Large deviation principle at speed n^d for the random metric in first-passage percolation

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April 14, 2024

Abstract

We consider the standard first passage percolation model on $\mathbb{Z}^d$ with bounded and bounded away from zero weights. We show that the rescaled passage time $\widetilde{\mathbf{T}}_{n,X}$ restricted to a compact set X satisfies a large deviation principle (LDP) at speed n^d in a space of geodesic metrics, i.e. an estimation of the form $\mathbb{P}\Big(\widetilde{\mathbf{T}}_{n,X} \approx D\Big) \approx \exp\big(-I(D)n^d\big)$ for any metric D. Moreover, $I(D)$ can be written as the integral over X of an elementary cost. Consequences include LDPs at speed n^d for the point-point passage time, the face-face passage time and the random ball of radius n. Our strategy consists in proving the existence of $\lim_{n\to\infty} -\frac{1}{n^d}\log\mathbb{P}\Big(\widetilde{\mathbf{T}}_{n,[0\,,1]^d} \approx g\Big)$ for any norm g with a multidimensional subadditivity argument, then using this result as an elementary building block to estimate $\mathbb{P}\Big(\widetilde{\mathbf{T}}_{n,X} \approx D\Big)$ for any metric D.

1 Introduction

1.1 Framework

First passage percolation. Let $d \geq 2$ be an integer. Let $\mathbb{E}^d$ the set of all non-oriented nearest-neighbour edges in $\mathbb{Z}^d$. A finite sequence $\pi \coloneqq (x_0, \dots, x_r)$ of elements of $\mathbb{Z}^d$ is called a *discrete path* if for all $i \in [\![0\,, r-1]\!]$, $(x_i, x_{i+1}) \in \mathbb{E}^d$.

Let ν be a probability distribution on $[0\,, \infty[$ and denote by a and b the infimum and supremum of its support respectively. We consider a family $(\tau_e)_{e\in\mathbb{E}^d}$ of i.i.d. random variables with distribution ν. The variable τ_e is called the *passage time along the edge* e. Given a discrete path $\pi = (x_0, \dots, x_r)$, the *passage time along* π is defined as

$$\tau(\pi) \coloneqq \sum_{i=0}^{r-1} \tau_{(x_i,x_{i+1})}. \tag{1.1}$$

For $x, y \in \mathbb{Z}^d$, the *passage time between* x *and* y is defined as

$$\mathbf{T}(x,y) \coloneqq \inf_{x \overset{\pi}{\rightsquigarrow} y} \tau(\pi), \tag{1.2}$$

where the infimum is taken over all discrete paths whose endpoints are x and y. The map $\mathbf{T}(\cdot,\cdot)$ is a metric on $\mathbb{Z}^d$. We call *discrete geodesic* between x and y any minimizer in (1.2). A well-known result

(Equation 2.4 in [2]) states that under a moment condition on ν, there exists an homogeneous function μ on $\mathbb{R}^d$, known as the *time constant*, such that almost surely, for all $x \in \mathbb{Z}^d$,

$$\frac{\mathbf{T}(0, nx)}{n} \xrightarrow[n\to\infty]{} \mu(x). \tag{1.3}$$

The time constant is a norm if $\nu(\{0\}) < p_c(\mathbb{Z}^d)$, where $p_c(\mathbb{Z}^d)$ is the critical parameter for bond percolation in $\mathbb{Z}^d$. Otherwise $\mu(x) = 0$ for all $x \in \mathbb{Z}^d$. As a consequence of (1.3) the probability of an event of the form $\{\mathbf{T}(0, n\mathrm{e}_1) \le \zeta n\}$, with $\zeta < \mu(\mathrm{e}_1)$ or $\{\mathbf{T}(0, n\mathrm{e}_1) \ge \zeta n\}$, with $\zeta > \mu(\mathrm{e}_1)$ (the so-called lower tail and upper tail large deviation events) converges to 0 as $n \to \infty$. In 1984, Kesten [13] obtained estimates for the speed of convergence: there exists (Theorem 5.2 in [13]) a convex decreasing function $J :]a\,, \mu(\mathrm{e}_1)[\to]0\,, \infty[$ such that for all $a < \zeta < \mu(\mathrm{e}_1)$,

$$\lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}(\mathbf{T}(0, n\mathrm{e}_1) \le \zeta n) = J(\zeta). \tag{1.4}$$

Moreover (Theorem 5.9 in [13]), under the assumption $b < \infty$, for all $\mu(\mathrm{e}_1) < \zeta < b$,

$$0 < \varliminf_{n\to\infty} -\frac{1}{n^d} \log \mathbb{P}(\mathbf{T}(0, n\mathrm{e}_1) \ge \zeta n) \le \varlimsup_{n\to\infty} -\frac{1}{n^d} \log \mathbb{P}(\mathbf{T}(0, n\mathrm{e}_1) \ge \zeta n) < \infty. \tag{1.5}$$

These results can be generalized to any direction with minor proof adaptations. Under an additional regularity assumption on ν, Basu, Ganguly and Sly [3] proved in 2021 that the limit in (1.5) actually exists in the case $d = 2$, $x = \mathrm{e}_1$, and as stated in their paper, their arguments are still valid for any dimension and any $x \in \mathbb{R}^d \setminus \{0\}$. Although not explicitly stated, their theorem implies the existence of a rate function $I_{\mathrm{pp}} : [0\,, b] \to [0\,, \infty]$ with which the process $\left(\frac{\mathbf{T}(0, n\mathrm{e}_1)}{n}\right)_{n\ge1}$ satisfies the *large deviation principle* (LDP) at speed n^d. In other words, for all Borel sets $A \subseteq [0\,, b]$,

$$\inf_{\zeta\in\overline{A}} I_{\mathrm{pp}}(\zeta) \le \varliminf_{n\to\infty} -\frac{1}{a_n} \log \mathbb{P}\left(\frac{\mathbf{T}(0, n\mathrm{e}_1)}{n} \in A\right) \le \varlimsup_{n\to\infty} -\frac{1}{a_n} \log \mathbb{P}\left(\frac{\mathbf{T}(0, n\mathrm{e}_1)}{n} \in A\right) \le \inf_{\zeta\in\mathring{A}} I_{\mathrm{pp}}(\zeta). \tag{1.6}$$

Corollary 1.6 generalizes this LDP to any distribution with bounded support and a subcritical atom at 0.

Large deviations. We give here some general large deviations theory tools. See Dembo and Zeitouni (2009) [11] for the general theory.

Definition 1.1. Let $\mathcal{X}$ a Hausdorff topological space. We call *rate function* a lower semicontinuous map $I : \mathcal{X} \to [0\,, \infty]$, i.e. a map whose sublevels are closed. We further say that I is a *good rate function* if its sublevels are compact.

We say that a random process $(X_n)_{n\ge1}$ with values in $\mathcal{X}$ satisfies the *large deviation principle*, at speed a_n, with the rate function I if for every Borel set $A \subseteq \mathcal{X}$,

$$\inf_{x\in\overline{A}} I(x) \le \varliminf_{n\to\infty} -\frac{1}{a_n} \log \mathbb{P}(X_n \in A) \le \varlimsup_{n\to\infty} -\frac{1}{a_n} \log \mathbb{P}(X_n \in A) \le \inf_{x\in\mathring{A}} I(x). \tag{1.7}$$

In this article only the case $a_n = n^d$ is considered. Lemma 1.2 will be of constant use.

Lemma 1.2. *Let $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$ be a metric space and $(X_n)_{n\ge1}$ a random process with values in $\mathcal{X}$. Define, for all $x \in \mathcal{X}$,*

$$\overline{I}(x) \coloneqq \lim_{\varepsilon\to0}{\uparrow} \varlimsup_{n\to\infty} -\frac{1}{n^d} \log \mathbb{P}(\mathrm{d}_{\mathcal{X}}(x, X_n) \le \varepsilon)$$
$$\textit{and } \underline{I}(x) \coloneqq \lim_{\varepsilon\to0}{\uparrow} \varliminf_{n\to\infty} -\frac{1}{n^d} \log \mathbb{P}(\mathrm{d}_{\mathcal{X}}(x, X_n) \le \varepsilon).$$

Then

(i) $\overline{I}$ *and* $\underline{I}$ *are rate functions on* $\mathcal{X}$*.*

(ii) For every open set $U \subseteq \mathcal{X}$*,*

$$\overline{\lim_{n\to\infty}} -\frac{1}{n^d}\log \mathbb{P}(X_n \in U) \leq \inf_{x\in U} \overline{I}(x). \tag{1.8}$$

(iii) For every compact set $K \subseteq \mathcal{X}$*,*

$$\underline{\lim_{n\to\infty}} -\frac{1}{n^d}\log \mathbb{P}(X_n \in K) \geq \min_{x\in K} \underline{I}(x). \tag{1.9}$$

Although they are straightforward adaptations of the proof of Theorem 4.1.11 in [11], we prove them in Appendix A for completeness, and because the formalisms are quite different. Note that on a compact metric space, with the notations of Lemma 1.2, $I := \overline{I} = \underline{I}$ implies that $(X_n)_{n\geq 1}$ satisfies a LDP at speed n^d, with the rate function I.

1.2 Main theorem

The aim of this paper is to prove that the rescaled metric $\mathbf{T}$ restricted to a box satisfies a LDP with a good rate function I_X (i.e. a function whose sublevels are compact) at speed n^d. Unless specified otherwise we work under the following assumption.

Assumption 1. *The bounds of* ν*'s support satisfy* $0 < a < b < \infty$*.*

We endow $\mathbb{R}^d$ with the norm defined by

$$\|x\| = \|x\|_1 := \sum_{i=1}^{d} |x_i|, \tag{1.10}$$

for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. We define d as the metric associated with this norm. Let $\mathbb{S}$ denote the unit sphere for this norm and $\mathrm{B}(z, r)$ (resp. $\overline{\mathrm{B}}(z, r)$) the open (resp. closed) ball of center z and radius r. Let $\mathcal{K}$ denote the set of all compact convex subsets of $\mathbb{R}^d$ with nonempty interior. Elementary properties of such subsets are gathered in Appendix B.

The rescaled metric. Given a polygonal path $\pi = (x_0, \ldots, x_r)$ with $\forall i \in [\![0\,, r]\!], x_i \in \mathbb{R}^d$, we generalize (1.1) by defining

$$\tau(\pi) := \sum_{i=0}^{r-1} \tau(x_i, x_{i+1}), \tag{1.11}$$

where for all $z, z' \in \mathbb{R}^d$,

$$\tau(z, z') := \begin{cases} \tau_{v,v'}\|z - z'\| & \text{if there exist adjacent vertices } v, v' \in \mathbb{Z}^d \text{ such that } z, z' \in [v\,, v'], \\ b\|z - z'\| & \text{otherwise.} \end{cases} \tag{1.12}$$

The passage time between two points $x, y \in \mathbb{R}^d$ is generalized to noninteger points by

$$\mathbf{T}(x, y) := \inf_{x \overset{\pi}{\rightsquigarrow} y} \tau(\pi), \tag{1.13}$$

where the infimum is taken over all finite sequences whose endpoints are x and y. Note that it differs from the usual generalization, which consists in replacing x and y by their projections on $\mathbb{Z}^d$. For all $X \in \mathcal{K}$, $x, y \in X$, we define

$$\mathbf{T}_X(x,y) \coloneqq \inf_{\substack{x \overset{\pi}{\rightsquigarrow} y \\ \pi \subseteq X}} \tau(\pi). \tag{1.14}$$

The maps $\mathbf{T}$ and $\mathbf{T}_X$ are metrics on $\mathbb{R}^d$ and X respectively. It is shown in Proposition 2.5 that the maps defined in (1.2) and (1.13) indeed coincide on integer points. Likewise, if $X = \prod_{i=1}^d [t_i\,, t_i']$ with $t_i, t_i' \in \mathbb{R}$ and $t_i < t_i'$, and $x, y \in X \cap \mathbb{Z}^d$ then restricting the infimum in the definition of $\mathbf{T}_X(x,y)$ on discrete paths only defines the same object.

For all $n \geq 1$, we also define the *rescaled* versions of $\mathbf{T}$ and $\mathbf{T}_X$ as

$$\begin{aligned} \widetilde{\mathbf{T}}_{n,X} : X \times X &\longrightarrow [0\,, \infty[\qquad \text{and} \quad \widetilde{\mathbf{T}}_n : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow [0\,, \infty[\\ (x,y) &\longmapsto \frac{1}{n}\mathbf{T}_{nX}(nx, ny) \qquad\qquad (x,y) \longmapsto \frac{1}{n}\mathbf{T}(nx, ny). \end{aligned} \tag{1.15}$$

The limit space. The random variable $\widetilde{\mathbf{T}}_{n,X}$ belongs to the space $\mathcal{D}_X$ defined below (see Proposition 2.4).

Definition 1.3. Let $X \in \mathcal{K} \cup \left\{\mathbb{R}^d\right\}$. We define the set of *admissible metrics* $\mathcal{D}_X$ as the set of all metrics D on X such that

(i) For all $x, y \in X$,
$$a\|x - y\| \leq D(x,y) \leq b\|x - y\|. \tag{1.16}$$

(ii) The metric space (X, D) is *geodesic*, i.e. for all $x, y \in X$, there exists an isometry $\sigma : [0\,, D(x,y)] \to X$ which maps 0 to x and $D(x,y)$ to y.

We call such a function σ a *D-geodesic* (or simply *geodesic* when there is no ambiguity) between x and y. A geodesic may be seen as a continuous path linking x to y, whose D-length (see (1.41)) is minimal, with a parametrization chosen such that the travel speed (with respect to the metric D) is 1. For all $X \in \mathcal{K}$, $\mathcal{D}_X$ is endowed with the *uniform distance* defined for every $D_1, D_2 \in \mathcal{D}_X$ by

$$\mathfrak{d}_\infty(D_1, D_2) \coloneqq \max_{(x,y) \in X^2} |D_1(x,y) - D_2(x,y)|, \tag{1.17}$$

which makes it compact (see Proposition 2.1), and its Borel σ-algebra $\mathcal{B}(\mathcal{D}_X)$. For all $D_1, D_2 \in \mathcal{D}_X$ we denote by $D_1 \leq D_2$ the assertion

$$\forall x, y \in X, \quad D_1(x,y) \leq D_2(x,y), \tag{1.18}$$

which defines a partial order on $\mathcal{D}_X$.

Definition 1.4. Let $\mathcal{N}$ denote the set of norms g on $\mathbb{R}^d$ such that $a\|\cdot\| \leq g \leq b\|\cdot\|$. If D is a metric on $X \in \mathcal{K}$ associated with such a norm, i.e. $D(x,y) = g(x-y)$ for all $x, y \in X$, we will identify D and g. Let $\mathcal{N}^*$ denote the subset of $\mathcal{N}$ consisting of norms g such that $\forall x \neq 0, g(x) < b\|x\|$.

One key property of a metric $D \in \mathcal{D}_X$ is that around almost every $z \in X$, D "behaves like" a norm $(\operatorname{grad} D)_z \in \mathcal{N}$ (see Subsection 2.4).

Result statement.

Theorem 1.5. *Let $X \in \mathcal{K}$. Under Assumption 1, the process $\left(\widetilde{\mathbf{T}}_{n,X}\right)_{n\geq 1}$ satisfies the large deviation principle at speed n^d with a good rate function $I_X : \mathcal{D}_X \to [0\,,\infty]$, i.e. for all $A \in \mathcal{B}(\mathcal{D}_X)$,*

$$\inf_{D\in\overline{A}} I_X(D) \leq \varliminf_{n\to\infty} -\frac{1}{n^d}\log \mathbb{P}\Big(\widetilde{\mathbf{T}}_{n,X} \in A\Big) \leq \varlimsup_{n\to\infty} -\frac{1}{n^d}\log \mathbb{P}\Big(\widetilde{\mathbf{T}}_{n,X} \in A\Big) \leq \inf_{D\in\mathring{A}} I_X(D). \tag{1.19}$$

Moreover, I_X is nondecreasing for the partial order defined in (1.18) *and for all $D \in \mathcal{D}_X$,*

$$I_X(D) = \int_X I_{[0\,,1]^d}((\operatorname{grad} D)_z)\mathrm{d}z. \tag{1.20}$$

1.3 Applications

By the so-called *contraction principle* (see e.g. Theorem 4.2.1 in [11]), Theorem 1.5 implies a LDP at speed n^d for the image of the process $(\widetilde{\mathbf{T}}_{n,X})_{n\geq 1}$ through a continuous map (see Lemma 6.1). We give three examples of such processes.

Point-point passage time.

Corollary 1.6. *Assume that $0 \leq a < b < \infty$ and $\nu(\{0\}) < p_c(\mathbb{Z}^d)$. Let $x \in \mathbb{R}^d \setminus \{0\}$. Then the process $\left(\frac{\mathbf{T}(0,nx)}{n}\right)_{n\geq 1}$ satisfies the LDP at speed n^d with a good rate function $I_{\mathrm{pp}(x)} : [a\|x\|\,, b\|x\|] \to [0\,,\infty]$. Moreover, $I_{\mathrm{pp}(x)}$ is continuous and nondecreasing. In particular, for all $a\|x\| \leq \zeta < b\|x\|$,*

$$\lim_{n\to\infty} -\frac{1}{n^d}\log \mathbb{P}\left(\frac{\mathbf{T}(0,nx)}{n} \geq \zeta\right) = I_{\mathrm{pp}(x)}(\zeta). \tag{1.21}$$

Note that by the definition of the time constant (1.3) and the order of the probability of the upper tail large deviation events (1.5), for all $x \in \mathbb{R}^d \setminus \{0\}$ and $a\|x\| \leq \zeta \leq b\|x\|$,

$$I_{\mathrm{pp}(x)}(\zeta) > 0 \Leftrightarrow \zeta > \mu(x).$$

Crossing times. For all $n \geq 1$, the *rescaled crossing times* of the box $[0\,,n]^d$ are defined as

$$\widetilde{\mathbf{T}}_{\mathrm{cross}}(n) \coloneqq \Big(\widetilde{\mathbf{T}}_{n,[0\,,1]^d}(H_1, H_1'), \ldots, \widetilde{\mathbf{T}}_{n,[0\,,1]^d}(H_d, H_d')\Big), \tag{1.22}$$

where $H_i \coloneqq \Big\{x = (x_1,\ldots,x_d) \in [0\,,1]^d \;\Big|\; x_i = 0\Big\}$ and $H_i' \coloneqq \Big\{x = (x_1,\ldots,x_d) \in [0\,,1]^d \;\Big|\; x_i = 1\Big\}$.

Corollary 1.7. *Assume that $0 \leq a < b < \infty$. The process $\Big(\widetilde{\mathbf{T}}_{\mathrm{cross}}(n)\Big)_{n\geq 1}$ satisfies the LDP at speed n^d with the good rate function*

$$\begin{aligned} I_{\mathrm{cross}} : [a\,,b]^d &\longrightarrow [0\,,\infty] \\ \zeta = (\zeta_1,\ldots,\zeta_d) &\longmapsto I^+_{[0\,,1]^d}\Big(g^\zeta\Big), \end{aligned} \tag{1.23}$$

where g^ζ is the seminorm defined by

$$g^\zeta(u) \coloneqq \max_{1\leq i\leq d}(\zeta_i|u_i|), \tag{1.24}$$

and $I^+_{[0\,,1]^d}$ is a function on seminorms defined in Lemma 6.3 which coincide with $I_{[0\,,1]^d}$ on elements of $\mathcal{N}$ under Assumption 1. Moreover:

(i) I_{cross} *is continuous on* $[a\,,b]^d$.

(ii) I_{cross} *is nondecreasing on* $[a\,,b]^d$ *for the componentwise order, i.e. for all* $\zeta = (\zeta_1, \dots, \zeta_d)$ *and* $\zeta' = (\zeta'_1, \dots, \zeta'_d)$ *in* $[a\,,b]^d$, *if* $\zeta_i \le \zeta'_i$ *for all* i, *then* $I_{\mathrm{cross}}(\zeta) \le I_{\mathrm{cross}}(\zeta')$.

(iii) I_{cross} *is separately convex on* $[a\,,b]^d$, *i.e. for all* $1 \le i \le d$ *and* $\zeta_1, \dots, \zeta_{i-1}, \zeta_{i+1}, \dots, \zeta_d \in [a\,,b]$, *the function*

$$t \mapsto I_{\mathrm{cross}}(\zeta_1, \dots, \zeta_{i-1}, t, \zeta_{i+1}, \dots, \zeta_d)$$

is convex on $[a\,,b]$.

(iv) $I_{\mathrm{cross}}(b, \dots, b) < \infty$ *if and only if* $\nu(\{b\}) > 0$.

In particular, for all $\zeta \in [a\,,b[^d$,

$$\lim_{n\to\infty} -\frac{1}{n^d} \log \mathbb{P}\left(\bigcap_{i=1}^{d} \Big\{ \widetilde{\mathbf{T}}_{n,[0\,,1]^d}(H_i, H'_i) \ge \zeta_i \Big\} \right) = I_{\mathrm{cross}}(\zeta). \tag{1.25}$$

Note that Corollaries 1.6 and 1.7 are stated in more general context than Theorem 1.5. For each result, we first treat the case $a > 0$ then take the limit when $a \to 0$ with a standard truncation argument. The assumption $\nu(\{0\}) < p_c(\mathbb{Z}^d)$ in Corollary 1.6 is used to show that with high probability, any geodesic from 0 to nx is included in a box of linear size, which is crucial to apply Theorem 1.5.

Rescaled growing ball. For all $n \ge 1$, let us consider the *rescaled growing ball*

$$\widetilde{\mathbf{B}}(n) := \frac{1}{n}\Big\{ x \in \mathbb{R}^d \;\Big|\; \mathbf{T}(0,x) \le n \Big\} = \Big\{ x \in \mathbb{R}^d \;\Big|\; \widetilde{\mathbf{T}}_n(0,x) \le 1 \Big\}. \tag{1.26}$$

Corollary 1.8. *Under Assumption 1, the process* $\big(\widetilde{\mathbf{B}}(n)\big)_{n\ge1}$ *satisifies the LDP at speed* n^d *with a good rate function, in the space of compact sets of* $\mathbb{R}^d$ *endowed with the Hausdorff distance defined as*

$$\mathfrak{d}_{\mathrm{H}}(K_1, K_2) := \inf\Big\{ \varepsilon > 0 \;\Big|\; K_1 \subseteq K_2 + \overline{\mathrm{B}}(0,\varepsilon) \text{ and } K_2 \subseteq K_1 + \overline{\mathrm{B}}(0,\varepsilon) \Big\}. \tag{1.27}$$

We don't know much about the rate function mentioned in Corollary 1.8 beyond the properties granted by the contraction principle.

1.4 Sketch of the proof

Let $X \in \mathcal{K}$. For all $D \in \mathcal{D}_X, n \ge 1$ and $\varepsilon > 0$, we define the large deviation event

$$\mathrm{LD}_{n,X}(D,\varepsilon) := \Big\{ \mathfrak{d}_\infty\Big(D, \widetilde{\mathbf{T}}_{n,X}\Big) \le \varepsilon \Big\}, \tag{1.28}$$

alongside with the *lower* and *upper rate functions*

$$\underline{I}_X(D) := \lim_{\varepsilon\to0}{\uparrow} \varliminf_{n\to\infty} -\frac{1}{n^d} \log \mathbb{P}(\mathrm{LD}_{n,X}(D,\varepsilon)), \tag{1.29}$$

$$\overline{I}_X(D) := \lim_{\varepsilon\to0}{\uparrow} \varlimsup_{n\to\infty} -\frac{1}{n^d} \log \mathbb{P}(\mathrm{LD}_{n,X}(D,\varepsilon)), \tag{1.30}$$

i.e. $\underline{I}_X$ and $\overline{I}_X$ are a special case of $\underline{I}$ and $\overline{I}$ as defined in Lemma 1.2, for $(\mathcal{X}, \mathrm{d}_{\mathcal{X}}) = (\mathcal{D}_X, \mathfrak{d}_\infty)$. The core of the proof consists in proving that $\underline{I}_X$ and $\overline{I}_X$ are actually equal, which implies the LDP thanks to Lemma 1.2.

Section 2 presents some topological preliminaries on the space of admissible metrics $\mathcal{D}_X$ and properties of its elements, including the definition and existence of the gradient of any $D \in \mathcal{D}_X$ at almost every point of X. We also present general methods for constructing and transforming metrics in $\mathcal{D}_X$.

In Section 3 it is shown that $\underline{I}_X$ and $\overline{I}_X$ are nondecreasing, which will imply that I_X is nondecreasing once the equality $\underline{I}_X = \overline{I}_X$ is proven. The general idea is that if $D_1 \leq D_2$, then a configuration in which $\widetilde{\mathbf{T}}_{n,X} \simeq D_2$ can be transformed into a configuration in which $\widetilde{\mathbf{T}}_{n,X} \simeq D_1$ by altering a subvolumic number of edge passage times.

In Section 4 we prove with a somewhat classic subadditive argument (see Section 1.5) that $\underline{I}_{[0,1]^d}$ and $\overline{I}_{[0,1]^d}$ coincide on $\mathcal{N}$, i.e. Theorem 1.9. We also study properties of $I_{[0,1]^d}$ on $\mathcal{N}$, namely Propositions 1.10 and 1.11.

Theorem 1.9. *Under Assumption 1, for all $g \in \mathcal{N}$, $\underline{I}_{[0,1]^d}(g) = \overline{I}_{[0,1]^d}(g)$. We will denote $I_{[0,1]^d}(g)$ their common value.*

Proposition 1.10. *Under Assumption 1, for all $g \in \mathcal{N}$,*

(i) $I_{[0,1]^d}(g) = \infty$ if and only if $g \in \mathcal{N} \setminus \mathcal{N}^*$ (i.e. $\exists x \neq 0, g(x) = b\|x\|$) and $\nu(\{b\}) = 0$.

(ii) $I_{[0,1]^d}(g) = 0$ if and only if $g \leq \mu$, where μ is defined by (1.3).

Proposition 1.11. *Under Assumption 1, the restriction of $I_{[0,1]^d}$ on $\mathcal{N}^*$ is continuous for the metric $\mathfrak{d}_\infty$ defined by (1.17). If furthermore $\nu(\{b\}) = 0$, then the restriction of $I_{[0,1]^d}$ on $\mathcal{N}$ is continuous.*

Section 5 concludes the proof, which essentially amounts to showing that for all $D \in \mathcal{D}_X$,

$$\overline{I}_X(D) \leq \int_X I_{[0,1]^d}(\mathrm{grad}\, D)_z \mathrm{d}z \leq \underline{I}_X(D). \tag{1.31}$$

The general idea is to approximate D by a metric whose gradient is constant on each element of a tile partition X then use Theorem 1.9 on each tile.

Section 6 is devoted to proving results of Subsection 1.3.

1.5 Open questions and related works

Lower tail large deviations. Theorem 1.5 does not provide a satisfying estimate for $\mathbb{P}\Big(\widetilde{\mathbf{T}}_{n,X} \simeq D\Big)$ for metrics $D \in \mathcal{D}_X$ such that $D \leq \mu$. Given (1.4), we conjecture that the appropriate speed for the study of such events is n.

Upper tail large deviations under milder assumptions. Chow and Zhang proved in 2003 [6] that under a finite exponential moment condition, the probability of the upper tail large deviation event for the face-face passage time, i.e. $\Big\{\widetilde{\mathbf{T}}_{n,[0,1]^d}(H_1, H_1') \geq \zeta\Big\}$ with $\zeta > \mu(\mathrm{e}_1)$ and the notations of Corollary 1.7, is of order $\exp\Big(-\Theta(n^d)\Big)$ and the existence of the rate function. Note that their result is neither a special case nor an extension of Corollary 1.7 because while the former is valid in a larger framework, the latter provide the LDP for all directions simultaneously.

Cranston, Gauthier and Mountford proved in 2009 [9] a criterion for $\mathbb{P}\Big(\frac{1}{n}\mathbf{T}(0, n\mathrm{e}_1) \geq \zeta\Big)$ with $\zeta > \mu(\mathrm{e}_1)$ to be of order $\exp\Big(-\Theta(n^d)\Big)$, for a certain family of distributions. On the other hand Cosco and Nakajima recently showed [7] that if $\nu([t\,, \infty[) \simeq \exp(-\alpha t^r)$, with $\alpha > 0$ and $0 < r \leq d$, then

$$\mathbb{P}\Big(\frac{1}{n}\mathbf{T}(0, n\mathrm{e}_1) \geq \zeta\Big) = \begin{cases} \exp(-\Theta(n^r)) & \text{if } 0 < r < d, \\ \exp\Big(-\Theta\Big(\frac{n^d}{(\log n)^{d-1}}\Big)\Big) & \text{if } r = d. \end{cases}$$

Moreover, they provide a description of the associated rate functions with a variational formula.

In particular, outside the bounded support assumption, for some distributions, the probability of different upper tail large deviation events may be of different order. Hence, for these distributions, a single LDP for the random metric as Theorem 1.5 cannot give appropriate estimates for the probability of all upper tail large deviation behaviours; rather, several LDPs, at different speeds, would be needed.

The subadditivity argument. The main idea of Chow and Zhang [6] to prove the existence of the limit

$$\lim_{n\to\infty} -\frac{1}{n^d} \log \mathbb{P}\Big(\widetilde{\mathbf{T}}_{[0\,,1]^d}(H_1, H_1') \geq \zeta\Big),$$

with $\zeta > \mu(\mathrm{e}_1)$, is to assemble k^d independent configurations on $[\![0\,, n]\!]^d$ satisfying $\Big\{\widetilde{\mathbf{T}}_{n,[0\,,1]^d}(H_1, H_1') \geq \zeta\Big\}$ to create a configuration on $[\![0\,, kn]\!]^d$ satisfying $\Big\{\widetilde{\mathbf{T}}_{kn,[0\,,1]^d}(H_1, H_1') \geq \zeta\Big\}$, leading to

$$\mathbb{P}\Big(\widetilde{\mathbf{T}}_{n,[0\,,1]^d}(H_1, H_1') \geq \zeta\Big)^{k^d} \leq \mathbb{P}\Big(\widetilde{\mathbf{T}}_{kn,[0\,,1]^d}(H_1, H_1') \geq \zeta\Big),$$

and the end is standard. However, this simple plan fails because there is no general link between the crossing times of the small boxes and the crossing time of the large one. The solution proposed by the authors is to consider a subset Z_n of $\Big\{\widetilde{\mathbf{T}}_{n,[0\,,1]^d}(H_1, H_1') \geq \zeta\Big\}$ with equivalent log-probability, such that configurations on $[\![0\,, n]\!]^d$ satisfying Z_n are compatible, in the sense that assembling them creates no "shortcuts".

To prove the existence of the limit

$$\lim_{n\to\infty} -\frac{1}{n^d} \log \mathbb{P}\Big(\frac{1}{n}\mathbf{T}(0, n\mathrm{e}_1) \geq \zeta\Big),$$

Basu, Ganguly and Sly [3] adopt a similar approach, with an added subtlety regarding the way configurations on the small boxes are assembled: each small box is fragmented into smaller tiles and the configuration on the large box is constructed so that, for example, its top left corner is made of the top left corner tiles of all the small boxes. To ensure that corresponding tiles are compatible, they essentially proceed as follows:

1. Prove that for a well-chosen (random but within a deterministic range) tile size, most of the tiles are *stable*, in the sense that the restriction of the random metric on each tile is similar to some (random, tile-dependent) norm.

2. Define Z_n as a subset of $\Big\{\frac{1}{n}\mathbf{T}(0, n\mathrm{e}_1) \geq \zeta\Big\}$ for which the good tile size and the norms on each tile are prescribed, chosen so that its log-probability is equivalent to $\mathbb{P}\Big(\frac{1}{n}\mathbf{T}(0, n\mathrm{e}_1) \geq \zeta\Big)$.

In this article, we adopt a somewhat different approach, as the subadditivity argument is only used in the proof of Theorem 1.9. The fact that we are, at this step, only interested in uniform environments makes our version of the argument less sophisticated than the one in [3], as tile compatibility is free. This enables us to avoid the technical difficulty of proving tile stability properties for the random metric that are uniform in the realization; this work is done in a rather simpler deterministic setting here.

Streams and maximal flows. The general philosophy of our proof is inspired by the recent work of Dembin and Théret [10] on large deviations of the streams and the maximal flows: in order to estimate the probability that there exists an admissible stream resembling a target stream, they first study the easier case where the target stream is uniform. They then use the estimate in the elementary case as a building block to get an estimate in the general case.

1.6 Notations

Elements of $\mathbb{R}^d$. We denote by $(\mathrm{e}_i)_{i=1}^d$ the canonical basis of $\mathbb{R}^d$ and $\langle \cdot, \cdot \rangle$ the standard inner product on $\mathbb{R}^d$. We denote by $\leq$ the coordinatewise order on $\mathbb{R}^d$. For all $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we define

$$\lfloor x \rfloor \coloneqq (\lfloor x_1 \rfloor, \dots, \lfloor x_d \rfloor) \text{ and } \lceil x \rceil \coloneqq (\lceil x_1 \rceil, \dots, \lceil x_d \rceil). \tag{1.32}$$

A sequence $(x_n)_{n \geq 1}$ of elements of $\mathbb{R}^d$ is said to have *monotone coordinates* if for all $i \in [\![1\,, d]\!]$, the sequence $(\langle x_n, \mathrm{e}_i \rangle)_{n \geq 1}$ is monotone.

