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Francis Wlazinski

► **To cite this version:**

| Francis Wlazinski. Procyclic morphisms. 2024. hal-04546070

HAL Id: hal-04546070

<https://hal.science/hal-04546070v1>

Preprint submitted on 16 Apr 2024

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Procyclic morphisms

Francis Wlazinski

April 15, 2024

Abstract

We generalise the notion of cyclic morphism and we study some properties of procyclic morphisms, in particular, for k -power freeness.

1 Words

An *alphabet* A is a finite set of symbols called *letters*. A *word* over A is a finite sequence of letters from A . The *empty word* ε is the empty sequence of letters. Given a non-empty word $u = a_1 \dots a_n$ with $a_i \in A$ for any integer i from 1 to n , the *length* of u denoted by $|u|$ is the integer n that is the number of letters of u . By convention, we have $|\varepsilon| = 0$.

$w = clebaba$ is a word sur $A = \{a, b, c, e, l\}$ of length $|w| = 7$.

$w = 011010011001011010010110011010011001$ is a word over $B = \{0, 1\}$. It is a prefix of Thue-Morse word.

Equipped with the concatenation operation, the set A^* of words over A is a free monoid with ε as neutral element and A as set of generators. We denote by A^+ the set of words of positive length over A , i.e., $A^+ = A^* \setminus \{\varepsilon\}$.

Since an alphabet with one element is limited interest to us, we always assume that the cardinality of considered alphabets is at least two.

In the rest of this paper, A and B are two alphabets such that $\text{Card}(A) \geq 2$ and $\text{Card}(B) \geq 2$. A word u is a *factor* of a word v if there exist two (possibly empty) words p and s such that $v = pus$. We denote $\text{Fcts}(v)$ the set of all factors of v . If $u \in \text{Fcts}(v)$, we also say that v *contains* the word u (as a factor). If $p = \varepsilon$, u is a *prefix* of v . If $s = \varepsilon$, u is a *suffix* of v . If $u \neq v$, u is a *proper factor* of v . If u , p and s are non-empty, u is an *internal factor* of v .

Let w be a non-empty word and let i, j be two integers such that $0 \leq i - 1 \leq j \leq |w|$. We denote by $w[i..j]$ the factor of w such that $|w[i..j]| = j - i + 1$ and $w = pw[i..j]s$ for two words s and p verifying $|p| = i - 1$. When $j > i$, $w[i..j]$ is simply the factor of w that starts at the i^{th} letter and ends at the j^{th} . Note that, when $j = i - 1$, we have $w[i..j] = \varepsilon$. When $i = j$, we also denote by $w[i]$ the factor $w[i..i]$ which is the i^{th} letter of w . In particular, $w[1]$ and $w[|w|]$ are respectively the first and the last letter of w .

Powers of a word are defined inductively by $u^0 = \varepsilon$, and for any integer $n \geq 1$, $u^n = uu^{n-1}$. Given an integer $k \geq 2$, since the case ε^k is of little interest, we call a *k-power* any word

u^k with $u \neq \varepsilon$. And a word is k -power-free if it does not contain any k -power as factor. A *primitive* word is a word which is not a ℓ -power of another word whatever the integer $\ell \geq 2$. When $k = 2$ (resp. $k = 3$), a k -power is also called a *square* (resp. a *cube*).

If a is a letter and if X is a (eventually empty) word, a word of the form $aXaXa$ is called an *overlap*. And a word of the form $aXXa$ is called a *framed-square* (Zolotov ([9]) called it a weak square).

Given an integer $k \geq 2$, a word is k -power-free (resp. overlap-free or framed-square-free) if it does not contain any factor that is a k -power (resp. an overlap or a framed-square).

Let us note that, for a word w of length lower or equal to three, w is square-free if and only if w is framed-square-free.

Let us first recall a well known result of Fine and Wilf [5, 6].

Theorem 1.1 *Let x and y be two words. If a power of x and a power of y have a common prefix of length at least equal to $|x| + |y| - \gcd(|x|, |y|)$ then x and y are powers of the same word.*

As a consequence of Theorem 1.1, we get:

Corollary 1.2 [2] *Let x and y be two words. If a power of x and a power of y have a common factor of length at least equal to $|x| + |y| - \gcd(|x|, |y|)$ then there exist two words t_1 and t_2 such that x is a power of t_1t_2 and y is a power of t_2t_1 with t_1t_2 and t_2t_1 primitive words. Furthermore, if $|x| > |y|$ then x is not primitive.*

Two words u and v are *conjugated* if $u = t_1t_2$ and $v = t_2t_1$ for two (possibly empty) words t_1 and t_2 . We also say that v is a *conjugate* of u . We trivially get that:

Lemma 1.3 *Every conjugate of a primitive word is primitive.*

The following proposition gives the well-known solutions (see [5]) to elementary equations on words and will be widely used in the following sections:

Proposition 1.4 *Let u, v, w be three words over A and let x, y be two letters in A .*

1. *If $vu = uw$ and $v \neq \varepsilon$, then there exist two words r and s over A , and an integer p such that $u = (rs)^p r$, $v = rs$ et $w = sr$.*
2. *If $vu = wv$, then there exist a word w over A , and two integers p and q such that $u = w^p$ and $v = w^q$.*
3. *If $xu = uy$, then $x = y$ and there exists an integer q such that $u = x^q$.*

As immediate consequences of Proposition 1.4(2), we get the two following results:

Lemma 1.5 [2, 4] *If a non-empty word v is an internal factor of vv , i.e., if there exist two non-empty words x and y such that $vv = xvy$, then there exist a non-empty word t and two integers $i, j \geq 1$ such that $x = t^i$, $y = t^j$, and $v = t^{i+j}$.*

Corollary 1.6 *Let u, v, w be three non-empty words of A^+ . If $uvw = wvu$ then there exist two words r and s and three integers i, j and k such that $u = (rs)^i r$, $v = (sr)^j s$ and $w = (rs)^k r$.*

We will consider another equations on words:

Lemma 1.7 *Let u, w be two words of A^* and let x, y be two letters of A . If $uxw = wyu$ then either $x = y$ or there exists two words u', w' of A^* such that $u'xw' = w'yu'$ and $|u'| + |w'| < |u| + |w|$.*

Proof.

If $u = \varepsilon$ or if $w = \varepsilon$, by Case 3 of Proposition 1.4, we get $x = y$.

