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Elegant vertex labelings with prime numbers

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Abstract

We consider graph labelings with an assignment of odd prime numbers to the vertices. Similarly to graceful graphs, a labeling is said to be elegant if the absolute differences between the labels of adjacent vertices describe exactly the first even numbers. The labels of an elegant tree with n vertices are the first n odd prime numbers and we want that the resulting edge labels are exactly the first even numbers up to $2n - 2$. We conjecture that each path is elegant and we give the algorithm with which we got elegant paths of n primes for all n up to $n = 3500$.

1 Introduction

In this paper, we adapt the notion of graceful graphs by considering an assignment of odd prime numbers to the vertices. Let $G = (V, E)$ be a graph, we look for labelings of the vertices with distinct odd primes which induce edge labelings with all even integers from 2 up to $2|E|$. For instance in the tree displayed in Fig. 1, the first twelve odd primes are assigned to the vertices and we get all the even positive integers up to 22. We call elegant any graph for which there exists such a labeling. We refer to Galian [2] for a very detailed survey about graph labelings.

Let us denote the increasing sequence of all odd prime numbers by $p_1, p_2, \dots, p_r, \dots$ and $\mathbb{P}_n = \{p_1, p_2, \dots, p_n\}$. We now precise the definition of an elegant graph.

Definition 1 *Let $G = (V, E)$ be an undirected graph without loop or multiple edge, with n vertices and r edges. We say that G is elegant if there exists an injective map $\varphi : V \rightarrow \{p_1, p_2, \dots, p_{r+1}\}$ such that the induced map*

$$\begin{aligned} \psi : E &\longrightarrow 2\mathbb{N}^* \\ e = uv &\longrightarrow \psi(e) = |\varphi(v) - \varphi(u)| \end{aligned}$$

is a one-to-one correspondence from E to $\{2, 4, 6, \dots, 2r\}$.

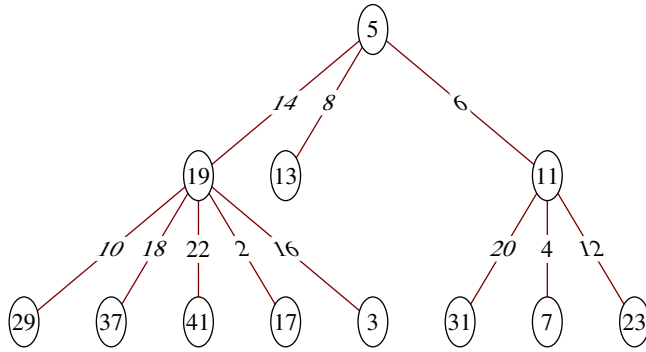


Figure 1: An elegant tree of 12 vertices.

The complete graphs up to K_4 are graceful and elegant: Their elegant labelings are respectively determined by $\varphi(V) = \{3, 5\}$, $\{5, 7, 11\}$ and $\{7, 11, 17, 19\}$. As in the case of graceful graphs, it seems that no other elegant complete graphs exists. We display an elegant labeling of the Petersen graph in Fig. 2.

Of course, if the number of vertices is too weak relatively to the maximal degree of the graph, then the graph is probably not (or cannot) be elegant. For instance, the star graphs S_n with $|V(S_n)| = n$ are not elegant as soon as the center has more than 8 adjacent vertices (in fact, we verified that the only elegant star graphs are S_2, S_3, S_5, S_6 and S_9). Nevertheless, in the case of trees, we could hope that for each integer $d \geq 2$, the answer to the following question A_d be positive.

Question A_d : Let $d \geq 2$ be an integer. Is there an integer N_d such that each tree of maximal degree d and with more than N_d vertices is elegant?

The answer to A_d is negative for d large enough that is an easy consequence of the prime number theorem, see for instance [1, 3]: Let us consider a symmetric and elegant tree T rooted at a vertex of degree d , with exactly d^k vertices of degree $d + 1$ at each level $k = 1, \dots, m - 1$ and with d^m leaves at level m . We have $n = |V(T)| = 1 + d + \dots + d^m$ and then $p_n \underset{d \rightarrow \infty}{\sim} md^m \log d$. Moreover, the absolute difference between labels of two adjacent vertices is less than $2n - 2 \underset{d \rightarrow \infty}{\sim} 2d^m$. We consider the path $v_0 v_1 \dots v_k \dots v_p$ on the tree T , from the vertex v_0 labeled by $\varphi(v_0) = 3$ and ended by the vertex v_p labeled by $\varphi(v_p) = p_n$. Since $p \leq 2m$, we get

