0, 0)$ or $(1, 0, -\frac{1}{2}, 0, -\frac{1}{2})$. We cannot apply Lemma 2 for the first choice since $k = \infty$ (the separant f_n vanishes). Let us consider the second choice. In this case, we find $k = r = 1$ and $A(q) = -q - 3$. Since $A(q)$ has no nonnegative integer roots, we may choose $\gamma = 0$ and we get $\beta = \delta = 5$. Applying Lemma 2, we conclude that the ODE has a single formal power series solution starting as follows.

$$\bar{y} = 1 - \frac{x^2}{4} - \frac{x^4}{48} + \dots$$

Example 4. Consider again (2). We have $n = 1$. The separant $f_n = x$ does not depend on y thus, for any series \bar{y} we have $k = r = 1$ and $A(q) = q + 4 - a$. If a is not an integer or $a = 0, 1, 2, 3$ then $A(q)$ has no nonnegative integer root and we may choose $\gamma = 0$, leading to $\beta = \delta = 5$.

For values of a which are not nonnegative integers, the polynomial system which must be satisfied by \bar{y} is $\bar{y}_0 = \dots = \bar{y}_4 = 0$ and $\bar{y} = 0$ is the unique corresponding series solution.

For nonnegative integer values of a less than 4 (say) $a = 3$, the polynomial system which must be satisfied by \bar{y} is $\bar{y}_0 = \bar{y}_1 = \bar{y}_2 = \bar{y}_4 = 0$. The coefficient \bar{y}_3 may be any number and $\bar{y} = \bar{y}_3 x^3/6$ is the unique corresponding series solution. Observe that this case includes the zero solution.

For nonnegative integer values of a greater than or equal to 4, the least value we may choose is $\gamma = a - 3$ which leads to $\beta = \delta = a + 2$. The polynomial system which must be satisfied by \bar{y} imposes $\bar{y}_i = 0$ for $i = 0, \dots, a - 1, a + 1$. The coefficient \bar{y}_a may be any number and $\bar{y} = \bar{y}_a x^a/a!$ is the unique corresponding series solution. Again, this case includes the zero solution.

Example 5. Consider again (3). We have $n = 2$. The separant $f_n(x, y) = x y$. First consider the case $\bar{y}_0 = 0$. Looking at the ODE, we see that $\bar{y}_1 = -1$ necessarily. This leads us to $k = 2$, $r = 1$ and $A(q) = 1$. Choosing $\gamma = 0$ we get $\beta = 7$ and $\delta = 8$. The polynomial system which must be satisfied by \bar{y} admits the unique following solution. The corresponding series \bar{y} exists and is unique.

$$(\bar{y}_2, \dots, \bar{y}_7) = (0, -2, -12, -160, -3400, -106160).$$

Consider now the case $\bar{y}_0 \neq 0$. Then $k = r = 1$ and $A(q) = \bar{y}_0 q + 4 \bar{y}_0 + 1$. If $\bar{y}_0 \neq -1/a$ for every nonnegative integer a or $a = 1, 2, 3$ then $A(q)$ has no nonnegative integer root and we may choose $\gamma = 0$, leading to $\beta = 5$ and $\delta = 6$. The polynomial system which must be satisfied by \bar{y} is inconsistent if $a = 1, 2, 3$ else it admits a unique solution. In the latter case, the corresponding series \bar{y} exists and is unique.

If $\bar{y}_0 = -1/a$ for some nonnegative integer $a \geq 4$ then the least value we may choose is $\gamma = a - 3$ leading to $\beta = a + 2$ and $\delta = a + 3$. The polynomial system which must be satisfied by \bar{y} is necessarily inconsistent (see the analysis of (3)) proving that there does not exist any series \bar{y} .

4 Basic notions on regular chains

In the sequel, we will consider systems mixing differential polynomials and regular polynomials (expressing constraints on the series coefficients). Since a regular polynomial can always be viewed as a differential polynomial of order zero, strictly speaking, we do not need to distinguish regular polynomials from differential ones. However, though we consider that we are working in some differential polynomial ring $\mathcal{F}\{y_1, \dots, y_\nu\}$ endowed with a ranking [18, chap. II, sect. 8], we will perform the distinction, for a better legibility.