Exponents. Whenever several families of edge passage times are considered and distinguished by exponents, the associated metrics $\mathbf{T}$, $\widetilde{\mathbf{T}}_n$, etc. will be marked with the same exponents. For example, $\mathbf{T}^{(1)}$ is the process defined as (1.13), with $\tau_e^{(1)}$ instead of τ_e.

Cardinal and volume. We denote by $\#A$ the cardinal of the set A. If A is a Borel subset of $\mathbb{R}^d$ we denote by $\mathrm{Leb}(A)$ its Lebesgue measure.

Edges. Given an edge e, the sentence "$z \in e$" will mean "z belongs to the segment between endpoints of e". For all $A \subseteq \mathbb{R}^d$ we will denote $\mathrm{E}(A)$ the set of all edges $e \in \mathbb{E}^d$ included in A. We will denote $\mathrm{E}_{\mathrm{ext}}(A)$ the set of all edges $e \in \mathbb{E}^d$, seen as segments, intersecting A. For all discrete path $\alpha = (\alpha_j)_{j=0}^r$, we define

$$\mathrm{E}_{\mathrm{path}}(\alpha) \coloneqq \{(\alpha_j, \alpha_{j+1}), j \in [\![0\,, r-1]\!]\}, \tag{1.33}$$

which is consistent with the previous definition of $\mathrm{E}(\cdot)$.

Paths. Given a polygonal path $\gamma = (x_0, \dots, x_r)$, we will write "$x \overset{\gamma}{\rightsquigarrow} y$" as a shorthand for "$x_0 = x$ and $x_r = y$". We will write "$x \underset{X}{\overset{\gamma}{\rightsquigarrow}} y$" to further indicate that for all $0 \leq i \leq r$, $x_i \in X$. For all $0 \leq i_1 \leq i_2 \leq r$, we define

$$\gamma_{|[i_1\,, i_2]} \coloneqq (x_i)_{i_1 \leq i \leq i_2}. \tag{1.34}$$

We call *continuous path* a continuous function $\gamma : [0\,, T] \to \mathbb{R}^d$. The length of the segment on which a continuous path γ is defined will be denoted by T_γ. Given a continuous path γ, we will write "$x \overset{\gamma}{\rightsquigarrow} y$" as a shorthand for "$\gamma(0) = x$ and $\gamma(T_\gamma) = y$". We will write "$x \underset{X}{\overset{\gamma}{\rightsquigarrow}} y$" to further indicate that for all $t \in [0\,, T_\gamma]$, $\gamma(t) \in X$.

Tiles. For all $v \in \mathbb{Z}^d$ and $k \geq 1$, we define the set

$$\mathrm{Tile}(v,k) \coloneqq \frac{1}{k}\Big(v + [0\,,1]^d\Big). \tag{1.35}$$

For all $k \geq 1$ and $X \in \mathcal{K}$, we define

$$\mathrm{V}_k^{\mathrm{int}}(X) \coloneqq \Big\{ v \in \mathbb{Z}^d \;\Big|\; \mathrm{Tile}(v,k) \subseteq X \Big\} \tag{1.36}$$

$$\text{and } \mathrm{V}_k^{\mathrm{ext}}(X) \coloneqq \Big\{ v \in \mathbb{Z}^d \;\Big|\; \mathrm{Tile}(v,k) \cap X \neq \emptyset \Big\}. \tag{1.37}$$

They satisfy (see Lemma B.4)

$$\lim_{k\to\infty} \frac{\#\mathrm{V}_k^{\mathrm{ext}}(X)}{k^d} = \lim_{k\to\infty} \frac{\#\mathrm{V}_k^{\mathrm{int}}(X)}{k^d} = \mathrm{Leb}(X). \tag{1.38}$$

Metrics. Given $X \in \mathcal{K}$ and $D \in \mathcal{D}_X$, it will be useful to extend D to $\mathbb{R}^d$ by defining

$$\overline{D}(x,y) \coloneqq \min\Big(\min_{x',y' \in X} \big(b\|x-x'\| + D(x',y') + b\|y-y'\|\big), b\|x-y\| \Big). \tag{1.39}$$

It is a metric that verifies (1.16). Given $X \in \mathcal{K} \cup \Big\{\mathbb{R}^d\Big\}$, a metric D on X and two subsets $A, B \subseteq X$, we define

$$D(A,B) \coloneqq \inf_{\substack{x\in A \\ y \in B}} D(x,y). \tag{1.40}$$

When D is the metric associated with the norm $\|\cdot\|$, we denote by $\mathrm{d}(A,B)$ this quantity. If $\gamma : [0\,,T] \to X$ is a continuous path, we define the D-length of γ as

$$D(\gamma) \coloneqq \sup_{0 \leq t_0 < \cdots < t_r \leq T} \sum_{i=0}^{r-1} D(\gamma(t_i), \gamma(t_{i+1})). \tag{1.41}$$

By the triangle inequality, for all continuous paths $x \overset{\gamma}{\rightsquigarrow} y$, $D(\gamma) \geq D(x,y)$. If $D \in \mathcal{D}_X$, then for all $x, y \in X$, any geodesic $x \overset{\sigma}{\rightsquigarrow} y$ satisfies $D(\sigma) = D(x,y)$. Since σ is 1-Lipschitz for D, it is Lipschitz for $\|\cdot\|$ by (1.16). Consequently, if $D \in \mathcal{D}_X$, then for all $x, y \in X$,

$$D(x,y) = \min\Big\{ D(\gamma) \;\Big|\; x \underset{X}{\overset{\gamma}{\rightsquigarrow}} y, \gamma \text{ Lipschitz} \Big\}. \tag{1.42}$$

We call *diameter* of a subset $X \subseteq \mathbb{R}^d$ the quantity

$$\mathrm{diam}(X) \coloneqq \sup_{x,y\in X} \|x-y\|. \tag{1.43}$$

Absolutely homogeneous functions. Let $\mathcal{C}_{\mathrm{Hom}}\Big(\mathbb{R}^d, \mathbb{R}\Big)$ denote the set of all continuous and absolutely homogeneous functions $f : \mathbb{R}^d \to \mathbb{R}$, i.e. satisfying $f(\lambda x) = |\lambda| f(x)$ for all $x \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$. We endow this space with the norm defined as

$$\|f\|_{\mathrm{Hom}} \coloneqq \sup_{u \in \mathbb{S}} |f(u)|. \tag{1.44}$$

We endow $\mathcal{C}_{\mathrm{Hom}}\Big(\mathbb{R}^d, \mathbb{R}\Big)$ with the cylinder σ-algebra, which is also its Borel σ-algbera.

2 The space $\mathcal{D}_X$

In this section we work under Assumption 1. We gather deterministic results about metrics in $\mathcal{D}_X$ used throughout the article.

2.1 Compactness

This subsection aims to show Proposition 2.1.

Proposition 2.1. *For all $X \in \mathcal{K}$, the space $(\mathcal{D}_X, \mathfrak{d}_\infty)$ is compact.*

We first state a criterion for elements of $\mathcal{D}_X$ which is a direct application of the Hopf-Rinow theorem (see e.g. [5, Proposition 3.7]).

Lemma 2.2. *Let $X \in \mathcal{K} \cup \left\{\mathbb{R}^d\right\}$. Let D be a metric on X that satisfies* (1.16)*. Then $D \in \mathcal{D}_X$ (i.e. (X, D) is geodesic) if and only if for all $x, y \in X$ and $\varepsilon > 0$, there exists $z \in X$ such that*

$$\max(D(x,z), D(z,y)) \leq \frac{1}{2}D(x,y) + \varepsilon. \tag{2.1}$$

Moreover, in this case, for all $x, y \in X$, there exists $z \in X$ such that

$$D(x,z) = D(z,y) = \frac{1}{2}D(x,y). \tag{2.2}$$

Proof of Proposition 2.1. We first prove that $\mathcal{D}_X$ is closed in the space $\mathcal{C}(X^2, \mathbb{R})$ of continuous functions from X^2 to $\mathbb{R}$ for the uniform convergence. Let $(D_n)_{n\geq 1}$ be a sequence of elements of $\mathcal{D}_X$ converging to D. It is clear that D is a metric satisfying (1.16). In light of Lemma 2.2, it is therefore sufficient to prove (2.1). Let $x, y \in X$. For all $n \geq 1$, there exists $z_n \in X$ such that

$$D_n(x, z_n) = D_n(z_n, y) = \frac{1}{2}D_n(x,y).$$

Hence for all n,

$$\begin{aligned} D(x, z_n) &\leq D_n(x, z_n) + \mathfrak{d}_\infty(D, D_n) \\ &= \frac{1}{2}D_n(x,y) + \mathfrak{d}_\infty(D, D_n) \\ &\leq \frac{1}{2}D(x,y) + 2\mathfrak{d}_\infty(D, D_n), \end{aligned}$$

and the same goes for $D(z_n, y)$. Thus D satisfies (2.1) so $D \in \mathcal{D}_X$.

Note that for all $D \in \mathcal{D}_X, x, x', y, y' \in X$,

$$|D(x,y) - D(x',y')| \leq D(x,x') + D(y,y') \leq b(\|x' - x\| + \|y' - y\|).$$

Consequently, $\mathcal{D}_X$ is a uniformally bounded, closed and equicontinuous subset of $\mathcal{C}(X^2, \mathbb{R})$, therefore it is compact thanks to the Arzelà-Ascoli theorem. □

2.2 Length of the geodesics

Lemma 2.3 will be of constant use, as it essentially entails that if $D \in \mathcal{D}_X$ and $x, y \in X$, $D(x,y)$ only depends on a local environment around x, of radius $\mathrm{O}(\|x-y\|)$.

Lemma 2.3. *Let $X \in \mathcal{K} \cup \left\{\mathbb{R}^d\right\}$ and $D \in \mathcal{D}_X$. Let σ be a D-geodesic between x and y. Then its $\|\cdot\|$-length satisfies*

$$\|\sigma\| \leq \frac{b}{a}\|x-y\|. \tag{2.3}$$

In particular, for all $0 \leq t \leq D(x,y)$,

$$\|x - \sigma(t)\| \leq \frac{b}{a}\|x-y\|. \tag{2.4}$$

Likewise, for all $x, y \in X$ the infimums in (1.13) *and* (1.14) *may be restricted to finite sequences included in $\overline{\mathrm{B}}\left(x, \frac{b}{a}\|x-y\|\right)$ only.*

Proof. Let $0 = t_0 < t_1 < \ldots t_r = D(x,y)$. By definition of σ, for all $0 \leq i \leq r-1$,

$$D(\sigma(t_i), \sigma(t_{i+1})) = t_{i+1} - t_i.$$

The lower bound in (1.16) implies

$$a\|\sigma(t_i) - \sigma(t_{i+1})\| \leq t_{i+1} - t_i.$$

Summing over i and applying the upper bound in (1.16), we get

$$a\sum_{i=0}^{r-1}\|\sigma(t_i) - \sigma(t_{i+1})\| \leq D(\sigma) = D(x,y) \leq b\|x-y\|,$$

hence (2.3). The rest is analoguous. □

2.3 Properties of the passage time

In this subsection we show that the metric $\mathbf{T}_X$ belongs to $\mathcal{D}_X$ (Proposition 2.4), and that it is an extension of the usual passage time (Proposition 2.5).

Proposition 2.4. *For all $X \in \mathcal{K}$, $\mathbf{T}_X \in \mathcal{D}_X$. For all $n \geq 1$, $\mathbf{T} \in \mathcal{D}_{\mathbb{R}^d}$.*

Proof. Let $X \in \mathcal{K}$. Then $\mathbf{T}_X$ is clearly a metric that satisfies (1.16), so it suffices to prove that it satisfies (2.1). Let $x, y \in X$ and $\varepsilon > 0$. Without loss of generality, we can assume $x \neq y$. Let $\pi = (x = x_0, \ldots, x_r = y)$ be a finite sequence in X such that

$$\tau(\pi) \leq \mathbf{T}_X(x,y) + \varepsilon.$$

There exists $r_0 \in [\![0\,, r-1]\!]$ such that

$$\sum_{j=0}^{r_0-1}\tau(x_j, x_{j+1}) \leq \frac{1}{2}\tau(\pi) \text{ et } \sum_{j=0}^{r_0}\tau(x_j, x_{j+1}) > \frac{1}{2}\tau(\pi).$$

A straightforward consequence of the definition of τ on $\mathbb{R}^d \times \mathbb{R}^d$ (see (1.12)) is that for all $z, z' \in [x_{r_0}, x_{r_0+1}]$,

$$\tau(z, z') \leq \frac{\|z - z'\|}{\|x_{r_0} - x_{r_0+1}\|} \tau(x_{r_0}, x_{r_0+1});$$

hence, denoting by z the point on the segment $[x_{r_0}, x_{r_0} + 1]$ such that

$$\sum_{j=0}^{r_0-1} \tau(x_j, x_{j+1}) + \frac{\|x_{r_0} - z\|}{\|x_{r_0} - x_{r_0+1}\|} \tau(x_{r_0}, x_{r_0+1}) = \frac{1}{2}\tau(\pi),$$

we have

$$\sum_{j=0}^{r_0-1} \tau(x_j, x_{j+1}) + \tau(x_{r_0}, z) \leq \frac{1}{2}\tau(\pi) \text{ et } \tau(z, x_{r_0+1}) + \sum_{j=r_0+1}^{r} \tau(x_j, x_{j+1}) \leq \frac{1}{2}\tau(\pi).$$

Consequently,

$$\max(\mathbf{T}_X(x, z), \mathbf{T}_X(z, y)) \leq \frac{1}{2}\mathbf{T}_X(x, y) + \frac{1}{2n}\varepsilon,$$

which concludes the proof of the first part. The second part is analoguous. □

Proposition 2.5.

(i) The maps defined by (1.2) *and* (1.13) *coincide on* $\mathbb{Z}^d \times \mathbb{Z}^d$.

(ii) Assume that $X = [t_1, t_1'] \times \cdots \times [t_d, t_d']$ *and* $x, y \in X \cap \mathbb{Z}^d$. *Then the infimum in* (1.14) *is attained on a discrete path.*

Proof. We first show that the second part implies the first. By Lemma 2.3, for all $x, y \in \mathbb{Z}^d$,

$$\mathbf{T}(x, y) = \inf\left\{ \tau(\pi) \;\middle|\; x \overset{\pi}{\rightsquigarrow} y, \pi \subseteq x + \left[-\frac{b}{a}\|x - y\|, \frac{b}{a}\|x - y\| \right]^d \right\} = \mathbf{T}_{x + \left[-\frac{b}{a}\|x-y\|, \frac{b}{a}\|x-y\| \right]^d}(x, y),$$

thus the discrete path γ given by the second part with $X \coloneqq x + \left[-\frac{b}{a}\|x - y\|, \frac{b}{a}\|x - y\| \right]^d$ satisfies $\tau(\gamma) = \mathbf{T}(x, y)$.

Let us show the second part. The key property of X used here is that for all $z_1, z_2 \in X$,

$$\text{any finite sequence of points with monotone coordinates and endpoints } z_1 \text{ and } z_2 \text{ is included in } X. \tag{2.5}$$

Let $\gamma = (x = x_0, x_1, \ldots, x_r = y)$ be a finite sequence in X. It is sufficient to show that there exists a discrete path from x to y with passage time at most $\tau(\gamma)$.

Step 1: Removing points outside $\mathbb{E}^d$. For all $1 \leq j \leq r - 1$ such that $x_j \notin \bigcup_{e \in \mathbb{E}^d} e$,

$$\begin{aligned} \tau(x_{j-1}, x_{j+1}) &\leq b\|x_{j-1} - x_{j+1}\| \\ &\leq b\|x_{j-1} - x_j\| + b\|x_j - x_{j+1}\| \\ &= \tau(x_{j-1}, x_j) + \tau(x_j, x_{j+1}), \end{aligned}$$

thus removing such points from γ can only decrease the passage time. From now on we may therefore assume that for all $0 \leq j \leq r$, $x_j \in X \cap (\bigcup_{e \in \mathbb{E}^d} e)$.

Step 2: Inserting integer points. Let $0 \leq j \leq r-1$ such that x_j and x_{j+1} are not both integer points. Without loss of generality we may assume that $x_j \leq x_{j+1}$ for coordinate-wise order. All coordinates of x_j except possibly one are integer, so x_j and $\lfloor x_j \rfloor$ belong to a common edge. The same goes for x_{j+1} and $\lceil x_{j+1} \rceil$.

Case 2.1: Assume that $\lceil x_j \rceil \leq \lfloor x_{j+1} \rfloor$. Property (2.5) implies that $\lceil x_j \rceil, \lfloor x_{j+1} \rfloor \in X \cap \mathbb{Z}^d$. Moreover the sequence $\gamma_j \coloneqq (x_j, \lceil x_j \rceil, \lfloor x_{j+1} \rfloor, x_{j+1})$ satisfies

$$\tau(\gamma_j) \leq b\|x_j - x_{j+1}\| = \tau(x_j, x_{j+1}).$$

Case 2.2: Assume that $\lceil x_j \rceil \not\leq \lfloor x_{j+1} \rfloor$. Then there exists $1 \leq i \leq d$ and $k \in \mathbb{Z}$ such that

$$k < \langle x_j, \mathrm{e}_i \rangle, \langle x_{j+1}, \mathrm{e}_i \rangle < k+1,$$

and all other coordinates of x_j and x_{j+1} are integers. Thus x_j and x_{j+1} belong to the strip

$$\mathrm{E}_{i,k}(X) \coloneqq \bigcup_{\substack{v=(v_1,\dots,v_d) \in X \cap \mathbb{Z}^d \\ v_i = k}} [v\,, v + \mathrm{e}_i].$$

Consequently, by inserting the sequence γ_j between x_j and x_{j+1} for all j satisfying Case 2.1, one constructs a sequence $\gamma' \coloneqq (x = y_0, \dots, y_s = y)$ such that for all $0 \leq j \leq s-1$, at least one of the following assertions is true:

$$\text{Both } y_j \text{ and } y_{j+1} \text{ are integer points.} \tag{2.6}$$

$$\text{Both } y_j \text{ and } y_{j+1} \text{ belong to a common strip } \mathrm{E}_{i,k}(X). \tag{2.7}$$

Step 3: Deleting non integer points. Let $0 \leq j_1 \leq j_1 + 2 \leq j_2 \leq s$ be indices such that y_{j_1} and y_{j_2} are integer points but for all $j_1 < j < j_2$, y_j is not. We claim that the subsequence of γ' between j_1 and j_2 may be replaced by a sequence of points in $X \cap \mathbb{Z}^d$ with a lesser or equal passage time. For all $j_1 < j < j_2$, there exists a unique pair $(i(j), k(j))$ such that $y_j \in \mathrm{E}_{i(j),k(j)}(X)$. Hence, (2.7) yields the existence of a pair (i,k) such that for all $j_1 \leq j \leq j_2$, $y_j \in \mathrm{E}_{i,k}(X)$.

Case 3.1: Assume that $\langle y_{j_1} - y_{j_2}, \mathrm{e}_i \rangle = 0$. Then

$$\begin{aligned} b\|y_{j_1} - y_{j_2}\| &\leq \sum_{i' \neq i} \sum_{j=j_1}^{j_2-1} b|\langle y_j - y_{j+1}, \mathrm{e}_{i'} \rangle| \\ &\leq \sum_{j=j_1}^{j_2-1} \tau(y_j, y_{j+1}), \end{aligned}$$

therefore (y_{j_1}, y_{j_2}) is a suitable sequence (see Figure 1, top).

Case 3.2: Assume that $\langle y_{j_1} - y_{j_2}, \mathrm{e_i} \rangle = \pm 1$. We only treat the subcase $\langle y_{j_1} - y_{j_2}, \mathrm{e_i} \rangle = 1$. The other one is similar. We have

$$\sum_{j=j_1}^{j_2-1} \tau(y_j, y_{j+1}) \geq \sum_{j=j_1}^{j_2-1} \left(|\langle y_j - y_{j+1}, \mathrm{e}_i \rangle| \min_{v \in V(j_1,j_2)} \tau(v, v+\mathrm{e}_i) + \sum_{i' \neq i} b|\langle y_j - y_{j+1}, \mathrm{e}_{i'} \rangle| \right),$$

where $V(j_1, j_2) \coloneqq \{\lfloor y_j \rfloor, \quad j_1 \leq j \leq j_2\}$. Thus, by fixing $v_0 \in \operatorname*{arg\,min}_{v \in V(j_1,j_2)} \tau(v, v+\mathrm{e}_i)$,

$$\begin{aligned} \sum_{j=j_1}^{j_2-1} \tau(y_j, y_{j+1}) &\geq b\|y_j - v_0\| + \left(\sum_{j=j_1}^{j_2-1} |\langle y_j - y_{j+1}, \mathrm{e}_i \rangle| \right) \tau(v_0, v_0 + \mathrm{e}_i) + b\|v_0 + \mathrm{e}_i - y_{j+1}\| \\ &\geq b\|y_j - v_0\| + \tau(v_0, v_0 + \mathrm{e}_i) + b\|v_0 + \mathrm{e}_i - y_{j+1}\|. \end{aligned}$$

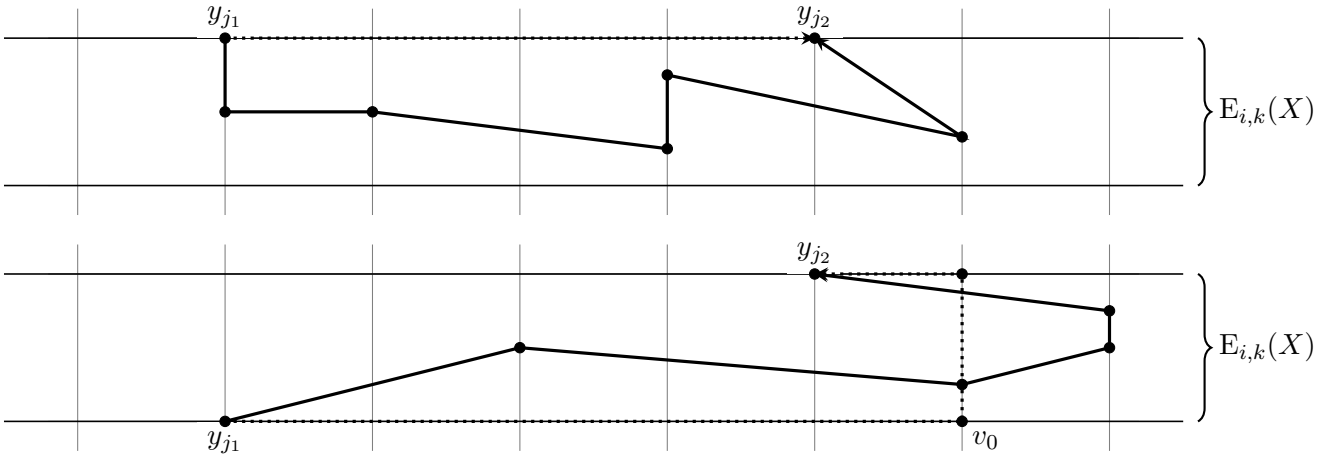


Figure 1: The subsequence of γ' between y_{j_1} and y_{j_2} (solid line) can be replaced by a quicker sequence (dotted lines) using only integer points.

Moreover the assumption on X implies that v_0 and $v_0 + \mathrm{e}_i$ are elements of X. Thus $(y_{j_1}, v_0, v_0 + \mathrm{e}_i, y_{j_2})$ is a suitable sequence (see Figure 1, bottom). Consequently, by considering every pair of indices (j_1, j_2) corresponding to consecutive integer points in γ', one constructs a sequence γ'' of points in $X \cap \mathbb{Z}^d$, with passage time at most $\tau(\gamma')$, whose endpoints are x and y. Finally, due to property (2.5), γ'' can be assumed to be a discrete path. □

2.4 Gradient of a metric

In this subsection $X \in \mathcal{K} \cup \left\{\mathbb{R}^d\right\}$ and $D \in \mathcal{D}_X$ are fixed. We define the gradient of D at a point $z \in X$. Proposition 2.9 and Proposition 2.10 essentially state that D can be reconstructed from its gradient and locally approximated by it, respectively. These two propositions are mere applications and reformulations of well known results around the so-called *metric derivative*, a tool initially introduced by Kircheim [14] to extend the area formula to functions taking values in a general metric space.

Lemma 2.6 will be of constant use in this subsection, as well as in Subsection 2.5. It is a consequence of the fact that *absolutely continuous* functions satisfy the fundamental theorem of calculus (see e.g. [15, Theorem 7.18]). The first part is a special case of Rademacher's theorem (see e.g. [12, Theorem 3.1.6]).

Lemma 2.6. *Let $\gamma : [0\,, T] \to \mathbb{R}^d$ be a Lipschitz path. Then γ is differentiable almost everywhere and for all $0 \leq t_1 \leq t_2 \leq T$,*

$$\gamma(t_2) - \gamma(t_1) = \int_{t_1}^{t_2} \gamma'(t)\mathrm{d}t. \tag{2.8}$$

Definition 2.7. The *gradient* of D at $z \in X$ is defined as the function

$$\begin{aligned} (\operatorname{grad} D)_z : \mathbb{R}^d &\longrightarrow [0\,, \infty[\\ u &\longmapsto \varlimsup_{h \to 0} \frac{\overline{D}(z, z + hu)}{|h|}, \end{aligned} \tag{2.9}$$

with $\overline{D}$ defined by (1.39). If $z \in \mathring{X}$, then $\overline{D}$ can be replaced by D in (2.9).

Proposition 2.8 (Elementary properties of the gradient)**.**

(i) Let $z \in X$. Then $(\operatorname{grad} D)_z$ is b-Lipschitz and absolutely homogeneous. In particular it belongs to the space $\mathcal{C}_{\mathrm{Hom}}\Big(\mathbb{R}^d, \mathbb{R}\Big)$ defined in Subsection 1.6.

(ii) Let $z \in X$. Then for all $u \in \mathbb{R}^d$,

$$a\|u\| \leq (\operatorname{grad} D)_z(u) \leq b\|u\|. \tag{2.10}$$

(iii) The map

$$\begin{aligned} X &\longmapsto \mathcal{C}_{\mathrm{Hom}}\Big(\mathbb{R}^d, \mathbb{R}\Big) \\ z &\longmapsto (\operatorname{grad} D)_z \end{aligned}$$

is measurable.

Proof. Notice that for all $u_1, u_2 \in \mathbb{R}^d, h \neq 0$,

$$|D(z, z+hu_1) - D(z, z+hu_2)| \leq D(z+hu_1, z+hu_2) \leq b|h|\|u_2 - u_1\|.$$

Hence $(\operatorname{grad} D)_z$ is b-Lipschitz. The rest is clear. □

Proposition 2.9. *For any Lipschitz path $\gamma : [0\,, T] \to X$,*

$$D(\gamma) = \int_0^T (\operatorname{grad} D)_{\gamma(t)}(\gamma'(t))\mathrm{d}t, \tag{2.11}$$

where $D(\gamma)$ is defined by (1.41)*. In particular, for all $x, y \in X$,*

$$D(x,y) = \min\left\{ \int_0^{T_\gamma} (\operatorname{grad} D)_{\gamma(t)}(\gamma'(t))\mathrm{d}t \;\middle|\; x \underset{X}{\overset{\gamma}{\rightsquigarrow}} y, \gamma \text{ Lipschitz} \right\} \tag{2.12}$$

and any geodesic from x to y attains the minimum.

Proof. Let $\gamma : [0\,, T] \longrightarrow X$ be a Lipschitz path from x to y. Theorem 4.1.6 in [1] implies that the limit

$$\dot{\gamma}(t) \coloneqq \lim_{h \to 0} \frac{D(\gamma(t), \gamma(t+h))}{|h|}$$

exists for almost every $0 \leq t \leq T$, and

$$D(\gamma) = \int_0^T \dot{\gamma}(t)\mathrm{d}t. \tag{2.13}$$

Let $0 \leq t \leq T$ be such that both $\dot{\gamma}(t)$ and $\gamma'(t)$ exist. We have

$$\begin{aligned} \big|D(\gamma(t), \gamma(t+h)) - D(\gamma(t), \gamma(t) + \gamma'(t)h)\big| &\leq D\big(\gamma(t+h), \gamma(t) + \gamma'(t)h\big) \\ &\leq b\big\|\gamma(t+h) - \gamma(t) - \gamma'(t)h\big\| = \mathrm{o}(h). \end{aligned}$$

Dividing by h and letting $h \to 0$ yields

$$(\operatorname{grad} D)_{\gamma(t)}(\gamma'(t)) = \dot{\gamma}(t). \tag{2.14}$$

The proposition follows from Equations (2.13) and (2.14). □

Proposition 2.10. *For almost every $z \in X$, $(\operatorname{grad} D)_z \in \mathcal{N}$, and there exists a function $\varepsilon : [0, \infty[\to [0, \infty[$ such that $\varepsilon(0^+) = 0$, for all $x, y \in X$,*

$$|D(x,y) - (\operatorname{grad} D)_z(y-x)| \leq \varepsilon(|x-z| + |y-z|) \cdot (|x-z| + |y-z|). \tag{2.15}$$

Proof. It is a special case of Theorem 2 in [14] for the identity map. However some minor adaptations need to be made, because their result is stated for maps taking values in a Banach space. The map

$$\begin{aligned} \iota : (\mathbb{R}^d, \overline{D}) &\longrightarrow \ell^\infty(\mathbb{R}^d) \\ x &\longmapsto \overline{D}(x, \cdot) - \overline{D}(0, \cdot) \end{aligned}$$

is an isometry from $\mathbb{R}^d$ to the space $\ell^\infty(\mathbb{R}^d)$ of bounded real functions on $\mathbb{R}^d$ endowed with the sup norm denoted by $\|\cdot\|_{\ell^\infty(\mathbb{R}^d)}$, therefore

$$\begin{aligned} f : (\mathbb{R}^d, \|\cdot\|) &\longrightarrow \ell^\infty(\mathbb{R}^d) \\ x &\longmapsto \iota(x) \end{aligned}$$

is Lipschitz and takes values in a Banach space. Theorem 2 in [14] hence implies that almost every $z \in \mathring{X}$ satisfies the following.

(i) For all $u \in \mathbb{R}^d$, the limit

$$\operatorname{MD}(f,z)(u) \coloneqq \lim_{\substack{h \to 0 \\ h > 0}} \frac{1}{h} \|f(z+hu) - f(z)\|_{\ell^\infty(\mathbb{R}^d)} \tag{2.16}$$

exists (the notation MD, for *metric derivative* is from [14]).

(ii) The function $\operatorname{MD}(f,z)(\cdot)$ is a seminorm.

(iii) There exists a function $\varepsilon : [0, \infty[\to [0, \infty[$ such that $\varepsilon(0^+) = 0$ and for all $x, y \in X$,

$$\Big| \|f(x) - f(y)\|_{\ell^\infty(\mathbb{R}^d)} - \operatorname{MD}(f,z)(y-x) \Big| \leq \varepsilon(|x-z| + |y-z|) \cdot (|x-z| + |y-z|). \tag{2.17}$$

Let us fix such a z. As $\operatorname{MD}(f,z)(\cdot)$ is a seminorm, Equation (2.16) can be rewritten as

$$\operatorname{MD}(f,z)(u) = \lim_{h \to 0} \frac{1}{|h|} \|f(z+hu) - f(z)\|_{\ell^\infty(\mathbb{R}^d)}. \tag{2.18}$$

Since ι is an isometry,

$$\operatorname{MD}(f,z)(u) = \lim_{h \to 0} \frac{1}{|h|} \overline{D}(z+hu, z) = \lim_{h \to 0} \frac{1}{|h|} D(z+hu, z) = (\operatorname{grad} D)_z(u),$$

and (2.17) can be rewritten as

$$|D(x,y) - \operatorname{MD}(f,z)(y-x)| \leq \varepsilon(|x-z| + |y-z|) \cdot (|x-z| + |y-z|).$$

Hence (2.15) and $(\operatorname{grad} D)_z$ is a seminorm. Finally, $(\operatorname{grad} D)_z$ satisfies (2.10) therefore $(\operatorname{grad} D)_z \in \mathcal{N}$. □

2.5 Building metrics

In this subsection we give some tools to manipulate and build admissible metrics: Lemma 2.11 essentially states that a metric $D \in \mathcal{D}_X$ can be defined by prescribing its gradient. Lemmas 2.13, 2.14 and 2.16 state that rescaling, translating or restricting an admissible metric yields an admissible metric. Lemma 2.15 states that stitching admissible metrics on different subsets of $\mathbb{R}^d$ yields an admissible metric on their reunion.

Lemma 2.11 (Prescribing the gradient of a metric)**.** *Recall the definition of $\mathcal{C}_{\mathrm{Hom}}\Big(\mathbb{R}^d, \mathbb{R}\Big)$ as the space of absolutely homogeneous functions from $\mathbb{R}^d$ to $\mathbb{R}$ in Subsection 1.6. Let*

$$\begin{aligned} g : X &\longrightarrow \mathcal{C}_{\mathrm{Hom}}\Big(\mathbb{R}^d, \mathbb{R}\Big) \\ z &\mapsto g_z \end{aligned}$$

be a measurable map such that for all $z \in X$, $a\|\cdot\| \leq g_z \leq b\|\cdot\|$. Consider

$$\begin{aligned} D : X^2 &\longrightarrow [0\,, \infty[\\ (x, y) &\longmapsto \inf\left\{ \int_0^{T_\gamma} g_{\gamma(t)}(\gamma'(t))\mathrm{d}t \;\middle|\; x \overset{\gamma}{\underset{X}{\rightsquigarrow}} y, \gamma \text{ } \textit{Lipschitz} \right\}. \end{aligned} \tag{2.19}$$

Then

(i) $D \in \mathcal{D}_X$.