$$md^m \log d \underset{d \rightarrow \infty}{\sim} |\varphi(v_0) - \varphi(v_p)| \leq \sum_{i=0}^{p-1} |\varphi(v_i) - \varphi(v_{i+1})| \leq 4md^m (1 + o(1))$$

which is impossible when d is large enough. Since $n \rightarrow \infty$ when $m \rightarrow \infty$, there exists a minimal integer d_1 such that A_d is false for all $d \geq d_1$.

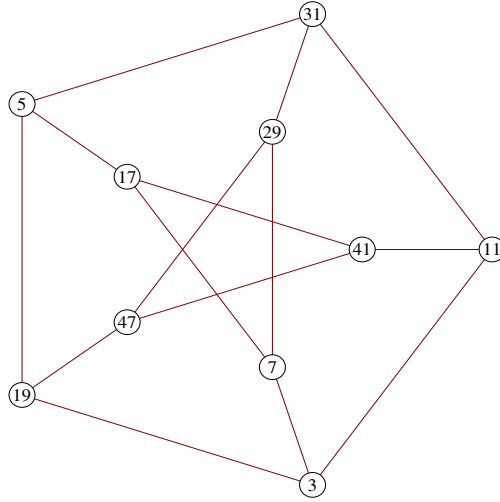


Figure 2: The Petersen graph is elegant.

As soon as $d = 3$, it is difficult to know if A_d is true or false (and we are far from the value $d = e^4$ which appears in the previous proof). Let us illustrate this point with an example: Let C_n be the regular caterpillar with n vertices of degree 3 and $n + 2$ leaves. On the one hand, up to $n = 25$ a trivial stochastic algorithm has given elegant labelings of C_n only for $n = 3, 5, 10, 18, 19, 20, 22$ but it is quite possible that C_n be elegant for all n large enough. Let us recall that Rosa [4] proved that all caterpillars are graceful. On the other hand, up to $n = 25$, as soon as we add a supplementary leaf w on any leaf $v \in V(C_n)$ or if we suppress anywhere one leaf of $V(C_n)$, then surprisingly our program finds in a few seconds that the modified tree is elegant.

Fortunately, the case $d = 2$ seems not to be resistant and we are confident that $N_2 = 2$:

Conjecture 1 *For all $n \geq 2$, the path of length n is elegant.*

Let us recall that a path of n vertices is elegant if there exists a permutation $\sigma \in S_n$ such that

$$\{|p_{\sigma(i+1)} - p_{\sigma(i)}|; 1 \leq i \leq n - 1\} = \{2, 4, 6, \dots, 2n - 2\}.$$

For instance, up to $n = 10$, the following labelings are elegant:

- $3 \underset{2}{-} 5$
- $5 \underset{2}{-} 3 \underset{4}{-} 7$
- $11 \underset{6}{-} 5 \underset{2}{-} 3 \underset{4}{-} 7$

- $13 \frac{-}{6} 7 \frac{-}{4} 11 \frac{-}{8} 3 \frac{-}{2} 5$
- $7 \frac{-}{2} 5 \frac{-}{6} 11 \frac{-}{8} 3 \frac{-}{10} 13 \frac{-}{4} 17$
- $13 \frac{-}{4} 17 \frac{-}{10} 7 \frac{-}{12} 19 \frac{-}{8} 11 \frac{-}{6} 5 \frac{-}{2} 3$
- $13 \frac{-}{10} 3 \frac{-}{4} 7 \frac{-}{2} 5 \frac{-}{14} 19 \frac{-}{8} 11 \frac{-}{12} 23 \frac{-}{6} 17$
- $11 \frac{-}{2} 13 \frac{-}{8} 5 \frac{-}{14} 19 \frac{-}{10} 29 \frac{-}{12} 17 \frac{-}{6} 23 \frac{-}{16} 7 \frac{-}{4} 3$
- $19 \frac{-}{12} 31 \frac{-}{14} 17 \frac{-}{6} 23 \frac{-}{10} 13 \frac{-}{16} 29 \frac{-}{18} 11 \frac{-}{8} 3 \frac{-}{4} 7 \frac{-}{2} 5$

2 Operations on admissible paths

In the sequel, a *path of l primes* $\mathcal{Q} = \mathcal{Q}_l = q_1 q_2 \dots q_l$ represents the labeling of the path of length l with the primes q_1, q_2, \dots, q_l in this order; we will identify the vertices of a path and their label q_i .