We consider a system

$$f_1, \dots, f_\rho, g_1, \dots, g_\sigma \tag{15}$$

where the f_i are differential equations and the g_i are polynomial equations. The leading derivatives of the f_i are derivatives of ρ different differential indeterminates y_1, \dots, y_ρ for which we seek formal power series solutions $\bar{y}_1, \dots, \bar{y}_\rho$.

Moreover, the f_i are pairwise partially reduced with respect to each other, so that $\{f_1, \dots, f_\rho\}$ forms a triangular set of pairwise partially autoreduced differential polynomials. The polynomials g_i provide constraints on the coefficients $\bar{y}_{i,j}$ where $i = 1, \dots, \rho$ and $j \geq 0$ of the formal power series. The f_i and the g_j thus depend on disjoint sets of symbols and may be considered separately.

We assume that the set $C = \{g_1, \dots, g_\sigma\}$ forms a squarefree regular chain (see [2] or the more recent [7, Definitions 6 and 24]).

The regular chain C defines a radical ideal $\mathfrak{a} = (C) : I_C^\infty$ which is the saturation of the ideal (C) generated by C by the multiplicative family generated by the initials of the g_i . The ideal \mathfrak{a} is radical because C is squarefree (see [19, 21], [16, Proposition 7.6] or [7, Proposition 25]). The ideal \mathfrak{a} is distinct from the unit ideal because C is a regular chain (see [7, Proposition 19] and references therein).

Given any polynomial p , the regular chain C permits to compute the *pseudoremainder of p by C* which is defined as follows (the leading variable of g_i is denoted x_i):

$$\text{prem}(p, C) = \begin{cases} \text{prem}(p, g_1, x_1) & (\sigma = 1) \\ \text{prem}(\text{prem}(p, g_\sigma, x_\sigma), C') & (C' = C \setminus \{g_\sigma\}) \end{cases}$$

and the *resultant of p by C* , which is defined as follows:

$$\text{res}(p, C) = \begin{cases} \text{res}(p, g_1, x_1) & (\sigma = 1) \\ \text{res}(\text{res}(p, g_\sigma, x_\sigma), C') & (C' = C \setminus \{g_\sigma\}) \end{cases}$$

The next proposition is a well-known property of regular chains. See [2, Theorem 6.1].

Proposition 1. *Let p be a polynomial and C be a regular chain defining an ideal \mathfrak{a} . Then $p \in \mathfrak{a}$ if and only if $\text{prem}(p, C) = 0$.*

The next proposition was proved in [8, Lemma 4] in the zerodimensional case. A variant of it is [31, Proposition 5.3]. See [7, Theorem 21, $\mathbf{1} \Rightarrow \mathbf{4}$].

Proposition 2. *Let p be a polynomial and C be a regular chain defining an ideal \mathfrak{a} . Then p is a zerodivisor modulo \mathfrak{a} if and only if $\text{res}(p, C) = 0$.*

The splitting cases mechanism. Since \mathfrak{a} is radical, it is equal to the intersection of its associated prime ideals $\mathfrak{a} = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_\pi$. If $\text{prem}(p, C) = 0$ then p belongs to all the associate prime ideals. If $\text{res}(p, C) = 0$ then p belongs to at least one of the associated prime ideals. If $\text{prem}(p, C) \neq 0$ and $\text{res}(p, C) = 0$ then it is possible to decompose \mathfrak{a} as an intersection $\mathfrak{a}_0 \cap \mathfrak{a}_1$ where \mathfrak{a}_0 (represented by a squarefree regular chain C_0) is the intersection of the associated prime ideals which contain p and \mathfrak{a}_1 (represented by a squarefree regular chain C_1) is the intersection of the other ones. This splitting cases mechanism is at the core of regular chain decomposition methods such as the one described in [22]. See also [4, Fig. 3]. Starting from a given regular chain C , it can only be performed finitely many (less than π) times.

On the variables the resultants depend on. Denote v_1, \dots, v_κ the variables occurring in $\text{res}(p, C)$. By the equidimensionality property of ideals defined by regular chains (this is actually even true for general triangular sets), these variables are algebraically independent modulo \mathfrak{p} where \mathfrak{p} is any associated prime ideal of \mathfrak{a} . See [7, section 5] and references therein.