(ii) The infimum in (2.19) *may be restricted to Lipschitz paths $x \overset{\gamma}{\rightsquigarrow} y$ such that $\|\gamma\| \leq \frac{b}{a}\|x - y\|$.*

(iii) For almost every $z \in X$,

$$(\operatorname{grad} D)_z \leq g_z. \tag{2.20}$$

(iv) If $z_0 \in \mathring{X}$ is such that g_{z_0} is a norm and $z \mapsto g_z$ is continuous at z_0 for $\|\cdot\|_{\mathrm{Hom}}$, then $g_{z_0} = (\operatorname{grad} D)_{z_0}$.

Proof. The fact that the integral in (2.19) is well-defined is a consequence of the measurability of $(z, u) \mapsto g_z(u)$, γ and γ'.

Proof of (i). The map D is clearly a metric on X. Let us show (1.16). Let $x, y \in X$. As X is convex, the affine path defined on $[0\,, 1]$ by $\gamma(t) \coloneqq (1-t)x + ty$ is Lipschitz and takes values in X, thus

$$D(x, y) \leq \int_0^1 g_{\gamma(t)}(\gamma'(t))\mathrm{d}t \leq \int_0^1 b\|\gamma'(t)\|\mathrm{d}t = b\|x - y\|.$$

For the other inequality, Lemma 2.6 implies that any Lipschitz path $x \overset{\gamma}{\underset{X}{\rightsquigarrow}} y$ satisfies

$$\int_0^{T_\gamma} g_{\gamma(t)}(\gamma'(t))\mathrm{d}t \geq \int_0^{T_\gamma} a\|\gamma'(t)\|\mathrm{d}t \geq a\left\|\int_0^{T_\gamma} \gamma'(t)\mathrm{d}t\right\| = a\|x - y\|.$$

In light of Lemma 2.2, it is sufficient to show that D satisfies (2.1). Let $x, y \in X$ and $\varepsilon > 0$. By definition of D there exists a Lipschitz path $x \overset{\gamma}{\underset{X}{\rightsquigarrow}} y$ such that

$$\int_0^{T_\gamma} g_{\gamma(t)}(\gamma'(t))\mathrm{d}t \leq D(x, y) + \varepsilon.$$

The intermediate value theorem implies the existence of $0 \le t_0 \le T_\gamma$ such that

$$\int_0^{t_0} g_{\gamma(t)}(\gamma'(t))\mathrm{d}t \le \frac{1}{2}D(x,y) + \frac{\varepsilon}{2}$$
$$\text{and } \int_{t_0}^{T_\gamma} g_{\gamma(t)}(\gamma'(t))\mathrm{d}t \le \frac{1}{2}D(x,y) + \frac{\varepsilon}{2}.$$

Thus

$$\max(D(x,\gamma(t_0)), D(\gamma(t_0),y)) \le \frac{1}{2}D(x,y) + \frac{\varepsilon}{2}, \tag{2.21}$$

therefore $D \in \mathcal{D}_X$.

Proof of (ii). It is a straightforward adaptation of the argument used in the proof of Lemma 2.3.

Proof of (iii). As the g_z and the $(\operatorname{grad} D)_z$ are continuous and absolutely homogeneous, it is sufficient to prove that for all u in a countable dense subset of $\mathbb{S}$,

$$\text{for almost every } z \in X, \quad (\operatorname{grad} D)_z(u) \le g_z(u). \tag{2.22}$$

For the sake of simplicity, we only treat the case $u = \mathrm{e}_1$. The proof may be adapted to any $u \in \mathbb{R}^d$. The boundary of X has zero Lebesgue measure (see Lemma B.3), thus it is sufficient to prove (2.22) for almost every $z \in \mathring{X}$. Let $z \in \mathring{X}$ and $\varepsilon > 0$ such that $z + [0\,,\varepsilon]^d \subseteq X$. Fubini's theorem yields

$$\int_{[0\,,\varepsilon]^d} g_{z+w}(\mathrm{e}_1)\mathrm{d}w = \int_{[0\,,\varepsilon]^{d-1}} \Big(\int_0^\varepsilon g_{z+t\mathrm{e}_1+w_{2:d}}(\mathrm{e}_1)\mathrm{d}t \Big) \mathrm{d}(w_2,\dots,w_d),$$

with the notation $w_{2:d} \coloneqq \sum_{i=2}^d w_i \mathrm{e}_i$. Moreover, it follows from (2.19) that for all $(w_2,\dots,w_d) \in [0\,,\varepsilon]^{d-1}$,

$$\int_0^\varepsilon g_{z+t\mathrm{e}_1+w_{2:d}}(\mathrm{e}_1)\mathrm{d}t \ge D(z+w_{2:d}, z+\varepsilon\mathrm{e}_1+w_{2:d}),$$

hence

$$\frac{1}{\varepsilon^d}\int_{[0\,,\varepsilon]^d} g_{z+w}(\mathrm{e}_1)\mathrm{d}w \ge \frac{1}{\varepsilon^{d-1}}\int_{[0\,,\varepsilon]^{d-1}} \frac{1}{\varepsilon} D(z+w_{2:d}, z+\varepsilon\mathrm{e}_1+w_{2:d})\mathrm{d}(w_2,\dots,w_d). \tag{2.23}$$

Lebesgue's differentiation theorem (see e.g. [15, Theorem 7.10]) implies that as $\varepsilon \to 0$, the left-hand side of (2.23) converges to $g_z(\mathrm{e}_1)$ for almost every $z \in \mathring{X}$. Besides, Proposition 2.10 implies that the right-hand side converges to $(\operatorname{grad} D)_z$ for almost every $z \in \mathring{X}$. This proves (2.22).

Proof of (iv). Let z_0 be a point of X as described in item (iv) of the lemma. Then $z \mapsto g_z$ admits a modulus of continuity ω at z_0, i.e. for all $z \in X$,

$$\|g_z - g_{z_0}\|_{\mathrm{Hom}} \le \omega(\|z - z_0\|),$$

and $\lim_{\varepsilon\to 0}\omega(\varepsilon) = 0$. Let $u \in \mathbb{S}$, $h \neq 0$. Let $z_0 \overset{\gamma}{\underset{X}{\rightsquigarrow}} z_0 + hu$ be a Lipschitz path such that $\|\gamma\| \le \frac{b|h|}{a}$. In particular, for all $0 \le T \le T_\gamma$,

$$\|\gamma(T) - z_0\| \le \frac{b|h|}{a}.$$

Consequently,

$$\begin{aligned}\left| \int_0^{T_\gamma} g_{\gamma(t)}(\gamma'(t))\mathrm{d}t - \int_0^{T_\gamma} g_{z_0}(\gamma'(t))\mathrm{d}t \right| &\le \int_0^{T_\gamma} \Big| g_{\gamma(t)}(\gamma'(t)) - g_{z_0}(\gamma'(t)) \Big| \mathrm{d}t \\ &\le \omega\Big(\frac{b|h|}{a}\Big) \int_0^{T_\gamma} \|\gamma'(t)\|\mathrm{d}t \\ &\le \omega\Big(\frac{b|h|}{a}\Big) \cdot \frac{b|h|}{a}.\end{aligned}$$

Hence

$$\left| D(z_0, z_0+hu) - \inf\left\{ \int_0^{T_\gamma} g_{z_0}(\gamma'(t))\mathrm{d}t \;\middle|\; z_0 \overset{\gamma}{\underset{X}{\rightsquigarrow}} z_0+hu, \gamma \text{ Lipschitz} \right\} \right| \leq \omega\Big(\frac{b|h|}{a}\Big) \cdot \frac{b|h|}{a}. \tag{2.24}$$

Besides, g_{z_0} is a norm so Jensen's inequality yields, for all Lipschitz paths γ,

$$\begin{aligned}\int_0^{T_\gamma} g_{z_0}(\gamma'(t))\mathrm{d}t &= T_\gamma \cdot \frac{1}{T_\gamma}\int_0^{T_\gamma} g_{z_0}(\gamma'(t))\mathrm{d}t \\ &\geq T_\gamma \cdot g_{z_0}\bigg(\frac{1}{T_\gamma}\int_0^{T_\gamma}\gamma'(t)\mathrm{d}t\bigg).\end{aligned}$$

A final application of Lemma 2.6 yields

$$\int_0^{T_\gamma} g_{z_0}(\gamma'(t))\mathrm{d}t \geq T_\gamma g_{z_0}\bigg(\frac{hu}{T_\gamma}\bigg) = |h| g_{z_0}(u). \tag{2.25}$$

Moreover any affine path from z_0 to z_0+hu is an equality case in Equation (2.25). Thus Equation (2.24) may be rewritten

$$\left| \frac{D(z_0, z_0+hu)}{|h|} - g_{z_0}(u) \right| \leq \omega\Big(\frac{b|h|}{a}\Big) \cdot \frac{b}{a}, \tag{2.26}$$

and letting $h \to 0$ concludes the proof. □

Remark 2.12. Even in the case where g_z is a norm for every $z \in X$, equality in (2.20) needs not to occur on a positive measure subset: take $X \coloneqq [0\,,1]^d$ and $g_z \coloneqq a\|\cdot\|$ if at least one coordinate of z is rational, $b\|\cdot\|$ otherwise. In this example $D = a\|\cdot\|$ but $g_z = b\|\cdot\|$ for almost every $z \in X$.

Lemma 2.13 (Scaling a metric)**.** *Let $D \in \mathcal{D}_X$ and $\lambda > 0$. Consider the metric on λX defined for all $x, y \in \lambda X$ by*

$$\mathrm{Sc}_\lambda(D)(x,y) \coloneqq \lambda D\Big(\frac{x}{\lambda}, \frac{y}{\lambda}\Big). \tag{2.27}$$

Then $\mathrm{Sc}_\lambda(D) \in \mathcal{D}_{\lambda X}$ and for all $z \in \lambda X$,

$$(\operatorname{grad} \mathrm{Sc}_\lambda(D))_z = (\operatorname{grad} D)_{\frac{z}{\lambda}}. \tag{2.28}$$

Proof. The metric $\mathrm{Sc}_\lambda(D)$ clearly satisfies (1.16). To show that it is a geodesic metric, consider $x, y \in \lambda X$. Let $\frac{x}{\lambda} \overset{\sigma}{\underset{X}{\rightsquigarrow}} \frac{y}{\lambda}$ a D-geodesic in the sense of Definition 1.3. Define

$$\begin{aligned}\tilde{\sigma} : [0\,, \mathrm{Sc}_\lambda(D)(x,y)] &\longrightarrow \lambda X \\ t &\longmapsto \lambda\sigma\Big(\frac{t}{\lambda}\Big).\end{aligned}$$

Then for all $0 \leq s \leq t \leq \mathrm{Sc}_\lambda(D)(x,y)$,

$$\begin{aligned}\mathrm{Sc}_\lambda(D)(\tilde{\sigma}(s), \tilde{\sigma}(t)) &= \lambda D\Big(\frac{\tilde{\sigma}(s)}{\lambda}, \frac{\tilde{\sigma}(t)}{\lambda}\Big) \\ &= \lambda D\Big(\sigma\Big(\frac{s}{\lambda}\Big), \sigma\Big(\frac{t}{\lambda}\Big)\Big).\end{aligned}$$

Since σ is an isometry from $\big[0\,, D(\frac{x}{\lambda}, \frac{y}{\lambda})\big]$ to (X, D),

$$\mathrm{Sc}_\lambda(D)(\tilde\sigma(s), \tilde\sigma(t)) = t - s,$$

thus $\tilde\sigma$ is an isometry from $[0\,, \mathrm{Sc}_\lambda(D)(x, y)]$ to $(\lambda X, \mathrm{Sc}_\lambda(D))$. Moreover, $\tilde\sigma(0) = x$ and $\tilde\sigma(\mathrm{Sc}_\lambda(D)(x, y)) = y$. In other words, $\tilde\sigma$ is a $\mathrm{Sc}_\lambda(D)$-geodesic from x to y, thus $\mathrm{Sc}_\lambda(D) \in \mathcal{D}_{\lambda X}$.

The equality (2.28) comes from the fact that for all $z \in \mathbb{R}^d, u \in \mathbb{S}$ and $h > 0$,

$$\frac{\overline{\mathrm{Sc}_\lambda(D)}(z, z + hu)}{h} = \frac{\overline{D}\Big(\frac{z}{\lambda}, \frac{z}{\lambda} + \frac{hu}{\lambda}\Big)}{\frac{h}{\lambda}}. \tag{2.29}$$

□

Lemma 2.14 (Translating a metric)**.** *Let $X \in \mathcal{K}$, $D \in \mathcal{D}_X$ and $z_0 \in X$. Consider the metric on $X + z_0$ defined for all $x, y \in X + z_0$ as*

$$\mathrm{Tr}_{z_0}(D)(x, y) \coloneqq D(x - z_0, y - z_0). \tag{2.30}$$

Then $\mathrm{Tr}_{z_0}(D) \in \mathcal{D}_{X+z_0}$ and for all $z \in X + z_0$,

$$(\mathrm{grad}\, \mathrm{Tr}_{z_0}(D))_z = (\mathrm{grad}\, D)_{z-z_0}. \tag{2.31}$$

Proof. The metric $\mathrm{Tr}_{z_0}(D)$ clearly satisfies (1.16). Moreover the image of a D-geodesic under $x \mapsto x+z_0$ is a $\mathrm{Tr}_{z_0}(D)$-geodesic, so $\mathrm{Tr}_{z_0}(D) \in \mathcal{D}_{X+z_0}$. The equality (2.31) comes from the fact that for all $z \in \mathbb{R}^d, u \in \mathbb{S}$ and $h > 0$,

$$\frac{\overline{\mathrm{Tr}_{z_0}(D)}(z, z + hu)}{h} = \frac{\overline{D}(z - z_0, z - z_0 + hu)}{h}. \tag{2.32}$$

□

Lemma 2.15 (Stitching metrics)**.** *Let $(X_v)_{v\in V}$ be a finite family of subsets in $\mathcal{K}$, each included in $X \in \mathcal{K}$. Let $(D_v)_{v\in V}$ be a family of metrics such that for all v, $D_v \in \mathcal{D}_{X_v}$. Consider the metric $D \in \mathcal{D}_X$ defined on X^2 by* (2.19)*, with*

$$g_z(u) \coloneqq \left(\min_{\substack{v\in V:\\ z\in X_v}} (\mathrm{grad}\, D_v)_z(u)\right) \vee (b\|u\|).$$

Then for all $z \in X_v \cap \Big(\bigcup_{v'\neq v} X_{v'}\Big)^{\mathrm{c}}$,

$$(\mathrm{grad}\, D)_z = (\mathrm{grad}\, D_v)_z. \tag{2.33}$$

Besides, for all $z \in X \cap (\bigcup_{v\in V} X_v)^{\mathrm{c}}$,

$$(\mathrm{grad}\, D)_z = b\|\cdot\|. \tag{2.34}$$

Proof. Recall that the infimum in Equation (2.19) can be restricted to paths included in $\overline{\mathrm{B}}\Big(x, \frac{b}{a}\|x - y\|\Big)$. In particular, by (2.12), for all $v \in V$ and $z \in X_v \cap \Big(\bigcup_{v'\neq v} X_{v'}\Big)^{\mathrm{c}}$, $\overline{D}$ and $\overline{D_v}$ coincide on a neighbourhood of z, hence (2.33). We prove (2.34) similarly. □

Lemma 2.16 (Restricting a metric)**.** *Let $Y, X \in \mathcal{K}$ be such that $Y \subseteq X$, and $D \in \mathcal{D}_X$. Define*

$$D_{|Y}(x,y) \coloneqq \inf\left\{ D(\gamma) \;\middle|\; x \overset{\gamma}{\underset{Y}{\rightsquigarrow}} y, \gamma \text{ Lipschitz} \right\}. \tag{2.35}$$

Then $D_{|Y} \in \mathcal{D}_Y$, and it is the minimum of the set of metrics $D' \in \mathcal{D}_Y$ such that for all $x, y \in Y$, $D'(x,y) \geq D(x,y)$. Moreover the gradients of D and $D_{|Y}$ coincide on $\mathring{Y}$. Finally, we have

$$\mathbf{T}_{X\,|Y} = \mathbf{T}_Y. \tag{2.36}$$

Proof. Indeed, $D_{|Y}$ is a metric satisfying (1.16), and a straightforward application of Lemma 2.2 yields $D_{|Y} \in \mathcal{D}_Y$.

Let $D' \in \mathcal{D}_Y$ be such that for all $x, y \in Y$, $D'(x,y) \geq D(x,y)$. Then for all Lipschitz paths γ included in Y, $D'(\gamma) \geq D(\gamma)$. Therefore (1.42) applied for the metric D' implies $D' \geq D_{|Y}$.

Let $z \in \mathring{Y}$. Lemma 2.3 implies the existence of a neighbourhood V of z such that for all $x \in X$, the D-geodesics from z to x are included in Y, therefore $D_{|Y}(z,x) = D(z,x)$, hence the equality of gradients.

Equation (2.36) remains to be shown. The inequality

$$\mathbf{T}_{X\,|Y} \leq \mathbf{T}_Y \tag{2.37}$$

is clear. Let us prove the converse inequality. Let σ be a $\mathbf{T}_{X\,|Y}$-geodesic. It suffices to show that

$$\mathbf{T}_Y(\sigma(0), \sigma(T_\sigma)) \leq \mathbf{T}_X(\sigma), \tag{2.38}$$

since (2.38) implies

$$\mathbf{T}_Y(\sigma(0), \sigma(T_\sigma)) \leq \mathbf{T}_{X\,|Y}(\sigma) = \mathbf{T}_{X\,|Y}(\sigma(0), \sigma(T_\sigma)).$$

By Lemma 2.3, for all $x, y \in \sigma$,

$$\mathbf{T}_X(x,y) = \mathbf{T}_{Y + \overline{\mathrm{B}}\left(0, \frac{b}{a}\|x-y\|\right)}(x,y).$$

In particular, for all $\varepsilon > 0$,

$$\mathbf{T}_X(\sigma) = \mathbf{T}_{Y+\overline{\mathrm{B}}(0,\varepsilon)}(\sigma) \geq \mathbf{T}_{Y+\overline{\mathrm{B}}(0,\varepsilon)}(\sigma(0), \sigma(T_\sigma)). \tag{2.39}$$

Besides, by continuity from above for the 1-dimensional Hausdorff measure $\mathcal{H}^1$,

$$\lim_{\varepsilon \to 0} \mathcal{H}^1\Big(\mathbb{E}^d \cap \Big((Y + \overline{\mathrm{B}}(0,\varepsilon)) \setminus Y\Big)\Big) = 0. \tag{2.40}$$

Let $\varepsilon > 0$ and $\pi = (x_0, \ldots, x_r)$ be a polygonal path from $\sigma(0)$ to $\sigma(T_\sigma)$ included in $Y + \overline{\mathrm{B}}(0,\varepsilon)$. We lower bound $\tau(\pi)$. Since for all $0 \leq i \leq r-1$ and $z \in [x_i\,, x_{i+1}]$,

$$\tau(x_i, z) + \tau(z, x_{i+1}) \leq \tau(x_i, x_{i+1}),$$

we may assume that π is self-avoiding and for all $0 \leq i \leq r-1$, $]x_i\,, x_{i+1}[$ is included in either Y or $(Y + \overline{\mathrm{B}}(0,\varepsilon)) \setminus Y$. Let $\hat{\pi}$ denote the path obtained from π by deleting points outside Y, i.e. informally by replacing excursions outside Y by segments. Then

$$\tau(\hat{\pi}) \leq \tau(\pi) + b\mathcal{H}^1\Big(\mathbb{E}^d \cap \Big((Y + \overline{\mathrm{B}}(0,\varepsilon)) \setminus Y\Big)\Big),$$

thus

$$\mathbf{T}_Y(\sigma(0), \sigma(T_\sigma)) \leq \mathbf{T}_{Y+\overline{\mathrm{B}}(0,\varepsilon)}(\sigma(0), \sigma(T_\sigma)) + b\mathcal{H}^1\Big(\mathbb{E}^d \cap \Big((Y + \overline{\mathrm{B}}(0,\varepsilon)) \setminus Y\Big)\Big).$$

Applying (2.40) gives

$$\mathbf{T}_Y(\sigma(0), \sigma(T_\sigma)) = \lim_{\varepsilon \to 0} \mathbf{T}_{Y+\overline{\mathrm{B}}(0,\varepsilon)}(\sigma(0), \sigma(T_\sigma)). \tag{2.41}$$

Combining (2.39) and (2.41) yields (2.38). □

2.6 The corridor lemma

Lemma 2.17 will be our main tool to get lower bounds for $\widetilde{\mathbf{T}}_{n,X}$. It is an adaptation of ideas used in the proof of Lemma 4.1 in Basu, Ganguly and Sly [3].

Lemma 2.17. *Let $X, X_1, \dots, X_K \in \mathcal{K}$ be such that for all $1 \le k \le K$, $X_k \subseteq X$. Let $D, D' \in \mathcal{D}_X$ and $X_0 \coloneqq X \setminus \bigcup_{k=1}^K X_k$. Let $0 < \delta_1 \le \operatorname{diam}(X)$ and $\delta_2 > 0$. Let $0 < \varepsilon < \frac{b}{2}$. Assume*

$$\forall 1 \le k_1 < k_2 \le K, \quad \inf_{\substack{x \in X_{k_1} \\ y \in X_{k_2}}} \|x - y\| \ge \delta_1, \tag{2.42}$$

$$\forall k \ge 1, \quad \forall x, y \in X_k, \quad D'|_{X_k}(x,y) \ge D(x,y) - \delta_2, \tag{2.43}$$

$$\forall \gamma :]0\,, T[\to X_0 \quad \textit{Lipschitz}, \quad D'(\gamma) \ge (b - \varepsilon)\|\gamma\|. \tag{2.44}$$

Then for all $x, y \in X$,

$$D'(x,y) \ge D(x,y) - 3\operatorname{diam}(X)\left(\varepsilon + \frac{\delta_2}{\delta_1}\right). \tag{2.45}$$

Proof. Let $x, y \in X$ and $x \overset{\sigma}{\rightsquigarrow} y$ be a D'-geodesic. The naive lower bound on $D'(x,y)$ obtained by decomposing σ in subpaths included in only one X_k, $0 \le k \le K$, and plugging in (2.43) or (2.44) to control each subpath is useless. Indeed, each application of (2.43) results in an additive error δ_2, but we have no bound on the number of subpaths as σ may for example go back and forth between X_0 and X_1 a large number of times. However a simple way to circumvent this obstacle is to replace σ with a slightly longer path that does not have such a pathological behaviour using (2.42) and (2.44).

Step 1: Regularizing σ. We call *excursion* any interval $]s\,, s'[$ such that $0 \le s < s' \le D'(x,y)$ and

$$\sigma(s), \sigma(s') \in \bigcup_{k=1}^K X_k, \quad \text{and} \quad \sigma\big(]s\,, s'[\big) \subseteq X_0.$$

We will denote by $\operatorname{Ex}(\sigma)$ the set of all excursions. Note that excursions are pairwise disjoint. We class them into two types, whether their endpoints belong to the same X_k or not:

$$\operatorname{Ex}_1(\sigma) \coloneqq \left\{]s\,, s'[\in \operatorname{Ex}(\sigma) \;\middle|\; \exists k \in [\![1\,, K]\!], \quad \sigma(s), \sigma(s') \in X_k \right\}$$
$$\operatorname{Ex}_2(\sigma) \coloneqq \operatorname{Ex}(\sigma) \setminus \operatorname{Ex}_1(\sigma).$$

Let us define the Lipschitz path

$$\begin{aligned} \gamma : [0\,, D'(x,y)] &\longrightarrow X \\ t &\longmapsto \begin{cases} \sigma(s) + \frac{t-s}{s'-s}(\sigma(s') - \sigma(s)) & \text{if } s < t < s' \text{ with }]s\,, s'[\in \operatorname{Ex}_1(\sigma), \\ \sigma(t) & \text{otherwise.} \end{cases} \end{aligned} \tag{2.46}$$

In other words, γ is the path obtained from σ by replacing first-type excursions by segments (see Figure 2, left).

The path γ satisfies

$$D'(\gamma) \le \frac{b}{b - \varepsilon} D'(x,y). \tag{2.47}$$

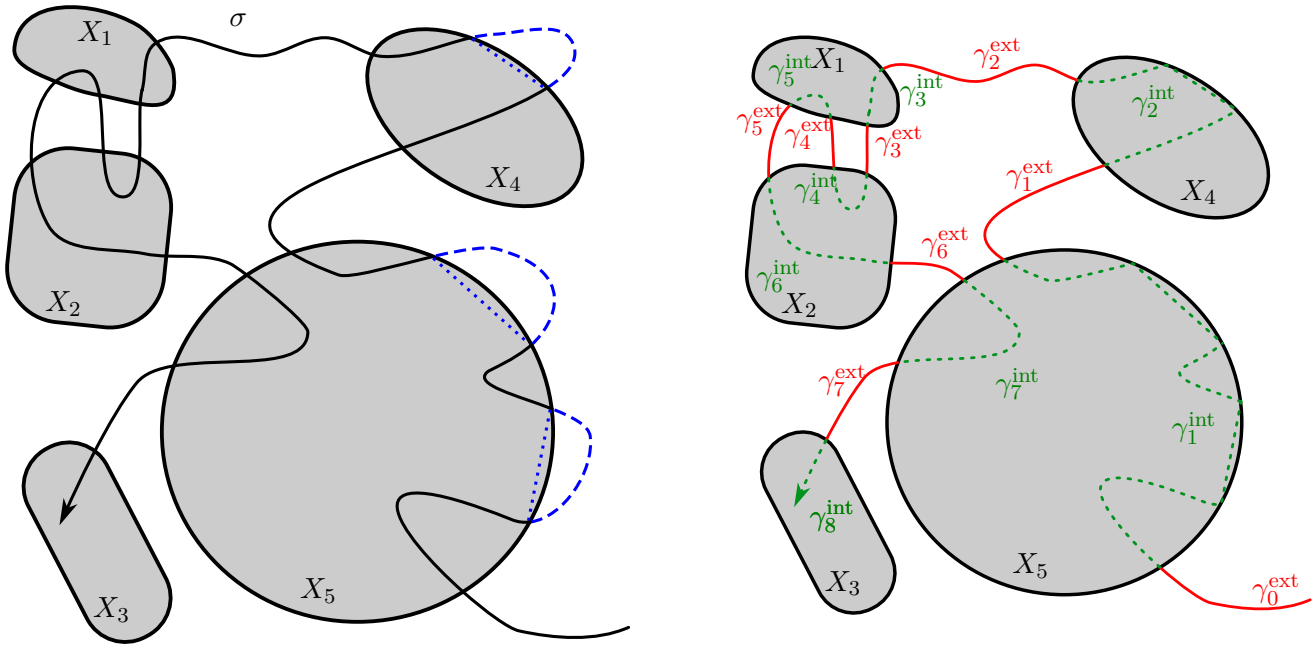


Figure 2: Left: in this example, σ has three first-type excursions (blue, dashed lines). The path γ is obtained by replacing σ by an affine path on each first-type excursion (blue, dotted lines). Right: The path γ contains only second-type excursions (red, solid lines) and subpaths included in a X_k for $k \in [\![1\,, K]\!]$ (green, dashed lines).

Indeed, consider a subdivision $(0 = t_0 < t_1 < \cdots < t_N = D'(x,y))$. It is sufficient to show that

$$\sum_{n=0}^{N-1} D'(\gamma(t_n), \gamma(t_{n+1})) \leq \frac{b}{b-\varepsilon} D'(x,y). \tag{2.48}$$

Without loss of generality, one may assume that $(t_k)_{k=0}^K$ can be written

$$\begin{aligned}\Big(&u_{0,0} < \cdots < u_{0,i_0} = s_1 = t_{1,0} < t_{1,1} < \cdots < t_{1,j_1} = s_1' \\ &= u_{1,0} < \cdots < u_{1,i_1} = s_2 < \cdots < s_L' \\ &= u_{L,0} < \cdots < u_{L,i_L}\Big),\end{aligned}$$

where the $u_{\ell,i}$ don't belong to any first-type excursion and for all $1 \leq \ell \leq L$, $]s_\ell\,, s_\ell'[$ is a first-type excursion. For all $0 \leq \ell \leq L, 0 \leq i \leq i_\ell - 1$,

$$D'(\gamma(u_{\ell,i}), \gamma(u_{\ell,i+1})) = D'(\sigma(u_{\ell,i}), \sigma(u_{\ell,i+1})). \tag{2.49}$$

Besides, for all $1 \leq \ell \leq L$, $\gamma_{|[s_\ell\,, s_\ell']}$ is an affine path, therefore (1.16) and (2.44) imply

$$\begin{aligned}\sum_{j=0}^{j_\ell - 1} D'(\gamma(t_{\ell,j}), \gamma(t_{\ell,j+1})) &\leq b\|\gamma(s_\ell) - \gamma(s_\ell')\| \\ &\leq \frac{b}{b-\varepsilon} D'(\gamma(s_\ell), \gamma(s_\ell')).\end{aligned} \tag{2.50}$$

Equations (2.49) and (2.50) yield (2.47).

Moreover γ can be decomposed as

$$x = x_0' \overset{\gamma_0^{\mathrm{ext}}}{\rightsquigarrow} x_1 \overset{\gamma_1^{\mathrm{int}}}{\rightsquigarrow} x_1' \overset{\gamma_1^{\mathrm{ext}}}{\rightsquigarrow} x_2 \overset{\gamma_2^{\mathrm{int}}}{\rightsquigarrow} \dots \overset{\gamma_r^{\mathrm{ext}}}{\rightsquigarrow} x_{r+1} = y,$$

where each path γ_i^{int} is included in one of the subsets X_k, whereas the paths γ_i^{ext} do not intersect any of the X_k except possibly at their endpoints, and for all $1 \le i \le r-1$, x_i' and x_{i+1} belong to different X_k (see Figure 2, right). The paths γ_0^{ext} and γ_r^{ext} may be empty.

Besides, (2.42) and (2.44) imply that for all $i \in [\![1\,, r-1]\!]$,

$$(b-\varepsilon)\delta_1 \le (b-\varepsilon)\Big\|\gamma_i^{\mathrm{ext}}\Big\| \le D'\Big(\gamma_i^{\mathrm{ext}}\Big).$$

Summing over i and applying (1.16), we get

$$(r-1)(b-\varepsilon)\delta_1 \le \sum_{i=1}^{r-1} D'\Big(\gamma_i^{\mathrm{ext}}\Big) \le D'(\sigma) \le b\|x-y\| \le b\,\mathrm{diam}(X).$$

Consequently,

$$r \le \frac{b\,\mathrm{diam}(X)}{(b-\varepsilon)\delta_1} + 1 \le \frac{3\,\mathrm{diam}(X)}{\delta_1}. \tag{2.51}$$

Step 2: Lower bounding $D'(\gamma)$. Hypothesis (2.43) and Inequality (2.51) yield

$$\begin{aligned}\sum_{i=1}^{r} D'\Big(\gamma_i^{\mathrm{int}}\Big) &\ge \sum_{i=1}^{r} D(x_i, x_{i+1}') - r\delta_2 \\ &\ge \sum_{i=1}^{r} D(x_i, x_{i+1}') - \frac{3\,\mathrm{diam}(X)\delta_2}{\delta_1}. \end{aligned}\tag{2.52}$$

Besides, (2.44) and (1.16) yield

$$\begin{aligned}\sum_{i=0}^{r} D'\Big(\gamma_i^{\mathrm{ext}}\Big) &\ge \sum_{i=0}^{r} (b-\varepsilon)\|x_i' - x_{i+1}\| \\ &\ge \frac{b-\varepsilon}{b}\sum_{i=0}^{r} D(x_i', x_{i+1}). \end{aligned}\tag{2.53}$$

Inequalities (2.52) and (2.53) imply

$$D'(\gamma) \ge \frac{b-\varepsilon}{b}\left(\sum_{i=1}^{r} D(x_i, x_{i+1}') + \sum_{i=0}^{r} D(x_i', x_{i+1})\right) - \frac{3\,\mathrm{diam}(X)\delta_2}{\delta_1}.$$

Thus, by triangle inequality,

$$D'(\gamma) \ge \frac{b-\varepsilon}{b} D(x,y) - \frac{3\,\mathrm{diam}(X)\delta_2}{\delta_1}.$$

Therefore, by (2.47),

$$\begin{aligned} D'(x,y) &\ge \frac{(b-\varepsilon)^2}{b^2} D(x,y) - \frac{3(b-\varepsilon)\,\mathrm{diam}(X)\delta_2}{b\delta_1}, \\ &= D(x,y) + \frac{-2\varepsilon b + \varepsilon^2}{b^2} D(x,y) - \frac{3(b-\varepsilon)\,\mathrm{diam}(X)\delta_2}{b\delta_1}. \\ &\ge D(x,y) - \frac{2\varepsilon}{b} D(x,y) - \frac{3\,\mathrm{diam}(X)\delta_2}{\delta_1}. \end{aligned}\tag{2.54}$$

It follows by (1.16) that

$$D'(x,y) \geq D(x,y) - 2\varepsilon \operatorname{diam}(X) - \frac{3\operatorname{diam}(X)\delta_2}{\delta_1}, \tag{2.55}$$

which concludes the proof. □

3 Monotonicity of the lower and upper rate functions

In this section $X \in \mathcal{K}$ is fixed and ν satisfies Assumption 1. The main result of the section is Proposition 3.1, which states the monotonicity of $\underline{I}_X$ and $\overline{I}_X$ (see (1.29) and (1.30)). It is proven in Subsections 3.1 and 3.2.