Definition 2 Let $n \geq 2$ be a given integer.

1. We say that a path $\mathcal{Q}_l = q_1 q_2 \dots q_l$ is admissible if the primes $q_i \in \mathbb{P}_n$ are distinct and if the set E_l of the $l - 1$ gaps $|q_{i+1} - q_i|$ is a subset of cardinal $l - 1$ of $\{2, 4, \dots, 2n - 2\}$.
2. With regard to a path $\mathcal{Q}_l = q_1 q_2 \dots q_l$ with $l < n$, a prime p in \mathbb{P}_n is said to be free if it is not a vertex of \mathcal{Q}_l . A gap $2k \leq 2n - 2$ is said to be free if it does not belong to E_l .

We also adopt the notations :

- If $\mathcal{Q} = q_1 q_2 \dots q_l$ then $\overline{\mathcal{Q}} = q_l q_{l-1} \dots q_1$.
- If $\mathcal{Q}_1 = q_1 q_2 \dots q_l$ and $\mathcal{Q}_2 = q_{l+1} q_{l+2} \dots q_p$, then $\mathcal{Q}_1 \mathcal{Q}_2 = q_1 \dots q_l q_{l+1} \dots q_p$ is the *concatenation* of \mathcal{Q}_1 and \mathcal{Q}_2 . When we want to add a prime to a path $\mathcal{Q} = \mathcal{Q}_1 \mathcal{Q}_2$ respectively on the left end, between \mathcal{Q}_1 and \mathcal{Q}_2 or on the right end, we note these paths $p\mathcal{Q}$, $\mathcal{Q}_1 p \mathcal{Q}_2$ or $\mathcal{Q} p$.
- If $\mathcal{Q} = q_1 q_2 \dots q_l$ then $f(\mathcal{Q}) = q_1$, $\ell(\mathcal{Q}) = q_l$ and $\text{length}(\mathcal{Q}) = l$.
- If $\mathcal{Q} = q_1 q_2 \dots q_l$ is admissible, $F_p(\mathcal{Q})$ is the set of free primes for \mathcal{Q} and $F_g(\mathcal{Q})$ is the set of free gaps for \mathcal{Q} .

In the algorithm described in Section 3, we randomly apply transformations on admissible paths of length $l < n$ primes in order to find other admissible paths. Our aim is either to improve the length l by adding a prime to the path, or to substitute a prime of the path for a free prime, or simply to shuffle the path. Proposition 1 gives two elementary tools A1-A2 and A3-A4 for shuffling an admissible path:

Proposition 1 *Let $n \geq 3$ and $\mathcal{Q} = \mathcal{Q}_1\mathcal{Q}_2$ be an admissible path of $l < n$ primes. We denote by $\delta = |f(\mathcal{Q}_2) - \ell(\mathcal{Q}_1)|$ the gap between \mathcal{Q}_1 and \mathcal{Q}_2 .*

A1. *If $|f(\mathcal{Q}_2) - f(\mathcal{Q}_1)| \in F_g(\mathcal{Q})$, then the path $\mathcal{Q}^* = \overline{\mathcal{Q}_1}\mathcal{Q}_2$ is admissible and*

$$F_g(\mathcal{Q}^*) = F_g(\mathcal{Q}) \cup \{\delta\} \setminus \{|f(\mathcal{Q}_2) - f(\mathcal{Q}_1)|\}.$$

A2. *If $|\ell(\mathcal{Q}_2) - \ell(\mathcal{Q}_1)| \in F_g(\mathcal{Q})$, then the path $\mathcal{Q}^* = \mathcal{Q}_1\overline{\mathcal{Q}_2}$ is admissible and*

$$F_g(\mathcal{Q}^*) = F_g(\mathcal{Q}) \cup \{\delta\} \setminus \{|\ell(\mathcal{Q}_2) - \ell(\mathcal{Q}_1)|\}.$$

A3. *If $|\ell(\mathcal{Q}_2) - f(\mathcal{Q}_1)| \in F_g(\mathcal{Q})$, then the path $\mathcal{Q}^* = \mathcal{Q}_2\mathcal{Q}_1$ is admissible and*

$$F_g(\mathcal{Q}^*) = F_g(\mathcal{Q}) \cup \{\delta\} \setminus \{|\ell(\mathcal{Q}_2) - f(\mathcal{Q}_1)|\}.$$