If $|u| = |w|$ then $u = w$ and $x = y$.

So, we may assume $|u| \neq |w|$ and, by a symmetric way and without loss of generality, we may assume that $|u| > |w| > 0$.

By Case 2 of Proposition 1.4, the equality $u(xw) = (wy)u$ implies that there exist two words r and s and an integer p such that $wy = rs$, $xw = sr$ and $u = (rs)^p r$.

If $|s| = 0$ then $wy = r = xw$. By Case 3 of Proposition 1.4, we get $x = y$.

If $|s| \geq 1$ then, from the two equalities, $wy = rs$ and $xw = sr$, we get that s starts by x and ends with y . If $|s| = 1$ then $s = y = x$. If $|s| \geq 2$ then there exists then a word s' such that $s = xs'y$. So, we get $wy = rxs'y$ and $xw = xs'yr$, that is, $rxs' = w = s'yr$ with $|rs'| < |w| < |u| + |w|$.

We obtain the result taking $u' = r$ and $w' = s'$. \square

Corollary 1.8 *Let u, w be two words of A^* and let x, y be two letters of A . If $uxw = wyu$ then $x = y$.*

Corollary 1.8 is a consequence of Lemma 1.7. Indeed, considering the length of the words, there does not exist a strictly decreasing (infinite) sequence of natural integers.

Corollary 1.9 *Let u, w be two words of A^* and let x, y be two letters of A . If $xuw = wuy$ then $x = y$.*

Proof.

If $u = \varepsilon$ or if $w = \varepsilon$ (that is $|w| = 0$), by Case 3 of Proposition 1.4, we get that $x = y$.

So, let us assume that $|w| \geq 1$ and that $u \neq \varepsilon$. From the equality $xuw = wuy$, we get that w starts by x and ends with y . If $|w| = 1$, we get $w = x = y$. If $|w| \geq 2$, there exists a word w' such that $w = xw'y$. The equality $xuw = wuy$ becomes $xuxw'y = xw'yuy$, i.e., $uxw' = w'yu$. And, by Corollary 1.8, it follows that $x = y$. \square

Corollary 1.10 *Let u, w be two words of A^* and let x, y be two letters of A . If $xuw = wuy$ then $x = y$.*

Proof.

If $u = \varepsilon$ or if $w = \varepsilon$ (that is $|w| = 0$), we get $x = y$ by Case 3 of Proposition 1.4.

If $|u| = |w|$, we get $u = w$ and, always by Case 3 of Proposition 1.4, it follows that $x = y$.

By Case 1 of Proposition 1.4, the equality $(xu)w = w(yu)$ implies that there exist two words r and s and an integer p such that $xu = rs$, $yu = sr$ and $w = (rs)^p r$.

If $|r| = 0$ or if $|s| = 0$ then $xu = yu$ and we get $x = y$.

If $|s| \geq 1$ and $|r| \geq 1$ then, from the two equalities $xu = rs$ and $yu = sr$, we get that r starts by x and s starts by y . There exist two words r' and s' such that $r = xr'$ and $s = ys'$.

So, we get $xu = xr'ys'$, $yu = ys'xr'$ and $u = r'ys' = s'xr'$. And, by Corollary 1.8, it implies that $x = y$. \square

Proposition 1.11 *Let t be a primitive word. If there exist an integer $\alpha \geq 2$ and two words v and ℓ such that $|v| \geq |t|$ and $v\ell v$ is a factor of t^α then $v\ell$ is a power of a conjugate of t . In particular, $|v\ell| \equiv 0 \pmod{|t|}$ and, if $\ell \neq \varepsilon$, then $v\ell$ is not primitive.*

Proof.

The word $v\ell v$ is a common factor of $(v\ell)^2$ and of t^α . Moreover, since $|v| \geq |t|$, we get $|v\ell v| \geq |v\ell| + |t|$. By Corollary 1.2, there exist two words t_1 and t_2 and two integers β and γ such that $t = (t_1 t_2)^\beta$ and $v\ell = (t_2 t_1)^\gamma$ with $t_1 t_2$ and $t_2 t_1$ primitive words. Since t is a primitive word, we get $\beta = 1$. If $\ell \neq \varepsilon$, we get $|v\ell| > |t|$ and so $v\ell$ is not primitive, that is, $\gamma \geq 2$. \square

2 Canvas of a word

Definition 2.1 *Let $1 \leq n \leq 9$ be an integer and let δ_n be the map from $\llbracket 0; n \rrbracket^+$ to $\llbracket 0; n \rrbracket^*$ defined in such a way: for a word $|w| \geq 2$, $\delta_n(w)$ is the word \underline{w} where, for any integer i from 1 to $|w| - 1$, the letter $w[i]$ is the remainder of the Euclidean division of $w[i + 1] - w[i]$ by $n + 1$. In other words, we have $\underline{w}[i] = w[i + 1] - w[i] \pmod{n + 1}$.*

If $w = 0112102$ then $\underline{w} = \delta_2(w) = 101222$.

If $\mu = 0110\ 1001\ 1001\ 0110\ 1001\ 0110\ 1001\ 1001\ 0110\ 0110\ 1001\ 0110\ 1001$ then $\underline{\mu} = \delta_1(\mu) = 1011101\ \mathbf{0}\ 1011101\ \mathbf{1}\ 1011101\ \mathbf{0}\ 1011101\ \mathbf{0}\ 1011101\ \mathbf{0}\ 1011101\ \mathbf{1}\ 1011101$.

If $f = 010010100100101001010$ then $\underline{f} = \delta_1(f) = 11\ \mathbf{0}\ 1111\ \mathbf{0}\ 11\ \mathbf{0}\ 1111\ \mathbf{0}\ 1111$.

We extend this definition to letters: by convention, for a letter x in A , $\delta_n(x) = \varepsilon$.

We can also extend, in a natural way, this definition to infinite words.

Let us remark that $|\underline{w}| = |w| - 1$ and that, if w does not contain any square of letters then \underline{w} does not contain any 0. Moreover, this map δ_n is not a morphism. More precisely, if $u, v \in \llbracket 0; n \rrbracket^*$ with $u = u_1 u_2 \dots u_p$ and $v = v_1 v_2 \dots v_q$, we have $\delta_n(uv) = \delta_n(u) \delta_n(u_p v_1) \delta_n(v)$ that is $\delta_n(uv) = \delta_n(u) \delta_n(u[|u|]v[1]) \delta_n(v)$.