Proposition 3.1. *The maps $\underline{I}_X$ and $\overline{I}_X$ are nondecreasing on $\mathcal{D}_X$.*

A useful consequence of Proposition 3.1 is that $\mathrm{LD}_n(D,\varepsilon)$ may be replaced in (1.29) by

$$\mathrm{LD}^+_{n,X}(D,\varepsilon) \coloneqq \Big\{\forall x,y \in X, \quad \widetilde{\mathbf{T}}_{n,X}(x,y) \geq D(x,y) - \varepsilon\Big\}. \tag{3.1}$$

Corollary 3.2. *For all $D \in \mathcal{D}_X$,*

$$\lim_{\varepsilon\to 0}\uparrow \varliminf_{n\to\infty} -\frac{1}{n^d}\log \mathbb{P}\Big(\mathrm{LD}^+_{n,X}(D,\varepsilon)\Big) = \underline{I}_X(D). \tag{3.2}$$

Proof of Corollary 3.2. Let $D \in \mathcal{D}_X$. For all $\varepsilon > 0$ the inclusion $\mathrm{LD}_n(D,\varepsilon) \subseteq \mathrm{LD}^+_n(D,\varepsilon)$ is clear, thus

$$\lim_{\varepsilon\to 0}\uparrow \varliminf_{n\to\infty} -\frac{1}{n^d}\log \mathbb{P}\Big(\mathrm{LD}^+_{n,X}(D,\varepsilon)\Big) \leq \underline{I}_X(D). \tag{3.3}$$

Consider the set $K_\varepsilon \coloneqq \{D' \in \mathcal{D}_X \mid \forall x,y \in X, D'(x,y) \geq D(x,y) - \varepsilon\}$. As K_ε is compact, Lemma 1.2 implies

$$\begin{aligned}\varliminf_{n\to\infty} -\frac{1}{n^d}\log \mathbb{P}\Big(\mathrm{LD}^+_{n,X}(D,\varepsilon)\Big) &= \varliminf_{n\to\infty} -\frac{1}{n^d}\log \mathbb{P}\Big(\widetilde{\mathbf{T}}_{n,X} \in K_\varepsilon\Big)\\ &\geq \min_{D'\in K_\varepsilon} \underline{I}_X(D').\end{aligned}$$

For all $\varepsilon > 0$, let D'_ε be a minimizer of $\underline{I}_X$ on K_ε. By compactness, there exists a decreasing sequence (ε_k), converging to 0, such that D'_{ε_k} converges to a metric D'_0 in $\mathcal{D}_X$. Since $\underline{I}_X$ is lower semicontinuous, letting $k \to \infty$ yields

$$\begin{aligned}\lim_{\varepsilon\to 0}\uparrow \varliminf_{n\to\infty} -\frac{1}{n^d}\log \mathbb{P}\Big(\mathrm{LD}^+_{n,X}(D,\varepsilon)\Big) &\geq \varliminf_{k\to\infty} \underline{I}_X\Big(D'_{\varepsilon_k}\Big)\\ &\geq \underline{I}_X(D'_0)\\ &\geq \min_{D'\geq D} \underline{I}_X(D').\end{aligned}$$

Applying Proposition 3.1 to simplify the right-hand side, we get

$$\lim_{\varepsilon\to 0}\uparrow \varliminf_{n\to\infty} -\frac{1}{n^d}\log \mathbb{P}\Big(\mathrm{LD}^+_{n,X}(D,\varepsilon)\Big) \geq \underline{I}_X(D). \tag{3.4}$$

□

3.1 The highway method

Let $D, D_0 \in \mathcal{D}_X$ be such that $D_0 \geq D$. Let $((x_p, y_p))_{p\geq 1}$ be a dense sequence in X^2 and, for all $p \geq 1$, a D-geodesic $x_p \overset{\sigma_p}{\rightsquigarrow} y_p$. We consider the sequence of maps $D_p : X^2 \to \mathbb{R}^+$ defined recursively by the relation

$$D_{p+1}(x, y) \coloneqq \min_{x',y'\in\sigma_{p+1}} (D_p(x, x') + D(x', y') + D_p(y', y)) \wedge D_p(x, y). \tag{3.5}$$

Informally, D_p is the metric obtained from D_0 by adding the network of highways $\{\sigma_1, \ldots, \sigma_p\}$, thus reducing the passage time. The key arguments in the proof of Proposition 3.1 are:

1. The limiting metric for a large number of highways is D (Lemma 3.3).
2. The cost of adding one highway is negligible (Lemma 3.4).

Lemma 3.3. *For all $p \geq 0$, $D_p \in \mathcal{D}_X$. Moreover the sequence $(D_p)_{p\geq 0}$ is nonincreasing and converges to D.*

Proof. It is straightforward to see that for all $p \geq 0$, $b\|\cdot\| \geq D_p \geq D_{p+1} \geq D$.

We show by induction that for all $p \geq 0$, $D_p \in \mathcal{D}_X$. Let $p \geq 0$. Assume that $D_p \in \mathcal{D}_X$. Let $x, y, z \in X$. The hardest case to consider when proving the triangle inequality for D_{p+1} between x, y and z is when there exists $x', y', y'', z'' \in \sigma_{p+1}$ such that

$$D_{p+1}(x, y) = D_p(x, x') + D(x', y') + D_p(y', y) \text{ and } D_{p+1}(y, z) = D_p(y, y'') + D(y'', z'') + D_p(z'', z). \tag{3.6}$$

Thanks to the definition of D_{p+1}, the triangle inequality for D and $D_p \geq D$,

$$\begin{aligned} D_{p+1}(x, z) &\leq D_p(x, x') + D(x', z') + D_p(z', z) \\ &\leq D_p(x, x') + D(x', y') + D(y', y) + D(y, y'') + D(y'', z') + D_p(z', z) \\ &\leq D_p(x, x') + D(x', y') + D_p(y', y) + D_p(y, y'') + D(y'', z') + D_p(z', z) \\ &= D_{p+1}(x, y) + D_{p+1}(y, z). \end{aligned}$$

The other cases are analogous. Hence D_{p+1} is a metric. Moreover a direct application of Lemma 2.2 shows that $D_{p+1} \in \mathcal{D}_X$.

As $\mathcal{D}_X$ is compact (see Proposition 2.1), to prove the convergence, it is sufficient to show that D is the unique adherence value of $(D_p)_{p\geq 0}$. For all $k \geq 1$ and $p \geq k$, $D_p(x_k, y_k) = D(x_k, y_k)$, thus any adherence value of $(D_p)_{p\geq 0}$ must coincide with D on a dense subset of X^2, therefore must be equal to D by continuity. □

Lemma 3.4. *There exists a constant C_1, depending only on a, b and X such that for all $p \geq 0, 0 < \delta \leq 1$, for large enough n,*

$$-\frac{1}{n^d} \log \mathbb{P}(\mathrm{LD}_{n,X}(D_{p+1}, C_1\delta)) \leq -\frac{1}{n^d} \log \mathbb{P}\Big(\mathrm{LD}_{n,X}(D_p, \delta^2)\Big) - \frac{C_1}{n^{d-1}} \log\Big(\frac{\nu([a\,, a+\delta])}{2}\Big). \tag{3.7}$$

We prove this lemma in Subsection 3.2.

Proof of Proposition 3.1. Let $p \geq 0$ and $0 < \delta \leq 1$. Taking the inferior limit in n in (3.7), we get

$$\varliminf_{n\to\infty} -\frac{1}{n^d} \log \mathbb{P}(\mathrm{LD}_{n,X}(D_{p+1}, C_1\delta)) \leq \varliminf_{n\to\infty} -\frac{1}{n^d} \log \mathbb{P}\Big(\mathrm{LD}_{n,X}(D_p, \delta^2)\Big).$$

Letting $\delta \to 0$ gives

$$\underline{I}_X(D_{p+1}) \leq \underline{I}_X(D_p). \tag{3.8}$$

Thanks to Lemma 3.3 and the lower semicontinuity of $\underline{I}_X$, we get

$$\underline{I}_X(D) = \underline{I}_X\Big(\lim_{p\to\infty} D_p\Big) \leq \varliminf_{p\to\infty} \underline{I}_X(D_p).$$

Applying (3.8), we obtain

$$\underline{I}_X(D) \leq \underline{I}_X(D_0). \tag{3.9}$$

We proceed analogously for $\overline{I}_X$. □

3.2 Proof of Lemma 3.4

Without loss of generality it is sufficient to treat the case $p = 0$. Let $0 < \delta < 1$. The idea is to build a configuration satisfying $\mathrm{LD}_{n,X}(D_1, C_1\delta)$ by modifying certain edge passage times in a realization of $\mathrm{LD}_{n,X}(D_0, \delta^2)$ in order to "create" the geodesic σ_1. We first prove a technical lemma which essentially states that an homothety of a geodesic can be well approximated by a discrete path with a linearly upper bounded number of edges. In the whole subsection, the sentence "for large enough $n, \mathcal{P}_n$" will mean "there exists $n_0 \geq 1$, depending on a, b, X and δ such that $\forall n \geq n_0, \mathcal{P}_n$".

Lemma 3.5. *For every $D \in \mathcal{D}_X$, every D-geodesic $x \overset{\sigma}{\rightsquigarrow} y$ and every $\lambda > 0$, there exists a discrete path $\alpha = (\alpha_j)_{j=0}^{p}$ and a surjective, nondecreasing, right-continuous map $\mathbf{j} : [0\,, D(x,y)] \to [\![0\,, p]\!]$ such that*

$$\forall 0 \leq t \leq D(x,y), \quad \Big\|\alpha_{\mathbf{j}(t)} - \lambda\sigma(t)\Big\| \leq d+1 \tag{3.10}$$

$$\forall 0 \leq t_1 < t_2 \leq D(x,y), \quad \mathbf{j}(t_2) - \mathbf{j}(t_1) \leq \lambda \int_{t_1}^{t_2} \|\sigma'(u)\| \mathrm{d}u + d. \tag{3.11}$$

Lemma 2.6 guarantees that the integral in (3.11) is well-defined. Note that α needs not to be included in λX. The inequality (3.11), along with (1.16), implies

$$\mathbf{j}(t_2) - \mathbf{j}(t_1) \leq \frac{\lambda}{a}(t_2 - t_1) + d. \tag{3.12}$$

In particular,

$$p \leq \frac{\lambda b}{a}\,\mathrm{diam}(X) + d. \tag{3.13}$$

Proof. For all $0 \leq t \leq D(x,y)$, we write $\sigma(t) \coloneqq (\sigma_1(t), \sigma_2(t), \ldots, \sigma_d(t))$. The idea of the approximation is to replace $\lambda\sigma_i(t)$ by the last visited integer. On instants t such that several coordinates of $\sigma(t)$ are integer, this approximation may have a jump between two non adjacent vertices of $\mathbb{Z}^d$. Since we aim to build a discrete path, the existence of such t causes a minor problem. Applying a small translation neutralizes this obstacle.

We claim that there exists $z \in \mathrm{B}(0,1)$ such that for all $0 \leq t \leq D(x,y)$,

$$\#\Big\{i \in [\![1\,, d]\!] \;\Big|\; \lambda\sigma_i(t) + z_i \in \mathbb{Z}\Big\} \leq 1. \tag{3.14}$$

Indeed, let $1 \le i_1 < i_2 \le d$ and $v \in \mathbb{Z}^2$. Consider

$$\begin{aligned} A(i_1,i_2,v) &\coloneqq \Big\{ z \in \mathrm{B}(0,1) \;\Big|\; \exists 0 \le t \le D(x,y), \quad \lambda\sigma_{i_1}(t) + z_{i_1} = v_1 \text{ and } \lambda\sigma_{i_2}(t) + z_{i_2} = v_2 \Big\} \\ &\subseteq \{ v - (\lambda\sigma_{i_1}(t), \lambda\sigma_{i_2}(t)), \quad 0 \le t \le D(x,y) \}. \end{aligned}$$

Besides, $t \mapsto (\lambda\sigma_{i_1}(t), \lambda\sigma_{i_2}(t))$ is Lipschitz therefore its image has Lebesgue measure zero (by Lipschitz property it may be covered by a family of K balls of radius $\mathrm{O}(1/K)$). Hence the set $\bigcup_{\substack{2 \le i_1 < i_2 \le d \\ v \in \mathbb{Z}^2}} A(i_1,i_2,v)$ of points $z \in \mathrm{B}(0,1)$ such that (3.14) fails has measure zero.

Consider the map

$$\begin{aligned} \hat\sigma : [0\,, D(x,y)] &\longrightarrow \mathbb{Z}^d \\ t &\longmapsto (\hat\sigma_1(t), \dots, \hat\sigma_d(t)), \end{aligned}$$

where

$$\hat\sigma_i(t) = \begin{cases} \lfloor \lambda\sigma_i(0) + z_i \rfloor & \text{if } (\lambda\sigma_i([0\,,t]) + z_i) \cap \mathbb{Z} = \emptyset, \\ \lambda\sigma_i(\sup\{ s \in [0\,,t] \mid \lambda\sigma_i(s) + z_i \in \mathbb{Z} \} e) + z_i & \text{otherwise.} \end{cases}$$

In other words, up to a translation, $\hat\sigma_i(t)$ is an approximation of $\lambda\sigma(t)$ by the last visited integer, except before the first visit. The map $\hat\sigma \coloneqq (\hat\sigma_1, \dots, \hat\sigma_d)$ is right-continuous. Moreover, (3.14) implies that its jumps have norm 1. Let p be the total number of jumps of $\hat\sigma$ and $\mathbf{j}(t)$ be the number of jumps before the instant t. It is straightforward to see that $\mathbf{j}$ is nondecreasing, right-continuous and surjective on $[\![0\,, p]\!]$. For all $j \in [\![0\,, p]\!]$, define

$$\alpha_j \coloneqq \hat\sigma(t_j),$$

where t_j is any element of $\mathbf{j}^{-1}(j)$ (e.g. the instant of the j^{th} jump). The inequality (3.10) is clear.

Let $1 \le i \le d$. For all pairs $u_1 < u_2$ of consecutive jump instants of $\hat\sigma_i$, Lemma 2.6 yields

$$\begin{aligned} 1 &= |\hat\sigma_i(u_1) - \hat\sigma_i(u_2)| \\ &= \lambda |\sigma_i(u_1) - \sigma_i(u_2)| \\ &= \lambda \left| \int_{u_1}^{u_2} \sigma_i'(u) \mathrm{d}u \right| \\ &\le \lambda \int_{u_1}^{u_2} |\sigma_i'(u)| \mathrm{d}u. \end{aligned}$$

Thus, for all $0 < t_1 < t_2 \le D(x,y)$, by denoting $t_1 \le u_1 < \dots u_r \le t_2$ the jump instants of $\hat\sigma_i$ in $[t_1\,, t_2]$, we have

$$\begin{aligned} \#\{ u \in [t_1\,, t_2] \mid \hat\sigma_i(u^-) \ne \hat\sigma_i(u) \} &= r - 1 + 1 \\ &\le \lambda \int_{u_1}^{u_r} |\sigma_i'(u)| \mathrm{d}u + 1 \\ &\le \lambda \int_{t_1}^{t_2} |\sigma_i'(u)| \mathrm{d}u + 1. \end{aligned}$$

Summing over i we obtain (3.11). □

Let $z_0 \in \mathring{X}$, $n \ge \frac{1}{\delta^3}$ and $(\alpha, \mathbf{j})$ the pair given by Lemma 3.5 with parameters $\sigma = \sigma_1$, $x = x_1$, $y = y_1$ and $\lambda = n(1-\delta^2)$. For all $0 \le j \le p$, define

$$\hat\alpha_j \coloneqq \alpha_j + \left\lfloor n\delta^2 z_0 \right\rfloor.$$

Since α is a discrete path, $\hat{\alpha}$ is also a discrete path. Lemma B.2 implies the existence a constant $C_2 > 0$, depending only on X and z_0, such that

$$\mathrm{d}\Big((1-\delta^2)X + \delta^2 z_0, \mathbb{R}^d \setminus X\Big) \geq C_2\delta^2,$$

therefore

$$\mathrm{d}\Big(n(1-\delta^2)X + n\delta^2 z_0, \mathbb{R}^d \setminus nX\Big) \geq C_2 n\delta^2.$$

In particular $\hat{\alpha} \subseteq nX$ for large enough n. Consider independent random variables τ^{B} and τ^{G} (B for Box and G for Geodesic) following the distribution $\nu^{\otimes \mathbb{E}^d}$, along with a i.i.d. family $(X_e)_{e\in \mathrm{E}_{\mathrm{path}}(\hat{\alpha})}$ of Bernoulli variables with parameter 1/2. We assume $(X_e)_{e\in \mathrm{E}_{\mathrm{path}}(\hat{\alpha})}$ to be independent of $(\tau^{\mathrm{B}}, \tau^{\mathrm{G}})$. For all $e \in \mathbb{E}^d$, we define

$$\tau_e \coloneqq \begin{cases} X_e\tau_e^{\mathrm{G}} + (1-X_e)\tau_e^{\mathrm{B}} & \text{if } e \in \mathrm{E}_{\mathrm{path}}(\hat{\alpha}), \\ \tau_e^{\mathrm{B}} & \text{otherwise.} \end{cases} \tag{3.15}$$

The family $\tau = (\tau_e)_{e\in\mathbb{E}^d}$ also follows the distribution $\nu^{\otimes\mathbb{E}^d}$. The core of the proof is Lemma 3.6.

Lemma 3.6. *There exists a constant C_3, depending only on a, b and X such that for all large enough n, there exists a random τ^{B}-measurable set $E_{\mathrm{fast}} \subseteq \mathrm{E}_{\mathrm{path}}(\hat{\alpha})$ such that*

$$\Big\{\tau^{\mathrm{B}} \in \mathrm{LD}_{n,X}(D_0,\delta^2)\Big\} \cap \left(\bigcap_{e\in\mathrm{E}_{\mathrm{path}}(\hat{\alpha})} \{X_e = \mathbb{1}_{E_{\mathrm{fast}}}(e)\}\right) \cap \left(\bigcap_{e\in\mathrm{E}_{\mathrm{path}}(\hat{\alpha})} \Big\{\tau_e^{\mathrm{G}} \leq a+\delta\Big\}\right) \subseteq \{\tau \in \mathrm{LD}_{n,X}(D_1, C_3\delta)\}. \tag{3.16}$$

Indeed, (3.13) yields

$$\begin{aligned} \#\mathrm{E}_{\mathrm{path}}(\hat{\alpha}) = p &\leq \frac{n(1-\delta^2)b}{a}\operatorname{diam}(X) + d \\ &\leq \frac{nb}{a}\operatorname{diam}(X) + d \\ &\leq C_4 n, \end{aligned} \tag{3.17}$$

where C_4 only depends on a, b and $\operatorname{diam}(X)$. Consequently,

$$\mathbb{P}\left(\bigcap_{e\in\mathrm{E}_{\mathrm{path}}(\hat{\alpha})} \Big\{\tau^{\mathrm{G}} \leq a+\delta\Big\}\right) \geq \nu([a\,, a+\delta])^{C_4 n}$$

$$\text{and a.s,} \quad \mathbb{P}\left(\bigcap_{e\in\mathrm{E}_{\mathrm{path}}(\hat{\alpha})} \{X_e = \mathbb{1}_{E_{\mathrm{fast}}}(e)\} \,\middle|\, \tau^{\mathrm{B}}\right) = \left(\frac{1}{2}\right)^{\#\mathrm{E}_{\mathrm{path}}(\hat{\alpha})} \geq \left(\frac{1}{2}\right)^{C_4 n}.$$

Thus taking probabilities on both sides of (3.16) leads to

$$\mathbb{P}\Big(\mathrm{LD}_{n,X}(D_0,\delta^2)\Big) \cdot \left(\frac{\nu([a\,, a+\delta])}{2}\right)^{C_4 n} \leq \mathbb{P}(\mathrm{LD}_{n,X}(D_1, C_3\delta)).$$

As $\mathrm{LD}_{n,X}(D_1,\varepsilon)$ is nondecreasing in ε, we have proven (3.7) with $C_1 \coloneqq C_3 \vee C_4$. This concludes the proof of Lemma 3.4. □

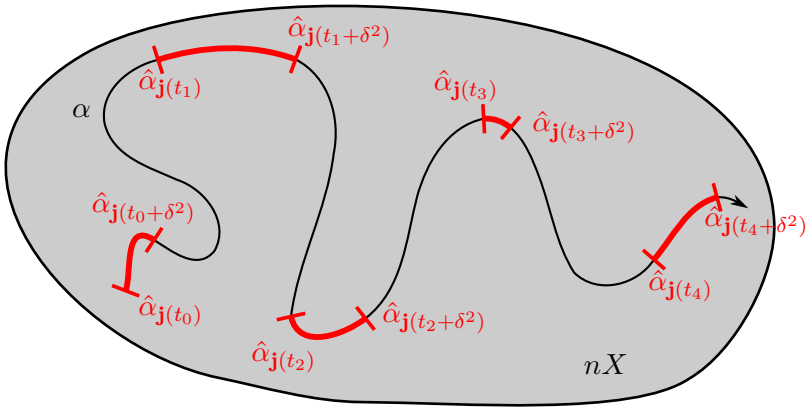


Figure 3: Edges in E_{fast} are represented by the thick red lines.

Proof of Lemma 3.6 . Let us consider a realization of the event $\left\{\tau^{\mathrm{B}} \in \mathrm{LD}_{n,X}(D_0, \delta^2)\right\}$. The general idea behind the construction of E_{fast} is to set the passage time of all edges along $\hat{\alpha}$ between $\hat{\alpha}_{\mathbf{j}(0)}$ to $\hat{\alpha}_{\mathbf{j}(\delta^2)}$ to a, choose $t_1 \geq \delta^2$ adequately, set the passage time of all edges along $\hat{\alpha}$ between $\hat{\alpha}_{\mathbf{j}(t_1)}$ to $\hat{\alpha}_{\mathbf{j}(t_1+\delta^2)}$ to a and so on. The t_k are chosen as small as possible, but such that the modified random metric never gets too far below the target time. We will denote by E_{fast} the set of modified edges (see Figure 3).

Preliminary step: $\hat{\alpha}$ well approaches σ_1. There exists a constant C_5, depending only on X and z_0, such that for large enough n, for all $0 \leq t \leq D(x,y)$,

$$\left\| \frac{\hat{\alpha}_{\mathbf{j}(t)}}{n} - \sigma_1(t) \right\| \leq C_5 \delta^2. \tag{3.18}$$

Indeed, for all $0 \leq t \leq D(x,y)$, triangle inequality and (3.10) yield

$$\begin{aligned}
\left\| \frac{\hat{\alpha}_{\mathbf{j}(t)}}{n} - \sigma_1(t) \right\| &= \frac{1}{n} \left\| \hat{\alpha}_{\mathbf{j}(t)} - n\sigma_1(t) \right\| \\
&\leq \frac{1}{n} \left[\left\| \hat{\alpha}_{\mathbf{j}(t)} - \alpha_{\mathbf{j}(t)} \right\| + \left\| \alpha_{\mathbf{j}(t)} - n(1-\delta^2)\sigma_1(t) \right\| + \left\| n(1-\delta^2)\sigma_1(t) - n\sigma_1(t) \right\| \right] \\
&\leq \frac{1}{n} \left[\left(n\delta^2 \|z_0\| + 1 \right) + (d+1) + n\delta^2 \|\sigma_1(t)\| \right].
\end{aligned}$$

Thus for large enough n, for all $0 \leq t \leq D(x,y)$,

$$\left\| \frac{\hat{\alpha}_{\mathbf{j}(t)}}{n} - \sigma_1(t) \right\| \leq (2\|z_0\| + \mathrm{diam}(X) + 1)\delta^2,$$

hence (3.18). We define

$$C_6 := 2bC_5 + 2 + \frac{b}{a}. \tag{3.19}$$

Step 1: Building E_{fast}. We claim that there exists a τ^{B}-measurable finite sequence $(0 = t_0 < t_1 < \cdots < t_K \le D(x,y))$ such that $D(x,y) - t_K \le C_6\delta$, and

$$\forall 1 \le k \le K, \qquad \delta^2 \le t_k - t_{k-1} \le C_6\delta, \tag{3.20}$$

$$\forall 1 \le k \le K, \quad \forall t \ge t_k, \qquad \widetilde{\mathbf{T}}^{(k)}_{n,X}\left(\frac{\hat\alpha_{\mathbf{j}(t_{k-1})}}{n}, \frac{\hat\alpha_{\mathbf{j}(t)}}{n}\right) \ge (t - t_{k-1})(1-\delta), \tag{3.21}$$

$$\forall 1 \le k \le K, \qquad \widetilde{\mathbf{T}}^{(k)}_{n,X}\left(\frac{\hat\alpha_{\mathbf{j}(t_{k-1})}}{n}, \frac{\hat\alpha_{\mathbf{j}(t_k)}}{n}\right) \le t_k - t_{k-1}, \tag{3.22}$$

where

$$\tau_e^{(k)} \coloneqq \begin{cases} a & \text{if } e \in \bigcup_{l=0}^{k-1} \mathrm{E}_{\mathrm{path}}\left(\hat\alpha_{|[\mathbf{j}(t_l)\,,\mathbf{j}(t_l+\delta^2)]}\right) \\ \tau_e^{\mathrm{B}} & \text{otherwise.} \end{cases} \tag{3.23}$$

Indeed let us assume that $t_0, \dots, t_q$ have already been defined and satisfy (3.20), (3.21) and (3.22) for all $1 \le k \le q$. Notice that if $q = 0$, we do not assume anything. If $D(x,y) - t_q \le C_6\delta$ then the construction is over with $K \coloneqq q$. Otherwise two cases need to be considered: either the latest modification has made the passage time far below the target time, and we need to skip an appropriate number of edges along $\hat\alpha$ before the next modification, or the passage time is not far below the target time, and we resume modifying the edge passage times without any skip.

Case 1: there exists $t \ge t_q + \delta^2$ such that

$$\widetilde{\mathbf{T}}^{(q+1)}_{n,X}\left(\frac{\hat\alpha_{\mathbf{j}(t_q)}}{n}, \frac{\hat\alpha_{\mathbf{j}(t)}}{n}\right) < (t - t_q)(1-\delta). \tag{3.24}$$

Let

$$t_{q+1} \coloneqq \sup\left\{ t \ge t_q + \delta^2 \;\middle|\; \widetilde{\mathbf{T}}^{(q+1)}_{n,X}\left(\frac{\hat\alpha_{\mathbf{j}(t_q)}}{n}, \frac{\hat\alpha_{\mathbf{j}(t)}}{n}\right) < (t - t_q)(1-\delta) \right\}.$$

We first show that (3.20) is true for $k = q+1$, i.e. $t_{q+1} \le t_q + C_6\delta < D(x,y)$. Indeed let $t > t_q + C_6\delta$. Consider a $\widetilde{\mathbf{T}}^{(q+1)}_{n,X}$-geodesic $\frac{\hat\alpha_{\mathbf{j}(t_q)}}{n} \overset{\gamma}{\rightsquigarrow} \frac{\hat\alpha_{\mathbf{j}(t)}}{n}$. Let z its last point belonging to $\frac{1}{n} \bigcup_{e \in \mathrm{E}_{\mathrm{path}}\left(\hat\alpha_{|[0\,,\mathbf{j}(t_q)]}\right)} e$, with the convention on edges given in Subsection 1.6 (see Figure 4). There exists $0 \le s \le t_q$ such that $z \in \left[\frac{\hat\alpha_{\mathbf{j}(s)}}{n}, \frac{\hat\alpha_{\mathbf{j}(s)+1}}{n}\right]$. Inequality (3.12) implies that for large enough n,

$$\mathbf{j}(t_q + \delta^2) - \mathbf{j}(t_q) \le \frac{n(1-\delta^2)\delta^2}{a} + d \le \frac{n\delta^2}{a}, \tag{3.25}$$

therefore the subpath of γ whose endpoints are z and $\frac{\hat\alpha_{\mathbf{j}(t)}}{n}$ use at most $\frac{n\delta^2}{a}$ edges e such that $\tau_e^{\mathrm{B}} \ne \tau_e^{(q+1)}$. Consequently, for large enough n,

$$\begin{aligned} \widetilde{\mathbf{T}}^{(q+1)}_{n,X}\left(\frac{\hat\alpha_{\mathbf{j}(t_q)}}{n}, \frac{\hat\alpha_{\mathbf{j}(t)}}{n}\right) &\ge \widetilde{\mathbf{T}}^{(q+1)}_{n,X}\left(z, \frac{\hat\alpha_{\mathbf{j}(t)}}{n}\right) \\ &\ge \widetilde{\mathbf{T}}^{\mathrm{B}}_{n,X}\left(z, \frac{\hat\alpha_{\mathbf{j}(t)}}{n}\right) - \frac{b\delta^2}{a} \\ &\ge \widetilde{\mathbf{T}}^{\mathrm{B}}_{n,X}\left(\frac{\hat\alpha_{\mathbf{j}(s)}}{n}, \frac{\hat\alpha_{\mathbf{j}(t)}}{n}\right) - \frac{b}{n} - \frac{b\delta^2}{a}. \end{aligned} \tag{3.26}$$

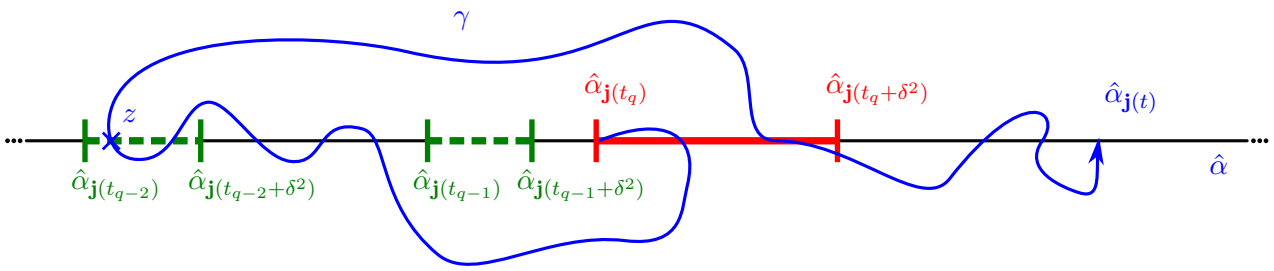


Figure 4: For the sake of simplicity, the set nX is not represented and $\hat{\alpha}$ is represented as a straight line. The newly modified edges are represented by the thick, red, solid line while the other modified edges are represented by the thick, green, dashed lines. Edges used by the portion of γ (blue curved line) lying between z and $\hat{\alpha}_{\mathbf{j}(t)}$ are either unmodified or newly modified.

Besides, since we work with a realization of $\left\{\tau^{\mathrm{B}} \in \mathrm{LD}_{n,X}(D_0, \delta^2)\right\}$ and $D_0 \geq D$,

$$\widetilde{\mathbf{T}}^{\mathrm{B}}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(s)}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t)}}{n}\right) \geq D\left(\frac{\hat{\alpha}_{\mathbf{j}(s)}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t)}}{n}\right) - \delta^2.$$

The triangle inequality then yields

$$\widetilde{\mathbf{T}}^{\mathrm{B}}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(s)}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t)}}{n}\right) \geq D(\sigma_1(s), \sigma_1(t)) - D\left(\sigma_1(s), \frac{\hat{\alpha}_{\mathbf{j}(s)}}{n}\right) - D\left(\sigma_1(t), \frac{\hat{\alpha}_{\mathbf{j}(t)}}{n}\right) - \delta^2.$$

Applying (3.18) and (1.16) on the second and third terms we obtain

$$\widetilde{\mathbf{T}}^{\mathrm{B}}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(s)}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t)}}{n}\right) \geq D(\sigma_1(s), \sigma_1(t)) - (2bC_5 + 1)\delta^2. \tag{3.27}$$

Combining (3.26) and (3.27) gives

$$\begin{aligned}\widetilde{\mathbf{T}}^{(q+1)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_q)}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t)}}{n}\right) &\geq D(\sigma_1(s), \sigma_1(t)) - \frac{b}{n} - \frac{b\delta^2}{a} - (2bC_5 + 1)\delta^2 \\ &\geq D(\sigma_1(s), \sigma_1(t)) - C_6\delta^2\end{aligned}$$

for large enough n. Moreover, σ_1 is a D-geodesic therefore $D(\sigma_1(s), \sigma_1(t)) = t - s$. Consequently,

$$\widetilde{\mathbf{T}}^{(q+1)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_q)}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t)}}{n}\right) \geq (t - t_q) - C_6\delta^2.$$

Hence

$$\begin{aligned}\widetilde{\mathbf{T}}^{(q+1)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_p)}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t)}}{n}\right) - (t - t_q)(1 - \delta) &\geq (t - t_q) - C_6\delta^2 - (t - t_q)(1 - \delta) \\ &\geq \delta(t - t_q - C_6\delta) \geq 0.\end{aligned} \tag{3.28}$$

In other words, for all $t \geq t_q + C_6\delta$, inequality (3.24) fails, thus (3.20) holds.