A4. *If $|\ell(\mathcal{Q}_2) - f(\mathcal{Q}_1)| = \delta$, then the path $\mathcal{Q}^* = \mathcal{Q}_2\mathcal{Q}_1$ is admissible and*

$$F_g(\mathcal{Q}^*) = F_g(\mathcal{Q}).$$

Remark 1 (Insertion of a free gap) In Algorithm 1 described in Section 3 and when we get an admissible transformation from \mathcal{Q} to \mathcal{Q}^* with a property A_i , we try to insert a free prime in \mathcal{Q}^* : If r is a new free prime for \mathcal{Q}^* , we can try to insert it directly with Proposition 2. If a gap δ has become free, we can try to insert in \mathcal{Q}^* by Proposition 2, any prime $r = s \pm \delta \in F_p(\mathcal{Q})$ if $\mathcal{Q}^* = \mathcal{Q}_1\mathcal{Q}_2$ with $s = \ell(\mathcal{Q}_1)$ or $s = f(\mathcal{Q}_2)$. Obviously, we also test in the algorithm if one of the paths $r\mathcal{Q}$ or $\mathcal{Q}r$ is admissible.

For instance, let us consider $n = 7$ and the admissible path $\mathcal{Q}_6 = 7-19-17-11-3-13$. The last free gap is $4 = |11 - 7|$ and we can apply the transformation given by A1 with $\mathcal{Q}_1 = 7-19-17$ and $\mathcal{Q}_2 = 11-3-13$, we get $\mathcal{Q}^* = \overline{\mathcal{Q}_1}\mathcal{Q}_2 = 17-19-7-11-3-13$ and the last free gap is now $\delta = 6$. The last free prime is $r = 5 = 11 - \delta$ and we can insert r by the third point of Proposition 2 with $\mathcal{Q}_1 = 17-19-17-11$ and $\mathcal{Q}_2 = 3-13$: We get an admissible and elegant path of seven primes with the admissible transformation $\mathcal{Q}_6 = \mathcal{Q}_1\mathcal{Q}_2 \rightarrow \mathcal{Q}_7 = \mathcal{Q}_1r\overline{\mathcal{Q}_2} = 17-19-7-11-5-13-3$.

Proposition 2 (Insertion of a free prime) *Let $n \geq 3$ and $\mathcal{Q} = \mathcal{Q}_1\mathcal{Q}_2$ be an admissible path of $l < n$ primes and let $r \in F_p(\mathcal{Q})$. We denote by \mathbf{C} the condition*

$$\mathbf{C} : \begin{cases} (|r - p| \in F_g(\mathcal{Q}) \text{ and } |r - p| \neq |q - r| \in F_g(\mathcal{Q})) \\ \quad \text{or} \\ (|r - p| \in F_g(\mathcal{Q}) \text{ and } |q - r| = \delta) \\ \quad \text{or} \\ (|q - r| \in F_g(\mathcal{Q}) \text{ and } |r - p| = \delta). \end{cases}$$

If \mathbf{C} is true for $\delta = |f(\mathcal{Q}_2) - \ell(\mathcal{Q}_1)|$ with

1. $p = \ell(\mathcal{Q}_1)$ and $q = f(\mathcal{Q}_2)$, then the path $\mathcal{Q}^* = \mathcal{Q}_1r\mathcal{Q}_2$ is admissible;
2. $p = f(\mathcal{Q}_1)$ and $q = f(\mathcal{Q}_2)$, then the path $\mathcal{Q}^* = \overline{\mathcal{Q}_1}r\mathcal{Q}_2$ is admissible;
3. $p = \ell(\mathcal{Q}_1)$ and $q = \ell(\mathcal{Q}_2)$, then the path $\mathcal{Q}^* = \mathcal{Q}_1r\overline{\mathcal{Q}_2}$ is admissible.