A word w is fully determined by only one $w[i]$ and \underline{w} . And, in particular, if $u[1] = v[1]$ and if $\underline{u} = \underline{v}$ then $u = v$. We say that \underline{w} is the *canvas* of the word w . And, from this canvas, we can get other words changing one of the letters of w .

$w_2 = 1220210$ is the word of $\llbracket 0; 2 \rrbracket^*$ obtained from $\underline{w} = 101222$ starting by the letter 1.

$w_3 = 1223131$ is the word of $\llbracket 0; 3 \rrbracket^*$ obtained from $\underline{w} = 101222$ starting by the letter 1.

$f_2 = 012201200122012001201$ is the word of $\llbracket 0; 2 \rrbracket^*$ obtained from $\underline{f} = \delta_1(f) = 11011110110111101111$ starting by the letter 0.

$\mu_1 = 0010011101110010011100100$ is the word of $\llbracket 0; 1 \rrbracket^*$ obtained from $\mu = 011010011001011010010110$ starting by the letter 0.

For any letter $a \in \llbracket 0; n \rrbracket$ and for any word $C \in \llbracket 0; p \rrbracket^*$ where $p \leq n$, we denote $(a, C)_n$ the word of $\llbracket 0; n \rrbracket^*$ whose canvas is C and that starts by a . In particular, if $w \in \llbracket 0; n \rrbracket^*$, we wrote $w = (w[1], \underline{w})_n$.

Let us remark that, if two words u and v have the same canvas, then, for any integer i from 1 to $|u|$, the difference $u[i] - v[i]$ is constant (equal to $u[1] - v[1]$) modulo $n + 1$.

The relation "having the same canvas" is an equivalence relation on words. The point here is not to study quotient sets. Similarly, the study of words whose canvas would have a particular form (for instance the Thue-Morse word, the Fibonacci word or more generally the Sturmian words) could be interesting. Another idea could be to study the different patterns of a word in an alphabet larger than the word itself. But the ways to explore are too numerous to state them or, even less, to attempt an approach of one of them in this article. We only will focus on morphisms and powers.

Definition 2.1 can be extended to any alphabet:

Definition 2.2 *Let $n \geq 1$ be an integer, let $A = (a_0, a_1, a_2, \dots, a_n)$ be an ordered alphabet and let δ_n be the map from A^* to A^* defined by $\delta_n(w) = \underline{w}$ where, for any integer i from 1 to $|w| - 1$, if $w[i] = a_r$ and $w[i + 1] = a_s$, then we get $\underline{w}[i] = a_t$ where $t \equiv s - r \pmod{n + 1}$.*

If $A = \{a, b, c\}$ and $w = abacca$ then $\underline{w} = bccab$.

Let us note that, for us, there are simply $n + 1$ different **symbols** in the alphabet $\llbracket 0; n \rrbracket$.

In the rest of the paper, $N \geq 1$, $n \geq 1$ and $p \geq 1$ are three integers.

For any word w of $\llbracket 0; N \rrbracket^*$, we will denote $\text{Somme}(w)_{N+1}$ the remainder of the Euclidean division of $\sum_{i=1}^{|w|} w[i]$ by $N + 1$.

Proofs of the following lemmas are left to the reader.

Lemme 2.3 *For any word w in $\llbracket 0; N \rrbracket^*$, we have $\text{Somme}(\underline{w})_{N+1} = w[|w|] - w[1] \pmod{N+1}$. In particular, w starts and ends with the same letter if and only if $\text{Somme}(\underline{w})_{N+1} = 0$.*

Lemme 2.4 *Let $k \geq 2$ be an integer.*

A word w in $\llbracket 0; N \rrbracket^$ contains a k -power if and only if \underline{w} contains a factor of the form $(Xy)^{k-1}X$ with X a word and y a letter such that $\text{Somme}(Xy)_{N+1} = 0$.*

Corollary 2.5 *Let $k \geq 2$ be an integer and let w be a word in $\llbracket 0; N \rrbracket^*$.*

If \underline{w} contains a $[(N + 1) \times k]$ -power then w contains a k -power.

Proof.

It comes directly from Lemmas 2.3 and 2.4 and from the fact that, for any non-empty word v , we have $\text{Somme}(v^{N+1})_{N+1} = 0$. \square

3 Morphismes

Let A and B be two alphabets. A *morphism* f from A^* to B^* is a map from A^* to B^* such that $f(uv) = f(u)f(v)$ for all the words u, v over A . When B has no importance, we simply say that f is defined over A or that f is defined on A^* .

Let $k \geq 2$ be an integer. A morphism f is *k-power-free* if the images of all k -power-free words are also k -power-free. When $k = 2$ (resp. $k = 3$), a k -power-free morphism is called a square-free morphism (resp. a cube-free morphism).

In a same way, a morphism f over A is *framed-square-free* (resp. *overlap-free*) if the image of any framed-square-free (resp. overlap-free) word is also framed-square-free (resp. overlap-free).

A morphism f over A is *k-power-free* (resp. framed-square-free or overlap-free) up to ℓ if the image of any word w such that $|w| \leq \ell$ by f is k -power-free (resp. framed-square-free or overlap-free).

A morphism f over A is called *prefix* (resp. *suffix*) if, for any letters a and b in A , the word $f(a)$ is not a prefix (resp. not a suffix) of $f(b)$. A morphism is *bifix* if it is prefix and suffix. A morphism f is non-erasing if $f(a) \neq \varepsilon$ for any letters a .

Let us note that a prefix (resp. suffix) morphism is injectif and is non-erasing.

A morphism f of A^* to B^* is a *ps-morphism* if the equalities $f(a) = ps$, $f(b) = ps'$, and $f(c) = p's$ with $a, b, c \in A$ (eventually $c = b$) and p, s, p' , and $s' \in B^*$ imply $b = a$ or $c = a$.

Let us note that a *ps*-morphism is bifix.

Lemma 3.1 comes directly from the definition of a prefix morphism or a suffix morphism.

Lemma 3.1 *Let f be a bifix morphism on an alphabet A and let u, v, w , and t be four words over A .*

The equality $f(u) = f(v)p$ where p is a prefix of $f(w)$ implies $u = vw'$ for a prefix w' of w such that $f(w') = p$.

Symmetrically, the equality $f(u) = sf(v)$ where s is suffix of $f(t)$ implies $u = t'v$ for a suffix t' of t such that $f(t') = s$.

