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LOCAL AND GLOBAL NOTIONS OF VISIBILITY WITH RESPECT TO KOBAYASHI DISTANCE, A COMPARISON

NIKOLAI NIKOLOV, AHMED YEKTA ÖKTEN, PASCAL J. THOMAS

ABSTRACT. In this note, we introduce the notion of visible boundary points with respect to Kobayashi distance for domains in \mathbb{C}^n . Following the work of A.D. Sarkar, we obtain additive and multiplicative localization results about Kobayashi distance near visible boundary points. Then using the additive localization result, we show that the visibility property with respect to Kobayashi distance is a local property of the boundary points and that it does not depend on the domain.

1. INTRODUCTION

The definition of invariant distances in complex analysis stems from the properties of holomorphic mappings, which are defined on open sets. In one or several dimensions, the question of extension of those mappings to the boundary of the open set (and to which boundary?) is of interest. The work of Balogh and Bonk [BB] introduced the metric property of Gromov hyperbolicity into this subject, along with an identification of the Gromov boundary with the Euclidean boundary, to provide another proof of the Fefferman extension theorem for mappings of strictly pseudoconvex domains.

Another geometric property that geodesic spaces may have, which can serve as a sort of substitute to Gromov hyperbolicity, is visibility. Informally speaking, a metric space satisfies visibility property if geodesics joining points approaching distinct points on the boundary pass through a compact set depending on those distinct boundary points. However, in general, it is not known whether geodesics for the Kobayashi-Royden pseudometric exist. Bharali and Zimmer [BZ] introduced a wider notion of visibility which holds for almost-geodesics with respect to the Kobayashi-Royden metric. They also established some sufficient conditions for it in terms of the growth of the Kobayashi distance and Kobayashi-Royden pseudometric.

This was followed by extensions of those results in [BM], and other examples of sufficient and necessary conditions for visibility in [BNT] and [CMS]. Most of the results in those papers

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relied on conditions which were local in terms of the Euclidean boundary of the domain, that is to say that e.g. to prove visibility of a domain Ω (with respect to $\partial\Omega$) one requires properties to be tested on $\Omega \cap U_p$, for all $p \in \partial\Omega$, where U_p is an appropriate neighborhood of p in \mathbb{C}^n . As the Kobayashi distance depends on the domain, [BNT] needed some localization results about Kobayashi distances along the lines of [FR].

It is then natural to conjecture that visibility can always be localized, that is to say, that a domain Ω is visible with respect to the Kobayashi-Royden metric of Ω and to $\partial\Omega$ (we will define this precisely later) if and only if for any $p \in \partial\Omega$, there exists a neighborhood U_p of p such that $\Omega \cap U_p$ is visible with respect to the Kobayashi-Royden metric of $\Omega \cap U_p$ and to $U_p \cap \partial\Omega$. This problem was first studied in [BGNT] and the authors were able to localize visibility under global assumptions such as Gromov hyperbolicity.

The goal of this note is to show without any global assumptions that visibility is indeed a local condition, depending on the boundary near a point. The main tool of the proof is additive localization of Kobayashi distance near visible points. These localization results were established in Sarkar [S1] which motivated the first version of this work [NÖT], which in turn was used to slightly improve the published version of Sarkar's work [S2].

The paper is organized as follows.

In Section 2, we recall the definitions about Kobayashi distance and Kobayashi-Royden metric. Further, we introduce the notion of *visible point* and relate it to the visibility definitions in the literature.

In Section 3, we prove Theorem 9 which gives additive localization for Kobayashi distance near visible points. Our proof follows the proof given in [S2], however only using local hypotheses. We also prove Theorem 14, which gives multiplicative localization of Kobayashi lengths near visible points.

In Section 4, we prove Theorem 15 that shows that local visibility at a boundary point is equivalent to global visibility at that point. In particular, it leads to Theorem 16 which tells us that if a boundary point is visible for a domain, then it is visible for any domain which locally looks like the initial one.

2. VISIBLE POINTS AND VISIBILITY PROPERTY

Let Ω be a domain in \mathbb{C}^n , $z, w \in \Omega$ and $v \in \mathbb{C}^n$. Recall that the Kobayashi pseudodistance k_Ω is the largest pseudodistance which does not exceed the Lempert function

$$l_\Omega(z, w) := \tanh^{-1} \tilde{l}_\Omega(z, w),$$

where Δ is the unit disc and $\tilde{l}_\Omega(z, w) := \inf\{|\alpha| : \exists \varphi \in \mathcal{O}(\Delta, \Omega) \text{ with } \varphi(0) = z, \varphi(\alpha) = w\}$.

Also recall the definition of Kobayashi-Royden pseudometric,

$$\kappa_\Omega(z; v) = \inf\{|\alpha| : \exists \varphi \in \mathcal{O}(\Delta, \Omega) \text{ with } \varphi(0) = z, \alpha\varphi'(0) = v\}.$$

The Kobayashi-Royden length of an absolutely continuous curve $\gamma : I \rightarrow \Omega$ is defined as

$$l_\Omega^k(\gamma) := \int_I \kappa_\Omega(\gamma(t), \gamma'(t)) dt.$$

By [R, V], it turns out that k_Ω is the integrated form of the Kobayashi-Royden pseudometric, that is, $k_\Omega(z, w) = \inf l_\Omega^\kappa(\gamma)$ where the infimum is taken over all absolutely continuous curves joining z to w .

We say that a domain $\Omega \subset \mathbb{C}^n$ is hyperbolic if k_Ω is a distance. This holds, for instance, for any bounded domain and for convex domains containing no affine complex lines.

In order to study the local behaviour of invariant metrics, we introduce a notion of hyperbolicity at boundary points. In fact, our definition is a generalization to boundary points of the characterization of hyperbolicity given in [NP, Proposition 3.1].