5 The case of series depending on parameters

In this section we focus on the case of a single squarefree regular chain $C = \{g_1, \dots, g_\sigma\}$. This regular chain is supposed to arise from a general system Ω , simplified through a regular chain decomposition algorithm such as the one described in [22] and implemented in the Maple `RegularChains` package or the `RosenfeldGroebner` of the Maple and Python `DifferentialAlgebra` packages.

This general system Ω from which C is supposed to be computed involves two types of equations:

1. prolongation equations i.e. polynomial equations obtained by differentiation and evaluation of the differential polynomials f_1, \dots, f_ρ ,
2. other polynomial equations introducing further constraints on the coefficients of the formal power series.

Let us focus on the prolongation equations. Assume that the leading derivative of each f_i is $y_i^{(n_i)}$ for $i = 1, \dots, \rho$. Denote μ_1, \dots, μ_ρ the prolongation bounds used to build the general system Ω so that the prolongation equations of Ω are the following ones, for $i = 1, \dots, \rho$:

$$f_i^{(p_i)}(0, \bar{y}_{i,j}), \quad p_i = 0, \dots, \mu_i.$$

Assumption on prolongation bounds. The overall management of the prolongation bounds μ_1, \dots, μ_ρ is a complicated issue, which is not addressed in this paper. In the sequel, we are sometimes going to assume that μ_1, \dots, μ_ρ are “large enough” without any further details, to avoid the description of a general decomposition algorithm.

Denote \bar{y}_i the initial value encoding formal power series (with coefficients $\bar{y}_{i,j}$) for $i = 1, \dots, \rho$ and consider Definition 1. We apply it over the differential polynomials f_i with separants s_i , for $i = 1, \dots, \rho$ (we rename these separants, denoted f_n in the Definition, for legibility) modulo the ideal \mathfrak{a} defined by the squarefree regular chain $C = \{g_1, \dots, g_\sigma\}$.

We assume each $s_i(x, \bar{y}_j)$ to be nonzero modulo \mathfrak{a} (we are using here the assumption that the prolongation bounds are “large enough” so that C contains all the relevant prolongation equations). We moreover assume that the first coefficient $c_{i,0}$ of each $s_i(x, \bar{y}_j)$ is regular modulo \mathfrak{a} .

Thus the valuations k_i (see Definition 1) are well defined and the polynomials $A_i(q)$ (see Definition 2) are not identically zero modulo \mathfrak{a} . The nonnegative integers r_i are well defined also.

We assume that all polynomials $A_i(q)$ are regular modulo \mathfrak{a} i.e. that they have at least one coefficient regular modulo \mathfrak{a} . Thanks to the splitting cases mechanism and the regularity assumption on the coefficients $c_{i,0}$, this assumption is easy to satisfy.

Fix some $i = 1, \dots, \rho$. Denote $P_i = \text{prem}(A_i(q), C)$ and $R_i = \text{res}(A_i(q), C)$. These polynomials depend on q and some other variables constrained by the regular chain C . The fact that $A_i(q)$ is regular modulo \mathfrak{a} implies that they are not identically zero.

Denote V the algebraic variety of the ideal \mathfrak{a} , with coordinates in the algebraically closed field \mathcal{F} . Since C is a regular chain, the ideal \mathfrak{a} is distinct from the unit ideal and, since we are solving equations in an algebraically closed field, the algebraic variety V is nonempty.

Case 1. Assume that $P_i(q^*) = 0$ and that $q^* \in \mathbb{N}$ is the maximum nonnegative integer root of P_i . Thanks to the specifications of the pseudo-remainder algorithm, there exists a power product h of initials of elements of C such that $h A_i(q) = P_i(q)$ modulo \mathfrak{a} . Since the elements of C do not depend on q , we see that the assumption $P_i(q^*) = 0$ is equivalent to $A_i(q^*) \in \mathfrak{a}$. Thus for all tuples of initial values $(\bar{y}_{j,k}) \in V$, the defining equation (13) of $\bar{y}_{i, n_i + 2k_i + 2 + q^* - r_i}$ vanishes identically. Let us assume that the prolongation bounds are large enough in the sense that $\mu_i \geq n_i + 2k_i + 2 + q^* - r_i$. Then, using the fact that V is nonempty and unless the regular chain C contains a dedicated equation for $\bar{y}_{i, n_i + 2k_i + 2 + q^* - r_i}$, we see that this coefficient can be chosen freely: the differential system has infinitely many formal power series solutions.