Lemma 3.2 [2, 4] *If f is not a ps-morphism then f is not a k-power-free morphism for any integer $k \geq 2$.*

Assuming that $f(\bar{w}) = pu^k s$ for a factor \bar{w} of a word w and a non-empty word u , and assuming that \bar{w} contains a factor w_0 such that $|f(w_0)| = |u|$, if f is a ps-morphism, Lemma 3.4 states that \bar{w} necessarily contains a k -power w'^k such that $f(w')$ is a conjugate of u . We will say that $f(w)$ contains a synchronized k -power u^k or that $f(w)$ and u^k are synchronized. More precisely:

Definition 3.3 Let $k \geq 2$ be an integer. Let f be a morphism from A^* to B^* , let w be a word over A , and let u be a non-empty word over B such that $f(w)$ contains the k -power u^k . Let \bar{w} be a shortest factor of w whose image by f contains u^k , i.e., $f(\bar{w}) = pu^ks$ with $|p| < |f(\bar{w}[1])|$ and $|s| < |f(\bar{w}[\lceil \bar{w} \rceil])|$.

We say that $f(w)$ and u^k are synchronized if there exist three words w_0, w_1 , and w_2 such that $|f(w_0)| = |u|$ and $\bar{w} = w_1w_0w_2$ with $p = \varepsilon$ if $w_1 = \varepsilon$, and $s = \varepsilon$ if $w_2 = \varepsilon$.

Let f be the morphism from $\{a, b, c\}^*$ to $\{a, b, c, d\}^*$ defined by $f(a) = abcd$, $f(b) = ac$ and $f(c) = d$.

If $w = abcbaa$, then we get $f(w) = ab(cda)^3cabcdabcd$. Taking $u = cda$, $\bar{w} = abcba$ and $w_0 = bc$, we get that $f(w)$ and u^3 are synchronized.

Let us remark that f is not a suffix morphism and so it is not k -power-free $k(\geq 2)$.

Lemma 3.4 [8] Let $k \geq 2$ be an integer, let f be a ps-morphism, and let w be a word such that $f(w) = pu^ks$ with $|p| < |f(w[1])|$ and $|s| < |f(w[\lceil w \rceil])|$. If $f(w)$ and u^k are synchronized then w starts or ends with a k -power of the form w_0^k with $f(w_0)$ and u conjuguated words.

Let us recall some results of Leconte:

Theorem 3.5 [3] Let f be a morphism from A^* to B^* . If f is square-free up to three then f is k -power-free for any integer $k \geq 4$.

Theorem 3.6 [3] Let f be a morphism from A^* to B^* . If f is square-free up to three and if, for any letter a in A , the word $f(a^2)$ does not contain any cube, then f is k -power-free for any integer $k \geq 3$.

Definition 3.7 We say that two morphisms f and g from $\llbracket 0; n \rrbracket^*$ to $\llbracket 0; p \rrbracket^*$ are equivalent if there exist a permutation σ_1 of $\llbracket 0; n \rrbracket$ et a permutation σ_2 of $\llbracket 0; p \rrbracket$ such that $f \circ \sigma_1 = \sigma_2 \circ g$.

The morphisms f, g from $\{0, 1, 2\}^*$ to $\{0, 1\}^*$ defined by $f(0) = 011$, $f(1) = 101$, $f(2) = 110$, $g(0) = 010$, $g(1) = 001$ and $g(2) = 100$ are equivalent.

Remark 3.8 The reader can easily verify that "be equivalent" is a equivalence relation on morphisms.

Lemme 3.9 Let f and g be two equivalent morphisms. Then, for any integer $k \geq 2$, f is k -power-free if and only if g is k -power-free.

Proof.

By symmetry, we only have to show that f is not k -power-free implies that g is not k -power-free.

Let us assume that $f \circ \sigma_1 = \sigma_2 \circ g$ and that $f(w) = pu^ks$ with $u \neq \varepsilon$ and w k -power-free. The word $\sigma_1^{-1}(w)$ is k -power-free and $g(\sigma_1^{-1}(w)) = \sigma_2^{-1}(\sigma_2(g(\sigma_1^{-1}(w))) = \sigma_2^{-1}(f(w)) = \sigma_2^{-1}(p)(\sigma_2^{-1}(u))^k\sigma_2^{-1}(s)$ with $\sigma_2^{-1}(u) \neq \varepsilon$. \square

Lemma 3.10 *Let f be a morphism from A^* to B^* and let a, b be two different letters of A . When f is a square-free morphism or a framed-square-free morphism, if $f(a)$ ends (respectively starts) with $x \in A^+$ then $f(b)$ does not start (respectively does not end) with x . When f is an overlap-free morphism, if $f(a)$ ends (respectively starts) with $x \in A^+$ then $f(b)$ does not end (respectively does not start) with x .*

Proof.

We only have to consider $f(ab)$, $f(ba)$, $f(baa)$ or $f(aab)$. \square

Definition 3.11 [1, 9] *An endomorphism f of $\llbracket 0; n \rrbracket^*$ is cyclic if f commutes with the permutation $\sigma = (0\ 1\ 2\ \dots\ n)$.*

The map f from $\{0, 1, 2\}^*$ to $\{0, 1, 2\}^*$ defined by $f(0) = 2102$, $f(1) = 0210$ and $f(2) = 1021$ is a cyclic morphism.

We extend this definition here.

Definition 3.12 *We say that a morphism from $\llbracket 0; n \rrbracket^*$ to $\llbracket 0; p \rrbracket^*$ is procyclic if $\delta_p(f(a)) = \delta_p(f(b))$ for any letters a and b of A . This common value will be called the canvas of the morphism.*

In other words, a morphism is procyclic if all the images of the letters have the same canvas. Since, for all a in A , we have $f(a) = ((f(a))[1], C)_p$, a procyclic morphism f from $\llbracket 0; n \rrbracket^*$ to $\llbracket 0; p \rrbracket^*$ is defined by its canvas C and the set of the first letters (or any another fixed letter) of the images of all the letters in $\llbracket 0; n \rrbracket$ that is $\{(f(\ell))[1] \mid \ell \in \llbracket 0; n \rrbracket\}$.

If a morphism f is already defined, we only use $\underline{f(a)} = \underline{f(b)}$ for all $a, b \in A$ to characterize the fact that f is procyclic.