Definition 1. *Let Ω be a domain in \mathbb{C}^n , $p \in \partial\Omega$. If Ω is bounded, we say that Ω is hyperbolic at p for any $p \in \partial\Omega$. If Ω is unbounded, we say that Ω is hyperbolic at p if we have*

$$(1) \quad \liminf_{z \rightarrow p, w \rightarrow \infty} l_\Omega(z, w) > 0.$$

For $A, B \subset \Omega$ we denote $l_\Omega(A, B) := \inf_{a \in A, b \in B} l_\Omega(a, b)$ and $k_\Omega(A, B) := \inf_{a \in A, b \in B} k_\Omega(a, b)$. By definition, (1) holds if and only if we can find a bounded neighbourhood U of p and another neighbourhood $V \subset\subset U$ of p such that we have

$$(2) \quad l_\Omega(\Omega \cap V, \Omega \setminus U) > 0.$$

In fact, the latter condition holds on any bounded domain Ω , for any two neighbourhoods $V \subset\subset U$ of $p \in \partial\Omega$. This explains why we choose to say that any bounded domain is hyperbolic at any of its boundary points.

One can observe that this property is even stronger.

Proposition 2. *Let Ω be an unbounded domain in \mathbb{C}^n and $p \in \partial\Omega$. Then Ω is hyperbolic at p if and only if there exists a bounded neighbourhood V of p such that for any other two neighbourhoods of p satisfying $V' \subset\subset U' \subset V$ we have*

$$(3) \quad k_\Omega(\Omega \cap V', \Omega \setminus U') > 0.$$

Proof. It is clear that if (3) holds, then we have (2) with V', U' playing the part of V, U , hence (1). So we will prove the converse.

Let $V \subset\subset U$ be two neighbourhoods of p such that U is bounded and (2) holds. Recall Royden's localization lemma [JP, Proposition 13.2.10]: if Ω is a domain in \mathbb{C}^n and D is any subdomain, we have

$$(4) \quad \tilde{l}_\Omega(z, \Omega \setminus D) \kappa_{\Omega \cap D}(z; v) \leq \kappa_\Omega(z; v) \quad z \in \Omega \cap V, \quad v \in \mathbb{C}^n.$$

Note that by (2), taking $D = U$ above, (4) implies that there exists a $C > 0$ such that

$$(5) \quad \kappa_{\Omega \cap U}(z; v) \leq C \kappa_\Omega(z; v) \quad z \in \Omega \cap V, \quad v \in \mathbb{C}^n.$$

Let $V' \subset\subset U' \subset V$ be any two neighbourhoods of p . Since the Kobayashi distance is the integrated version of the Kobayashi-Royden metric, $k_\Omega(\Omega \cap V', \Omega \setminus U') = \inf_{\gamma \in \Gamma} l_\Omega^\kappa(\gamma)$, where Γ is the set of absolutely continuous curves $\gamma : I \rightarrow \Omega$ joining points in $\Omega \cap V'$ to points in $\Omega \setminus U'$. Due to connectivity of curves, it is clear that for any curve in Γ , we can find another curve in Γ of shorter or equal length whose image is contained in \bar{V} . Therefore

$k_{\Omega}(\Omega \cap V', \Omega \setminus U') = \inf_{\gamma \in \Gamma'} l_{\Omega}^{\kappa}(\gamma)$, where $\Gamma' \subset \Gamma$ is the set of such curves whose image lie in \bar{V} . On the other hand, since U is bounded, $\Omega \cap U$ is bounded, so

$$\inf_{\gamma \in \Gamma'} l_{\Omega \cap U}^{\kappa}(\gamma) \geq k_{\Omega \cap U}(\Omega \cap V', \Omega \setminus U') = c > 0.$$

By (5) we obtain

$$k_{\Omega}(\Omega \cap V', \Omega \setminus U') = \inf_{\gamma \in \Gamma} l_{\Omega}^{\kappa}(\gamma) = \inf_{\gamma \in \Gamma'} l_{\Omega}^{\kappa}(\gamma) \geq \frac{\inf_{\gamma \in \Gamma'} l_{\Omega \cap U}^{\kappa}(\gamma)}{C} \geq \frac{c}{C}.$$

□

One may argue as above to provide a new proof of [NP, Proposition 3.1].

Note that hyperbolicity does not imply local hyperbolicity without some assumptions. To see this, one may study the Kobayashi distance on the domain $D := \{(z, w) \in \mathbb{C}^2 : z \in \Delta \setminus \{0\}, |zw| < 1\}$ which is biholomorphic to $\Delta \setminus \{0\} \times \Delta$, hence pseudoconvex and hyperbolic. It is not difficult to see that D is not hyperbolic at points $\{(0, z) : z \in \mathbb{C}\} \subset \partial D$.

Definition 3. *Let Ω be a domain in \mathbb{C}^n . For $\lambda \geq 1$ and $\epsilon \geq 0$ we say that an absolutely continuous curve $\gamma : I \rightarrow \Omega$ is a (λ, ϵ) -geodesic for Ω if for all $t_1 \leq t_2 \in I$ we have that*

$$l_{\Omega}^{\kappa}(\gamma|_{[t_1, t_2]}) \leq \lambda k_{\Omega}(\gamma(t_1), \gamma(t_2)) + \epsilon.$$

We claim that in the case where Ω above is hyperbolic, this definition implies the definition of (λ, ϵ) -almost geodesics given in [BZ]. To see this notice that on hyperbolic domains any absolutely continuous curve can be reparametrized with respect to Kobayashi-Royden length, that is, we can parametrize γ so that $l_{\Omega}^{\kappa}(\gamma|_{[t_1, t_2]}) = |t_1 - t_2|$ for all $t_1 \leq t_2 \in I$. In particular, we have $\kappa_{\Omega}(\gamma(t); \gamma'(t)) = 1$ almost everywhere so $\kappa_{\Omega}(\gamma(t); \gamma'(t)) \leq \lambda$ almost everywhere. This can be shown with arguments similar to the arguments in the proof of [BZ, Proposition 4.4.]. Due to this parametrization and the definition of (λ, ϵ) -geodesics we obtain

$$k_{\Omega}(\gamma(t_1), \gamma(t_2)) \leq \lambda k_{\Omega}(\gamma(t_1), \gamma(t_2)) + \epsilon \leq \lambda l_{\Omega}^{\kappa}(\gamma|_{[t_1, t_2]}) + \epsilon = \lambda |t_1 - t_2| + \epsilon$$

and

$$k_{\Omega}(\gamma(t_1), \gamma(t_2)) \geq \lambda^{-1} l_{\Omega}^{\kappa}(\gamma|_{[t_1, t_2]}) - \lambda^{-1} \epsilon = \lambda^{-1} |t_1 - t_2| - \lambda^{-1} \epsilon \geq \lambda^{-1} |t_1 - t_2| - \epsilon$$

so the claim follows.