Inequality (3.21) for $k = q + 1$ is a direct consequence of $\mathbf{j}$'s right-continuity and the fact that for all $t \in]t_{q+1}\,, D(x, y)]$,

$$\widetilde{\mathbf{T}}^{(q+1)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_q)}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t)}}{n}\right) \geq (t - t_q)(1 - \delta).$$

To show inequality (3.22) for $k = q+1$, it suffices to notice that

$$\begin{aligned}\widetilde{\mathbf{T}}^{(q+1)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_q)}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t_{q+1})}}{n}\right) &\le \widetilde{\mathbf{T}}^{(q+1)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_q)}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t^-_{q+1})}}{n}\right) + \frac{b}{n}\\ &\le (t_{q+1} - t_q)(1-\delta) + \frac{b}{n}\\ &\le t_{q+1} - t_q\end{aligned}$$

for large enough n.

Case 2: for all $t \ge t_q + \delta^2$, inequality (3.24) *fails.* We define $t_{q+1} \coloneqq t_q + \delta^2$. Only inequality (3.22) is non trivial. Edges along $\hat{\alpha}[\mathbf{j}(t_q)\,, \mathbf{j}(t_{q+1})]$ have passage time a in the configuration $\tau^{(q+1)}$, therefore

$$\widetilde{\mathbf{T}}^{(q+1)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_q)}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t_{q+1})}}{n}\right) \le \frac{a}{n}(\mathbf{j}(t_{q+1}) - \mathbf{j}(t_q)).$$

Applying (3.12) we get

$$\begin{aligned}\widetilde{\mathbf{T}}^{(q+1)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_q)}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t_{q+1})}}{n}\right) &\le \frac{a}{n}\cdot\left(\frac{n(1-\delta^2)}{a}(t_{q+1} - t_q) + d\right)\\ &= (1-\delta^2)(t_{q+1} - t_q) + \frac{ad}{n}\\ &\le t_{q+1} - t_q\end{aligned}$$

for large enough n, thus (3.22) is proved for $k = q+1$.

We define

$$E_{\text{fast}} \coloneqq \bigcup_{k=0}^{K-1} \mathrm{E}_{\text{path}}\Big(\hat{\alpha}\big|_{[\mathbf{j}(t_k)\,, \mathbf{j}(t_k+\delta^2)]}\Big). \tag{3.29}$$

The integer K is the number of steps in the construction. By the lower bound in (3.20),

$$K \le \frac{D(x_1, y_1)}{\delta^2} < \infty. \tag{3.30}$$

Step 2: Estimating the passage time between two milestones. We claim that for all $0 \le k_1 < k_2 \le K$,

$$t_{k_2} - t_{k_1} - C_7\delta \le \widetilde{\mathbf{T}}^{(K)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_{k_1})}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t_{k_2})}}{n}\right) \le t_{k_2} - t_{k_1}, \tag{3.31}$$

where C_7 is a constant depending only on a, b and X.

Let $0 \le k_1 < k_2 \le K$. The triangle inequality for $\widetilde{\mathbf{T}}^{(K)}_{n,X}$ and the inequality $\widetilde{\mathbf{T}}^{(K)}_{n,X} \le \widetilde{\mathbf{T}}^{(k)}_{n,X}$ for all $0 \le k \le K$ yield

$$\begin{aligned}\widetilde{\mathbf{T}}^{(K)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_{k_1})}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t_{k_2})}}{n}\right) &\le \sum_{k=k_1+1}^{k_2} \widetilde{\mathbf{T}}^{(K)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_{k-1})}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t_k)}}{n}\right),\\ &\le \sum_{k=k_1+1}^{k_2} \widetilde{\mathbf{T}}^{(k)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_{k-1})}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t_k)}}{n}\right).\end{aligned}$$

Applying inequality (3.22) gives the upper bound in (3.31).

We now turn to the lower bound. Consider a $\widetilde{\mathbf{T}}_{n,X}^{(K)}$-geodesic $\frac{\hat{\alpha}_{\mathbf{j}(t_{k_1})}}{n} \overset{\gamma}{\rightsquigarrow} \frac{\hat{\alpha}_{\mathbf{j}(t_{k_2})}}{n}$. Let z denote the first point of γ belonging to

$$\frac{1}{n} \bigcup_{l=k_1+1}^{K-1} \mathrm{E}_{\mathrm{path}}\Big(\hat{\alpha}_{|[\mathbf{j}(t_l)\,,\mathbf{j}(t_l+\delta^2)]}\Big).$$

Note that z always exists, since γ takes the value $\frac{\hat{\alpha}_{\mathbf{j}(t_{k_2})}}{n}$. There exists $t_{k_1+1} \leq t \leq D(x_1, y_1)$ such that

$$z \in \left[\frac{\hat{\alpha}_{\mathbf{j}(t)}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t)+1}}{n}\right].$$

The edges used by γ before reaching z have the same passage time in configurations $\tau^{(k_1+1)}$ and $\tau^{(K)}$, therefore

$$\widetilde{\mathbf{T}}_{n,X}^{(K)}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_{k_1})}}{n}, z\right) = \widetilde{\mathbf{T}}_{n,X}^{(k_1+1)}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_{k_1})}}{n}, z\right).$$

Since $\left\|z - \frac{\hat{\alpha}_{\mathbf{j}(t)}}{n}\right\| \leq \frac{1}{n}$,

$$\widetilde{\mathbf{T}}_{n,X}^{(K)}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_{k_1})}}{n}, z\right) \geq \widetilde{\mathbf{T}}_{n,X}^{(k_1+1)}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_{k_1})}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t)}}{n}\right) - \frac{b}{n}.$$

Applying (3.21), we get

$$\widetilde{\mathbf{T}}_{n,X}^{(K)}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_{k_1})}}{n}, z\right) \geq (t - t_{k_1})(1-\delta) - \frac{b}{n}. \tag{3.32}$$

In the easier case where $t \geq t_{k_2}$, inequality (3.32) implies

$$\widetilde{\mathbf{T}}_{n,X}^{(K)}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_{k_1})}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t_{k_2})}}{n}\right) \geq (t_{k_2} - t_{k_1})(1-\delta) - \delta^3 \tag{3.33}$$

for large enough n, which is stronger than the desired bound. Otherwise there exists $k_1 + 1 \leq k \leq k_2 - 1$ such that $t_k \leq t \leq t_k + \delta^2$. Every edge along $\hat{\alpha}_{|[\mathbf{j}(t_k)\,,\mathbf{j}(t)]}$ has a passage time a in configuration $\tau^{(K)}$, thus inequality (3.12) yields

$$\begin{aligned}
\widetilde{\mathbf{T}}_{n,X}^{(K)}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_k)}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t)}}{n}\right) &\leq \frac{a}{n}(\mathbf{j}(t) - \mathbf{j}(t_k)) \\
&\leq (1-\delta^2)(t - t_k) + \frac{ad}{n} \\
&\leq t - t_k + \delta^3
\end{aligned} \tag{3.34}$$

for large enough n. Since γ is a geodesic,

$$\widetilde{\mathbf{T}}_{n,X}^{(K)}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_{k_1})}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t_{k_2})}}{n}\right) = \widetilde{\mathbf{T}}_{n,X}^{(K)}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_{k_1})}}{n}, z\right) + \widetilde{\mathbf{T}}_{n,X}^{(K)}\left(z, \frac{\hat{\alpha}_{\mathbf{j}(t_{k_2})}}{n}\right). \tag{3.35}$$

Since $\left\|z - \frac{\hat{\alpha}_{\mathbf{j}(t)}}{n}\right\| \leq \frac{1}{n}$, the triangle inequality yields

$$\widetilde{\mathbf{T}}_{n,X}^{(K)}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_{k_1})}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t_{k_2})}}{n}\right) \geq \widetilde{\mathbf{T}}_{n,X}^{(K)}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_{k_1})}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t)}}{n}\right) + \widetilde{\mathbf{T}}_{n,X}^{(K)}\left(\frac{\hat{\alpha}_{\mathbf{j}(t)}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t_{k_2})}}{n}\right) - \frac{2b}{n}. \tag{3.36}$$

Applying again the triangle inequality yields

$$\begin{aligned}\widetilde{\mathbf{T}}^{(K)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_{k_1})}}{n},\frac{\hat{\alpha}_{\mathbf{j}(t_{k_2})}}{n}\right) \geq \widetilde{\mathbf{T}}^{(K)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_{k_1})}}{n},\frac{\hat{\alpha}_{\mathbf{j}(t)}}{n}\right) + \widetilde{\mathbf{T}}^{(K)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_k)}}{n},\frac{\hat{\alpha}_{\mathbf{j}(t_{k_2})}}{n}\right) \\ - \widetilde{\mathbf{T}}^{(K)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_k)}}{n},\frac{\hat{\alpha}_{\mathbf{j}(t)}}{n}\right) - \frac{2b}{n}.\end{aligned} \tag{3.37}$$

In the right-hand side of (3.37), lower bounding the first term with (3.21) and the third with (3.34) gives

$$\begin{aligned}\widetilde{\mathbf{T}}^{(K)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_{k_1})}}{n},\frac{\hat{\alpha}_{\mathbf{j}(t_{k_2})}}{n}\right) &\geq (1-\delta)(t-t_{k_1}) + \widetilde{\mathbf{T}}^{(K)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_k)}}{n},\frac{\hat{\alpha}_{\mathbf{j}(t_{k_2})}}{n}\right) + (t_k - t) - \delta^3 - \frac{2b}{n} \\ &= \widetilde{\mathbf{T}}^{(K)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_k)}}{n},\frac{\hat{\alpha}_{\mathbf{j}(t_{k_2})}}{n}\right) + (1-\delta)(t_k - t_{k_1}) - \delta(t - t_k) - \delta^3 - \frac{2b}{n}.\end{aligned}$$

As $t - t_k \leq \delta^2$, for large enough n,

$$\widetilde{\mathbf{T}}^{(K)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_{k_1})}}{n},\frac{\hat{\alpha}_{\mathbf{j}(t_{k_2})}}{n}\right) \geq \widetilde{\mathbf{T}}^{(K)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_k)}}{n},\frac{\hat{\alpha}_{\mathbf{j}(t_{k_2})}}{n}\right) + (1-\delta)(t_k - t_{k_1}) - 3\delta^3. \tag{3.38}$$

It remains to get a lower bound on $\widetilde{\mathbf{T}}^{(K)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_k)}}{n},\frac{\hat{\alpha}_{\mathbf{j}(t_{k_2})}}{n}\right)$. Using recursively the same arguments one gets a lower bound analogous to (3.38) or (in at most K steps) finally analogous to (3.33). This leads to

$$\widetilde{\mathbf{T}}^{(K)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_{k_1})}}{n},\frac{\hat{\alpha}_{\mathbf{j}(t_{k_2})}}{n}\right) \geq (1-\delta)(t_{k_2} - t_{k_1}) - 3K\delta^3.$$

Applying (3.30) and $t_{k_2} - t_{k_1} \leq D(x_1, y_1)$, hence the lower bound in (3.31) is proven.

Step 3: Extending (3.31) **beyond milestones.** We show that for all $z_1, z_2 \in X$,

$$D_1(z_1,z_2) - C_8\delta \leq \widetilde{\mathbf{T}}^{(K)}_{n,X}(z_1,z_2) \leq D_1(z_1,z_2) + C_8\delta, \tag{3.39}$$

where C_8 may only depend on a, b and X.

Let $z_1, z_2 \in X$. We start with the lower bound. Let $z_1 \overset{\gamma}{\rightsquigarrow} z_2$ be a $\widetilde{\mathbf{T}}^{(K)}_{n,X}$-geodesic. If γ does not intersect $\frac{1}{n}E_{\mathrm{fast}}$, then as $\tau^{\mathrm{B}} \in \mathrm{LD}_{n,X}(D_0, \delta^2)$,

$$\widetilde{\mathbf{T}}^{(K)}_{n,X}(\gamma) = \widetilde{\mathbf{T}}^{\mathrm{B}}_{n,X}(\gamma) \geq D_0(z_1,z_2) - \delta^2 \geq D_1(z_1,z_2) - \delta^2,$$

thus the lower bound in (3.39) is proven. Otherwise γ can be decomposed as

$$z_1 \overset{\gamma_{\mathrm{L}}}{\rightsquigarrow} z_{\mathrm{L}} \overset{\gamma_{\mathrm{M}}}{\rightsquigarrow} z_{\mathrm{R}} \overset{\gamma_{\mathrm{R}}}{\rightsquigarrow} z_2,$$

where z_{L} and z_{R} are respectively the first and last visit of γ in $\frac{1}{n}E_{\mathrm{fast}}$. We have

$$\begin{aligned}\widetilde{\mathbf{T}}^{(K)}_{n,X}(\gamma) &= \widetilde{\mathbf{T}}^{(K)}_{n,X}(\gamma_{\mathrm{L}}) + \widetilde{\mathbf{T}}^{(K)}_{n,X}(\gamma_{\mathrm{M}}) + \widetilde{\mathbf{T}}^{(K)}_{n,X}(\gamma_{\mathrm{R}}) \\ &= \widetilde{\mathbf{T}}^{\mathrm{B}}_{n,X}(\gamma_{\mathrm{L}}) + \widetilde{\mathbf{T}}^{(K)}_{n,X}(\gamma_{\mathrm{M}}) + \widetilde{\mathbf{T}}^{\mathrm{B}}_{n,X}(\gamma_{\mathrm{R}}),\end{aligned}$$

thus

$$\widetilde{\mathbf{T}}^{(K)}_{n,X}(\gamma) = \widetilde{\mathbf{T}}^{\mathrm{B}}_{n,X}(z_1,z_{\mathrm{L}}) + \widetilde{\mathbf{T}}^{(K)}_{n,X}(z_{\mathrm{L}},z_{\mathrm{R}}) + \widetilde{\mathbf{T}}^{\mathrm{B}}_{n,X}(z_{\mathrm{R}},z_2). \tag{3.40}$$

As $\tau^{\mathrm{B}} \in \mathrm{LD}_{n,x}(D_0, \delta^2)$,

$$\widetilde{\mathbf{T}}^{\mathrm{B}}_{n,X}(z_1, z_{\mathrm{L}}) \geq D_0(z_1, z_{\mathrm{L}}) - \delta^2 \geq D_1(z_1, z_{\mathrm{L}}) - \delta^2 \tag{3.41}$$

and

$$\widetilde{\mathbf{T}}^{\mathrm{B}}_{n,X}(z_{\mathrm{R}}, z_2) \geq D_0(z_{\mathrm{R}}, z_2) - \delta^2 \geq D_1(z_{\mathrm{R}}, z_2) - \delta^2. \tag{3.42}$$

By definition of z_{L} there exists $t_{\mathrm{L}} \in \{t_k, 1 \leq k \leq K-1\}$ and $t_{\mathrm{L}} \leq s_{\mathrm{L}} \leq t_{\mathrm{L}} + \delta^2$ such that $z_{\mathrm{L}} \in \left[\frac{\hat{\alpha}_{\mathbf{j}(s_{\mathrm{L}})}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(s_{\mathrm{L}})+1}}{n}\right]$. Therefore (3.12) yields

$$\left\| z_{\mathrm{L}} - \frac{\hat{\alpha}_{\mathbf{j}(t_{\mathrm{L}})}}{n} \right\| \leq \left(\frac{1}{a} + 1\right)\delta^2. \tag{3.43}$$

Applying (3.18) gives

$$\|z_{\mathrm{L}} - \sigma_1(t_{\mathrm{L}})\| \leq \left(C_5 + \frac{1}{a} + 1\right)\delta^2. \tag{3.44}$$

We define t_{R} similarly and z_{R} satisfies analogous inequalities. Triangle inequality, (3.43) and (1.16) yield

$$\widetilde{\mathbf{T}}^{(K)}_{n,X}(z_{\mathrm{L}}, z_{\mathrm{R}}) \geq \widetilde{\mathbf{T}}^{(K)}_{n,X}\left(\frac{\hat{\alpha}_{\mathbf{j}(t_{\mathrm{L}})}}{n}, \frac{\hat{\alpha}_{\mathbf{j}(t_{\mathrm{R}})}}{n}\right) - 2b\left(\frac{1}{a} + 1\right)\delta^2.$$

Applying (3.31), we get

$$\widetilde{\mathbf{T}}^{(K)}_{n,X}(z_{\mathrm{L}}, z_{\mathrm{R}}) \geq |t_{\mathrm{L}} - t_{\mathrm{R}}| - 2b\left(\frac{1}{a} + 1\right)\delta^2 - C_7\delta. \tag{3.45}$$

Besides, the triangle inequality, (3.44) and (1.16) yield

$$D_1(z_{\mathrm{L}}, z_{\mathrm{R}}) \leq D_1(\sigma_1(t_{\mathrm{L}}), \sigma_1(t_{\mathrm{R}})) + 2b\left(C_5 + \frac{1}{a} + 1\right)\delta^2.$$

By definition of D_1 (see (3.5)), $D_1(\sigma_1(t_{\mathrm{L}}), \sigma_1(t_{\mathrm{R}})) = D(\sigma_1(t_{\mathrm{L}}), \sigma_1(t_{\mathrm{R}}))$. Moreover, σ_1 is a D-geodesic, therefore

$$D_1(z_{\mathrm{L}}, z_{\mathrm{R}}) \leq |t_{\mathrm{L}} - t_{\mathrm{R}}| + 2b\left(C_5 + \frac{1}{a} + 1\right)\delta^2. \tag{3.46}$$

Combining (3.45) and (3.46), we get

$$\widetilde{\mathbf{T}}^{(K)}_{n,X}(z_{\mathrm{L}}, z_{\mathrm{R}}) \geq D_1(z_{\mathrm{L}}, z_{\mathrm{R}}) - 2b\left(C_5 + \frac{2}{a} + 2\right)\delta^2 - C_7\delta. \tag{3.47}$$

Inequalities (3.41), (3.42) and (3.47) give a lower bound for each term in (3.40), leading to the lower bound in (3.39).

We now turn to the upper bound in (3.39). Recall the definition of D_1 given in (3.5). If $D_1(z_1, z_2) = D_0(z_1, z_2)$ then

$$\widetilde{\mathbf{T}}^{(K)}_{n,X}(z_1, z_2) \leq \widetilde{\mathbf{T}}^{\mathrm{B}}_{n,X}(z_1, z_2) \leq D_0(z_1, z_2) + \delta^2 = D_1(z_1, z_2) + \delta^2. \tag{3.48}$$

Otherwise, there exists $z_{\mathrm{L}}, z_{\mathrm{R}} \in \sigma_1$ such that $D_1(z_1, z_2) = D_0(z_1, z_{\mathrm{L}}) + D(z_{\mathrm{L}}, z_{\mathrm{R}}) + D_0(z_{\mathrm{R}}, z_2)$, therefore

$$\begin{aligned}
\widetilde{\mathbf{T}}^{(K)}_{n,X}(z_1, z_2) &\leq \widetilde{\mathbf{T}}^{(K)}_{n,X}(z_1, z_{\mathrm{L}}) + \widetilde{\mathbf{T}}^{(K)}_{n,X}(z_{\mathrm{L}}, z_{\mathrm{R}}) + \widetilde{\mathbf{T}}^{(K)}_{n,X}(z_{\mathrm{R}}, z_2) \\
&\leq \widetilde{\mathbf{T}}^{\mathrm{B}}_{n,X}(z_1, z_{\mathrm{L}}) + \widetilde{\mathbf{T}}^{(K)}_{n,X}(z_{\mathrm{L}}, z_{\mathrm{R}}) + \widetilde{\mathbf{T}}^{\mathrm{B}}_{n,X}(z_{\mathrm{R}}, z_2) \\
&\leq D_0(z_1, z_{\mathrm{L}}) + \widetilde{\mathbf{T}}^{(K)}_{n,X}(z_{\mathrm{L}}, z_{\mathrm{R}}) + D_0(z_{\mathrm{R}}, z_2) + 2\delta^2.
\end{aligned} \tag{3.49}$$

Besides, there exist $0 \leq s_{\mathrm{L}}, s_{\mathrm{R}} \leq D(x_1, y_1)$ such that $z_{\mathrm{L}} = \sigma_1(s_{\mathrm{L}})$ and $z_{\mathrm{R}} = \sigma_1(s_{\mathrm{R}})$. The upper bound in (3.20), along with $D(x_1, y_1) - t_K \leq C_6\delta$, imply the existence of $t_{\mathrm{L}}, t_{\mathrm{R}} \in \{t_k, k \in [\![1, K]\!]\}$ such that $|t_{\mathrm{L}} - s_{\mathrm{L}}|, |t_{\mathrm{R}} - s_{\mathrm{R}}| \leq C_6\delta$. Proceeding as for (3.47), we show that the second term in (3.49) is upper bounded by $D(z_{\mathrm{L}}, z_{\mathrm{R}}) + \mathrm{O}(\delta) = D_1(z_{\mathrm{L}}, z_{\mathrm{R}}) + \mathrm{O}(\delta)$, hence the upper bound in (3.39) is proven.

Step 4: Conclusion. On the event

$$\Big\{\tau^{\mathrm{B}} \in \mathrm{LD}_{n,X}(D_0,\delta^2)\Big\} \cap \left(\bigcap_{e\in \mathrm{E}_{\mathrm{path}}(\hat{\alpha})} \{X_e = \mathbb{1}_{E_{\mathrm{fast}}}(e)\}\right) \cap \left(\bigcap_{e\in \mathrm{E}_{\mathrm{path}}(\hat{\alpha})} \Big\{\tau^{\mathrm{G}} \le a+\delta\Big\}\right),$$

configuration τ and $\tau^{(K)}$ agree on all edges except those in E_{fast} where they may differ up to δ. Consequently, for large enough n,

$$\mathfrak{d}_\infty\Big(\widetilde{\mathbf{T}}^{(K)}_{n,X}, \widetilde{\mathbf{T}}_{n,X}\Big) \le \frac{\delta \# E_{\mathrm{fast}}}{n}.$$

Moreover, $\# E_{\mathrm{fast}}$ is bounded by $\#\hat{\alpha}$, thus by (3.13), for large enough n,

$$\# E_{\mathrm{fast}} \le \frac{n(1-\delta^2)b}{a}\operatorname{diam}(X) + d \le \frac{nb}{a}\operatorname{diam}(X).$$

Consequently, for large enough n,

$$\mathfrak{d}_\infty\Big(\widetilde{\mathbf{T}}^{(K)}_{n,X}, \widetilde{\mathbf{T}}_{n,X}\Big) \le \frac{b\delta}{a}\operatorname{diam}(X). \tag{3.50}$$

Combining inequalities (3.39) and (3.50) concludes the proof. □

4 The elementary rate function

In this section we prove Theorem 1.9, Proposition 1.10 and Proposition 1.11. We work under Assumption 1.

4.1 Existence

In this subsection we fix $g \in \mathcal{N}$ and prove Theorem 1.9, which is a consequence of Lemma 4.1.

Lemma 4.1. *There exists a constant $C_9 > 0$, depending only on a, b and d, such that for all $\varepsilon > 0$ and $0 < \delta \le 1$, for large enough n, for large enough m,*

$$-\frac{1}{m^d}\log \mathbb{P}\Big(\mathrm{LD}_{m,[0\,,1]^d}(g, C_9(\varepsilon+\delta))\Big) \le -\frac{1}{n^d}\log \mathbb{P}\Big(\mathrm{LD}_{n,[0\,,1]^d}(g,\delta^2)\Big) - C_9\delta \log \nu([b-\varepsilon\,,b]). \tag{4.1}$$

Proof of Theorem 1.9. Let $\varepsilon > 0$ and $0 < \delta \le \varepsilon \wedge 1$. For all $m \ge 1$, $\mathrm{LD}_{m,[0\,,1]^d}(g, C_9(\varepsilon+\delta)) \subseteq \mathrm{LD}_{m,[0\,,1]^d}(g, 2C_9\varepsilon)$. Hence (4.1) implies that for large enough n, for large enough m,

$$-\frac{1}{m^d}\log \mathbb{P}\Big(\mathrm{LD}_{m,[0\,,1]^d}(g, 2C_9\varepsilon)\Big) \le -\frac{1}{n^d}\log \mathbb{P}\Big(\mathrm{LD}_{n,[0\,,1]^d}(g,\delta^2)\Big) - C_9\delta \log \nu([b-\varepsilon\,,b]).$$

Considering the superior limit in m then the inferior limit in n, we get

$$\varlimsup_{m\to\infty} -\frac{1}{m^d}\log \mathbb{P}\Big(\mathrm{LD}_{m,[0\,,1]^d}(g, 2C_9\varepsilon)\Big) \le \varliminf_{n\to\infty} -\frac{1}{n^d}\log \mathbb{P}\Big(\mathrm{LD}_{n,[0\,,1]^d}(g,\delta^2)\Big) - C_9\delta \log \nu([b-\varepsilon\,,b]). \tag{4.2}$$

Letting $\delta \to 0$ then $\varepsilon \to 0$ gives $\overline{I}_{[0\,,1]^d}(g) \le \underline{I}_{[0\,,1]^d}(g)$. This concludes the proof. □

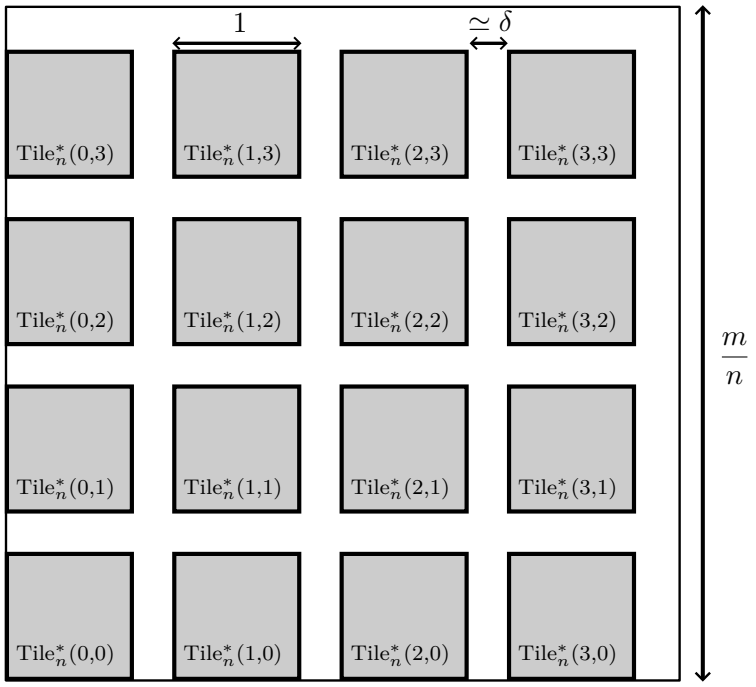


Figure 5: Illustration of the tiles defined by (4.3) in the case $d = 2$, $k = 4$.

Proof of Lemma 4.1. Fix $\varepsilon > 0$ and $0 < \delta \leq 1$. Let $n, m \geq 1$ and k be the only integer such that $nk(1+\delta) < m \leq n(k+1)(1+\delta)$. We will build an event included in $\mathrm{LD}_{m,[\![0\,,1]\!]^d}(g, C_9(\varepsilon+\delta))$ from k^d independent realizations of $\mathrm{LD}_{n,[\![0\,,1]\!]^d}(g,\delta^2)$.

For all $v \in [\![0\,, k-1]\!]^d$, let

$$\mathrm{Tile}_n^*(v) \coloneqq [0\,,1]^d + \frac{\lfloor n(1+\delta)\rfloor}{n} v \tag{4.3}$$

(see Figure 5). The set $n\,\mathrm{Tile}_n^*(v)$ is an *integer* translation of $[\![0\,,n]\!]^d$ therefore by stationarity of $(\tau_e)_{e\in\mathbb{E}^d}$,

$$\mathbb{P}\Big(\mathrm{LD}_{n,\mathrm{Tile}_n^*(v)}(g,\delta^2)\Big) = \mathbb{P}\Big(\mathrm{LD}_{n,[\![0\,,1]\!]^d}(g,\delta^2)\Big). \tag{4.4}$$

Moreover, for large enough n, the variables $\Big(\mathbf{T}_{n,\mathrm{Tile}_n^*(v)}\Big)_{v\in[\![0\,,k-1]\!]^d}$ live on pairwise disjoint subsets of $\mathbb{E}^d$ therefore are independent. Let Corridor denote the set of edges included in $[\![0\,,m]\!]^d$ but not in any $n\,\mathrm{Tile}_n^*(v)$. This set satisfies

$$\#\,\mathrm{Corridor} = \#\mathrm{E}\Big([\![0\,,m]\!]^d\Big) - k^d \#\mathrm{E}\Big([\![0\,,n]\!]^d\Big) \leq d(m+1)^d - dk^d(n-1)^d. \tag{4.5}$$

We define the favorable event

$$\mathrm{Fav}^* \coloneqq \left(\bigcap_{v\in[\![0\,,k-1]\!]^d} \mathrm{LD}_{n,\mathrm{Tile}_n^*(v)}(g,\delta^2)\right) \cap \left(\bigcap_{e\in\mathrm{Corridor}} \{\tau_e \geq b-\varepsilon\}\right). \tag{4.6}$$

Applying (4.4) and (4.5) gives

$$\begin{aligned} -\frac{1}{m^d}\log\mathbb{P}(\mathrm{Fav}^*) &\leq -\left(\frac{k}{m}\right)^d \log\mathbb{P}\Big(\mathrm{LD}_{n,[\![0\,,1]\!]^d}(g,\delta^2)\Big) - \frac{d(m+1)^d - dk^d(n-1)^d}{m^d}\log\nu([b-\varepsilon\,,b]) \\ &\leq -\frac{1}{n^d}\log\mathbb{P}\Big(\mathrm{LD}_{n,[\![0\,,1]\!]^d}(g,\delta^2)\Big) - C_{10}\delta\log\nu([b-\varepsilon\,,b]), \end{aligned} \tag{4.7}$$

for large enough n, large enough m, where C_{10} only depends on d.