In order to substitute a prime p of a path \mathcal{Q} for a free prime, we can consider 36 transformations $A_5, A_6, \dots, A_{40} : \mathcal{Q} \in X \rightarrow \mathcal{Q}^* \in Y$ where $X = \{q\mathcal{Q}_1\mathcal{Q}_2, \mathcal{Q}_1q\mathcal{Q}_2, \mathcal{Q}_1\mathcal{Q}_2q\}$ and $Y = \{r\mathcal{Q}_1\mathcal{Q}_2, r\overline{\mathcal{Q}_1}\mathcal{Q}_2, r\mathcal{Q}_1\overline{\mathcal{Q}_2}, r\overline{\mathcal{Q}_1}\overline{\mathcal{Q}_2}, \mathcal{Q}_1r\mathcal{Q}_2, \mathcal{Q}_1r\overline{\mathcal{Q}_2}, \overline{\mathcal{Q}_1}r\mathcal{Q}_2, \overline{\mathcal{Q}_1}r\overline{\mathcal{Q}_2}, \mathcal{Q}_1\mathcal{Q}_2r, \mathcal{Q}_1\overline{\mathcal{Q}_2}r, \overline{\mathcal{Q}_1}\mathcal{Q}_2r, \overline{\mathcal{Q}_1}\overline{\mathcal{Q}_2}r\}$. The following proposition details only the transformations $\mathcal{Q} = \mathcal{Q}_1q\mathcal{Q}_2 \rightarrow \mathcal{Q}^* = \mathcal{Q}_1r\mathcal{Q}_2$ and $\mathcal{Q} = \mathcal{Q}_1q\mathcal{Q}_2 \rightarrow \mathcal{Q}^* = \overline{\mathcal{Q}_1}r\mathcal{Q}_2$, but it is trivial to find for which conditions the other transformations A_i give admissible paths.

Proposition 3 *Let $n \geq 3$ and $\mathcal{Q} = \mathcal{Q}_1q\mathcal{Q}_2$ be an admissible path of $l < n$ primes. Let r be a free prime.*

1. *The path $\mathcal{Q}^* = \mathcal{Q}_1r\mathcal{Q}_2$ is admissible if $|r - \ell(\mathcal{Q}_1)| \neq |f(\mathcal{Q}_2) - r|$ and*

$$\{|r - \ell(\mathcal{Q}_1)|, |f(\mathcal{Q}_2) - r|\} \subset F_g(\mathcal{Q}) \cup \{|q - \ell(\mathcal{Q}_1)|, |f(\mathcal{Q}_2) - q|\}.$$

2. *The path $\mathcal{Q}^* = \overline{\mathcal{Q}_1}r\mathcal{Q}_2$ is admissible if $|r - f(\mathcal{Q}_1)| \neq |f(\mathcal{Q}_2) - r|$ and*

$$\{|r - f(\mathcal{Q}_1)|, |f(\mathcal{Q}_2) - r|\} \subset F_g(\mathcal{Q}) \cup \{|q - \ell(\mathcal{Q}_1)|, |f(\mathcal{Q}_2) - q|\}.$$

3 An algorithm for finding elegant paths

We have found elegant labelings for all paths up to $n = 3500$ vertices with the algorithm we describe below. In Algorithm 1, we construct incrementally a sequence of paths: from a path \mathcal{Q}_l of l primes, we try either to add a prime or to modify the path \mathcal{Q}_l without improving its length. The integer n being fixed, our aim is to obtain $l = n$.

The first part 1-4 of Algorithm 1 is a trivial greedy algorithm, we simply try to add free primes on the ends of \mathcal{Q}_l . In the While statement of the step 5, we intensively and randomly use the transformations A_i given in Section 2. When m reaches N without giving an elegant path of length n , we quit and we start a new run of Algorithm 1. With $N = 40n$, we get with Algorithm 1, all the elegant paths of n primes from $n = 2$ up to $n = 200$ in less than 5 seconds with one core of a 3.6 GHz processor.

If $n \geq 200$, we accelerate the calculations with Algorithm 2 in which we have chosen $N = 20n$ et $c_0 = 20$: when a run reaches $l \geq n - \delta$, we do not give up the path if $l < n$ but we suppress the right end q_l of the path \mathcal{Q}_l . We take for instance $\delta = 1$ for $n = 200$ and $\delta = 5$ from $n = 2000$ up to $n = 3500$. This can be seen as a perturbation on a non-optimal and rigid configuration. When $n \in [2000, 2100]$, the average run-time to find one elegant path with Algorithm 2 is 4 minutes; when $n \in [3400, 3500]$, this average run-time becomes 40 minutes. It is surprising to find a solution so easily among $3500! > 10^{10000}$ permutations.