Unless noted otherwise, we will assume that any (λ, ϵ) -geodesic on hyperbolic domains is parametrized as a (λ, ϵ) -geodesic in the sense of [BZ].

For brevity, we will say ϵ -geodesics for $(1, \epsilon)$ -geodesics. As the Kobayashi distance is given as the infimum of the Kobayashi-Royden lengths of the curves, we see that for any $\epsilon > 0$ and any two points, we can find ϵ -geodesics (hence, (λ, ϵ) -geodesics) joining them. Also, using the triangle inequality, one may observe that for ϵ -geodesics, if the condition in the definition is satisfied with the endpoints, then it holds everywhere.

Let us introduce the notion of λ -visible points. Informally, being a λ -visible point means that a (λ, ϵ) -geodesic with an extremity “near” that point avoids the boundary immediately. Explicitly:

Definition 4. Let $\Omega \subset \mathbb{C}^n$, $\lambda \geq 1$ and $\epsilon \geq 0$. We say that $p \in \partial\Omega$ is a (λ, ϵ) -visible point for Ω if we have the following property: for any bounded neighbourhood U of p , there exist $V \subset\subset U$ and a compactum $K_{\lambda, \epsilon} \subset\subset \Omega$, such that if $\gamma : I \rightarrow \Omega$ is a (λ, ϵ) -geodesic for Ω which joins a point in $\Omega \cap V$ to a point in $\Omega \setminus U$, then $\gamma(I) \cap K_{\lambda, \epsilon} \neq \emptyset$.

We say that $p \in \partial\Omega$ is a λ -visible point for Ω if p is (λ, ϵ) -visible for Ω for all $\epsilon \geq 0$.

We also say that p is a *weakly visible point* for Ω if it is 1-visible for Ω and p is a *visible point* for Ω if it is λ -visible for Ω for all $\lambda \geq 1$.

It is clear that if p is a λ -visible point for Ω and $\lambda' \leq \lambda$ then p is a λ' -visible point for Ω .

Let $\Omega \subset \mathbb{C}^n$ be any domain and $z, w, o \in \Omega$. We recall the definition of the Gromov product with respect to Kobayashi distance

$$(z|w)_o^\Omega := \frac{1}{2}(k_\Omega(z, o) + k_\Omega(w, o) - k_\Omega(z, w)).$$

Definition 5. Let $\Omega \subset \mathbb{C}^n$, $o \in \Omega$, $p \in \partial\Omega$. We say that p satisfies the Gromov property for Ω if for any $q \neq p$ in $\partial\Omega$, we have that

$$\limsup_{z \rightarrow p, w \rightarrow q} (z|w)_o^\Omega < \infty.$$

We say that p satisfies the weak Gromov property for Ω if for any $o \in \Omega$ there exists a constant $c \leq 0$ such that for any $q \neq p$ in $\bar{\Omega}$ we have

$$\liminf_{z \rightarrow p, w \rightarrow q} (k_\Omega(z, w) - k_\Omega(z, o)) \geq c.$$

It is clear that the Gromov property does not depend on the choice of the base point o , and that if p satisfies the Gromov property for Ω , then it satisfies the weak Gromov property for Ω .

Being a λ -visible point leads to the following conditions about growth of Kobayashi distance.

Proposition 6. Let $\Omega \subset \mathbb{C}^n$ be a hyperbolic domain and $p \in \partial\Omega$.

- (1) Suppose that p is a λ -visible point for Ω . Then p satisfies the Gromov property for Ω .
- (2) Suppose that p is a λ -visible point for Ω . Then Ω is hyperbolic at p .

Proof. The first statement follows from the proof of [BNT, Proposition 2.4].

To see that the second statement holds, note that by assumption for any bounded neighbourhood U of p , we can find another neighbourhood of p , $V \subset\subset U$ and compact set $K \subset\subset \Omega$, where any ϵ -geodesic joining points in $\Omega \cap V$ and $\Omega \setminus U$ meets K . It is well known that on hyperbolic domains the distance of any compact set to the boundary is bounded below. Therefore by shrinking V if necessary and taking ϵ small enough, we obtain

$$k_\Omega(\Omega \cap V, \Omega \setminus U) \geq k_\Omega(\Omega \cap V, K) - \epsilon > 0.$$

It follows by Proposition 2 that Ω is hyperbolic at p . □

It is worth noting that by the proof of [BNT, Proposition 2.5], if Ω is complete hyperbolic, that is, if (Ω, k_Ω) is complete as a metric space, then p satisfies the Gromov property if and only if it is a weakly visible point. Moreover, statement (1) in Proposition 6 holds even when the domain is not hyperbolic.

Definition 7. Let Ω, D be two hyperbolic domains in \mathbb{C}^n which have a common boundary point p , and assume that both Ω and D are hyperbolic at p . We say that Ω is equivalent to D at p if there exists a bounded neighbourhood $U \subset \mathbb{C}^n$ of p such that $U \cap \Omega = U \cap D$.

It is easy to see the definition above indeed is an equivalence relation. Denote the equivalence class of Ω at p by $[\Omega]_p$. In some sense, these equivalence classes can be considered as “germs” of domains at points in \mathbb{C}^n .

We recall the definition of visible pairs. Let $\Omega \subset \mathbb{C}^n$, $p \neq q \in \partial\Omega$. $\{p, q\}$ is said to be a *visible pair* if there exists neighbourhoods U_p, U_q of p and q respectively such that if $\gamma : I \rightarrow \Omega$ is a (λ, ϵ) -geodesic joining a point in U_p to point in U_q , then $\gamma(I)$ intersects a compact set $K_{\lambda, \epsilon} \subset \subset \Omega$. Ω enjoys the *visibility property* if for any $p \neq q \in \partial\Omega$ we have that $\{p, q\}$ is a visible pair.

With the same spirit, one can define λ -visible pairs and λ -visibility property if one restricts the definition above to (λ', ϵ) -geodesics where $\lambda' \leq \lambda$.

Proposition 8. Let $\Omega \subset \mathbb{C}^n$, $\lambda \geq 1$. A point $p \in \partial\Omega$ is a λ -visible for Ω if and only if for any $q \in \partial\Omega$ such that $q \neq p$, $\{p, q\}$ is a λ -visible pair. Consequently, Ω enjoys λ -visibility property if and only if any $p \in \partial\Omega$ is a λ -visible point for Ω .