Case 2. Assume that $R_i(q^*) = 0$ and that $q^* \in \mathbb{N}$ is the maximum nonnegative integer root of R_i . There exists some polynomial u such that $u A_i(q) = R_i(q)$ (see [3, Lemma 4.14]). Since the elements of C do not depend on q , we see that u does not either and the assumption $R_i(q^*) = 0$ is equivalent to $\text{res}(A_i(q^*), C) = 0$ i.e. is equivalent to $A_i(q^*)$ is a zerodivisor modulo \mathfrak{a} . Thanks to the splitting cases mechanism, this case is easily reduced to the other cases.

Case 3. Assume that $R_i(q)$ has no nonnegative integer root.

Subcase 3.1. If $R_i(q)$ only depends on q so that $R_i(q) \in \mathcal{F}$ for each $q \in \mathbb{N}$ then any tuple of initial values $(\bar{y}_{j,k}) \in V$ can be prolonged to a unique formal power series solution of the differential system.

Subcase 3.2 If $R_i(q)$ depends on some coefficients $\bar{y}_{j,k}$ (at least one) then these coefficients lie in a positive dimension algebraic subvariety of V (see “*On the variables the resultant depends on*” in Section 4). Thus this subvariety contains nonsingular points. According to Lemma 3 (the polynomials W_j and the prime ideal \mathfrak{p} of the Lemma correspond to the polynomials $A_i(q)$ for $q \in \mathbb{N}$ and any associated prime ideal of \mathfrak{a}), some tuples of initial values $(\bar{y}_{j,k}) \in V$ — but possibly not all of them — can be prolonged to formal power series solutions of the differential system.

6 Computation of the nonnegative integer roots

The various cases of the above section require the computation of the nonnegative integer roots of the polynomials $P_i(q)$ and $R_i(q)$ for $i = 1, \dots, \rho$. These polynomials belong to $\mathcal{F}[Y][q]$ where Y denotes a finite set of variables $\bar{y}_{j,k}$. Let $P(q)$ be any of them. Decompose

$$P(q) = a_1 m_1 + \dots + a_\tau m_\tau$$

where the m_j are pairwise distinct power products of variables in Y and the a_j are polynomials in $\mathcal{F}[q]$.

Proposition 3. *A nonnegative integer annihilates $P(q)$ if and only if it annihilates all the polynomials $a_j(q)$ for $j = 1, \dots, \tau$ hence their gcd.*

The computation of the gcd is algorithmic thanks to the Euclidean algorithm in $\mathcal{F}[q]$. We are thus reduced to the problem of computing the nonnegative integer roots of a univariate polynomial $G(q)$. If $G(q)$ has complex coefficients, its real roots are the common roots of the two polynomials $\Re(G(q))$ and $\Im(G(q))$, which have real coefficients. The nonnegative integer roots of a polynomial with real coefficients can be computed by a real root isolation algorithm, based on sign variations, such as [9].

Observe that this algorithm may be optimized by stopping the isolation process over any interval which is small enough (e.g. has length less than 1). The integer roots in such intervals can then easily be determined by evaluation (this optimization is important since it avoids the cost of isolating roots belonging to a tiny cluster).

Another optimization consists in evaluating some of the variables belonging to Y before the real root isolation process (this can even be done before the computation of the pseudo-remainder or the resultant). One then obtains an overset of the set of the sought roots. The irrelevant roots of the overset can then be discarded by evaluation.