The morphism f from $\{0, 1, 2\}^*$ to $\{0, 1, 2, 3\}^*$ defined by $f(0) = 0123$, $f(1) = 3012$ and $f(2) = 1230$ is a procyclic morphism with canvas 111.

Remark 3.13 *The Thue-Morse morphism ($\bar{\mu} : \{0; 1\}^* \rightarrow \{0; 1\}^*; 0 \mapsto 01$ and $1 \mapsto 10$) is a procyclic morphism with canvas 1. It is also a cyclic endomorphism.*

Example 3.14 *In this example, we are interested in the word of Thue-Morse μ which is obtained by infinitely iterating the Thue-Morse morphism from 0. In other words, $\mu = \lim_{n \rightarrow \infty} \bar{\mu}^n(0)$. Let us note that, for any integer $n \geq 0$, $\bar{\mu}^n(0)$ ends with 0 if n is even and ends with 1 if n is odd. Moreover, $\bar{\mu}^n(1)$ always starts by 1.*

Let $\underline{\mu_n}$ be the canvas of the morphism $\bar{\mu}^n$, i.e., $\underline{\mu_n} = \underline{\bar{\mu}^n(0)} = \underline{\bar{\mu}^n(1)}$ and let t be the sequence defined by $t_n = 0$ if n is even and $t_n = 1$ if n is odd.

So, for all integers $n \geq 2$, we get that $\underline{\mu_n} = \underline{\bar{\mu}^{n-1}(0)\bar{\mu}^{n-1}(1)} = \underline{\mu_{n-1}t_n\mu_{n-1}}$: it means that $\underline{\mu_n}$ is a bordered word (a sesquipower).

Remark 3.15 *A procyclic morphism from $\llbracket 0; n \rrbracket^*$ to $\llbracket 0; p \rrbracket^*$ is defined by its canvas and a map from $\llbracket 0; n \rrbracket$ to $\llbracket 0; p \rrbracket$ corresponding to the first letter (or another fixed one) of the image of the letters of $\llbracket 0; n \rrbracket$.*

Remark 3.16 *Let us remark that a procyclic morphism is not necessarily an endomorphism and that, by definition, it is uniform.*

Let us also note that a cyclic endomorphism is a procyclic morphism.

Remark 3.17 *As in the example 3.14, the study of the canvas of a word generated by a procyclic endomorphism surely leads us to sesquipowers.*

4 Procyclic morphisms and k -power

Lemme 4.1 *Let $k \geq 2$ be an integer. A procyclic k -power-free endomorphism on $\llbracket 0; n \rrbracket^*$ is equivalent to a cyclic endomorphism.*

Proof.

Let A be the alphabet $\llbracket 0; n \rrbracket$. Let f be a procyclic k -power-free endomorphism on A^* with canvas C and let a, b be two letters of A .

Since f is procyclic, if $f(a)$ and $f(b)$ start by the same letter then $f(a) = f(b)$. So, if $a \neq b$, we get that $f(a^{k-1}b)$ contains a k -power. It would mean that f is not a k -power-free morphism: a contradiction with the hypotheses.

So, we have $a \neq b \Rightarrow (f(a))[1] \neq (f(b))[1]$. In other words, $\{(f(\ell))[1] \mid \ell \in A\} = A$. It implies that there exists a permutation τ of $\llbracket 0; n \rrbracket$ such that, for any letter a in $\llbracket 0; n \rrbracket$, we have that $(f \circ \tau)(a)$ starts by a .

The endomorphism $g = f \circ \tau$ is also procyclic with canvas C .

Taking $\sigma = (012 \dots n)$, we get $(\sigma \circ g)(a) = \sigma((f \circ \tau)(a)) = \sigma((a, C)_n) = (\sigma(a), C)_n = (f \circ \tau)(\sigma(a)) = (g \circ \sigma)(a)$. \square

Lemme 4.2 *Let f be a procyclic square-free endomorphism on $\llbracket 0; n \rrbracket^*$. Then, for any letter a in $\llbracket 0; n \rrbracket$, $f(a)$ starts and ends with the same letter.*

Proof.

Let us assume that f is L -uniform with $L \geq 1$ and let a, b be two letters in $A = \llbracket 0; n \rrbracket$.

As in the proof of Lemma 4.1, if $f(a)$ and $f(b)$ start or end by the same letter then $f(a) = f(b)$ and $f(ab)$ is a square. Since f is square-free, it only happens when $a = b$.

So $\{(f(\ell))[1] \mid \ell \in A\} = A = \{(f(\ell))[L] \mid \ell \in A\}$.

Let us now assume that $f(a)$ ends by the letter x . For any letter $b \neq a$, if $f(b)$ started by x then $f(ab)$ would contain the square xx , i.e., f is not square-free: this is a contradiction with the hypotheses. Since $\{(f(\ell))[L] \mid \ell \in A\} = A$, it means that $f(a)$ starts by x . \square

Remark 4.3 *Lemma 4.2 is not suitable for procyclic framed-square-free endomorphisms. For instance, the endomorphism on $\llbracket 0; 1 \rrbracket^*$ defined by $f(0) = 01 = f(1)$ is a procyclic morphism which is not square-free since $f(01) = (01)^2$. But it is a framed-square-free morphism since the image of any word satisfies $(01)^i$ for an integer i : it does not contain any framed-square. And yet, $f(0)$ starts and ends with different letters.*

Lemma 4.4 *Let f be a procyclic morphism from $\llbracket 0; n \rrbracket^*$ to $\llbracket 0; p \rrbracket^*$ with canvas C . Let X, Y and Z be the images of some letters by f .*

If we have the equality $XYb = SZP$ where b is a letter, S is a non-empty suffix of the image of a letter by f and P is a prefix of the image of a letter by f with $|P| \geq 2$, then $\underline{X[[X]]Y[1]} = \underline{Y[[Y]]b} = \underline{S[[S]]Z[1]} = \underline{Z[[Z]]P[1]}$

When $f(w)$ and a k -power are not synchronized, we do have the condition $|P| \geq 2$.

Proof.

If we denote $\alpha = \underline{X[[X]]Y[1]}$, $\beta = \underline{Y[[Y]]b}$, $\gamma = \underline{S[[S]]Z[1]}$ and $\delta = \underline{Z[[Z]]P[1]}$, from $XYb = SZP$, we get that $C\alpha C\beta = \underline{XYb} = \underline{SZP} = C_2\gamma C\delta C_1$ where C_1 is a prefix of C and C_2 is a suffix of C .