Proof. It is clear that being visible at p is a stronger assumption than any $\{p, q\}$ being visible pairs. So, we will prove the converse.

We suppose that p is not a visible point for Ω . Then, there exists a bounded neighbourhood U of p , and sequences $z_n \rightarrow p$, $w_n \in \Omega \setminus U$ and a sequence of (λ, ϵ) -geodesics $\gamma_n : I_n \rightarrow \Omega$ joining z_n to w_n which eventually avoid any compact set in Ω . By connectivity of geodesics, each γ_n must meet $(\partial U) \cap \Omega$. Set $\tau_n := \inf\{t \in I_n : \gamma_n(t) \in (\partial U) \cap \Omega\}$ and $w'_n = \gamma_n(\tau_n)$. By construction, $\gamma_n|_{[0, \tau_n]}$ are (λ, ϵ) -geodesics for Ω joining z_n to w'_n which tend to $\partial\Omega$. Moreover, by our assumption passing to a subsequence if necessary we have $w'_n \rightarrow r \in \partial\Omega \cap \bar{U}$. This shows that $\{p, r\} \subset \partial\Omega$ is not a visible pair. \square

Proposition 8 shows, in some sense, that visibility for a domain is a local property of the boundary points. But the (almost) geodesics involved in the definition are constructed using the global metric; we will remove this restriction later, to give conditions depending only on the equivalence class of Ω at each boundary point.

3. LOCALIZATION OF KOBAYASHI DISTANCE NEAR VISIBLE POINTS

Let Ω be a domain in \mathbb{C}^n , $p \in \partial\Omega$. We say that k_Ω satisfies *additive (resp. multiplicative) localization at p* if there exists neighbourhoods $V \subset \subset U$ of p , and $C \geq 0$ such that

$$k_{\Omega \cap V}(z, w) \leq k_\Omega(z, w) + C, \text{ resp. } k_{\Omega \cap V}(z, w) \leq Ck_\Omega(z, w)$$

for any $z, w \in \Omega \cap V$. We would like to note that whenever we discuss such a property we will assume that $z, w \in \Omega \cap V$ belong to the same component of $\Omega \cap U$.

This section is devoted to proving additive and multiplicative localization results near visible points. Although our results are given with weaker assumptions, they are of the same character as those in [S2]. This is essentially due to Proposition 8, which relates the notion of visible

points to the notion of visible pairs. Even though our proofs closely follow [S2], we discuss the arguments in detail to show the local nature, and also to simplify some steps of proofs given in [S2].

We are ready to introduce the following theorem, which is an analogue of [S2, Theorem 1.1, Theorem 1.3].

Theorem 9. *Let $\Omega \subset \mathbb{C}^n$ be a hyperbolic domain, $p \in \partial\Omega$ and $\lambda \geq 1$. Assume that one or both of the followings hold.*

- (1) *p is a λ -visible point for Ω .*
- (2) *There exists a bounded neighbourhood of p , $U \subset \mathbb{C}^n$ such that p is a λ -visible point for $\Omega \cap U$ and Ω is hyperbolic at p .*

Then, there exists another neighbourhood $V \subset\subset U$ of p , and a constant $C \geq 0$ such that for any $z, w \in \Omega \cap V$ we have

$$k_{\Omega \cap U}(z, w) \leq k_{\Omega}(z, w) + C.$$

In order to prove Theorem 9 we will provide a localization result about Kobayashi-Royden lengths. Our result is an analogue of [S2, Lemma 3.5, Lemma 3.6].

Lemma 10. *Let $\Omega \subset \mathbb{C}^n$ be a hyperbolic domain and $p \in \partial\Omega$.*

- (1) *If p is a λ -visible point for Ω for some $\lambda \geq 1$, then for any bounded neighbourhood U of p there exists $V \subset\subset U$ and a constant $C \geq 0$ such that for all (λ, ϵ) -geodesics $\gamma : I \rightarrow \Omega \cap V$ for Ω we have*

$$l_{\Omega \cap U}^{\kappa}(\gamma) \leq l_{\Omega}^{\kappa}(\gamma) + C.$$

- (2) *If there exists a bounded neighbourhood of p , $U \subset \mathbb{C}^n$ and $\lambda \geq 1$ such that p is a λ -visible point for $\Omega \cap U$ and Ω is hyperbolic at p , then there exists a neighbourhood $V \subset\subset U$ and a constant $C \geq 0$ such that for all (λ, ϵ) -geodesics $\gamma : I \rightarrow \Omega \cap V$ for Ω we have*

$$l_{\Omega \cap U}^{\kappa}(\gamma) \leq l_{\Omega}^{\kappa}(\gamma) + C.$$

Proof of Lemma 10 requires the reparametrization of λ -geodesics in the sense of almost-geodesics. As noted earlier, this is possible on hyperbolic domains. As the later results depend on Lemma 10 we see that hyperbolicity assumption is essential.

We will need the following crucial estimate given in [S2]. It follows from Royden's localization lemma, by comparing the Kobayashi pseudodistance to the Lempert function and using the inequality $\tanh(x) \geq 1 - e^{-x}$ for all $x \geq 0$.

Lemma 11. [S2, Lemma 3.1] *Let $\Omega \subset \mathbb{C}^n$ be a hyperbolic domain and $V \subset\subset U$ be neighbourhoods of $p \in \partial\Omega$ such that*

$$k_{\Omega}(\Omega \cap V, \Omega \setminus U) > 0.$$

Then there exists $C > 0$ such that the infinitesimal Kobayashi metric satisfies

$$(6) \quad \kappa_{\Omega \cap U}(x; v) \leq (1 + Ce^{-k_{\Omega}(x, \Omega \setminus U)})\kappa_{\Omega}(x; v)$$

for all $x \in \Omega \cap V$, $v \in \mathbb{C}^n$.

It follows by [S2] that C above can be taken to be $\coth(k_\Omega(\Omega \cap V, \Omega \setminus U))$.