7 Implementation

A prototype software is implemented in the Python `DifferentialAlgebra` (version 4.1) package [5] which can be installed using the standard `pip` Python package installing facility, on Linux and MacOS platforms. The following commands investigate our Example 2 from Section 2 using it. First load the package (which relies on the Python/sympy package).

```
[1]: from sympy import *
      from DifferentialAlgebra import *
      init_printing ()
```

The symbol x encodes the independent variable. The symbol q will be used later. The symbol y is used for the differential indeterminate.

```
[2]: x,q = var ('x,q')
     y = indexedbase ('y')
```

The symbols y_i are used for denoting the coefficients of the initial value encoding series \bar{y} .

```
[3]: params = [q] + [y[i] for i in range (9, -1, -1)]
     params
```

```
[3]: [q, y9, y8, y7, y6, y5, y4, y3, y2, y1, y0]
```

```
[4]: ybar = Add (*[y[i]*x**i/factorial(i) for i in range (0,5)])
     ybar
```

```
[4]:  $\frac{x^4 y_4}{24} + \frac{x^3 y_3}{6} + \frac{x^2 y_2}{2} + x y_1 + y_0$ 
```

When designing software for the problem addressed in this paper, in which many different symbols are needed, it is not straightforward to establish the relationship between the differential indeterminates and the coefficients of the formal power series. The design we have adopted consists in using Python dictionaries. The variable `point` associates one initial value encoding series (hence their coefficients) to each differential indeterminate. It lets also users decide whether they want to divide these coefficients by the factorials or not. The dictionary permits also to specify the expansion point.

```
[5]: point = { x:0, y:ybar }
     point
```

```
[5]: { x : 0, y :  $\frac{x^4 y_4}{24} + \frac{x^3 y_3}{6} + \frac{x^2 y_2}{2} + x y_1 + y_0$  }
```

The next command defines the differential polynomial ring with respect to which computations are going to be performed. It is $\mathbb{Q}[x, y_0, \dots, y_9, q]\{y\}$. The symbols q and the y_i are defined as constants: their derivatives are zero.

```
[6]: R = DifferentialRing (derivations = [x], blocks = [y, params],
                          parameters = params, notation = 'jet')
```

Let us assign to f the ODE (3) in the “jet” notation.

```
[7]: f = x*y*y[x,x] + y[x] + y**2 + 1
     f
```

```
[7]:  $x y_{x,x} y + y_x + y^2 + 1$ 
```

The next command defines the system (15) associated to our problem: it contains the ODE and one (simple) algebraic equation, assigning to y_0 the value $-\frac{1}{5}$. Such a system actually forms a regular differential chain C .

```
[8]: C = RegularDifferentialChain ([f, 5*y[0] + 1], R)
```

The next command computes the prerequisite data needed before prolongating the ODE and investigating the existence and uniqueness of formal power series solutions. It returns a tuple $(u, n, k, r, c_0, A(q))$ where $u = \dot{y}$ is the leading derivative of the ODE, $n = 2$ is its order and the other components are defined in Definitions 1 and 2. The last argument `edo=f` permits to compute the data for a specified element of C . Without this optional argument, a list of tuples is returned: one per ODE in C .

```
[9]: u, n, k, r, c0, A = C.prolongation_prerequisites (q, point, edo=f)
      u, n, k, r, c0, A
```

```
[9]: (yx,x, 2, 1, 1, y0, qy0 + 4y0 + 1)
```

The returned polynomial $A(q)$ is not simplified with respect to C (as is, it has no positive integer root). Let us compute its pseudo-remainder $P(q)$ with respect to C (the reduction process involves the algebraic constraints of C only).

```
[10]: P = C.prem (A)
      P
```

```
[10]: 1 - q
```

The next computation is not really needed: it permits us to show that the positive integer root algorithm described in Section 6 is implemented.

```
[11]: R.positive_integer_roots (P, q)
```

```
[11]: [1]
```

The above data permit to take $\gamma = 2$ and define $\beta = 2k + 2 + \gamma + r = 7$ and $\delta = n + 2k + 2 + \gamma = 8$ which permit to eventually prove the absence of formal power series solutions. We do not give the corresponding computations because this part of the software is not implemented.

Acknowledgement

The authors would like to thank Michel Petitot for fruitful discussions on this topic. The first author would like to thank the organizers of the CIMPA School *Algebraic and Tropical Methods for Solving Differential Equations*.

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