Furthermore, since $|C\alpha C\beta| = |C_2\gamma C\delta C_1|$, we get $|C| = |C_2C_1|$. In other words, $C = C_1C_2$ and the equality becomes $C_1C_2\alpha C_1C_2\beta = C_2\gamma C_1C_2\delta C_1$.

It means that $C_1 (\neq \varepsilon)$ ends with β . And, since $|C_1C_2\alpha| = |C_2\gamma C_1|$, from the prefix of the equality, we get $C_1C_2\alpha = C_2\gamma C_1$, that is, C_1 ends with α . In other words, we have $\alpha = \beta$.

There exists a word C'_1 such that $C_1 = C'_1\alpha$ and so $C'_1\alpha C_2\alpha C'_1\alpha C_2\alpha = C_2\gamma C'_1\alpha C_2\delta C'_1\alpha$. By length criterions, we get the equalities $C'_1\alpha C_2 = C_2\gamma C'_1$ and $C'_1\alpha C_2 = C_2\delta C'_1$. We conclude with Corollary 1.8: $\alpha = \gamma$ and $\alpha = \delta$. \square

We show in an identical way:

Lemma 4.5 *Let f be a procyclic morphism from $\llbracket 0; n \rrbracket^*$ to $\llbracket 0; p \rrbracket^*$ with canvas C . Let X, Y and Z be the images of some letters by f .*

If we have $bXY = SZP$ where b is a letter, S is the suffix of the image of a letter by f with $|S| \geq 2$ and P is the non-empty prefix of the image of a letter by f , then $\underline{bX[1]} = \underline{X[[X]]Y[1]} = \underline{S[[S]]Z[1]} = \underline{Z[[Z]]P[1]}$

Corollary 4.6 *Let $\ell \geq 2$ and $L \geq 2$ be two integers and let f be a L -uniform procyclic morphism from $\llbracket 0; n \rrbracket^*$ to $\llbracket 0; p \rrbracket^*$ with canvas C . Let $x_0, x_1 \dots x_{\ell+1}, x'_0, x'_1, \dots x'_\ell$ be some letters of $\llbracket 0; n \rrbracket$.*

If we have the equality $s_0f(x_1) \dots f(x_\ell)p_{\ell+1} = s'_0f(x'_1) \dots f(x'_{\ell-1})p'_\ell$ with $f(x_0) = p_0s_0$, $f(x'_0) = p'_0s'_0$, $f(x_{\ell+1}) = p_{\ell+1}s_{\ell+1}$ and $f(x'_\ell) = p'_\ell s'_\ell$ with $p'_0, p_{\ell+1}, p'_\ell$ some non-empty words then the letters $(f(x_i)[L]f(x_{i+1})[1])_{i=1..l}$ and $(f(x'_i)[L]f(x'_{i+1})[1])_{i=0..l-1}$ are all the same letter α .

Moreover, if $|s_0| \neq 0$ then $\underline{f(x_0)[L]f(x_1)[1]} = \alpha$.

Proof.

Since $|s_0f(x_\ell)p_{\ell+1}| = |s'_0p'_\ell|$ and $|f(x_\ell)p_{\ell+1}| > |p'_\ell|$, we get $|s_0| < |s'_0|$. Let s''_0 be the suffix of s'_0 such that $s'_0 = s_0s''_0$. It means that s''_0 is also a suffix of $f(x'_0)$.

The equality becomes: $f(x_1) \dots f(x_\ell)p_{\ell+1} = s''_0f(x'_1) \dots f(x'_{\ell-1})p'_\ell$ (1)

So we get $f(x_1)f(x_2)x = s''_0f(x'_1)p$ where x is the first letter of $f(x_3)$ and p is a prefix of $f(x'_2)$. Since $|s_0| < |s'_0|$, we get $|s''_0| \leq |s'_0| < L$ and so $|p| \geq 2$. By Lemma 4.4, we get that $\underline{f(x_1)[L]f(x_2)[1]} = \underline{f(x_2)[L]f(x_3)[1]} = \underline{f(x'_0)[L]f(x'_1)[1]} = \underline{f(x'_1)[L]f(x'_2)[1]} = \alpha$.

If $|s_0| \neq 0$ then, using the last letter of s_0 , by Lemma 4.5, we get $\underline{f(x_0)[L]f(x_1)[1]} = \alpha$.

If $\ell \geq 3$, we remove the $L \times (\ell - 2)$ first letters of the words of Equation (1) to obtain recursively the result. \square

Considering a non-empty prefix of $f(x_\ell)$ in Corollary 4.6, we immediatly get:

Corollary 4.7 *Let $\ell \geq 2$ and $L \geq 2$ be two integers and let f be a L -uniform procyclic morphism from $\llbracket 0; n \rrbracket^*$ to $\llbracket 0; p \rrbracket^*$ with canvas C . Let $x_0, x_1 \dots x_\ell, x'_0, x'_1, \dots x'_\ell$ be some letters of $\llbracket 0; n \rrbracket$.*

If we have the equality $s_0 f(x_1) \dots f(x_{\ell-1}) p_\ell = s'_0 f(x'_1) \dots f(x'_{\ell-1}) p'_\ell$ with $|s_0| < |s'_0|$, $f(x_0) = p_0 s_0$, $f(x'_0) = p'_0 s'_0$, $f(x_\ell) = p_\ell s_\ell$, $f(x'_\ell) = p'_\ell s'_\ell$ and p'_0, p_ℓ, p'_ℓ non-empty words then the letters $(f(x_i)[L]f(x_{i+1})[1])_{i=1..l-1}$ and $(f(x'_i)[L]f(x'_{i+1})[1])_{i=0..l-1}$ are all the same letter α .

Moreover, if $|s_0| \neq 0$ then $f(x_0)[L]f(x_1)[1] = \alpha$.

Lemme 4.8 *Let $k \geq 2$ be an integer and let f be a procyclic morphism from $\llbracket 0; n \rrbracket^*$ to $\llbracket 0; p \rrbracket^*$ with canvas C .*

If there exist two words π and σ , a non-empty word u and a word w of length greater or equal to $2k + 2$ such that $f(w) = \pi u^k \sigma$ with $|\pi| < |f(w[1])|$ and $|\sigma| < |f(w[|w|])|$ and if $f(w)$ and u^k are not synchronized then there exists a letter α , two integers $2 \leq i < j$ and two words t_1 and t_2 such that $f(w) = (C\alpha)^{|w|-1} C$, $C\alpha = (t_1 t_2)^i$ and $\underline{uu[1]} = (t_2 t_1)^j$ with $t_1 t_2, t_2 t_1$ primitive words and j is not a multiple of i .