Proof of Lemma 10. Let U be as in Lemma 10. Proposition 2 and our assumption in (2) of Lemma 10 imply that in both cases, there exists a neighbourhood $V \subset\subset U$ of p such that

$$k_\Omega(\Omega \cap V, \Omega \setminus U) > 0.$$

Let V be such a set, assume that $z, w \in \Omega \cap V$ and $\gamma : I \rightarrow \Omega \cap V$ is a (λ, ϵ) -geodesic for Ω joining them. By a direct calculation, using (6) we see that

$$\begin{aligned} l_{\Omega \cap U}^\kappa(\gamma) &= \int_I \kappa_{\Omega \cap U}(\gamma(t); \gamma'(t)) dt \leq \int_I (1 + C e^{-k_\Omega(\gamma(t), \Omega \setminus U)}) \kappa_\Omega(\gamma(t); \gamma'(t)) dt \\ (7) \quad &\leq l_\Omega^\kappa(\gamma) + C \int_I e^{-k_\Omega(\gamma(t), \Omega \setminus U)} \kappa_\Omega(\gamma(t); \gamma'(t)) dt \end{aligned}$$

Set $E := \int_I e^{-k_\Omega(\gamma(t), \Omega \setminus U)} \kappa_\Omega(\gamma(t); \gamma'(t)) dt$. We will show that E is uniformly bounded above for any such curve.

We first assume that p is a λ -visible point for Ω .

By Proposition 6, p satisfies the weak Gromov property for Ω . Notice that this property implies that for any bounded neighbourhood U of p we can find another neighbourhood $V \subset\subset U$ such that for a given $o \in \Omega$, we have a constant $c \leq 0$ such that

$$k_\Omega(z, w) \geq k_\Omega(z, o) + c$$

for all $z \in \Omega \cap V$, $w \in \Omega \setminus U$.

In particular, shrinking V if necessary and fixing $o \in \Omega \cap V$ we can find a constant $c \leq 0$ satisfying

$$k_\Omega(\gamma(t), \Omega \setminus U) := \inf_{w \in \Omega \setminus U} k_\Omega(\gamma(t), w) \geq k_\Omega(\gamma(t), o) + c$$

for all $t \in I$. Thus, we obtain

$$(8) \quad E \leq C' \int_I e^{-k_\Omega(\gamma(t), o)} \kappa_\Omega(\gamma(t); \gamma'(t)) dt.$$

We reparametrize γ so that it becomes an $(1, \epsilon)$ -almost geodesic for κ_Ω , i.e. $\kappa_\Omega(\gamma(t); \gamma'(t)) = 1$ almost everywhere. Choose $t_0 \in I$ with $k_\Omega(\gamma(t_0), o) \leq k_\Omega(\gamma(t), o)$ for all $t \in I$. Thus

$$k_\Omega(\gamma(t), o) \geq \frac{1}{2} (k_\Omega(\gamma(t_0), o) + k_\Omega(\gamma(t), o)) \geq \frac{1}{2} k_\Omega(\gamma(t_0), \gamma(t)) \geq \frac{1}{2} \left(\frac{|t - t_0|}{\lambda} - \epsilon \right).$$

Using this fact again, by (8) we obtain

$$E \leq C'' \int_I e^{-\frac{|t-t_0|}{2\lambda}} \kappa_\Omega(\gamma(t); \gamma'(t)) dt \leq 2C'' \int_{\mathbb{R}} e^{-\frac{t}{2\lambda}} dt \leq C'''$$

so by (7) we have

$$l_{\Omega \cap U}^\kappa(\gamma) \leq l_\Omega^\kappa(\gamma) + C''''.$$

This finishes the case where p is λ -visible for Ω .

Now we assume p is λ -visible for $\Omega \cap U$ and Ω is hyperbolic at p .

By the hyperbolicity assumption at p , choosing V as before we deduce that (7) still holds and

we want to show that $E := \int_I e^{-k_\Omega(\gamma(t), \Omega \setminus U)} \kappa_\Omega(\gamma(t); \gamma'(t)) dt$ is uniformly bounded. To do so, we take a $W \subset\subset U$ satisfying $V \subset\subset W$ and clearly we have that

$$(9) \quad k_\Omega(\gamma(t), \Omega \setminus U) \geq k_\Omega(\gamma(t), \Omega \setminus W).$$

Claim. There are constants c_1, c_2 so that $k_\Omega(\gamma(t), \Omega \setminus W) \geq c_1 k_{\Omega \cap U}(\gamma(t), \Omega \setminus W) + c_2$.

This is proven in [S2] but for convenience we repeat the arguments.

Note that as Ω is hyperbolic at p , by Proposition 2 shrinking V, W if necessary we can find a constant $C \geq 1$ so that

$$(10) \quad k_{\Omega \cap U}(z; v) \leq C k_\Omega(z; v) \quad z \in \Omega \cap \overline{W}, v \in \mathbb{C}^n$$

Fix $\delta, \epsilon > 0$. For each t , choose $w_t \in \Omega \setminus W$ such that $k_\Omega(\gamma(t), w_t) \leq k_\Omega(\gamma(t), \Omega \setminus W) + \delta$ and let $\sigma_t : I_t \rightarrow \Omega$ be an ϵ -geodesic for Ω joining $\gamma(t)$ to w_t . Set $\tau_t := \inf\{t' \in I_t : \sigma_t(t') \in \Omega \setminus W\}$ and $w'_t := \sigma_t(\tau_t)$ and set $\sigma'_t := \sigma_t|_{[0, \tau_t]}$.

Then by (10) we obtain

$$\begin{aligned} k_{\Omega \cap U}(\gamma(t), \Omega \setminus W) &\leq k_{\Omega \cap U}(\gamma(t), w'_t) \leq l_{\Omega \cap U}^\kappa(\sigma'_t) \leq C l_\Omega^\kappa(\sigma'_t) \\ &\leq C l_\Omega^\kappa(\sigma_t) \leq C k_\Omega(\gamma(t), \Omega \setminus W) + C\epsilon + C\delta. \end{aligned}$$

The estimate above finishes the proof of the claim.

Now, by (9) and the claim we obtain

$$E \leq C' \int_I e^{-C'' k_{\Omega \cap U}(\gamma(t), \Omega \setminus W)} \kappa_\Omega(\gamma(t); \gamma'(t)) dt.$$

To continue, one can repeat the proof of the case where p is λ -visible for Ω by applying the weak Gromov property of p with respect to $\Omega \cap U$ instead of Ω to see that also in this case E is bounded.