Algorithm 1

1. Randomly choose a prime q_1 in \mathbb{P}_n and set $l := 1$, $\mathcal{Q}_l := q_1$;
2. $l_1 := 0$;
3. **While** $l_1 \neq l$ and $l < n$ **do**
 - $l_1 := l_1 + 1$;
 - If we find $p \in F_p(\mathcal{Q}_l)$ s.t. $\mathcal{Q}_l p$ (or resp. $p\mathcal{Q}_l$) is admissible then we set $\mathcal{Q}_{l+1} := \mathcal{Q}_l p$ (or resp. $\mathcal{Q}_{l+1} := p\mathcal{Q}_l$) and $l := l + 1$;
4. **If** $l = n$ **then** return the elegant path \mathcal{Q}_n and **quit**.
5. **While** $l < n$ and $m < N$ **do**
 - Randomly choose $i \in \{1, 2, 3\}$.
 - Case** $i = 1$: Look for an admissible transformation $\mathcal{Q}^* = \overline{\mathcal{Q}_1}\mathcal{Q}_2$ or $\mathcal{Q}_1\overline{\mathcal{Q}_2}$ of \mathcal{Q}_l with A1 or A2. If we succeed in changing \mathcal{Q}_l , we set $\mathcal{Q}_l := \mathcal{Q}^*$. Then, we try to insert a free prime in this new path \mathcal{Q}_l with Remark 1 and Prop. 2; if we succeed in this, we set $l := l + 1$ and $\mathcal{Q}_l := \mathcal{Q}^*$, the path \mathcal{Q}^* having been given by Prop. 6.
 - Case** $i = 2$: **If** $|\ell(\mathcal{Q}_l) - f(\mathcal{Q}_l)| \in F_g(\mathcal{Q}_l)$ **then** randomly choose $u \in \{1, \dots, l - 1\}$ and set $\mathcal{Q}_1 := q_1 \cdots q_u$, $\mathcal{Q}_2 := q_{u+1} \cdots q_l$ and $\mathcal{Q}_l := \mathcal{Q}_2\mathcal{Q}_1$. Then, if we succeed in inserting a free prime in this new \mathcal{Q}_l with Remark 1 and Prop. 2, we set $l := l + 1$ and $\mathcal{Q}_l := \mathcal{Q}^*$, the path \mathcal{Q}^* having been given by Prop. 2.
 - Else** there exists $u \in \{1, \dots, l - 1\}$ s.t. $|q_{u+1} - q_u| = |\ell(\mathcal{Q}_l) - f(\mathcal{Q}_l)|$ and we set $\mathcal{Q}_l := \mathcal{Q}_2\mathcal{Q}_1$ with $\mathcal{Q}_1 = q_1 \cdots q_u$ and $\mathcal{Q}_2 = q_{u+1} \cdots q_l$.
 - Case** $i = 3$: Look for a transformation among A_5, \dots, A_{40} which gives an admissible path \mathcal{Q}^* . If we succeed in modifying \mathcal{Q}_l , we set $\mathcal{Q}_l := \mathcal{Q}^*$. If we succeed in inserting a free prime in this new \mathcal{Q}_l with Remark 1 and Prop. 2, we set $l := l + 1$ and $\mathcal{Q}_l := \mathcal{Q}^*$.
 - $m := m + 1$.
6. **Return** the path \mathcal{Q}_l which is elegant if $l = n$.

Algorithm 2

$l := 0;$
While $l < n$ **do**
 1. **Do Algorithm 1** which gives an admissible path \mathcal{Q}_l of length l ;
 2. **If** $l = n$ **then return** the elegant path \mathcal{Q}_n ;
 3. **If** $n - \delta \leq l < n$ **then**
 $c := 0;$
 While $l < n$ and $c < c_0$ **do**
 (a) Suppress the last prime q_l of \mathcal{Q}_l and set $l := l - 1$;
 (b) **Do Step 5 of Algorithm 1** and set $c := c + 1$;
 (c) **If** $l = n$ **then return** the elegant path \mathcal{Q}_n .

Let us detail an example of a possible run of **Algorithm 1** in the case $n = 11$.