From now on we assume that Fav* occurs. We first lower bound $\widetilde{\mathbf{T}}_{m,[\![0\,,1]\!]^d}$ using Lemma 2.17. For all $v_1 \neq v_2$,

$$\inf_{\substack{x\in n\,\mathrm{Tile}_n^*(v_1)\\ y\in n\,\mathrm{Tile}_n^*(v_2)}} \|x-y\| \geq n\delta - d. \tag{4.8}$$

Equation (2.36) in Lemma 2.16 implies that on the event $\mathrm{LD}_{n,\mathrm{Tile}_n^*(v)}(g,\delta^2)$, for all $v \in [\![0\,,k-1]\!]^d$, $x,y \in n\,\mathrm{Tile}_n^*(v)$,

$$\begin{aligned}\mathbf{T}_{[\![0\,,m]\!]^d}\big|_{n\,\mathrm{Tile}_n^*(v)}(x,y) &= \mathbf{T}_{n\,\mathrm{Tile}_n^*(v)}(x,y)\\ &\geq g(x-y) - n\delta^2.\end{aligned} \tag{4.9}$$

On the event $\bigcap_{e\in\mathrm{Corridor}}\{\tau_e \geq b-\varepsilon\}$ for all Lipschitz paths γ included in $[\![0\,,m]\!]^d \setminus \bigcup_{v\in[\![0\,,k-1]\!]^d} n\,\mathrm{Tile}_n^*(v)$,

$$\mathbf{T}_{[\![0\,,m]\!]^d}(\gamma) \geq (b-\varepsilon)\|\gamma\|. \tag{4.10}$$

Thanks to Lemma 2.17 with $\delta_1 \coloneqq \delta n - d$ and $\delta_2 \coloneqq n\delta^2$, we get, for all $x,y \in [\![0\,,m]\!]^d$,

$$\begin{aligned}\mathbf{T}_{[\![0\,,m]\!]^d}(x,y) &\geq g(x-y) - 3\,\mathrm{diam}\Big([\![0\,,m]\!]^d\Big)\left(\varepsilon + \frac{n\delta^2}{n\delta - d}\right)\\ &\geq g(x-y) - 4dm(\varepsilon+\delta)\end{aligned}$$

for large enough n, with fixed $\delta > 0$. Thus for all $x,y \in [\![0\,,1]\!]^d$,

$$\widetilde{\mathbf{T}}_{m,[\![0\,,1]\!]^d}(x,y) \geq g(x-y) - 4d(\varepsilon+\delta). \tag{4.11}$$

We now turn to the upper bound. For all $v \in [\![0\,,k-1]\!]^d$, for all $z,z' \in n\,\mathrm{Tile}_n^*(v)$, on the event $\mathrm{LD}_{n,\mathrm{Tile}_n^*(v)}(g,\delta^2)$,

$$\mathbf{T}_{[\![0\,,m]\!]^d}(z,z') \leq \mathbf{T}_{n\,\mathrm{Tile}_n^*(v)}(z,z') \leq g(z-z') + \delta^2 n. \tag{4.12}$$

Let $x,y \in [\![0\,,m]\!]^d$ and $\hat{x},\hat{y}$ their respective projections on $\bigcup_{v\in[\![0\,,k-1]\!]^d} n\,\mathrm{Tile}_n^*(v)$ (in case of non-unicity the choice does not matter). Consider the map

$$\begin{aligned}f : \bigcup_v n\,\mathrm{Tile}_n^*(v) &\longrightarrow [\![0\,,nk]\!]^d\\ z &\longmapsto z - (\lfloor n(1+\delta)\rfloor - n)v \qquad \text{if } z \in \mathrm{Tile}_n^*(v),\end{aligned}$$

which translates $n\,\mathrm{Tile}_n^*(v)$ onto $n\,\mathrm{Tile}(v,1)$, defined by (1.35). We claim that with fixed n and δ, for large enough m,

$$\|x - f(\hat{x})\| \leq 2dk(n\delta+1). \tag{4.13}$$

Indeed,

$$\begin{aligned}\|x-\hat{x}\| &\leq d[m-n-(k-1)\lfloor n(1+\delta)\rfloor]\\ &\leq d[m-n-(k-1)n(1+\delta)+(k-1)]\\ &\leq d[n(1+2\delta)+(k-1)],\end{aligned}$$

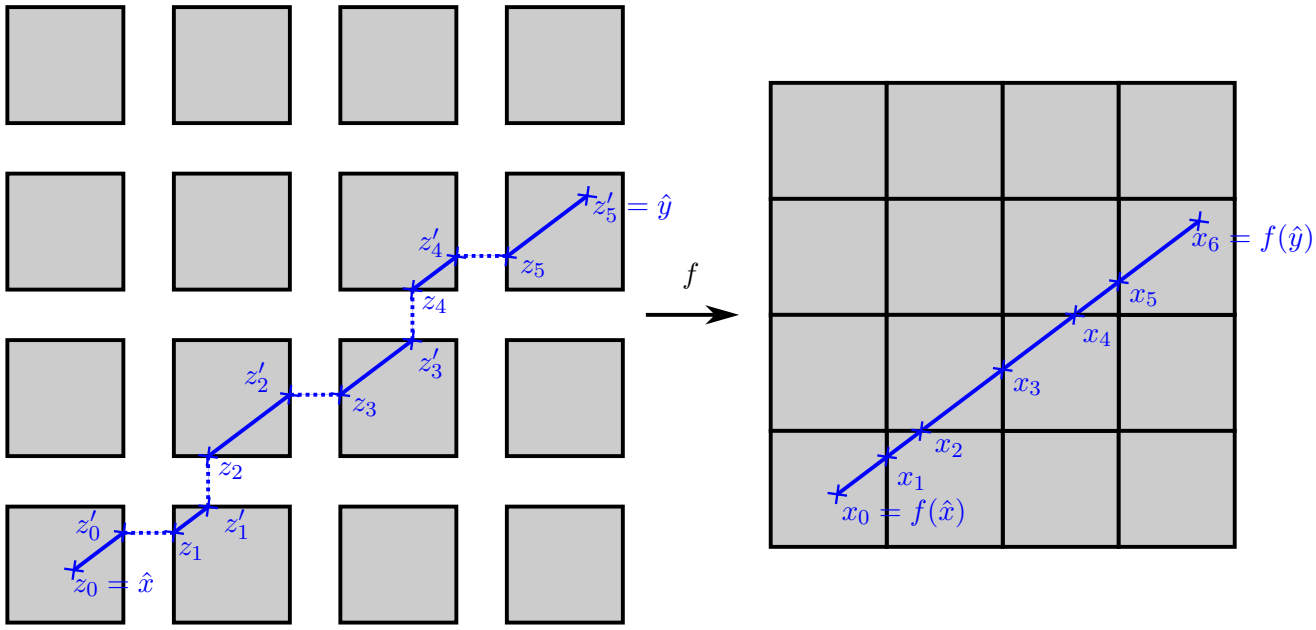


Figure 6: Construction of the points $(z_i)_{0\le i\le r}$ and $(z'_i)_{0\le i\le r-1}$ in the case $d = 2$, $k = 4$.

and

$$\begin{aligned}\|\hat{x} - f(\hat{x})\| &\le (\lfloor n(1+\delta)\rfloor - n)(k-1)d \\ &\le n\delta(k-1)d, \end{aligned} \tag{4.14}$$

therefore by triangle inequality,

$$\|x - f(\hat{x})\| \le d[n(1+2\delta) + (k-1) + n\delta(k-1)],$$

thus (4.13) for large enough m.

The segment from $f(\hat{x})$ to $f(\hat{y})$ can be decomposed as

$$[f(\hat{x})\,, f(\hat{y})] = [x_0\,, x_1] \cup [x_1\,, x_2] \cup \cdots \cup [x_{r-1}\,, x_r],$$

where each segment $[x_i\,, x_{i+1}]$ is included in a tile $n\,\mathrm{Tile}(v_i, 1)$, and the $(v_i)_{0\le i\le r-1}$ are pairwise distinct. Moreover the sequence $(v_i)_{0\le i\le r-1}$ may be chosen so that it has monotone coordinates. In particular, $r \le kd$. For all $0 \le i \le r-1$, let $[z_i\,, z'_i]$ denote the unique segment included in $n\,\mathrm{Tile}^*_n(v_i)$ whose image under f is $[x_i\,, x_{i+1}]$ (see Figure 6). The triangle inequality yields

$$\mathbf{T}_{[0\,,m]^d}(\hat{x}, \hat{y}) \le \sum_{i=0}^{r} \mathbf{T}_{[0\,,m]^d}(z_i, z'_i) + \sum_{i=0}^{r-1} \mathbf{T}_{[0\,,m]^d}(z'_i, z_{i+1}). \tag{4.15}$$

Inequality (4.12) gives

$$\begin{aligned}\sum_{i=0}^{r} \mathbf{T}_{[0\,,m]^d}(z_i, z'_i) &\le \sum_{i=0}^{r} \Big(g(z_i - z'_i) + \delta^2 n\Big) \\ &= \sum_{i=0}^{r} g(z_i - z'_i) + (r+1)\delta^2 n \\ &= \sum_{i=0}^{r} g(x_i - x_{i+1}) + (r+1)\delta^2 n,\end{aligned}$$

thus

$$\sum_{i=0}^{r} \mathbf{T}_{[0\,,m]^d}(z_i, z_i') \leq g(f(\hat{x}) - f(\hat{y})) + (r+1)\delta^2 n. \tag{4.16}$$

Consequently, using the triangle inequality, (4.13) and $g \leq b\|\cdot\|$ in (4.16), we get

$$\sum_{i=0}^{r} \mathbf{T}_{[0\,,m]^d}(z_i, z_i') \leq g(x-y) + 4bdk(n\delta+1) + (r+1)\delta^2 n. \tag{4.17}$$

Besides, for all $0 \leq i \leq r-1$,

$$\mathbf{T}_{[0\,,m]^d}(z_i', z_{i+1}) \leq b\|z_i' - z_{i+1}\| \leq bd(\delta n + 1),$$

thus

$$\sum_{i=0}^{r-1} \mathbf{T}_{[0\,,m]^d}(z_i', z_{i+1}) \leq rbd(\delta n + 1). \tag{4.18}$$

Combining (4.15), (4.17) and (4.18), we obtain

$$\mathbf{T}_{[0\,,m]^d}(\hat{x}, \hat{y}) \leq g(x-y) + 4bdk(n\delta+1) + (r+1)\delta^2 n + rbd(\delta n + 1).$$

Applying $r \leq kd$, we get

$$\mathbf{T}_{[0\,,m]^d}(\hat{x}, \hat{y}) \leq g(x-y) + 4bdk(n\delta+1) + (kd+1)\delta^2 n + kbd^2(\delta n + 1),$$

thus a final use of the triangle inequality and (4.14) leads to

$$\begin{aligned}\mathbf{T}_{[0\,,m]^d}(x, y) &\leq g(x-y) + 2bdn\delta(k-1) + 4bdk(n\delta+1) + (kd+1)\delta^2 n + kbd^2(\delta n + 1) \\ &\leq g(x-y) + 6bdk(n\delta+1) + (kd+1)\delta^2 n + kbd^2(\delta n + 1)\end{aligned}$$

for large enough m. Consequently, for all $\delta > 0$, for large enough n, for large enough m, for all $x, y \in [0\,,1]^d$,

$$\widetilde{\mathbf{T}}_{m,[0\,,1]^d}(x, y) \leq g(x-y) + 2d(6b + bd + \delta)\delta. \tag{4.19}$$

The lemma is a consequence of (4.7), (4.11) and (4.19). □

4.2 Characterizing the cases $I_{[0\,,1]^d}(g) < \infty$ and $I_{[0\,,1]^d}(g) = 0$: proof of Proposition 1.10

Thanks to Proposition 3.1, it is sufficient to prove Lemmas 4.2 and 4.3.

Lemma 4.2.

(i) For all $a \leq \zeta \leq b$,

$$I_{[0\,,1]^d}(\zeta\|\cdot\|) \leq -d \log \nu([\zeta\,, b]). \tag{4.20}$$

(ii) If $g \in \mathcal{N}$ satisfies $\|g\|_{\mathrm{Hom}} \geq b - \eta$, then

$$I_{[0\,,1]^d}(g) \geq -\log 2 - \frac{1}{2} \log \nu([b - 4\eta\,, b]). \tag{4.21}$$

Proof. Proof of (i). Let $a \le \zeta \le b$. Note the inclusion

$$\bigcap_{e \in \mathrm{E}(\llbracket 0\,,n \rrbracket^d)} \{\tau_e \ge \zeta\} \subseteq \Big\{ \widetilde{\mathbf{T}}_{n,[0\,,1]^d} \ge \zeta \|\cdot\| \Big\}.$$

Consequently, Corollary 3.2 yields (4.20).

Proof of (ii). Assume that $\|g\|_{\mathrm{Hom}} \ge b - \eta$. Then there exists $u \in \mathbb{S}$ such that $g(u) \ge b - \eta$. By convexity and symmetry there exists e_i such that $g(\mathrm{e}_i) \ge b - \eta$; without loss of generality we assume $i = 1$. The argument consists in proving that on a large deviation event around g, a volumic number of edges must have a passage time close to b. For all $v \in \llbracket 0\,, n \rrbracket^{d-1}$, let π_v denote the segment from $(v, 0)$ to (v, n). Since $\mathrm{LD}_n(g, \eta) \subseteq \bigcap_v \{\tau(\pi_v) \ge n(b - 2\eta)\}$ and the $(\tau(\pi_v))_v$ are i.i.d.,

$$\mathbb{P}(\mathrm{LD}_n(g, \eta)) \le \mathbb{P}(\tau(\pi_0) \ge n(b - 2\eta))^{n^{d-1}}.$$

Besides, if at least $n/2$ indices $i \in \llbracket 0\,, n - 1 \rrbracket$ are such that $\tau(i\mathrm{e}_1, (i + 1)\mathrm{e}_1) < b - 4\eta$, then

$$\tau(\pi_0) < \frac{n}{2}(b - 4\eta) + \frac{nb}{2} = n(b - 2\eta).$$

Consequently,

$$\begin{aligned} \mathbb{P}(\tau(\pi_0) \ge n(b - 2\eta)) &\le \mathbb{P}\left(\bigcup_{\substack{A \subseteq \llbracket 0\,, n-1 \rrbracket \\ \#A = \lfloor n/2 \rfloor}} \bigcap_{i \in A} \{\tau(i\mathrm{e}_1, (i + 1)\mathrm{e}_1) \ge b - 4\eta\} \right) \\ &\le 2^n \nu([b - 4\eta\,, b])^{\lfloor n/2 \rfloor}, \end{aligned} \tag{4.22}$$

therefore

$$I_{[0\,,1]^d}(g) \ge \varliminf_{n \to \infty} -\frac{1}{n^d} \log \mathbb{P}(\mathrm{LD}_n(g, \eta)) \ge -\log 2 - \frac{1}{2} \log \nu([b - 4\eta\,, b]). \tag{4.23}$$

□

Lemma 4.3.

(i) $I_{[0\,,1]^d}(\mu) = 0$.

(ii) Let $g \in \mathcal{N}$. If there exists $u \in \mathbb{R}^d$ such that $g(u) > \mu(u)$, then $I_{[0\,,1]^d}(g) > 0$.

Proof. Proof of (i). Let $x, y \in [0\,, 1]^d$ and $\varepsilon > 0$. By triangle inequality and (1.16),

$$\widetilde{\mathbf{T}}_n(x, y) \ge \widetilde{\mathbf{T}}_n\Big(\frac{\lfloor nx \rfloor}{n}, \frac{\lfloor ny \rfloor}{n} \Big) - \frac{2bd}{n}.$$

Consequently, by definition of μ (see (1.3)),

$$\mathbb{P}\Big(\widetilde{\mathbf{T}}_n(x, y) \ge \mu(x - y) - \varepsilon \Big) \xrightarrow[n \to \infty]{} 1. \tag{4.24}$$

Besides, for large enough k, for all $n \ge 1$,

$$\bigcap_{x, y \in \frac{1}{k}\llbracket 0\,, k \rrbracket^d} \Big\{ \widetilde{\mathbf{T}}_n(x, y) \ge \mu(x - y) - \varepsilon \Big\} \subseteq \mathrm{LD}^+_{n,[0\,,1]^d}(\mu, 2\varepsilon). \tag{4.25}$$

By (4.24), with fixed k, the probability of the left hand-side of (4.25) converges to 1 as $n \to \infty$. We conclude with Corollary 3.2.

Proof of (ii). We adapt the arguments used by Kesten to prove a bound on the probability of the upper tail large deviation event for the point-point passage time, in the direction e_1 [13, Equation (5.13)]. Assume that there exists $u \in \mathbb{R}^d$ such that $g(u) > \mu(u)$. By homogeneity we can assume $u \in \mathbb{S}$. For all $n, k \geq 1$, define the tilted box

$$\mathrm{Box}(n,k) \coloneqq \Big\{ tu + x, x \in \overline{\mathrm{B}}(0,n) \cap u^{\perp}, t \in [0\,, nk] \Big\},$$

where $u^{\perp}$ the orthogonal complement of u. As in [13, bottom of page 198], one shows that there exists $n \geq 1$ such that for all $k \geq 1$,

$$\frac{\mathbb{E}\Big[\mathbf{T}_{\mathrm{Box}(n,k)}(0, nku)\Big]}{nk} < g(u).$$

Consequently, by Talagrand's inequality (see e.g. [2, Theorem 3.13]), there exists $\varepsilon > 0$ such that

$$\varliminf_{k\to\infty} -\frac{1}{nk} \log \mathbb{P}\left(\frac{\mathbf{T}_{\mathrm{Box}(n,k)}(0, nku)}{nk} \geq g(u) - 2\varepsilon \right) > 0 \tag{4.26}$$

Moreover, there exists $C_{11} > 0$ such that for all $k \geq 1$, there exists $v_1, \ldots, v_{\lfloor C_{11} k^{d-1} \rfloor} \in \mathbb{Z}^d$, such that the boxes $(\mathrm{Box}(n,k) + v_i)_{1 \leq i \leq \lfloor C_{11} k^{d-1} \rfloor}$ are pairwise disjoint and included in $[0\,, 2nk]^d$. This implies the inclusion

$$\mathrm{LD}_{2nk,[0\,,1]^d}(g,\varepsilon) \subseteq \bigcap_{i=1}^{\lfloor C_{11} k^{d-1} \rfloor} \left\{ \frac{\mathbf{T}_{v_i + \mathrm{Box}(n,k)}(v_i, v_i + nku)}{nk} \geq g(u) - 2\varepsilon \right\},$$

and the terms of the intersection are independent. Consequently, by stationarity,

$$-\frac{1}{nk^d} \log \mathbb{P}\Big(\mathrm{LD}_{2nk,[0\,,1]^d}(g,\varepsilon) \Big) \geq -\frac{\Big\lfloor C_{11} k^{d-1} \Big\rfloor}{nk^d} \log \mathbb{P}\left(\frac{\mathbf{T}_{\mathrm{Box}(n,k)}(0, nku)}{nk} \geq g(u) - 2\varepsilon \right).$$

Letting $k \to \infty$ and applying (4.26), we get $I_{[0\,,1]^d}(g) > 0$. □

4.3 Continuity of the rate function on $\mathcal{N}^*$: proof of Proposition 1.11

We already know by Lemma 1.2 (i) that the restriction of $I_{[0\,,1]^d}$ on $\mathcal{N}^*$ is lower semicontinuous, thus it is sufficient for the first part of the proposition to show that this function is upper semicontinuous. This follows from Lemma 4.4.

Lemma 4.4. *Let $g \in \mathcal{N}^*$ and $0 < \eta \leq \frac{b - \|g\|_{\mathrm{Hom}}}{2}$. Then there exists a constant $C_{12} > 0$, depending only on d, such that for all $p \geq 1$,*

$$I_{[0\,,1]^d}\left(g + \frac{\eta}{p} \|\cdot\| \right) \leq I_{[0\,,1]^d}(g) - \frac{C_{12}}{p} \log \nu([b - \eta\,, b]). \tag{4.27}$$

Indeed, let (g_n) be a sequence of norms in $\mathcal{N}^*$ converging to $g \in \mathcal{N}^*$ and η as in Lemma 4.4. Let $p \geq 1$. For large enough n,

$$g_n \leq g + \frac{\eta}{p} \|\cdot\|,$$

therefore by (4.27) and Proposition 3.1,

$$\overline{\lim_{n\to\infty}} I_{[0\,,1]^d}(g_n) \le I_{[0\,,1]^d}(g) - \frac{C_{12}}{p}\log\nu([b-\eta\,,b]).$$

Letting $p\to\infty$, we get

$$\overline{\lim_{n\to\infty}} I_{[0\,,1]^d}(g_n) \le I_{[0\,,1]^d}(g), \tag{4.28}$$

i.e. the restriction of $I_{[0\,,1]^d}$ on $\mathcal{N}^*$ is upper semicontinuous.

To prove the second part of Proposition 1.11, note that by Proposition 1.10, if $\nu(\{b\}) = 0$ and $g \in \mathcal{N}\setminus\mathcal{N}^*$ then $I_{[0\,,1]^d}(g) = \infty$, thus upper semicontinuity at g is straightforward. □

Proof of Lemma 4.4. Let $p, k \ge 2$. For all $v \in \mathbb{Z}^d$, v is said to be *hard* if at least one of its coordinates is a multiple of p, and *soft* otherwise. Recall the definition of $\mathrm{Tile}(\cdot,\cdot)$ given by (1.35). Let

$$S(k,p) \coloneqq \bigcup_{\substack{v\in[\![0\,,kp-1]\!]^d\\ v \text{ soft}}} \mathrm{Tile}(v,kp) \text{ and } H(k,p) \coloneqq [0\,,1]^d \setminus S(k,p). \tag{4.29}$$

The choice of notations S and H refers to *soft* and *hard* respectively. Note that

$$\#\Big\{v\in[\![0\,,kp-1]\!]^d \;\Big|\; v \text{ hard}\Big\} = k^d\Big(p^d-(p-1)^d\Big). \tag{4.30}$$

We define the metric $D_k^{(p)}$ by prescribing its gradient as in Lemma 2.11, Equation (2.19) with $X=[0\,,1]^d$ and

$$g_z \coloneqq \begin{cases} g & \text{if } z\in S(k,p),\\ (b-\eta)\|\cdot\| & \text{if } z\in H(k,p).\end{cases} \tag{4.31}$$

We first show that

$$\underline{I}_{[0\,,1]^d}\Big(D_k^{(p)}\Big) \le I_{[0\,,1]^d}(g) - \frac{C_{12}}{p}\nu([b-\eta\,,b]), \tag{4.32}$$

where $C_{12}>0$ only depends on d. Our construction is similar to the one performed in the proof of Lemma 4.1, except for the soft/hard distinction and the fixed number of tiles involved. Let $0<\varepsilon,\delta<\frac14$ and $n\ge1$. Let $m\coloneqq nkp$. For all $v\in[\![0\,,kp-1]\!]^d$, we define the tile

$$\mathrm{Tile}_n^\dagger(v) \coloneqq v + \left[\frac{\lfloor n\delta\rfloor}{n}\,,\frac{\lceil n(1-\delta)\rceil}{n}\right]^d. \tag{4.33}$$

Note that it is not equal to the tile $\mathrm{Tile}_n^*(v)$ we defined as in (4.3). For large enough n, the variables $\Big(\mathbf{T}_{n,\mathrm{Tile}_n^\dagger(v)}\Big)_{v\in[\![0\,,kp-1]\!]^d}$ live on pairwise disjoint subsets of $\mathbb{E}^d$ so they are independent. Let Corridor denote the set of edges included in $[0\,,m]^d$ but not in any $n\,\mathrm{Tile}_n^\dagger(v)$. This set satisfies

$$\#\,\mathrm{Corridor} = \#\mathrm{E}\Big([\![0\,,m]\!]^d\Big) - k^dp^d\#\mathrm{E}\Big(n\,\mathrm{Tile}_n^\dagger(0)\Big) \le d(m+1)^d - dk^dp^d(n(1-2\delta)-2)^d. \tag{4.34}$$

We define the event

$$\mathrm{Fav}^\dagger \coloneqq \left(\bigcap_{\substack{v\in[\![0\,,kp-1]\!]^d\\ v \text{ soft}}} \mathrm{LD}_{n,\mathrm{Tile}_n^\dagger(v)}(g,\delta^2)\right) \cap \left(\bigcap_{\substack{v\in[\![0\,,kp-1]\!]^d\\ v \text{ hard}}}\ \bigcap_{e\in\mathrm{E}\left(n\,\mathrm{Tile}_n^\dagger(v)\right)} \{\tau_e \ge b-\eta\}\right)$$
$$\cap \left(\bigcap_{e\in\mathrm{Corridor}} \{\tau_e\ge b-\varepsilon\}\right).$$

By stationarity, for all $v \in [\![0\,, kp-1]\!]^d$,

$$\begin{aligned}\mathbb{P}\Big(\mathrm{LD}_{n,\mathrm{Tile}_n^\dagger(v)}(g,\delta^2)\Big) &= \mathbb{P}\left(\mathrm{LD}_{n,\left[0\,,\frac{\lceil n(1-\delta)\rceil}{n}-\frac{\lfloor n\delta\rfloor}{n}\right]^d}(g,\delta^2)\right)\\ &= \mathbb{P}\left(\mathrm{LD}_{\lceil n(1-\delta)\rceil-\lfloor n\delta\rfloor,[0\,,1]^d}\left(g,\frac{n\delta^2}{\lceil n(1-\delta)\rceil-\lfloor n\delta\rfloor}\right)\right).\end{aligned}$$

Consequently, for large enough n,

$$\mathbb{P}\Big(\mathrm{LD}_{n,\mathrm{Tile}_n^\dagger(v)}(g,\delta^2)\Big) \geq \mathbb{P}\Big(\mathrm{LD}_{\lceil n(1-\delta)\rceil-\lfloor n\delta\rfloor,[0\,,1]^d}\Big(g,\delta^2\Big)\Big). \tag{4.35}$$

By independence, stationarity, (4.30) and (4.34), for large enough n,

$$\begin{aligned}\mathbb{P}\Big(\mathrm{Fav}^\dagger\Big) \geq \mathbb{P}\Big(\mathrm{LD}_{\lceil n(1-\delta)\rceil-\lfloor n\delta\rfloor,[0\,,1]^d}(g,\delta^2)\Big)^{(pk)^d} \cdot \nu([b-\eta\,,b])^{dk^d\left(p^d-(p-1)^d\right)(n+1)^d}\\ \cdot \nu([b-\varepsilon\,,b])^{d\left((m+1)^d-k^dp^d(n(1-2\delta)-2)^d\right)}.\end{aligned} \tag{4.36}$$

Thanks to Lemma 2.17 with $X=[0\,,1]^d$, $D=D_k^{(p)}$, $D'=\widetilde{\mathbf{T}}_{m,[0\,,1]^d}$, $\delta_1=\frac{n\delta}{m}$ and $\delta_2=\frac{n\delta^2}{m}$, on the event $\mathrm{Fav}^\dagger$, for all $x,y\in[0\,,1]^d$,

$$\widetilde{\mathbf{T}}_{m,[0\,,1]^d}(x,y) \geq D_k^{(p)}(x,y) - 3\left(\varepsilon+\frac{n\delta^2}{m}\cdot\frac{m}{n\delta}\right) = D_k^{(p)}(x,y)-3(\varepsilon+\delta),$$

i.e. $\mathrm{Fav}^\dagger \subseteq \mathrm{LD}_m^+\Big(D_k^{(p)},3(\varepsilon+\delta)\Big)$. Taking the log in (4.36), multiplying by $-\frac{1}{m^d}$ and letting $n\to\infty$, we therefore get

$$\begin{aligned}\varliminf_{m\to\infty} -\frac{1}{m^d}\log\mathbb{P}\Big(\mathrm{LD}_m^+\Big(D_k^{(p)},3(\varepsilon+\delta)\Big)\Big) \leq (1-2\delta)^d I_{[0\,,1]^d}(g) - \frac{C_{12}}{p}\log\nu([b-\eta\,,b])\\ - C_{13}\delta\log\nu([b-\varepsilon\,,b]),\end{aligned} \tag{4.37}$$

where C_{12} and C_{13} only depends on d. Choosing ε small enough then δ small enough and applying Corollary 3.2, we obtain (4.32).

Let $D^{(p)}$ be an adherence value of $\left(D_k^{(p)}\right)_{k\geq 2}$ and $x,y\in[0\,,1]^d$. We claim that

$$D^{(p)}(x,y) \geq g(x-y) + \frac{\eta}{p}\|x-y\|. \tag{4.38}$$

Indeed, let $x \overset{\gamma}{\underset{[0\,,1]^d}{\rightsquigarrow}} y$ be a Lipschitz path. In order to lower bound $D^{(p)}(x,y)$, we bound the integral in (2.19). We have

$$\begin{aligned}\int_0^{T_\gamma} g_{\gamma(t)}(\gamma'(t))\mathrm{d}t &= \int_0^{T_\gamma}\Big(\mathbb{1}_{H(k,p)}(\gamma(t))(b-\eta)\|\gamma'(t)\| + \mathbb{1}_{S(k,p)}(\gamma(t))g(\gamma'(t))\Big)\mathrm{d}t\\ &= \int_0^{T_\gamma} g(\gamma'(t))\mathrm{d}t + \int_0^{T_\gamma}\mathbb{1}_{H(k,p)}(\gamma(t))\big[(b-\eta)\|\gamma'(t)\| - g(\gamma'(t))\big]\mathrm{d}t.\end{aligned}$$

Applying Jensen's inequality on the first term and $\|g\|_{\mathrm{Hom}}\leq b-2\eta$ on the second gives

$$\int_0^{T_\gamma} g_{\gamma(t)}(\gamma'(t))\mathrm{d}t \geq g(x-y) + \eta\int_0^{T_\gamma}\mathbb{1}_{H(k,p)}(\gamma(t))\|\gamma'(t)\|\mathrm{d}t. \tag{4.39}$$

Using the notations $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d)$, $\gamma(t) = (\gamma_1(t), \dots, \gamma_d(t))$, standard integral manipulations give

$$\begin{aligned}
\int_0^{T_\gamma} \mathbb{1}_{H(k,p)}(\gamma(t))\|\gamma'(t)\|\mathrm{d}t &= \sum_{i=1}^d \int_0^{T_\gamma} \mathbb{1}_{H(k,p)}(\gamma(t))|\gamma_i'(t)|\mathrm{d}t \\
&\geq \sum_{i=1}^d \int_0^{T_\gamma} \mathbb{1}_{p\mathbb{Z}}(\lfloor kp\gamma_i(t)\rfloor)|\gamma_i'(t)|\mathrm{d}t \\
&\geq \sum_{i=1}^d \left| \int_0^{T_\gamma} \mathbb{1}_{p\mathbb{Z}}(\lfloor kp\gamma_i(t)\rfloor)\gamma_i'(t)\right|\mathrm{d}t \\
&= \sum_{i=1}^d \left| \int_{x_i}^{y_i} \mathbb{1}_{p\mathbb{Z}}(\lfloor kps\rfloor)\right|\mathrm{d}s \\
&= \sum_{i=1}^d \mathrm{Leb}\left([x_i\,, y_i] \cap \left(\bigcup_{j=0}^{k-1} \left[\frac{j}{k}\,, \frac{jp+1}{kp}\right[\right)\right),
\end{aligned}$$

thus

$$\begin{aligned}
\int_0^{T_\gamma} \mathbb{1}_{H(k,p)}(\gamma(t))\|\gamma'(t)\|\mathrm{d}t &\geq \sum_{i=1}^d \frac{\lfloor k|x_i - y_i|\rfloor}{pk} \\
&\geq \frac{\|x-y\|}{p} - \frac{d}{pk}.
\end{aligned}$$

Combining this inequality with (4.39), we get

$$\int_0^{T_\gamma} g_{\gamma(t)}(\gamma'(t))\mathrm{d}t \geq g(x-y) + \frac{\eta\|x-y\|}{p} - \frac{\eta d}{pk}.$$

Taking the infimum with respect to γ then letting $k \to \infty$ yields (4.38). □

5 Proof of the main result

In this section $X \in \mathcal{K}$ is fixed and ν satisfies Assumption 1. We prove Theorem 1.5. Thanks to Lemma 1.2, the remark following it and the compactness of $\mathcal{D}_X$, it is sufficient to show that the upper and lower rate functions defined in (1.30) and (1.29) are respectively upper bounded and lower bounded by the integral in (1.20), i.e. Propositions 5.1 and 5.3. Note that for every $D \in \mathcal{D}_X$, the integrand in (1.20) is well-defined almost everywhere, since the gradient of D at almost every point belongs to $\mathcal{N}$ (Proposition 2.10) and the lower and upper rate functions agree on $\mathcal{N}$ (Theorem 1.9). Besides $\underline{I}_{[0\,,1]^d}$ is measurable as a limit of measurable functions, and $z \mapsto (\mathrm{grad}\, D)_z$ is also measurable (see Proposition 2.8), therefore the integrand in (1.20) is measurable. Recall the definitions (1.36) and (1.37).

5.1 Upper bounding the upper rate function

Proposition 5.1. *Let $D \in \mathcal{D}_X$ and $(g_z)_{z\in X} = ((\mathrm{grad}\, D)_z)_{z\in X}$ its gradient. Then*

$$\overline{I}_X(D) \leq \int_X I_{[0\,,1]^d}(g_z)\mathrm{d}z. \tag{5.1}$$

The general idea is to "build" the metric D by assembling together tiles on which the rescaled passage time resembles g_z. We actually use this procedure only in the case where D has a constant gradient on each tile at a certain scale then use the properties of the rate function to extend the result to any metric. We first need Lemma 5.2 to account for the case where X is not of the form $\prod_{i=1}^{d}[t_i\,,t_i']$.

Lemma 5.2. *Assume $X \subseteq [0\,,\lambda]^d$. Then for all $g \in \mathcal{N}$,*

$$\overline{I}_X(g) \leq \lambda^d I_{[0\,,1]^d}(g). \tag{5.2}$$

Proof. Let $g \in \mathcal{N}, n \geq 1, \delta > 0$. Consider the convex, compact set

$$X^{-\delta} \coloneqq \Big\{ z \in X \;\Big|\; \mathrm{d}(z, X^{\mathrm{c}}) \geq \delta \Big\}. \tag{5.3}$$

Lemma B.5 states that

$$\max_{z \in X} \mathrm{d}(z, X^{-\delta}) \underset{\delta \to 0}{\longrightarrow} 0. \tag{5.4}$$

From now on δ is chosen small enough so that $X^{-\delta}$ is not empty.

Assume that the event $\mathrm{LD}_{\lceil n\lambda \rceil, [0\,,1]^d}(g, \delta^2)$ occurs. Let $x, y \in nX$. We have

$$\mathbf{T}_{nX}(x,y) \geq \mathbf{T}_{[0\,,\lceil n\lambda \rceil]^d}(x,y) \geq g(x-y) - \lceil n\lambda \rceil \delta^2. \tag{5.5}$$

Let $\hat{x}$ and $\hat{y}$ be the projections of x and y on $nX^{-\delta}$. For all $0 \leq i \leq \left\lceil \frac{bd\,\mathrm{diam}(X)}{a\delta} \right\rceil =: r$, let $x_i \coloneqq \hat{x} + i(\hat{y}-\hat{x})/r$. By convexity all the x_i belong to $nX^{-\delta}$. Lemma 2.3 implies that for all $0 \leq i \leq r-1$, any $\mathbf{T}_{[0\,,\lceil n\lambda \rceil]^d}$-geodesic from x_i to x_{i+1} is included in $\overline{\mathrm{B}}(x_i, n\delta) \subseteq nX$. Consequently,

$$\mathbf{T}_{nX}(x_i, x_{i+1}) = \mathbf{T}_{[0\,,\lceil n\lambda \rceil]^d}(x_i, x_{i+1}) \leq \frac{g(\hat{x}-\hat{y})}{r} + \lceil n\lambda \rceil \delta^2.$$

The triangle inequality then yields

$$\mathbf{T}_{nX}(\hat{x}, \hat{y}) \leq g(\hat{x}-\hat{y}) + r\lceil n\lambda \rceil \delta^2,$$

hence, by (1.16),

$$\mathbf{T}_{nX}(x,y) \leq g(x-y) + 4bn \max_{z \in X} \mathrm{d}(z, X^{-\delta}) + r\lceil n\lambda \rceil \delta^2. \tag{5.6}$$

Inequalities (5.5) and (5.6), along with (5.4) imply that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for large enough n,

$$\mathbb{P}(\mathrm{LD}_{n,X}(g, \varepsilon)) \geq \mathbb{P}\Big(\mathrm{LD}_{\lceil n\lambda \rceil, [0\,,1]^d}(g, \delta^2)\Big), \tag{5.7}$$

thus (5.2). □

Proof of Proposition 5.1. We first prove the bound (5.1) for three particular cases with extra regularity on D.