By (7), the theorem follows. \square

Lemma 10 and its proof immediately lead to the following useful corollary.

Corollary 12. *Let Ω be a hyperbolic domain in \mathbb{C}^n , $p \in \partial\Omega$, and let $V \subset\subset U$ be two neighbourhoods of p such that U is bounded and we have*

$$k_\Omega(\Omega \cap V, \Omega \setminus U) > 0.$$

Let $z, w \in \Omega \cap V$ and assume that γ is an (λ, ϵ) -geodesic for Ω joining z to w whose image lies in $\Omega \cap V$. Then there exists an $\epsilon' \geq \epsilon$ such that γ is an (λ, ϵ') -geodesic for $\Omega \cap U$.

Proof. Observe that in the proof of Lemma 10 we showed that if γ is an (λ, ϵ) -geodesic for Ω that lies in $\Omega \cap V$ then there exists a constant $C \geq 0$ such that $l_{\Omega \cap U}^\kappa(\gamma) \leq l_\Omega^\kappa(\gamma) + C$ and C depends on U, V and λ .

Now, as γ is an (λ, ϵ) -geodesic for Ω we have

$$l_{\Omega \cap U}^\kappa(\gamma) \leq l_\Omega^\kappa(\gamma) + C \leq \lambda k_\Omega(z, w) + C + \epsilon \leq \lambda k_{\Omega \cap U}(z, w) + C + \epsilon.$$

Setting $\epsilon' = \epsilon + C$ we are done. \square

Proof of Theorem 9. We first assume that p is a λ -visible point for Ω .

Let U be a bounded neighbourhood of p . By Proposition 6, we have that Ω is hyperbolic at p so we can find a $V \subset\subset U$ satisfying

$$k_\Omega(\Omega \cap V, \Omega \setminus U) > 0.$$

We first assume that γ is an ϵ -geodesic for Ω which joins $z, w \in \Omega \cap V$. Observe that if the image of γ lies entirely in $\Omega \cap V$ by Lemma 10 we have

$$k_{\Omega \cap U}(z, w) \leq l_{\Omega \cap U}^\kappa(\gamma) \leq l_\Omega^\kappa(\gamma) + C \leq k_\Omega(z, w) + \epsilon + C$$

so in this case the localization holds.

Now to get a contradiction, we assume that there exists $z_n, w_n \subset \Omega \cap V$ such that

$$(11) \quad k_{\Omega \cap U}(z_n, w_n) - k_\Omega(z_n, w_n) \rightarrow \infty.$$

Let $\gamma_n : I_n \rightarrow \Omega$ be ϵ -geodesics for Ω joining z_n to w_n . By the observation above, by looking at a subsequence if necessary we observe that we must have $\gamma_n(I_n) \not\subset V$. Set $\tau_n^1 := \inf\{t \in I_n : \gamma_n(t) \in \Omega \setminus V\}$, $z'_n := \gamma_n(\tau_n^1)$ and $\tau_n^2 := \sup\{t \in I_n : \gamma_n(t) \in \Omega \setminus V\}$, $w'_n := \gamma_n(\tau_n^2)$. Let γ_n^1, γ_n^2 be the parts of γ_n that join z_n to z'_n and w_n to w'_n respectively.

As the curves γ_n are ϵ -geodesics for Ω , by the above we have

$$(12) \quad \begin{aligned} k_\Omega(z_n, w_n) + \epsilon &\geq l_\Omega^\kappa(\gamma) = l_\Omega^\kappa(\gamma_n^1) + l_\Omega^\kappa(\gamma_n^2) + l_\Omega^\kappa(\gamma_n \setminus \gamma_n^1 \cup \gamma_n^2) \\ &\geq l_{\Omega \cap U}^\kappa(\gamma_n^1) + l_{\Omega \cap U}^\kappa(\gamma_n^2) + k_\Omega(z'_n, w'_n) - 2C \\ &\geq k_{\Omega \cap U}(z_n, z'_n) + k_{\Omega \cap U}(w_n, w'_n) + k_\Omega(z'_n, w'_n) - 2C \end{aligned}$$

Suppose that $\{z'_n, w'_n\}_{n \in \mathbb{N}}$ is compact in $\Omega \cap U$. Then adding and subtracting $k_{\Omega \cap U}(z'_n, w'_n)$ to the above inequality, by the triangle inequality, (11) fails. Therefore, by looking at a subsequence if necessary, we assume $z'_n \rightarrow s, w'_n \rightarrow r$ and either s, r or both lie in $(\partial\Omega) \cap U$. As the approach is the same for all cases, we will provide the proof for the case where both $s, r \in (\partial\Omega) \cap U$.

Since p is a visible point for Ω , we can find $U' \subset\subset V$ such that any ϵ -geodesic for Ω joining points in $\Omega \cap U'$ to points in $\Omega \setminus V$ meets a compactum $K \subset\subset \Omega$. By looking at a subsequence if necessary, we assume that $z_n, w_n \in U'$, so weak visibility of p for Ω applies to the ϵ -geodesics $\{\gamma_n^1, \gamma_n^2\}_{n \in \mathbb{N}}$. Thus, we can take \tilde{z}_n, \tilde{w}_n so that they remain in the intersection of images of γ_n^1, γ_n^2 with K . By construction images of γ_n^1, γ_n^2 remain in $\Omega \cap V$ so we have $\{\tilde{z}_n, \tilde{w}_n\}_{n \in \mathbb{N}} \subset\subset \Omega \cap U$. A similar calculation to (12) leads to

$$\begin{aligned} k_\Omega(z_n, w_n) + \epsilon &\geq k_{\Omega \cap U}(z_n, \tilde{z}_n) + k_{\Omega \cap U}(w_n, \tilde{w}_n) + k_\Omega(\tilde{z}_n, \tilde{w}_n) - 2C \\ &\geq k_{\Omega \cap U}(z_n, \tilde{z}_n) + k_{\Omega \cap U}(w_n, \tilde{w}_n) + k_\Omega(\tilde{z}_n, \tilde{w}_n) + k_{\Omega \cap U}(\tilde{z}_n, \tilde{w}_n) - k_{\Omega \cap U}(\tilde{z}_n, \tilde{w}_n) - 2C \\ &\geq k_{\Omega \cap U}(z_n, w_n) - C', \end{aligned}$$

where the last inequality follows from the triangle inequality and the fact that $\{\tilde{z}_n, \tilde{w}_n\}_{n \in \mathbb{N}} \subset\subset \Omega \cap U$. We therefore conclude that in the case where p is λ -visible for Ω , such a sequence can not exist.