- **Steps 1 – 4 of Algorithm 1:** We randomly choose $q_1 = 5$, $q_2 = 7$, $q_3 = 3, \dots$, and we get

$$\mathcal{Q}_{10} := 5 \underset{2}{-} 7 \underset{4}{-} 3 \underset{16}{-} 19 \underset{12}{-} 31 \underset{20}{-} 11 \underset{18}{-} 29 \underset{8}{-} 37 \underset{14}{-} 23 \underset{6}{-} 17,$$

the last free prime is 13 and the free gap is 10.

- **Step 5 of Algorithm 1:** We randomly choose values for i in $\{1, 2, 3\}$:
 – $i = 1$: The gap $|17 - 7| = 10$ is free and we apply the transformation $\mathcal{Q} = \mathcal{Q}_1 \mathcal{Q}_2 \rightarrow \mathcal{Q}^* = \mathcal{Q}_1 \overline{\mathcal{Q}_2}$ with $\mathcal{Q}_1 = 5 - 7$ and $\mathcal{Q}_2 = 3 - 19 - 31 - 11 - 29 - 37 - 23 - 17$. We get

$$\mathcal{Q}_{10} := 5 \underset{2}{-} 7 \underset{10}{-} 17 \underset{6}{-} 23 \underset{14}{-} 37 \underset{8}{-} 29 \underset{18}{-} 11 \underset{20}{-} 31 \underset{12}{-} 19 \underset{16}{-} 3,$$

the last free gap is now 4.

- $i = 3$: We apply the transformation $\mathcal{Q} = \mathcal{Q}_1 \mathcal{Q}_2 q \rightarrow \mathcal{Q}^* = \mathcal{Q}_1 r \overline{\mathcal{Q}_2}$ with $\mathcal{Q}_1 = 5 - 7 - 17$, $\mathcal{Q}_2 = 23 - 37 - 29 - 11 - 31 - 19$, $q = 3$ and $r = 13$. We find

$$\mathcal{Q}_{10} := 5 \underset{2}{-} 7 \underset{10}{-} 17 \underset{4}{-} 13 \underset{6}{-} 19 \underset{12}{-} 31 \underset{20}{-} 11 \underset{18}{-} 29 \underset{8}{-} 37 \underset{14}{-} 23,$$

the free prime is now 3 and the free gap is 16.

- $i = 2$: Since the gap between the two ends is $|23 - 5| = 18 \notin F_g(\mathcal{Q})$, we apply $\mathcal{Q} = \mathcal{Q}_1 \mathcal{Q}_2 \rightarrow \mathcal{Q}^* = \mathcal{Q}_2 \mathcal{Q}_1$ with $\mathcal{Q}_1 = 5 - 7 - 17 - 13 - 19 - 31 - 11$ and $\mathcal{Q}_2 = 29 - 37 - 23$. We get

$$\mathcal{Q}_{10} := 29 \underset{8}{-} 37 \underset{14}{-} 23 \underset{18}{-} 5 \underset{2}{-} 7 \underset{10}{-} 17 \underset{4}{-} 13 \underset{6}{-} 19 \underset{12}{-} 31 \underset{20}{-} 11.$$

- $i = 1$: The gap $|29 - 13| = 16$ is free and we apply the transformation $Q = Q_1 Q_2 \rightarrow Q^* = \overline{Q_1} Q_2$ with $Q_1 = 29 - 37 - 23 - 5 - 7$ and $Q_2 = 17 - 13 - 19 - 31 - 11$. We get

$$Q_{10} := 17 \underset{10}{-} 7 \underset{2}{-} 5 \underset{18}{-} 23 \underset{14}{-} 37 \underset{8}{-} 29 \underset{16}{-} 13 \underset{6}{-} 19 \underset{12}{-} 31 \underset{20}{-} 11,$$

and the last free gap is 4. We can place the prime 3 between 7 and 5 and we get an elegant path of eleven primes:

$$Q_{11} := 17 \underset{10}{-} 7 \underset{4}{-} 3 \underset{2}{-} 5 \underset{18}{-} 23 \underset{14}{-} 37 \underset{8}{-} 29 \underset{16}{-} 13 \underset{6}{-} 19 \underset{12}{-} 31 \underset{20}{-} 11.$$

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References

- [1] T. M. Apostol. *Introduction to Analytic Number Theory*, Springer-Verlag 1976, New York.
- [2] J. A. Gallian. A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics* 5, #DS6, 2005.
- [3] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*, 6th Edition, Oxford University Press, Oxford, 2008.
- [4] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs*, (Internat. Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris (1967) 349-355.