Case 1: Constant, noncritical gradient on each tile. Assume that there exists $k \geq 1$ and a family $(g^v)_{v \in \mathrm{V}_k^{\mathrm{ext}}(X)}$ of elements of $\mathcal{N}^*$ such that D is the metric defined like in Lemma (2.15), with $X_v = \mathrm{Tile}(v,k) \cap X$ and $D_v = g^v$. Let $0 < \delta, \varepsilon \leq 1$ and $n \geq 1$. For all $v \in [\![0\,, k-1]\!]^d$, we define

$$\mathrm{Tile}^{\ddagger}(v,k) \coloneqq (\mathrm{Tile}(v,k) \cap X)^{-\delta}, \tag{5.8}$$

with the notation defined in (5.3). We assume that δ is small enough for all the tiles $\mathrm{Tile}^{\ddagger}(v,k)$, with $v \in [\![0\,, k-1]\!]^d$, to be non empty. Consider the set

$$\mathrm{Corridor} \coloneqq \mathrm{E}_{\mathrm{ext}}(nX) \setminus \left(\bigcup_{v \in \mathrm{V}_k^{\mathrm{ext}}(X)} \mathrm{E}\Big(n\,\mathrm{Tile}^{\ddagger}(v,k)\Big) \right),$$

where $\mathrm{E}_{\mathrm{ext}}(nX)$ is defined in Subsection 1.6. It satisfies

$$\#\,\mathrm{Corridor} \leq \#\Big(\mathrm{V}_k^{\mathrm{ext}}(X)\Big) \left[\left(\frac{n}{k} + 2\right)^d - \left(\frac{n}{k} - 2n\delta - 2\right)^d \right]. \tag{5.9}$$

Consider the event

$$\mathrm{Fav}^{\ddagger} \coloneqq \left(\bigcap_{v \in \mathrm{V}_k^{\mathrm{ext}}(X)} \mathrm{LD}_{n,\mathrm{Tile}^{\ddagger}(v,k)}(g^v, \delta^2) \right) \cap \left(\bigcap_{e \in \mathrm{Corridor}} \{\tau_e \geq b - \varepsilon\} \right). \tag{5.10}$$

By independence and (5.9),

$$\mathbb{P}\Big(\mathrm{Fav}^{\ddagger}\Big) \geq \left(\prod_{v \in \mathrm{V}_k^{\mathrm{ext}}(X)} \mathbb{P}\Big(\mathrm{LD}_{n,\mathrm{Tile}^{\ddagger}(v,k)}(g^v, \delta^2)\Big) \right) \cdot \nu([b-\varepsilon\,, b])^{\#\mathrm{V}_k^{\mathrm{ext}}(X)\left[\left(\frac{n}{k}+2\right)^d - \left(\frac{n}{k}-2n\delta-2\right)^d\right]}. \tag{5.11}$$

We follow the same strategy as in Lemma 4.1. Assume that the event $\mathrm{Fav}^{\ddagger}$ occurs. Lemma 2.17 implies that for all $x, y \in X$,

$$\widetilde{\mathbf{T}}_{n,X}(x,y) \geq D(x,y) - 3\,\mathrm{diam}(X)(\varepsilon + \delta). \tag{5.12}$$

We define

$$\eta = \eta(\delta) \coloneqq \max_{v \in \mathrm{V}_k^{\mathrm{ext}}(X)} \max_{x \in X} \mathrm{d}\Big(x, \mathrm{Tile}^{\ddagger}(v,k)\Big). \tag{5.13}$$

By Lemma B.5,

$$\lim_{\delta \to 0} \eta(\delta) = 0. \tag{5.14}$$

Let $(x_\ell)_{1 \leq \ell \leq L}$ be a finite family of points in X such that the balls $\mathrm{B}(x_\ell, \varepsilon)$ cover X. By definition of D and convexity of the $(g^v)_{v \in \mathrm{V}_k^{\mathrm{ext}}(X)}$, for all $1 \leq \ell, \ell' \leq L$, there exist sequences $(x_\ell = y_0(\ell,\ell'), \ldots, y_{r(\ell,\ell')}(\ell,\ell') = x_{\ell'}) \in X^{r(\ell,\ell')+1}$ and $(v_0(\ell,\ell'), \ldots, v_{r(\ell,\ell')-1}(\ell,\ell')) \in \mathrm{V}_k^{\mathrm{ext}}(X)^{r(\ell,\ell')}$ such that for all $0 \leq j \leq r(\ell,\ell') - 1$, both $y_j(\ell,\ell')$ and $y_{j+1}(\ell,\ell')$ belong to $\mathrm{Tile}(v_j(\ell,\ell'),k)$, and

$$D(x_\ell, x_{\ell'}) \geq \sum_{j=0}^{r(\ell,\ell')-1} g^{v_j(\ell,\ell')}(y_j(\ell,\ell') - y_{j+1}(\ell,\ell')) - \varepsilon. \tag{5.15}$$

Let $\mathbf{r} \coloneqq \max_{1\le \ell,\ell'\le L} r(\ell,\ell')$ and fix $1 \le \ell, \ell' \le L$. We claim that

$$\widetilde{\mathbf{T}}_{n,X}(x_\ell, x_{\ell'}) \le D(x_\ell, x_{\ell'}) + \varepsilon + (\mathbf{r}+1)\Big[\delta^2 + 4b\eta(\delta)\Big]. \tag{5.16}$$

The arguments are similar to the ones used in the proof of (4.19). To lighten the notations we omit the dependancy in (ℓ,ℓ') in the proof of this claim. For all $0 \le j \le r$, let p_j denote the projection onto $\mathrm{Tile}^{\ddagger}(v_j,k)$. By triangle inequality,

$$\begin{aligned}\widetilde{\mathbf{T}}_{n,X}(x_\ell, x_{\ell'}) \le \widetilde{\mathbf{T}}_{n,X}(x_\ell, p_0(x_\ell)) + \sum_{j=0}^{r} \widetilde{\mathbf{T}}_{n,X}(p_j(y_j), p_j(y_{j+1})) \\ + \sum_{j=0}^{r-1} \widetilde{\mathbf{T}}_{n,X}(p_j(y_{j+1}), p_{j+1}(y_{j+1})) + \widetilde{\mathbf{T}}_{n,X}(p_r(x_{\ell'}), x_{\ell'}).\end{aligned} \tag{5.17}$$

By definition of $\mathrm{Fav}^{\ddagger}$, the second term in the right-hand side of (5.17) satisfies

$$\sum_{j=0}^{r} \widetilde{\mathbf{T}}_{n,X}(p_j(y_j), p_j(y_{j+1})) \le \sum_{j=0}^{r} \Big[g^{v_j}(p_j(y_j) - p_j(y_{j+1})) + \delta^2\Big].$$

Applying the definition of η and (1.16), we obtain

$$\sum_{j=0}^{r} \widetilde{\mathbf{T}}_{n,X}(p_j(y_j), p_j(y_{j+1})) \le \sum_{j=0}^{r} \Big[g^{v_j}(y_j - y_{j+1}) + \delta^2 + 2b\eta\Big],$$

thus, by (5.15),

$$\begin{aligned}\sum_{j=0}^{r} \widetilde{\mathbf{T}}_{n,X}(p_j(y_j), p_j(y_{j+1})) &\le \sum_{j=0}^{r} g^{v_j}(y_j - y_{j+1}) + (\mathbf{r}+1)(\delta^2 + 2b\eta) \\ &\le D(x_\ell, x_{\ell'}) + \varepsilon + (\mathbf{r}+1)(\delta^2 + 2b\eta).\end{aligned} \tag{5.18}$$

Applying the definition of η and (1.16) to the three other terms in the right-hand side of (5.17), we get

$$\widetilde{\mathbf{T}}_{n,X}(x_\ell, x_{\ell'}) \le D(x_\ell, x_{\ell'}) + \varepsilon + (\mathbf{r}+1)(\delta^2 + 2b\eta) + 2b\eta + 2b\mathbf{r}\eta, \tag{5.19}$$

hence (5.16). Equations (5.12) and (5.16), alongside with the fact that the balls $\mathrm{B}(x_\ell, \varepsilon)$ cover X and the limit (5.14) imply that for all $\varepsilon > 0$, for small enough $\delta > 0$, for all $n \ge 1$,

$$\mathrm{Fav}^{\ddagger} \subseteq \mathrm{LD}_{n,X}(D, C_{14}\varepsilon), \tag{5.20}$$

where C_{14} only depends on $\mathrm{diam}(X)$.

From (5.11) and (5.20) we deduce

$$\begin{aligned}&\overline{\lim_{n\to\infty}} -\frac{1}{n^d} \log \mathbb{P}(\mathrm{LD}_{n,X}(D, C_{14}\varepsilon)) \\ &\le \sum_{v \in \mathrm{V}_k^{\mathrm{ext}}(X)} \overline{I}_{\mathrm{Tile}^{\ddagger}(v,k)}(g^v) - \frac{\#\mathrm{V}_k^{\mathrm{ext}}(X)}{k^d}\Big(1 - (1-2k\delta)^d\Big) \log \nu([b-\varepsilon\,, b]).\end{aligned}$$

By stationarity and (5.2), we get

$$\begin{aligned}&\overline{\lim_{n\to\infty}} -\frac{1}{n^d} \log \mathbb{P}(\mathrm{LD}_{n,X}(D, C_{14}\varepsilon)) \\ &\le \frac{1}{k^d} \sum_{v \in \mathrm{V}_k^{\mathrm{ext}}(X)} I_{[0\,,1]^d}(g^v) - \frac{\#\mathrm{V}_k^{\mathrm{ext}}(X)}{k^d}\Big(1 - (1-2k\delta)^d\Big) \log \nu([b-\varepsilon\,, b]).\end{aligned}$$

Letting $\delta \to 0$ then $\varepsilon \to 0$ yields

$$\overline{I}_X(D) \leq \frac{1}{k^d} \sum_{v \in \mathrm{V}_k^{\mathrm{ext}}(X)} I_{[0,1]^d}(g^v).$$

Lemma 2.15 implies that for all z in the interior of $\mathrm{Tile}(v,k) \cap X$, $g_z = g^v$, hence

$$\overline{I}_X(D) \leq \int_X I_{[0,1]^d}(g_z)\mathrm{d}z + \mathrm{Leb}\left(\left(\bigcup_{v \in \mathrm{V}_k^{\mathrm{ext}}(X)} \mathrm{Tile}(v,k)\right) \setminus X\right) \max_{v \in \mathrm{V}_k^{\mathrm{ext}}(X)} I_{[0,1]^d}(g^v). \tag{5.21}$$

Let K be a multiple of k. Then D may be seen as a metric built from uniform tiles of size $1/K$ rather than $1/k$, thus inequality (5.21) may be enhanced to

$$\overline{I}_X(D) \leq \int_X I_{[0,1]^d}(g_z)\mathrm{d}z + \mathrm{Leb}\left(\left(\bigcup_{v \in \mathrm{V}_K^{\mathrm{ext}}(X)} \mathrm{Tile}(v,k)\right) \setminus X\right) \max_{v \in \mathrm{V}_k^{\mathrm{ext}}(X)} I_{[0,1]^d}(g^v).$$

Note that the maximum in the second term is considered over $\mathrm{V}_k^{\mathrm{ext}}(X)$ rather than $\mathrm{V}_K^{\mathrm{ext}}(X)$. It is finite thanks to Proposition 1.10. Letting $K \to \infty$ and applying (1.38) yield (5.1).

Case 2: Continuous, non critical gradient. Assume that there exists a continuous map

$$\begin{aligned} X &\longrightarrow \mathcal{N} \\ z &\longmapsto \hat{g}_z \end{aligned}$$

such that $b' \coloneqq \sup_{z \in X} \|\hat{g}_z\|_{\mathrm{Hom}} < b$ and D satisfies (2.19) with $(\hat{g}_z)_{z \in X}$. In this case, by Lemma 2.11 (iv), for all $z \in \mathring{X}$, $\hat{g}_z = g_z$, thus (5.1) is equivalent to

$$\overline{I}_X(D) \leq \int_X I_{[0,1]^d}(\hat{g}_z)\mathrm{d}z. \tag{5.22}$$

For all $k \geq 1$, we consider the metric $D_k \in \mathcal{D}_X$ defined as in Lemma 2.15, with $X_v = \mathrm{Tile}(v,k) \cap X$ and $D_v = g_v^{(k)} \coloneqq \hat{g}_z$, with z being any fixed point in $\mathrm{Tile}(v,k) \cap X$, for all $v \in \mathrm{V}_k^{\mathrm{ext}}(X)$. Then D_k falls under Case 1, therefore

$$\overline{I}_X(D_k) \leq \int_X I_{[0,1]^d}\left(g^{(k)}_{\frac{1}{k}\lfloor kz \rfloor}\right)\mathrm{d}z. \tag{5.23}$$

Besides $g^{(k)}_{\frac{1}{k}\lfloor kz \rfloor}$ converges to $\hat{g}_z$ in $\mathcal{C}_{\mathrm{Hom}}\left(\mathbb{R}^d, \mathbb{R}\right)$ as $k \to \infty$, uniformly on X. Since the restriction of $I_{[0,1]^d}$ on the compact $\{g \in \mathcal{N} \mid a\|\cdot\| \leq g \leq b'\|\cdot\|\}$ is continuous and bounded (see Propositions 1.10 and 1.11),

$$\int_X I_{[0,1]^d}\left(g^{(k)}_{\frac{1}{k}\lfloor kz \rfloor}\right)\mathrm{d}z \xrightarrow[k \to \infty]{} \int_X I_{[0,1]^d}(\hat{g}_z)\mathrm{d}z. \tag{5.24}$$

Moreover, the convergence of $g^{(k)}_{\frac{1}{k}\lfloor kz \rfloor}$ to $\hat{g}_z$ and Lemma 2.11 (ii) imply the convergence of D_k to D. Thus by lower semicontinuity of $\overline{I}_X$, letting $k \to \infty$ in (5.23) yields (5.22).

Case 3: Non critical gradient. Assume that $b' \coloneqq \sup_{z\in X}\|g_z\|_{\mathrm{Hom}} < b$. We use a routine convolution argument (see e.g. [4, Theorem 4.22]) to regularize $z \mapsto g_z$. Let (ξ^n) be a sequence of test functions from $\mathbb{R}^d$ to $[0\,,\infty]$, of integral 1, such that the support of ξ^n is included in $\overline{\mathrm{B}}(0,1/n)$. For all $z \in X$ and $u \in \mathbb{R}^d$, we consider the convolution

$$g_z^n(u) \coloneqq \int_{\mathbb{R}^d} g_{z-s}(u)\xi^n(s)\mathrm{d}s \le b'\|u\|, \tag{5.25}$$

with the convention $g_{z-s} = b'\|\cdot\|$ if $z - s \notin X$. For all $u \in \mathbb{R}^d$, as $n \to \infty$, $g_\cdot^n(u)$ converges to $g_\cdot(u)$ in $\mathrm{L}^1(X, \mathbb{R})$, thus almost everywhere on X along a subsequence. By a standard diagonal argument, this convergence is still true for all $u \in \mathbb{Q}^d$ almost everywhere on X. As the g_z and g_z^n are b-Lipschitz, there exists an extraction φ such that for almost every $z \in X$,

$$g_z^{\varphi(n)} \xrightarrow[n\to\infty]{\|\cdot\|_{\mathrm{Hom}}} g_z. \tag{5.26}$$

Let $D^{\varphi(n)}$ denote the metric defined by (2.19) with the function $z \mapsto g_z^{\varphi(n)}$. Since $\mathcal{D}_X$ is compact (see Proposition 2.1) there exists an extraction ψ such that $D^{\varphi\circ\psi(n)}$ converges to some $D' \in \mathcal{D}_X$. We claim that $D' \ge D$. Indeed, for all $x, y \in X$ and Lipschitz paths $x \overset{\gamma}{\underset{X}{\rightsquigarrow}} y$, Fubini's theorem yields

$$\begin{aligned}\int_0^{T_\gamma} g_{\gamma(t)}^{\varphi\circ\psi(n)}(\gamma'(t))\mathrm{d}t &= \int_0^{T_\gamma}\left(\int_{\mathbb{R}^d} g_{\gamma(t)-s}(\gamma'(t))\xi^{\varphi\circ\psi(n)}(s)\mathrm{d}s\right)\mathrm{d}t\\ &= \int_{\mathbb{R}^d}\left(\int_0^{T_\gamma} g_{\gamma(t)-s}(\gamma'(t))\mathrm{d}t\right)\xi^{\varphi\circ\psi(n)}(s)\mathrm{d}s.\end{aligned}$$

Applying (2.12), we get

$$\int_0^{T_\gamma} g_{\gamma(t)}^{\varphi\circ\psi(n)}(\gamma'(t))\mathrm{d}t \ge \int_{\mathbb{R}^d} D(x-s, y-s)\xi^{\varphi\circ\psi(n)}(s)\mathrm{d}s,$$

thus

$$D^{\phi\circ\psi(n)}(x,y) \ge \int_{\mathbb{R}^d} D(x-s, y-s)\xi^{\varphi\circ\psi(n)}(s)\mathrm{d}s. \tag{5.27}$$

By a standard regularization argument (see [4, Proposition 4.21]) the right-hand side of (5.27) converges to $D(x, y)$ as $n \to \infty$, thus $D'(x, y) \ge D(x, y)$.

Besides, for all $n \ge 1$, $z \mapsto g_z^{\varphi\circ\psi(n)}$ is continuous on X and $\sup_{z\in X}\left\|g_z^{\varphi\circ\psi(n)}\right\|_{\mathrm{Hom}} < b$ by (5.25), therefore by (5.22) (see Case 2),

$$\overline{I}_X(D^{\varphi\circ\psi(n)}) \le \int_X I_{[0\,,1]^d}(g_z^{\varphi\circ\psi(n)})\mathrm{d}z.$$

The restriction of $I_{[0\,,1]^d}$ on $\{g \in \mathcal{N} \mid a\|\cdot\| \le g \le b'\|\cdot\|\}$ is continuous and bounded (see Proposition 1.11), therefore by (5.26),

$$\varlimsup_{n\to\infty} \overline{I}_X(D^{\varphi\circ\psi(n)}) \le \int_X I_{[0\,,1]^d}(g_z)\mathrm{d}z.$$

Applying lower semicontinuity and Proposition 3.1, we obtain

$$\overline{I}_X(D) \le \overline{I}_X(D') \le \int_X I_{[0\,,1]^d}(g_z)\mathrm{d}z. \tag{5.28}$$

General case. Let D^n denote the metric defined by (2.19) with

$$g_z^n := \frac{n-1}{n} g_z + \frac{a}{n} \|\cdot\|. \tag{5.29}$$

Equation (2.12) implies that for all Lipschitz paths $x \overset{\gamma}{\underset{X}{\rightsquigarrow}} y$,

$$\begin{aligned} \int_0^{T_\gamma} g^n_{\gamma(t)}(\gamma'(t))\mathrm{d}t &= \frac{n-1}{n} \int_0^{T_\gamma} g_{\gamma(t)}(\gamma'(t))\mathrm{d}t + \frac{a}{n} \int_0^{T_\gamma} \|\gamma'(t)\|\mathrm{d}t \\ &\geq \frac{n-1}{n} D(x,y) + \frac{a}{n}\|x-y\|. \end{aligned} \tag{5.30}$$

Consequently, for all $z \in X$, $(\operatorname{grad} D^n)_z \geq g_z^n$. Besides, Lemma 2.11 implies the converse inequality, therefore $(\operatorname{grad} D^n)_z = g_z^n$. The metric D^n falls under Case 3, thus

$$\overline{I}_X(D^n) \leq \int_X I_{[0,1]^d}(g_z^n)\mathrm{d}z.$$

Letting $n \to \infty$ gives (5.1) by lower semicontinuity and monotone convergence. □

5.2 Lower bounding the lower rate function

Proposition 5.3. *Let $D \in \mathcal{D}_X$ and $(g_z)_{z\in X}$ its gradient. Then*

$$\underline{I}_X(D) \geq \int_X I_{[0,1]^d}(g_z)\mathrm{d}z. \tag{5.31}$$

Lemma 5.4 provides a link between the lower rate function evaluated on a metric and on a rescaled, translated version of it, as defined by (2.27) and (2.30).

Lemma 5.4. *Let $k \geq 1, v \in \mathbb{Z}^d$ and $D \in \mathcal{D}_{\mathrm{Tile}(v,k)}$. Then*

$$\underline{I}_{\mathrm{Tile}(v,k)}(D) \geq \frac{1}{k^d} \underline{I}_{[0,1]^d}(\mathrm{Tr}_{-v}(\mathrm{Sc}_k(D))). \tag{5.32}$$

Proof. Let $\varepsilon > 0$, $m \geq 1$ and n be the unique integer such that $nk \leq m < (n+1)k$. Note that for all $x \in nk\,\mathrm{Tile}(v,k)$,

$$\left\| \frac{m}{nk} \cdot x - x \right\| \leq \frac{\|x\|}{n} \leq \|v\| + d. \tag{5.33}$$

In particular,

$$nk\,\mathrm{Tile}(v,k) \subseteq m\,\mathrm{Tile}(v,k) + \overline{\mathrm{B}}(0, \|v\| + d). \tag{5.34}$$

Consider the event

$$\mathrm{Fav}^{**} := \mathrm{LD}_{m,\mathrm{Tile}(v,k)}(D,\varepsilon) \cap \left(\bigcap_{e \in \mathrm{E}_{\mathrm{ext}}\left(m\,\mathrm{Tile}(v,k) + \overline{\mathrm{B}}\left(0, \|v\|+d\right)\right) \setminus \mathrm{E}_{\mathrm{ext}}(m\,\mathrm{Tile}(v,k))} \{\tau_e \geq b - \varepsilon\} \right).$$

One easily checks that

$$\varliminf_{m\to\infty} -\frac{1}{m^d} \log \mathbb{P}(\mathrm{Fav}^{**}) = \varliminf_{m\to\infty} -\frac{1}{m^d} \log \mathbb{P}\Big(\mathrm{LD}_{m,\mathrm{Tile}(v,k)}(D,\varepsilon)\Big). \tag{5.35}$$

Assume that the event Fav^{**} occurs. Let $x, y \in nk\,\mathrm{Tile}(v,k)$. By (5.34),

$$\mathbf{T}_{nk\,\mathrm{Tile}(v,k)}(x,y) \geq \mathbf{T}_{m\,\mathrm{Tile}(v,k)+\overline{\mathrm{B}}\left(0,\|v\|+d\right)}(x,y).$$

Triangle inequality and (1.16) yield

$$\mathbf{T}_{nk\,\mathrm{Tile}(v,k)}(x,y) \geq \mathbf{T}_{m\,\mathrm{Tile}(v,k)+\overline{\mathrm{B}}\left(0,\|v\|+d\right)}\left(\frac{mx}{nk},\frac{my}{nk}\right) - 2b(\|v\|+d). \tag{5.36}$$

Let $\frac{mx}{nk} \overset{\sigma}{\rightsquigarrow} \frac{my}{nk}$ be a $\mathbf{T}_{m\,\mathrm{Tile}(v,k)+\overline{\mathrm{B}}\left(0,\|v\|+d\right)}$-geodesic and γ the path obtained by replacing excursions of σ outside $m\,\mathrm{Tile}(v,k)$ by segments, as in (2.46). Since edges in $\mathrm{E}_{\mathrm{ext}}\Big(m\,\mathrm{Tile}(v,k)+\overline{\mathrm{B}}(0,\|v\|+d)\Big) \setminus \mathrm{E}_{\mathrm{ext}}(m\,\mathrm{Tile}(v,k))$ have a passage time greater than $b-\varepsilon$,

$$\mathbf{T}_{m\,\mathrm{Tile}(v,k)+\overline{\mathrm{B}}\left(0,\|v\|+d\right)}\left(\frac{mx}{nk},\frac{my}{nk}\right) = \mathbf{T}_X(\sigma) \geq \frac{b-\varepsilon}{b}\mathbf{T}_X(\gamma) \geq \frac{b-\varepsilon}{b}\mathbf{T}_{m\,\mathrm{Tile}(v,k)}\left(\frac{mx}{nk},\frac{my}{nk}\right), \tag{5.37}$$

with the definition (2.35). Combining (5.36) and (5.37), we get

$$\begin{aligned}\mathbf{T}_{nk\,\mathrm{Tile}(v,k)}(x,y) &\geq \frac{b-\varepsilon}{b}\mathbf{T}_{m\,\mathrm{Tile}(v,k)}\left(\frac{mx}{nk},\frac{my}{nk}\right) - 2b(\|v\|+d)\\ &\geq \frac{m(b-\varepsilon)}{b}\left[D\left(\frac{x}{nk},\frac{y}{nk}\right)-\varepsilon\right] - 2b(\|v\|+d),\end{aligned}$$

thus for small enough $\varepsilon > 0$ and large enough m (thus large n), for all $x, y \in \mathrm{Tile}(v,k)$,

$$\begin{aligned}\widetilde{\mathbf{T}}_{nk,\mathrm{Tile}(v,k)}(x,y) &\geq \frac{b-\varepsilon}{b}[D(x,y)-\varepsilon] - \frac{2b}{nk}(\|v\|+d)\\ &\geq D(x,y) - \frac{\varepsilon}{b}\operatorname{diam}(\mathrm{Tile}(v,k)) - \varepsilon - \left(1+\frac{1}{n}\right)\cdot\frac{2b}{m}(\|v\|+d),\end{aligned}$$

therefore

$$\mathrm{Fav}^{**} \subseteq \mathrm{LD}^{+}_{nk,\mathrm{Tile}(v,k)}\left(D, C_{15}\left(\varepsilon + \frac{\|v\|+d}{m}\right)\right) = \mathrm{LD}^{+}_{n,\mathrm{Tile}(v,1)}\left(\mathrm{Sc}_k(D), kC_{15}\left(\varepsilon+\frac{\|v\|+d}{m}\right)\right), \tag{5.38}$$

where C_{15} only depends on b and LD^{+} is defined in (3.1). Equations (5.35) and (5.38) lead to

$$\underline{I}_{\mathrm{Tile}(v,k)}(D) \geq \frac{1}{k^d}\underline{I}_{\mathrm{Tile}(v,1)}(\mathrm{Sc}_k(D)). \tag{5.39}$$

We conclude by stationarity. □

Proof of Proposition 5.3. Let $k \geq 1$, $\varepsilon > 0$. Define

$$X_k \coloneqq \bigcup_{v \in \mathrm{V}_k^{\mathrm{int}}(X)} \mathrm{Tile}(v,k). \tag{5.40}$$

Note that for all $z \in \mathring{X}$, $z \in X_k$ for large enough k. The idea is to show that a large deviation event around D is included in an intersection of large deviation events around restrictions of D (see (2.35)) on tiles of the type $\mathrm{Tile}(v,k)$. We first need to make minor adjustments on the boundary of tiles to ensure independence. For all $v \in \mathrm{V}_k^{\mathrm{int}}(X)$, let $\mathrm{E}_{\mathrm{bd}}(n\,\mathrm{Tile}(v,k))$ denote the set of edges belonging to $\mathrm{E}_{\mathrm{ext}}(n\,\mathrm{Tile}(v,k))$, but not included in the *interior* of $n\,\mathrm{Tile}(v,k)$. Consider a family of $\#(\mathrm{V}_k^{\mathrm{int}}(X))+1$

independent configurations $\Big(\tau, \big(\tau^{(v)}\big)_{v\in \mathrm{V}_k^{\mathrm{int}}(X)}\Big)$, each with distribution $\nu^{\otimes \mathbb{E}^d}$. For all $v \in \mathrm{V}_k^{\mathrm{int}}(X), e \in \mathbb{E}^d$, define

$$\tau_e^{[v]} \coloneqq \begin{cases} \tau_e^{(v)} & \text{if } e \in \mathrm{E}_{\mathrm{bd}}(n\,\mathrm{Tile}(v,k)), \\ \tau_e & \text{otherwise.} \end{cases} \tag{5.41}$$

Note that $\Big(\big(\tau_e^{[v]}\big)_{e\in \mathrm{E}_{\mathrm{ext}}(n\,\mathrm{Tile}(v,k))}, v \in \mathrm{V}_k^{\mathrm{int}}(X)\Big)$ are independent. Consider the event

$$\mathrm{Fav}^{\dagger\dagger} \coloneqq \Big\{\tau \in \mathrm{LD}^+_{n,X}(D,\varepsilon)\Big\} \cap \left(\bigcap_{v\in \mathrm{V}_k^{\mathrm{int}}(X)} \bigcap_{e\in \mathrm{E}_{\mathrm{bd}}(n\,\mathrm{Tile}(v,k))} \Big\{\tau_e^{(v)} \geq b-\varepsilon\Big\}\right).$$

By independence,

$$\mathbb{P}\Big(\mathrm{Fav}^{\dagger\dagger}\Big) \geq \mathbb{P}\Big(\mathrm{LD}^+_{n,X}(D,\varepsilon)\Big) \cdot \nu([b-\varepsilon\,,b])^{\#\left(\mathrm{V}_k^{\mathrm{int}}(X)\right)\cdot\#\left(\mathrm{E}_{\mathrm{bd}}(n\,\mathrm{Tile}(0,k))\right)}. \tag{5.42}$$

Let $v \in \mathrm{V}_k^{\mathrm{int}}(X)$ and $n \geq 1$. Equation (2.3) implies that any $\widetilde{\mathbf{T}}_{n,X}$-geodesic or $\widetilde{\mathbf{T}}^{[v]}_{n,X}$-geodesic σ has length at most $\frac{b\,\mathrm{diam}(X)}{a}$. Besides, for all $e \in \mathbb{E}^d$,

$$\tau_e^{[v]} \geq \tau_e - \varepsilon,$$

thus

$$\mathrm{Fav}^{\dagger\dagger} \subseteq \left\{\tau^{[v]} \in \mathrm{LD}^+_{n,\mathrm{Tile}(v,k)}\Big(D_{|\mathrm{Tile}(v,k)}, \Big(\frac{b\,\mathrm{diam}(X)}{a}+1\Big)\varepsilon\Big)\right\}.$$

Consequently,

$$\mathrm{Fav}^{\dagger\dagger} \subseteq \bigcap_{v\in \mathrm{V}_k^{\mathrm{int}}(X)} \left\{\tau^{[v]} \in \mathrm{LD}^+_{n,\mathrm{Tile}(v,k)}\Big(D_{|\mathrm{Tile}(v,k)}, \Big(\frac{b\,\mathrm{diam}(X)}{a}+1\Big)\varepsilon\Big)\right\}. \tag{5.43}$$

The events in the intersection are independent. Applying (5.42), we get

$$\begin{aligned} -\frac{1}{n^d}\log \mathbb{P}\Big(\mathrm{LD}^+_{n,X}(D,\varepsilon)\Big) - \frac{\#(\mathrm{V}_k^{\mathrm{int}}(X))\cdot\#(\mathrm{E}_{\mathrm{bd}}(n\,\mathrm{Tile}(0,k)))}{n^d}\log\nu([b-\varepsilon\,,b]) \\ \geq \sum_{v\in \mathrm{V}_k^{\mathrm{int}}(X)} -\frac{1}{n^d}\log\mathbb{P}\left(\mathrm{LD}^+_{n,\mathrm{Tile}(v,k)}\Big(D_{|\mathrm{Tile}(v,k)}, \Big(\frac{b\,\mathrm{diam}(X)}{a}+1\Big)\varepsilon\Big)\right). \end{aligned}$$

Thanks to Corollary 3.2, letting $n \to \infty$ then $\varepsilon \to 0$, we obtain

$$\underline{I}_X(D) \geq \sum_{v\in \mathrm{V}_k^{\mathrm{int}}(X)} \underline{I}_{\mathrm{Tile}(v,k)}\Big(D_{|\mathrm{Tile}(v,k)}\Big).$$

Applying Lemma 5.4 yields

$$\begin{aligned} \underline{I}_X(D) &\geq \frac{1}{k^d}\sum_{v\in \mathrm{V}_k^{\mathrm{int}}(X)} \underline{I}_{[0\,,1]^d}\Big(\mathrm{Tr}_{-v}\Big(\mathrm{Sc}_k\Big(D_{|\mathrm{Tile}(v,1)}\Big)\Big)\Big) \\ &\geq \int_{X_k} \underline{I}_{[0\,,1]^d}\Big(\mathrm{Tr}_{-v_k(z)}\Big(\mathrm{Sc}_k\Big(D_{|\mathrm{Tile}(v_k(z),1)}\Big)\Big)\Big)\mathrm{d}z, \end{aligned} \tag{5.44}$$

where $v_k(z)$ is such that $z \in \mathrm{Tile}(v_k(z), k)$, chosen in a measurable way in case of non-unicity. Proposition 2.10 implies that for almost every $z \in X$, for all $\varepsilon > 0$, for large enough k, for all $x, y \in [0\,,1]^d$,

$$\begin{aligned}\mathrm{Tr}_{-v_k(z)}(\mathrm{Sc}_k(D))(x,y) &= kD\Big(\frac{x+v_k(z)}{k}, \frac{y+v_k(z)}{k}\Big)\\ &\geq k\Big[g_z\Big(\frac{x}{k}-\frac{y}{k}\Big) - \Big(\Big\|\frac{x+v_k(z)}{k} - z\Big\| + \Big\|\frac{y+v_k(z)}{k} - z\Big\|\Big)\varepsilon\Big]\\ &\geq g_z(x-y) - 2\varepsilon d.\end{aligned}$$

Consequently, for almost every $z \in X$, for all $\varepsilon > 0$, for large enough k,

$$\underline{I}_{[0\,,1]^d}\Big(\mathrm{Tr}_{-v_k(z)}\Big(\mathrm{Sc}_k\Big(D_{|\mathrm{Tile}(v_k(z),1)}\Big)\Big)\Big) \geq \varliminf_{n\to\infty} -\frac{1}{n^d}\log\mathbb{P}\Big(\mathrm{LD}^+_{n,[0\,,1]^d}(g_z, 3d\varepsilon)\Big).$$

Letting $k \to \infty$ then $\varepsilon \to 0$ and applying Corollary 3.2 again, we get, for almost every $z \in X$,

$$\varliminf_{k\to\infty} \underline{I}_{[0\,,1]^d}\Big(\mathrm{Tr}_{-v_k(z)}\Big(\mathrm{Sc}_k\Big(D_{|\mathrm{Tile}(v_k(z),1)}\Big)\Big)\Big) \geq I_{[0\,,1]^d}(g_z). \tag{5.45}$$

We conclude the proof by applying Fatou's lemma to (5.44) and using (5.45). □

6 Proof of the corollaries

Lemma 6.1 gives a LDP for any process that is the image of $(\widetilde{\mathbf{T}}_{n,X})_{n\geq 1}$ through a continuous map. We recall that it is a special case of the contraction principle (see e.g. [11, Theorem 4.2.1]).