We now assume that p is λ -visible for $\Omega \cap U$ and Ω is hyperbolic at p .

We choose $V \subset\subset U$ such that $k_\Omega(\Omega \cap V, \Omega \setminus U) > 0$. To get a contradiction, we assume that localization fails. Thus, we can find points z_n, w_n in $\Omega \cap V$ such that

$$k_{\Omega \cap U}(z_n, w_n) - k_\Omega(z_n, w_n) \rightarrow \infty.$$

Repeating the construction above we see that ϵ -geodesics joining z_n to w_n must leave $\Omega \cap V$ and (12) holds also on this case. We may apply a similar visibility argument however we claim that using the weak Gromov property of p with respect to $\Omega \cap U$ is enough. To see this observe that by (12) we have

$$(13) \quad k_\Omega(z_n, w_n) \geq k_{\Omega \cap U}(z_n, z'_n) + k_{\Omega \cap U}(w_n, w'_n) - 2C.$$

Fix $o \in \Omega \cap V$ and shrink V if necessary to get by the weak Gromov property of p with respect to $\Omega \cap U$ a constant $c \leq 0$ such that

$$k_{\Omega \cap U}(z_n, z'_n) \geq k_{\Omega \cap U}(z_n, o) + c \text{ and } k_{\Omega \cap U}(w_n, w'_n) \geq k_{\Omega \cap U}(w_n, o) + c.$$

Having this in mind and continuing (13), by the triangle inequality we obtain

$$\begin{aligned} k_\Omega(z_n, w_n) &\geq k_{\Omega \cap U}(z_n, z'_n) + k_{\Omega \cap U}(w_n, w'_n) - 2C \\ &\geq k_{\Omega \cap U}(z_n, o) + k_{\Omega \cap U}(w_n, o) - 2C + 2c \geq k_{\Omega \cap U}(z_n, w_n) - 2C + 2c. \end{aligned}$$

We see that such a sequence cannot exist. This finishes the proof of this case. Hence, we have the theorem. \square

Theorem 9 gives the following corollary, which can be seen as a converse to Corollary 12.

Corollary 13. *Let Ω be a hyperbolic domain and $p \in \partial\Omega$. Let $V \subset\subset U$ be two neighbourhoods of $p \in \partial\Omega$ such that additive localization holds for U, V , that is, there exists a $C \geq 0$ such that*

$$k_{\Omega \cap U}(z, w) \leq k_\Omega(z, w) + C$$

for all $z, w \in \Omega \cap V$. For any $\epsilon > 0$, there exists an $\epsilon' \geq \epsilon$ such that any (λ, ϵ) -geodesic $\gamma : I \rightarrow \Omega \cap V$ for $\Omega \cap U$ is a (λ, ϵ') -geodesic for Ω .

Proof. Let γ be as above and let z, w be endpoints of the image of γ . Then by Theorem 9 we obtain

$$l_\Omega^\kappa(\gamma) \leq l_{\Omega \cap U}^\kappa(\gamma) \leq \lambda \kappa_{\Omega \cap U}(z, w) + \epsilon \leq \lambda \kappa_\Omega(z, w) + \lambda C + \epsilon.$$

Setting $\epsilon' := \epsilon + \lambda C$, we are done. \square

One may observe that the weak Gromov property plays a key role in the proofs above. In fact, statements (2) of both Theorem 9 and Lemma 10 still hold when one replaces "being a visible point for" with "satisfying the weak Gromov property for". Moreover, statement (1) of Lemma 10 also holds under the same change, however with the additional assumption that Ω being hyperbolic at p . It is unclear that if conditions in statement (1) of Theorem 9 can be relaxed.

As a conclusion for this section we will present a new multiplicative localization result.

Theorem 14. *Let $\Omega \subset \mathbb{C}^n$ be a hyperbolic domain, $p \in \partial\Omega$. Assume that one or both of the followings hold.*

- (1) p is a λ -visible point for Ω .

- (2) *There exists a neighbourhood of p , $U \subset \mathbb{C}^n$ such that p is a λ -visible point for $\Omega \cap U$ and Ω is hyperbolic at p .*

Then there exists a neighbourhood $V \subset\subset U$ of p and a constant $C \geq 1$ such that

$$k_{\Omega \cap U}(z, w) \leq Ck_{\Omega}(z, w)$$

for all $z, w \in \Omega \cap V$.

Proof. Due to Proposition 6 and the assumption in (2) on both cases we can find a neighbourhood $V \subset\subset U$ of p such that

$$k_{\Omega}(\Omega \cap V, \Omega \setminus U) > 0.$$

Let V be such a set. It follows from the proof of Lemma 10 that, if there are ϵ_n -geodesics γ_n joining $z \in \Omega \cap V$ to $w \in \Omega \cap V$ such that $\epsilon_n \rightarrow 0$ and the images of γ_n remain in $\Omega \cap V$ there is a constant $C \geq 1$ such that

$$k_{\Omega \cap U}(z, w) \leq Ck_{\Omega}(z, w).$$

so in this case multiplicative localization holds.

Let $V' \subset\subset V$ be a neighbourhood of p such that there exists a constant $C_{V'} \geq 0$ such that we have

$$k_{\Omega \cap U}(z, w) \leq k_{\Omega}(z, w) + C_{V'}$$

for any $z, w \in \Omega \cap V'$. Further, assume that $k_{\Omega}(\Omega \cap V', \Omega \setminus V) =: c \geq 0$. Note that we can choose such a neighbourhood due to Proposition 2 and Theorem 9.