Proof.

For all integers j between 0 and k , let i_j be the smallest integer such that $|f(w[1..i_j])| \geq |\pi u^j|$. Since f is L -uniform for an integer L and since $|w| \geq 2k + 2$, we have $|u^k| \geq |f(w[2..|w| - 1])| \geq 2k \times L$. Since $f(w)$ and u^k are not synchronized, we have $|u| > 2L$ and $1 = i_0 < i_1 < i_2 < \dots < i_{k-1} < i_k = |w|$. For all integers $0 \leq q \leq k$, there exist some words p_q and s_q such that $f(w[i_q]) = p_q s_q$ and, for all integers $1 \leq q \leq k$, we get $u = s_{q-1} f(w[i_{q-1} + 1..i_q - 1]) p_q$ with $p_q \neq \varepsilon$ (by definition of i_q). Furthermore, $s_0 \neq \varepsilon$, $p_0 = \pi$ and $s_k = \sigma$.

The previous situation can be summed up by Figure 1.

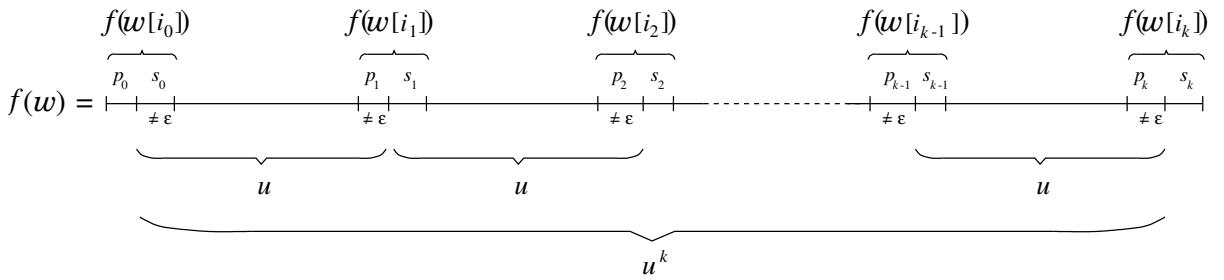


Figure 1: Decomposition of a k -power

If, for any integer $1 \leq q \leq k$, we had $i_q \leq i_{q-1} + 2$ then we would have $|w| = i_k \leq 1 + 2k$: this is a contradiction with the hypotheses. It means that $|w[i_q + 1..i_{q+1} - 1]| \geq 2$.

Since $f(w)$ and u^k are not synchronized, we get $s_i \neq s_j$ when $i \neq j$. Let ℓ be the integer such that $|s_\ell| = \min_{0 \leq i \leq k-1} |s_i|$. Since, for any integer $1 \leq q \leq k$, we have $|u| = |s_{q-1}f(w[i_{q-1} + 1..i_q - 1])p_q|$, it implies that $|w[i_\ell + 1..i_{\ell+1} - 1]| \geq |w[i_q + 1..i_{q+1} - 1]| \geq 2$.

If $\ell \neq k - 1$, then $u = s_\ell f(w[i_\ell + 1..i_{\ell+1} - 1])p_{\ell+1} = s_{\ell+1} f(w[i_{\ell+1} + 1..i_{\ell+2} - 1])p_{\ell+2}$. By a length criterion, the integers $i_{\ell+2} - i_{\ell+1}$ and $i_{\ell+1} - i_\ell$ can only differ by 1. By Corollaries 4.6 et 4.7, the letters $(f(w[q])[L]f(w[q+1])[1])_{q=i_\ell..i_{\ell+2}-1}$ are the same and are all denoted by α . The words $\underline{uu} = \underline{uu[1]} \underline{u} = \underline{u} \underline{u[u]u[1]} \underline{u}$ and $C\alpha$ have a common factor of length $|\underline{uu}| \geq |\underline{uu[1]}| + |C\alpha|$.

By Corollary 1.2, there exist two words t_1 and t_2 et two integers i and j such that $C\alpha = (t_1 t_2)^i$ and $\underline{uu[1]} = (t_2 t_1)^j$ with $t_1 t_2$ and $t_2 t_1$ primitive words.

If j is a multiple of i (in particular when $i = 1$), since $L = |C\alpha|$ and $|u| = |\underline{uu[1]}|$, we get that $|u|$ is a multiple of L and so that $f(w)$ and u^k are synchronized: this is a contradiction with the hypotheses. Moreover, since $|u| > L$, we get $j > i$.

The case $\ell = k - 1$ is solved in a same way, considering $u = s_{\ell-1} f(w[i_{\ell-1} + 1..i_\ell - 1])p_\ell = s_\ell f(w[i_{\ell+1} + 1..i_{\ell+2} - 1])p_{\ell+2}$. \square

We already got a test-set to verify whether a uniform morphism is k -power-free:

Proposition 4.9 [7] *A uniform morphism defined on A is k -power-free if and only if it is k -power-free up to $k \times \text{Card}(A) + k + 1$.*

We get a lower bound for procyclic morphisms.

Proposition 4.10 *A procyclic morphism from $\llbracket 0; n \rrbracket^*$ to $\llbracket 0; p \rrbracket^*$ is k -power-free if and only if it is k -power-free up to $\ell_k = \max \left\{ 2k + 2; \left\lceil \frac{1 + (n + 1) \times k}{2} \right\rceil \right\}$.*

Proof.

Let us assume that f is a procyclic morphism from $\llbracket 0; n \rrbracket^*$ to $\llbracket 0; p \rrbracket^*$ with canvas C .

By contradiction, we assume that f is k -power-free up to ℓ_k but that there exists a k -power-free word w of length greater or equal to $\ell_k + 1$ such that $f(w)$ contains a k -power. More precisely, we assume that w is the smallest word such that $f(w)$ contains the k -power u^k , that is, $f(w) = pu^k s$ with $|p| < |f(w[1])|$ and $|s| < |f(w[|w|])|$.

Since w does not contain any k -power, by Lemma 3.4, $f(w)$ and u^k are not synchronized. And, since $|w| \geq 2k + 2$, by Lemma 4.8, there exist a letter α , two integers $2 \leq i < j$ et two words t_1 and t_2 such that $\underline{f(w)} = (C\alpha)^{|w|-1} C$, $C\alpha = (t_1 t_2)^i$ and $\underline{uu[1]} = (t_2 t_1)^j$ with $t_1 t_2$ and $t_2 t_1$ primitive words.

Since $i \geq 2$, it follows that $\underline{f(w)}$ contains $(t_1 t_2)^{2|w|-2} (t_1 t_2) = (t_1 t_2)^{2|w|-1}$.

Since $|w| \geq \left\lceil \frac{1 + (n + 1) \times k}{2} \right\rceil$, we get $2|w| - 1 \geq (n + 1) \times k$. By Corollary 2.5, it means that w contains a k -power: a final contradiction. \square

5 Square-free morphisms and framed-square-free morphisms

Non-synchronisation and definition of a ps-morphism with square-free morphisms lead us to the following lemma:

Lemme 5.1 *Let f be a morphism from A^* to B^* .*

If there exist three letters x, y and z of A such that $f(x) = sp$ with s a non-empty suffix of $f(y)$ and p a non-empty prefix of $f(z)$ with $x \neq y$ or $x \neq z$ then f is not square-free.

Proof.

We only have to consider $f(yx)$ or $f(xz)$. \square

Proposition 5.2 *A uniform square-free morphism is framed-square-free.*

Remark 5.3 *The converse is false as shows by the example of Remark 4.3.*

Proof.

Let f a L -uniform (with $L \geq 1$) morphism from A^* to B^* .

By contraposition, we assume that f is not framed-square-free. We will show that f is not square-free, that is, there exists a square-free word whose image contains a square.

In other words, we assume that there exists a framed-square-free word w of A^* such that $f(w) = pbUUbs$ with $b \in B$ and $U \in B^*$.

Let us assume that the length of w is minimal (the image of a word of length lower than $|w|$ does not contain any framed-square). It implies that $|p| < |f(w[1])|$ and $|s| < |f(w[|w|])|$.

Let us recall that, if $|w| \leq 3$, then w is square-free is equivalent to w is framed-square-free. It ends the proof in this case since the image of w contains a square.

Let us note that, if $L = 1$, then $f(w) = pbUUbs$ implies that w contains a framed-square. It is a contradiction with the hypotheses. So we will work with $L \geq 2$.

By Lemma 3.2, we also can assume that f is a ps-morphism.

Let us also note that, if $f(w)$ contains a square of letters, then this square is factor of the image of a word w_0 of one letter or of two letters that would be factor of w . So w_0 (framed-square-free) is square-free and $f(w_0)$ contains a square (of letters): we get that f is not square-free. It ends the proof in this case.

From now, let us assume that $|w| \geq 4$, that $U \neq \varepsilon$ and that, as w , $f(w)$ does not contain any square of letters.

From $|w| \geq 4$, we get that either $|pbU| \leq |f(w[1..2])|$ or $|Ubs| \leq |f(w[|w| - 1..|w|])|$. Since $|pb| \leq |f(w[1])|$ and $|Us| \leq |f(w[|w|])|$, we get that one of the two occurrences of U necessarily contains at least one image of a letter by f .

If $f(w)$ and U^2 are not synchronized, by Lemma 5.1, f is not square-free: it ends the proof in this case.

Since $f(w)$ and U^2 are synchronized and, by Lemma 3.4, w starts or ends with a word w_0^2 . Moreover, $f(w_0)$ is a conjugate of U .

If w starts with w_0^2 , since $|w_0| \geq 2$, there exist two words U_1 and U_2 such that $U = U_1U_2$ and $f(w_0) = pbU_1 = U_2U_1$. So $U_2 = pb$ and $f(w)$ ends with $Ubs = U_1U_2bs = U_1pbbs$ that contradicts the fact that $f(w)$ does not contain any square of letters.

In a same way, if w ends with w_0^2 , we get a contradiction with the hypotheses. \square

Remark 5.4 *An other example showing that the converse is false is given by Zolotov (Theorem 2 in [9]): The pro-cyclic endomorphism $\phi : \{0, 1, 2\}^* \rightarrow \{0, 1, 2\}^*$ defined by $\phi(0) = 01210$, $\phi(1) = 12021$ and $\phi(2) = 20102$ is framed-square-free.*

But it is not square-free since $\phi(101) = 120210121012021 = 120(2101)^22021$. Let us also remark that since $\phi(101)$ contains an overlap, the endomorphism ϕ is not overlap-free.

6 Power-free morphisms

Proposition 6.1 *If f is a morphism from A^* to B^* and if, for any $a \in A$, we have $f(a)$ square-free and $f(a)$ starts and ends with the same letter then $f(a^2)$ can not contain any cube.*

Proof.

By contradiction, let us assume that $f(aa) = \alpha_1u^3\alpha_2$ with $u \neq 0$.

If $|\alpha_1u^2| \leq |f(a)|$ then α_1u^2 is a prefix of $f(a)$: it contradicts the hypotheses.

If $|\alpha_1u| \geq |f(a)|$ then $u^2\alpha_2$ is a suffix of $f(a)$: it is again a contradiction with the hypotheses.

So $|\alpha_1u| < |f(a)| < |\alpha_1u^2|$. But $f(a)$ starts and ends with the same letter x . So xx is a factor of u which is factor of $f(a)$: a final contradiction with the hypotheses. \square

By Lemma 4.2, it follows that:

Corollary 6.2 *If f is a pro-cyclic square-free endomorphism then $f(a^2)$ can not contain any cube.*

And, by theorems 3.5 and 3.6, we immediatly get that:

Corollary 6.3 *A pro-cyclic square-free morphism is k -power-free for any integer $k \geq 4$.*

Corollary 6.4 *A pro-cyclic square-free endomorphism is k -power-free for any integer $k \geq 3$.*

Remark 6.5 *In the case where f is not an endomorphism, Corollary 6.3 can not be improved. For instance, let g be the pro-cyclic morphism defined by $g : \{0, 1, 2\}^* \rightarrow \{0, 1, 2, 3, 4, 5, 6\}^*$; $g(0) = 0304303$, $g(1) = 1415414$ and $g(2) = 2526525$. We get that g is square-free (and so k -power-free $k \geq 4$). But $g(00)$ contains $(30)^3$ and so g is not cube-free.*

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