We suppose that $\gamma : I \rightarrow \Omega$ is an ϵ -geodesic for Ω joining $z \in \Omega \cap V'$ to $w \in \Omega \cap V'$ where $\epsilon < c$ which leaves $\Omega \cap V$. Set $\tau := \inf\{t \in \Omega : \gamma(t) \in \Omega \cap \partial V\}$ and $w' := \gamma(\tau)$. Then,

$$c < k_{\Omega}(\Omega \cap V', \Omega \setminus V) \leq k_{\Omega}(z, w') \leq l_{\Omega}^k(\gamma|_{[0, \tau]}) \leq l_{\Omega}^k(\gamma) \leq k_{\Omega}(z, w) + \epsilon.$$

By above we obtain that $k_{\Omega}(z, w) \geq c - \epsilon$. As additive localization holds on V' , we have

$$\frac{k_{\Omega \cap U}(z, w)}{k_{\Omega}(z, w)} \leq 1 + \frac{C_{V'}}{k_{\Omega}(z, w)} \leq 1 + \frac{C_{V'}}{c - \epsilon}$$

so $k_{\Omega \cap U}(z, w) \leq C'k_{\Omega}(z, w)$. Setting $V := V'$ in the theorem, we are done. \square

By looking at the proof above, one may observe that if $k_{\Omega}(z, w)$ is bounded below, then additive localization is stronger than multiplicative localization. On the other hand, it is clear that if $k_{\Omega}(z, w)$ tends to 0, multiplicative localization is a much better tool to compare k_{Ω} with $k_{\Omega \cap U}$.

4. LOCAL VISIBILITY AND GLOBAL VISIBILITY

The goal of this section is to show that visibility is a local property. Unlike the results given in [BGNT], we do not have any global assumptions such as Gromov hyperbolicity. Notably, the key elements in our proofs are Corollary 12 and Corollary 13. We will first prove the following:

Theorem 15. *Let $\Omega \subset \mathbb{C}^n$ be a hyperbolic domain, $\lambda \geq 1$ and assume that Ω is hyperbolic at $p \in \partial\Omega$. Then the followings are equivalent.*

- (1) *p is a λ -visible point for Ω .*

- (2) *There exists a bounded neighbourhood U of p such that p is a λ -visible point for $\Omega \cap U$.*
 (3) *For any bounded neighbourhood U of p , p is a λ -visible point for $\Omega \cap U$.*

Proof. The implication that (3) \implies (2) is clear.

We will show that (2) implies (1) by contradiction. To see this assume that p is not λ -visible for Ω . Then by Proposition 8 we can find a point $q \in \partial\Omega$ such that $q \neq p$ and sequences $z_n \rightarrow p$, $w_n \rightarrow q$ and (λ, ϵ) -geodesics $\gamma_n : I_n \rightarrow \Omega$ joining z_n to w_n such that $\gamma_n(I_n)$ avoids K_n where the sequence $\{K_n\}_{n \in \mathbb{N}}$ satisfy $K_n \subset\subset K_{n+1} \subset\subset \Omega$ and $\Omega = \bigcup_n K_n$. Now, take another neighbourhood $V \subset\subset U$ of p . By choosing V small enough we assume that $w_n \notin V$ and by hyperbolicity of Ω at p , we can also assume that $k_\Omega(\Omega \cap V, \Omega \setminus U) > 0$. Set $\tau_n := \inf\{t \in I_n : \gamma_n(t) \in (\partial V) \cap \Omega\}$ and $w'_n := \gamma_n(\tau_n)$. Due to our assumption, passing to a subsequence if necessary, we assume $\lim_{n \rightarrow \infty} w'_n := r \in (\partial\Omega) \cap \bar{V}$.

Now, consider the curves $\gamma_n|_{[0, \tau_n]}$. By construction they are (λ, ϵ) -geodesics for Ω whose image remain in $\Omega \cap V$. By Corollary 12, there is an $\epsilon' > 0$ such they are (λ, ϵ') -geodesics for $\Omega \cap U$. We observe that for the pair $\{p, r\} \subset (\partial\Omega \cap U)$, we can find (λ, ϵ') -geodesics for $\Omega \cap U$, joining points tending to p and r respectively. By construction, they eventually avoid any compact set in $\Omega \cap U$. This shows that $\{p, r\}$ is not a λ -visible pair for $\Omega \cap U$. By Proposition 8, p is not a λ -visible point for $\Omega \cap U$. This is a contradiction, we must have (2) \implies (1).

Now we will show that (1) \implies (3). The proof is very similar to the one above.

Let U be any bounded neighbourhood of p . To get a contradiction we suppose that there exist sequences $z_n \rightarrow p$, $w_n \rightarrow q \in \partial\Omega \cap U$ with $q \neq p$, and (λ, ϵ) -geodesics for $\Omega \cap U$ $\gamma_n : I_n \rightarrow \Omega \cap U$ joining z_n to w_n eventually avoiding any compact set in $\Omega \cap U$.

By Theorem 9, we can find another neighbourhood $V \subset\subset U$ of p and a constant $C \geq 0$ satisfying

$$k_{\Omega \cap U}(z, w) \leq k_\Omega(z, w) + C$$

for any $z, w \in \Omega \cap V$.

By shrinking V if necessary, we assume that $w_n \notin \Omega \cap V$. As above, we consider the restrictions of image of γ_n to $\Omega \cap V$. Denote those curves γ'_n and set w'_n to be the point where γ'_n leaves V . Our assumption gives that by passing to a subsequence if necessary we have $w'_n \rightarrow r \in \bar{V} \cap \partial\Omega$. By Corollary 13, there exists an ϵ' such that each γ'_n is a (λ, ϵ') -geodesic for Ω . We observe that we can find (λ, ϵ') -geodesics for Ω joining $\{p, r\} \subset \partial\Omega$ which tend to $\partial\Omega$. Thus, $\{p, q\}$ is not a λ -visible pair for Ω . By Proposition 8, this contradicts with the fact that p is a λ -visible point for Ω , and we see that (1) \implies (3). We are done. \square

By Theorem 15 it is easy to see that λ -visibility of a boundary point is independent from the domain itself. More formally we have:

Theorem 16. *Let $\Omega \subset \mathbb{C}^n$ be a hyperbolic domain which is hyperbolic at $p \in \partial\Omega$. Then p is a λ -visible point for Ω if and only if for any $D \in [\Omega]_p$, we have that p is a λ -visible point for D .*

Proof. The second statement clearly implies the first.

To see the converse, we assume that p is visible for Ω . Let $D \in [\Omega]_p$. Then there exists a bounded neighbourhood U of p such that $\Omega \cap U = D \cap U$. Theorem 15 implies that p is a

visible point for $D \cap U$. By definition D is also hyperbolic at p so again by Theorem 15 we see that p is a visible point for D . We are done. \square

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