Lemma 6.1. *Let $X \in \mathcal{K}$. Under Assumption 1, for every Hausdorff topological space $\mathcal{Y}$ and continuous map $f : \mathcal{D}_X \to \mathcal{Y}$, the process $\Big(f(\widetilde{\mathbf{T}}_{n,X})\Big)_{n\geq 1}$ satisfies the large deviation principle with the good rate function*

$$\begin{aligned}&\mathcal{Y} \longrightarrow [0\,,\infty]\\ &y \longmapsto \min_{\substack{D\in\mathcal{D}_X:\\ f(D)=y}} I_X(D) = \min_{\substack{D\in\mathcal{D}_X:\\ f(D)=y}} \int_X I_{[0\,,1]^d}(\mathrm{grad}\, D)_z \mathrm{d}z.\end{aligned} \tag{6.1}$$

However, as Corollaries 1.6 and 1.7 are stated with assumptions on ν milder than Assumption 1, we need extra work consisting in taking the limit as $\alpha \to 0$ of the model constructed with the passage times

$$\tau_e^{(\alpha)} \coloneqq \tau_e \vee \alpha. \tag{6.2}$$

Let $\nu^{(\alpha)}$ denote their distribution. Let $\mathcal{D}_X^{(\alpha)}$ denote the space defined as in Definition 1.3, with $\alpha > 0$ instead of a. The set $\mathcal{N}^{(\alpha)}$ is defined likewise. Let $\mathcal{SN}$ denote the set of seminorms g such that $g \leq b\|\cdot\|$ and $\mathcal{SN}^*$ the set of seminorms g such that $g \leq (b-\eta)\|\cdot\|$, for some $\eta > 0$. Both are endowed with the topology induced by $\|\cdot\|_{\mathrm{Hom}}$ (see (1.44)). For every rate function introduced in this section we will mark with the exponent $\cdot^{(\alpha)}$ the corresponding function when ν is replaced by $\nu^{(\alpha)}$.

6.1 Preliminaries: limit of the elementary rate function

For all $g \in \mathcal{SN}$, we introduce

$$\overline{I}^+_{[0\,,1]^d}(g) \coloneqq \varlimsup_{\varepsilon\to 0}\uparrow \varlimsup_{n\to\infty} -\frac{1}{n^d}\log\mathbb{P}\Big(\mathrm{LD}^+_{n,X}(g,\varepsilon)\Big) \tag{6.3}$$

$$\text{and } \underline{I}^+_{[0\,,1]^d}(g) \coloneqq \varlimsup_{\varepsilon\to 0}\uparrow \varliminf_{n\to\infty} -\frac{1}{n^d}\log\mathbb{P}\Big(\mathrm{LD}^+_{n,X}(g,\varepsilon)\Big), \tag{6.4}$$

with the definition (3.1). The functions $\overline{I}^{+}_{[0,1]^d}$ and $\underline{I}^{+}_{[0,1]^d}$ are lower semicontinuous for the norm $\|\cdot\|_{\mathrm{Hom}}$. In this subsection we show that they are equal and have a simple expression using the elementary rate function under the distribution $\nu^{(\alpha)}$.

Lemma 6.2. *For all $g \in \mathcal{SN}$ and $\alpha > 0$,*

$$\overline{I}^{+,(\alpha)}_{[0,1]^d}(g) = \underline{I}^{+,(\alpha)}_{[0,1]^d}(g) = I^{(\alpha)}_{[0,1]^d}([\alpha\|\cdot\|] \vee g). \tag{6.5}$$

We denote by $I^{+,(\alpha)}_{[0,1]^d}(g)$ this number.

Proof. Let $g \in \mathcal{SN}$ and $\varepsilon > 0$. Theorem 1.5 with $X = [0,1]^d$ and the distribution $\nu^{(\alpha)}$, applied to the compact set $A_\varepsilon := \left\{ D \in \mathcal{D}^{(\alpha)}_{[0,1]^d} \;\middle|\; \forall x, y \in [0,1]^d, D(x,y) \geq g(x-y) - \varepsilon \right\}$ gives

$$\min_{D \in A_\varepsilon} I^{(\alpha)}_{[0,1]^d}(D) \leq \underline{\lim}_{n\to\infty} -\frac{1}{n^d} \log \mathbb{P}\left(\widetilde{\mathbf{T}}^{(\alpha)}_{n,[0,1]^d} \in A_\varepsilon \right) \leq \overline{\lim}_{n\to\infty} -\frac{1}{n^d} \log \mathbb{P}\left(\widetilde{\mathbf{T}}^{(\alpha)}_{n,[0,1]^d} \in A_\varepsilon \right) \leq \inf_{D \in \mathring{A}_\varepsilon} I^{(\alpha)}_{[0,1]^d}(D). \tag{6.6}$$

The norm $[\alpha\|\cdot\|] \vee g$ is in $\mathring{A}_\varepsilon$, therefore the upper bound in (6.6) implies

$$\lim_{\varepsilon\to 0}\uparrow \overline{\lim_{n\to\infty}} -\frac{1}{n^d} \log \mathbb{P}\left(\widetilde{\mathbf{T}}^{(\alpha)}_{n,[0,1]^d} \in A_\varepsilon \right) \leq I^{(\alpha)}_{[0,1]^d}([\alpha\|\cdot\|] \vee g). \tag{6.7}$$

Besides, by the lower bound in (6.6), there exists $D_\varepsilon \in A_\varepsilon$ such that

$$\underline{\lim}_{n\to\infty} -\frac{1}{n^d} \log \mathbb{P}\left(\widetilde{\mathbf{T}}^{(\alpha)}_{n,[0,1]^d} \in A_\varepsilon \right) \geq I^{(\alpha)}_{[0,1]^d}(D_\varepsilon).$$

By compactness of $\mathcal{D}^{(\alpha)}_{[0,1]^d}$, $(D_\varepsilon)_{\varepsilon>0}$ has an adherence value D as $\varepsilon \to 0$, thus by lower semicontinuity of $I^{(\alpha)}_{[0,1]^d}$,

$$\lim_{\varepsilon\to 0}\uparrow \underline{\lim}_{n\to\infty} -\frac{1}{n^d} \log \mathbb{P}\left(\widetilde{\mathbf{T}}^{(\alpha)}_{n,[0,1]^d} \in A_\varepsilon \right) \geq I^{(\alpha)}_{[0,1]^d}(D).$$

Since $D \geq [\alpha\|\cdot\|] \vee g$ and $I^{(\alpha)}_{[0,1]^d}$ is nondecreasing, we have

$$\lim_{\varepsilon\to 0}\uparrow \underline{\lim}_{n\to\infty} -\frac{1}{n^d} \log \mathbb{P}\left(\widetilde{\mathbf{T}}^{(\alpha)}_{n,[0,1]^d} \in A_\varepsilon \right) \geq I^{(\alpha)}_{[0,1]^d}([\alpha\|\cdot\|] \vee g). \tag{6.8}$$

The lemma is a consequence of (6.7) and (6.8). □

Lemma 6.3. *For all $g \in \mathcal{SN}$,*

$$\overline{I}^{+}_{[0,1]^d}(g) = \underline{I}^{+}_{[0,1]^d}(g) = \lim_{\alpha\to 0}\uparrow I^{+,(\alpha)}_{[0,1]^d}(g), \tag{6.9}$$

with uniform convergence on the compact subsets of $\mathcal{SN}^*$. We denote by $I^{+}_{[0,1]^d}(g)$ this number. Moreover, the restriction of $I^{+}_{[0,1]^d}$ on $\mathcal{SN}^*$ is continuous.

Proof. Let $g \le b\|\cdot\|$ be a seminorm. For all $0 < \alpha_1 < \alpha_2$, the inclusions

$$\Big\{\tau \in \mathrm{LD}_n^+(g,\varepsilon)\Big\} \subseteq \Big\{\tau^{(\alpha_1)} \in \mathrm{LD}_n^+(g,\varepsilon)\Big\} \subseteq \Big\{\tau^{(\alpha_2)} \in \mathrm{LD}_n^+(g,\varepsilon)\Big\},$$

hold, thus by Lemma 6.2,

$$\underline{I}^+_{[0\,,1]^d}(g) \ge \lim_{\alpha\to 0}\uparrow I^{+,(\alpha)}_{[0\,,1]^d}(g). \tag{6.10}$$

Let us show the converse inequality. Let $p \ge 1$. Consider the event

$$\mathrm{Fav}^{\ddagger\ddagger} \coloneqq \Big\{\tau^{(\alpha)} \in \mathrm{LD}^+_{n,[0\,,1]^d}(g,\varepsilon)\Big\} \cap \left(\bigcap_{\substack{e\in \mathrm{E}\left([0\,,n]^d\right)\\ e\,\text{hard}}} \Big\{\tau^{(\alpha)} \ge \frac{b}{2}\Big\}\right),$$

where for all $v = (v_1,\dots,v_d) \in \mathbb{Z}^d$ and $i \in [\![1\,,d]\!]$, the edge $e = (v, v+\mathrm{e}_i)$ is called *hard* if $v_i \in p\mathbb{Z}$. By the FKG inequality, there exists a constant C_{16}, depending only on d, such that

$$\overline{\lim_{n\to\infty}} -\frac{1}{n^d}\log\mathbb{P}\Big(\mathrm{Fav}^{\ddagger\ddagger}\Big) \le \overline{\lim_{n\to\infty}} -\frac{1}{n^d}\log\mathbb{P}\left(\tau^{(\alpha)} \in \mathrm{LD}^+_{n,[0\,,1]^d}(g,\varepsilon)\right) - \frac{C_{16}}{p}\log\nu\left(\left[\frac{b}{2}\,,b\right]\right). \tag{6.11}$$

Assume that $\mathrm{Fav}^{\ddagger\ddagger}$ occurs. Let $x,y \in [\![0\,,n]\!]^d$ and $x \overset{\gamma}{\rightsquigarrow} y$ be a discrete $\mathbf{T}_{[0\,,n]^d}$-geodesic. Since γ is self-avoiding, it uses at least $\left\lfloor \frac{\|\gamma\|}{p^d}\right\rfloor$ hard edges. Consequently,

$$\frac{b}{2}\cdot\left\lfloor\frac{\|\gamma\|}{p^d}\right\rfloor \le \mathbf{T}_{[0\,,n]^d}(\gamma) \le b\|x-y\|,$$

thus

$$\|\gamma\| \le 2(\|x-y\|+1)p^d \le 2(nd+1)p^d.$$

Since for all edges e, $\tau_e^{(\alpha)}$ and τ_e differ by at most α,

$$\mathbf{T}_{[0\,,n]^d}(\gamma) \ge \mathbf{T}^{(\alpha)}_{[0\,,n]^d}(\gamma) - 2\alpha(nd+1)p^d,$$

thus for all $x,y \in [\![0\,,n]\!]^d$,

$$\mathbf{T}_{[0\,,n]^d}(x,y) \ge \mathbf{T}^{(\alpha)}_{[0\,,n]^d}(x,y) - 2\alpha(nd+1)p^d.$$

This bound can be extended to all $x,y \in [0\,,n]^d$, up to a bounded additive error term thanks to (1.16). Consequently, for large enough n,

$$\mathrm{Fav}^{\ddagger\ddagger} \subseteq \Big\{\tau \in \mathrm{LD}^+_{n,[0\,,1]^d}\Big(g, 2\varepsilon + 2\alpha d p^d\Big)\Big\}. \tag{6.12}$$

Combining (6.11) and (6.12), we get

$$\overline{\lim_{n\to\infty}} -\frac{1}{n^d}\log\mathbb{P}\left(\mathrm{LD}^+_{n,[0\,,1]^d}\Big(g,2\varepsilon+2\alpha d p^d\Big)\right) \le \overline{\lim_{n\to\infty}} -\frac{1}{n^d}\log\mathbb{P}\Big(\tau^{(\alpha)} \in \mathrm{LD}^+_n(g,\varepsilon)\Big) - \frac{C_{16}}{p}\log\nu\left(\left[\frac{b}{2}\,,b\right]\right).$$

Thus, by definition of $\overline{I}^{+,(\alpha)}_{[0\,,1]^d}(g)$ and by (6.5),

$$\overline{\lim_{n\to\infty}} -\frac{1}{n^d}\log\mathbb{P}\left(\mathrm{LD}^+_{n,[0\,,1]^d}\Big(g,2\varepsilon+2\alpha d p^d\Big)\right) \le I^{+,(\alpha)}_{[0\,,1]^d}(g) - \frac{C_{16}}{p}\log\nu\left(\left[\frac{b}{2}\,,b\right]\right).$$

For small enough α, $2\alpha d p^d \le \varepsilon$, thus

$$\overline{\lim_{n\to\infty}} -\frac{1}{n^d}\log \mathbb{P}\Big(\mathrm{LD}^+_{n,[0\,,1]^d}(g,3\varepsilon)\Big) \le \lim_{\alpha\to 0}\uparrow I^{+,(\alpha)}_{[0\,,1]^d}(g) - \frac{C_{16}}{p}\log\nu\Big(\Big[\frac{b}{2}\,,b\Big]\Big).$$

Letting $p\to\infty$ and $\varepsilon\to 0$, we obtain

$$\overline{I}^+_{[0\,,1]^d}(g) \le \lim_{\alpha\to 0}\uparrow I^{+,(\alpha)}_{[0\,,1]^d}(g). \tag{6.13}$$

This concludes the proof of (6.9).

Let $\eta>0$. Lemma 4.4 and the monotonicity of the rate function imply that for all $\alpha>0$, for all norms $\alpha\|\cdot\| \le g_1, g_2 \le (b-2\eta)\|\cdot\|$ such that $\|g_1-g_2\|_{\mathrm{Hom}} \le \frac{\eta}{p}$,

$$\begin{aligned} I^{(\alpha)}_{[0\,,1]^d}(g_1) &\le I^{(\alpha)}_{[0\,,1]^d}\Big(g_2+\frac{\eta}{p}\|\cdot\|\Big)\\ &\le I^{(\alpha)}_{[0\,,1]^d}(g_2) - \frac{C_{12}}{p}\log\nu^{(\alpha)}([b-\eta\,,b]). \end{aligned}$$

Since the roles of g_1 and g_2 are symmetric, this gives

$$\Big|I^{(\alpha)}_{[0\,,1]^d}(g_1) - I^{(\alpha)}_{[0\,,1]^d}(g_2)\Big| \le -\frac{C_{12}}{p}\log\nu^{(\alpha)}([b-\eta\,,b]) \le -\frac{C_{12}}{p}\log\nu([b-\eta\,,b]).$$

Moreover, for all $g_1, g_2 \in \mathcal{SN}$,

$$\Big\|[\alpha\|\cdot\|]\vee g_1 - [\alpha\|\cdot\|]\vee g_2\Big\|_{\mathrm{Hom}} \le \|g_1-g_2\|_{\mathrm{Hom}}.$$

Applying (6.5), we deduce that on the set of seminorms g such that $g\le(b-2\eta)\|\cdot\|$, the functions $\Big(I^{+,(\alpha)}_{[0\,,1]^d}\Big)_{\alpha>0}$ form an equicontinuous family, hence we obtain the announced uniform convergence and continuity. □

For all $X\in\mathcal{K}$ and $D\in\mathcal{D}^{(\alpha)}_X$, we define

$$I^+_X(D) \coloneqq \lim_{\alpha'\to 0}\uparrow I^{(\alpha')}_X(D) = \lim_{\alpha'\to 0}\uparrow \int_X I^{(\alpha')}_{[0\,,1]^d}((\mathrm{grad}\,D)_z)\mathrm{d}z = \int_X I^+_{[0\,,1]^d}((\mathrm{grad}\,D)_z)\mathrm{d}z. \tag{6.14}$$

6.2 Point-point passage time: proof of Corollary 1.6

Fix $x\in\mathbb{R}^d\setminus\{0\}$. In this subsection, we assume that $0\le a<b<\infty$, with $\nu(\{0\})<p_c(\mathbb{Z}^d)$. In particular,

$$\alpha_0 \coloneqq \frac{1}{2}\inf_{u\in\mathbb{S}}\mu(u) > 0. \tag{6.15}$$

Let $C_{17}\coloneqq\frac{b\|x\|}{\alpha_0}$ and $X\coloneqq[-C_{17}\,,C_{17}]^d$. The Cox-Durrett shape theorem (see [8, Theorem 3] for the case $d=2$ and [13, Theorem 1.7] for the general case) implies that

$$\lim_{n\to\infty}\mathbb{P}\Big(\widetilde{\mathbf{T}}_{n,X}(0,\partial X)\ge b\|x\|\Big)=1. \tag{6.16}$$

Corollary 1.6 is a consequence of Lemmas 6.4 and 6.5.

Lemma 6.4. *The process* $\left(\frac{\mathbf{T}(0,nx)}{n}\right)$ *satisfies the LDP at speed* n^d *with the good rate function*

$$\begin{aligned} I_{\mathrm{pp}(x)} : [a\|x\|\,, b\|x\|] &\longrightarrow [0\,, \infty] \\ \zeta &\longmapsto \min_{\substack{D\in\mathcal{D}_X^{(\alpha_0)}:\\ D(0,x)\geq\zeta}} I_X^+(D). \end{aligned} \tag{6.17}$$

Lemma 6.5. *The function* $I_{\mathrm{pp}(x)}$ *is continuous on* $[a\|x\|\,, b\|x\|]$.

Proof of Lemma 6.4. For all $a\|x\| \leq \zeta \leq b\|x\|$, we define

$$\begin{aligned} \overline{I}^+_{\mathrm{pp}(x)}(\zeta) &\coloneqq \lim_{\varepsilon\to 0}\uparrow \varlimsup_{n\to\infty} -\frac{1}{n^d}\log\mathbb{P}\Big(\widetilde{\mathbf{T}}_{n,X}(0,x)\geq \zeta-\varepsilon\Big) \\ \text{and } \underline{I}^+_{\mathrm{pp}(x)}(\zeta) &\coloneqq \lim_{\varepsilon\to 0}\uparrow \varliminf_{n\to\infty} -\frac{1}{n^d}\log\mathbb{P}\Big(\widetilde{\mathbf{T}}_{n,X}(0,x)\geq \zeta-\varepsilon\Big). \end{aligned}$$

We claim that for all $0<\alpha\leq\alpha_0$ and $a\|x\|\leq\zeta\leq b\|x\|$,

$$I^{+,(\alpha)}_{\mathrm{pp}(x)}(\zeta) \coloneqq \overline{I}^{+,(\alpha)}_{\mathrm{pp}(x)}(\zeta) = \underline{I}^{+,(\alpha)}_{\mathrm{pp}(x)}(\zeta) = \min_{\substack{D\in\mathcal{D}_X^{(\alpha_0)}:\\ D(0,x)\geq\zeta}} I_X^{(\alpha)}(D). \tag{6.18}$$

Indeed let $\varepsilon>0$ and

$$A_\varepsilon \coloneqq \Big\{D\in\mathcal{D}_X^{(\alpha_0)} \mid D(0,x)\geq\zeta-\varepsilon\Big\}.$$

Since $\widetilde{\mathbf{T}}^{(\alpha)}_{n,X}$ satisfies the LDP with the rate function $I_X^{(\alpha)}$,

$$\min_{D\in A_\varepsilon} I_X^{(\alpha)}(D) \leq \varliminf_{n\to\infty} -\frac{1}{n^d}\log\mathbb{P}\Big(\widetilde{\mathbf{T}}^{(\alpha)}_{n,X}\in A_\varepsilon\Big) \leq \varlimsup_{n\to\infty} -\frac{1}{n^d}\log\mathbb{P}\Big(\widetilde{\mathbf{T}}^{(\alpha)}_{n,X}\in A_\varepsilon\Big) \leq \inf_{D\in\mathring{A}_\varepsilon} I_X^{(\alpha)}(D).$$

Besides, by the FKG inequality and the definition of α_0,

$$\mathbb{P}\Big(\widetilde{\mathbf{T}}^{(\alpha)}_{n,X}\in A_\varepsilon\Big) \underset{n\to\infty}{\sim} \mathbb{P}\Big(\widetilde{\mathbf{T}}^{(\alpha)}_{n,X}(0,x)\geq\zeta-\varepsilon\Big),$$

thus

$$\begin{aligned} \min_{D\in A_\varepsilon} I_X^{(\alpha)}(D) &\leq \varliminf_{n\to\infty} -\frac{1}{n^d}\log\mathbb{P}\Big(\widetilde{\mathbf{T}}^{(\alpha)}_{n,X}(0,x)\geq\zeta-\varepsilon\Big) \\ &\leq \varlimsup_{n\to\infty} -\frac{1}{n^d}\log\mathbb{P}\Big(\widetilde{\mathbf{T}}^{(\alpha)}_{n,X}(0,x)\geq\zeta-\varepsilon\Big) \leq \inf_{D\in\mathring{A}_\varepsilon} I_X^{(\alpha)}(D). \end{aligned}$$

Thanks to the lower semicontinuity of $I_X^{(\alpha)}$, the same arguments as the onesused for the proof of (6.7) and (6.8) give (6.18).

The same arguments as the ones used in the proof of Lemma 6.3 lead to

$$I^+_{\mathrm{pp}(x)}(\zeta) \coloneqq \overline{I}^+_{\mathrm{pp}(x)}(\zeta) = \underline{I}^+_{\mathrm{pp}(x)}(\zeta) = \lim_{\alpha\to 0}\uparrow I^{+,(\alpha)}_{\mathrm{pp}(x)}(\zeta).$$

We claim that

$$\lim_{\alpha\to 0}\uparrow I^{+,(\alpha)}_{\mathrm{pp}(x)}(\zeta) = \min_{\substack{D\in\mathcal{D}_X^{(\alpha_0)}:\\ D(0,x)\geq\zeta}} I_X^+(D) =: I_{\mathrm{pp}(x)}(\zeta). \tag{6.19}$$

Indeed, the inequality

$$\lim_{\alpha\to 0}\uparrow I^{+,(\alpha)}_{\mathrm{pp}(x)}(\zeta) \le \min_{\substack{D\in\mathcal{D}_X^{(\alpha_0)}:\\ D(0,x)\ge\zeta}} I^+_X(D)$$

is a direct consequence of $I_X^{(\alpha)} \le I_X^+$ and (6.18). Let us prove the converse inequality. For all $\alpha > 0$, the set

$$K_\alpha \coloneqq \left\{ D\in\mathcal{D}_X^{(\alpha_0)} \;\middle|\; D(0,x)\ge\zeta,\quad I_X^{(\alpha)}(D) \le \lim_{\alpha'\to 0}\uparrow I^{+,(\alpha')}_{\mathrm{pp}(x)}(\zeta)\right\}$$

is compact. Moreover if $\alpha_1 < \alpha_2$ then $I_X^{(\alpha_1)} \ge I_X^{(\alpha_2)}$, thus $K_{\alpha_1}\subseteq K_{\alpha_2}$. Hence the intersection K_0 of all the K_α contains at least one element D. We have

$$I_X^+(D) = \lim_{\alpha\to 0}\uparrow I_X^{(\alpha)}(D) \le \lim_{\alpha\to 0}\uparrow I^{+,(\alpha)}_{\mathrm{pp}(x)}(\zeta), \tag{6.20}$$

thus (6.19).

For all $a\|x\| \le \zeta \le b\|x\|$, define

$$\begin{aligned}\overline{I}_{\mathrm{pp}(x)}(\zeta) &\coloneqq \lim_{\varepsilon\to 0}\uparrow \overline{\lim_{n\to\infty}} -\frac{1}{n^d}\log\mathbb{P}\Big(\Big|\widetilde{\mathbf{T}}_n(0,x)-\zeta\Big|\le\varepsilon\Big)\\ \text{and } \underline{I}_{\mathrm{pp}(x)}(\zeta) &\coloneqq \lim_{\varepsilon\to 0}\uparrow \underline{\lim_{n\to\infty}} -\frac{1}{n^d}\log\mathbb{P}\Big(\Big|\widetilde{\mathbf{T}}_n(0,x)-\zeta\Big|\le\varepsilon\Big).\end{aligned}$$

Thanks to Lemma 1.2, it remains to show that

$$\overline{I}_{\mathrm{pp}(x)}(\zeta) = \underline{I}_{\mathrm{pp}(x)}(\zeta) = I^+_{\mathrm{pp}(x)}(\zeta). \tag{6.21}$$

The inequality

$$\underline{I}_{\mathrm{pp}(x)}(\zeta) \ge I^+_{\mathrm{pp}(x)}(\zeta) \tag{6.22}$$

is a direct consequence of the inclusion

$$\Big\{\Big|\widetilde{\mathbf{T}}_n(0,x)-\zeta\Big|\le\varepsilon\Big\} \subseteq \Big\{\widetilde{\mathbf{T}}_{n,X}(0,x)\ge\zeta-\varepsilon\Big\}.$$

To prove the converse inequality, first note that by the FKG inequality and (6.16),

$$\mathbb{P}\Big(\widetilde{\mathbf{T}}_{n,X}(0,x)\ge\zeta-\varepsilon\Big) \underset{n\to\infty}{\asymp} \mathbb{P}\Big(\widetilde{\mathbf{T}}_{n}(0,x)\ge\zeta-\varepsilon\Big),$$

thus

$$\overline{\lim_{n\to\infty}} -\frac{1}{n^d}\log\mathbb{P}\Big(\widetilde{\mathbf{T}}_{n,X}(0,x)\ge\zeta-\varepsilon\Big) = \overline{\lim_{n\to\infty}} -\frac{1}{n^d}\log\mathbb{P}\Big(\widetilde{\mathbf{T}}_{n}(0,x)\ge\zeta-\varepsilon\Big). \tag{6.23}$$

Let $n\ge 1$ be an integer large enough so that

$$\frac{1}{n}(a\|\lfloor nx\rfloor\| + bd) \le \zeta+\varepsilon \text{ and } \frac{b}{n}\le\varepsilon.$$

Let $\gamma = \Big(0=y_0,y_1,\dots,y_{\|\lfloor nx\rfloor\|}=\lfloor nx\rfloor\Big)$ be a discrete path from 0 to $\lfloor nx\rfloor$ with minimal number of edges. For all $R\in[\![0\,,\|\lfloor nx\rfloor\|]\!]$, we define the configuration $\tau^{[R]}$ by

$$\tau_e^{[R]} \coloneqq \begin{cases} a & \text{if } e\in \mathrm{E}_{\mathrm{path}}\Big(\gamma_{|[0\,,R]}\Big),\\ \tau_e & \text{otherwise.}\end{cases} \tag{6.24}$$

On the event $\Big\{\widetilde{\mathbf{T}}_n(0,x) \geq \zeta - \varepsilon\Big\}$, for large enough n,

$$\widetilde{\mathbf{T}}_n^{[0]}(0,x) = \widetilde{\mathbf{T}}_n(0,x) \geq \zeta - \varepsilon$$
$$\text{and } \widetilde{\mathbf{T}}_n^{[\|\lfloor nx\rfloor\|]}(0,x) \leq \frac{1}{n}(a\|\lfloor nx\rfloor\| + bd) \leq \zeta + \varepsilon.$$

Moreover, for all $R \in [\![0\,, \|\lfloor nx\rfloor\| - 1]\!]$,

$$0 \leq \widetilde{\mathbf{T}}_n^{[R]}(0,x) - \widetilde{\mathbf{T}}_n^{[R+1]}(0,x) \leq \frac{b}{n} \leq \varepsilon,$$

hence there exists a τ-measurable random integer R_0 such that

$$\Big|\widetilde{\mathbf{T}}_n^{[R_0]}(0,x) - \zeta\Big| \leq \varepsilon,$$

take for example the smallest R such that $\widetilde{\mathbf{T}}_n^{[R]}(0,x) \leq \zeta + \varepsilon$. Consider a configuration τ' with the same distribution as τ, and $(X_r)_{1\leq r\leq \|\lfloor nx\rfloor\|}$ a family of independent Bernoulli variables with parameter $1/2$, and assume the τ, τ' and (X_r) are independent. The configuration defined by

$$\tau_e^* \coloneqq \begin{cases} X_r\tau'_e + (1-X_r)\tau_e & \text{if } e = (y_{r-1}, y_r), \text{ with } 1 \leq r \leq \|\lfloor nx\rfloor\|, \\ \tau_e & \text{otherwise,} \end{cases} \tag{6.25}$$

has the same distribution as τ. Reasoning as in Step 4 of the proof of Lemma 3.6, on the event

$$\Big\{\widetilde{\mathbf{T}}_n(0,x) \geq \zeta - \varepsilon\Big\} \cap \left(\bigcap_{r=1}^{\|\lfloor nx\rfloor\|} \Big\{X_r = \mathbb{1}_{[\![1\,,R_0]\!]}(r)\Big\}\right) \cap \left(\bigcap_{r=1}^{\|\lfloor nx\rfloor\|} \Big\{\tau'_{(y_{r-1},y_r)} \leq a + \varepsilon\Big\}\right),$$

the configurations τ^* and $\tau^{[R_0]}$ agree on all edges except R_0 of them, where they may differ up to ε. Consequently,

$$\begin{aligned} \varlimsup_{n\to\infty} -\frac{1}{n^d}\log\mathbb{P}\Big(\widetilde{\mathbf{T}}_n(0,x) \geq \zeta - \varepsilon\Big) &\geq \varlimsup_{n\to\infty} -\frac{1}{n^d}\log\mathbb{P}\left(\Big|\widetilde{\mathbf{T}}_n(0,x) - \zeta\Big| \leq \left(1 + \frac{R_0}{n}\right)\varepsilon\right) \\ &\geq \varlimsup_{n\to\infty} -\frac{1}{n^d}\log\mathbb{P}\left(\Big|\widetilde{\mathbf{T}}_n(0,x) - \zeta\Big| \leq \left(1 + \frac{\|\lfloor nx\rfloor\|}{n}\right)\varepsilon\right). \end{aligned} \tag{6.26}$$

Combining (6.23) and (6.26) and letting $\varepsilon \to 0$, we get

$$\overline{I}_{\mathrm{pp}(x)}(\zeta) \leq I^+_{\mathrm{pp}(x)}(\zeta). \tag{6.27}$$

Consequently, (6.21) holds. This concludes the proof of the lemma. □

Proof of Lemma 6.5. Since $I_{\mathrm{pp}(x)}$ is nondecreasing and lower semicontinuous, it is sufficient to prove the right-continuity. Let $a\|x\| \leq \zeta < b\|x\|$. By (6.17) there exists $D \in \mathcal{D}_X^{(\alpha_0)}$ such that $D(0,x) \geq \zeta$ and

$$I_{\mathrm{pp}(x)}(\zeta) = I_X^+(D). \tag{6.28}$$

The idea is to build another metric $\hat{D}$ whose cost is slightly larger, such that $\hat{D}(0,x) > \zeta$, leading to

$$I_{\mathrm{pp}(x)}(\zeta^+) \leq I_{\mathrm{pp}(x)}(\zeta). \tag{6.29}$$

The details of the construction of $\hat{D}$ differ whether ν has an atom at b or not.

Case 1: $\nu(\{b\}) > 0$. Without loss of generality we assume that $x_i \geq 0$ for all $i \in [\![1\,,d]\!]$. For all $A \subseteq [\![1\,,d]\!]$, we define the sets

$$X_A \coloneqq \left\{ y \in X \;\middle|\; \forall i \in A, y_i \geq \frac{x_i}{2} \text{ and } \forall i \in [\![1\,,d]\!] \setminus A, y_i \leq \frac{x_i}{2} \right\}$$

and

$$X'_A \coloneqq X_A + \varepsilon \sum_{i \in A} x_i \mathrm{e}_i.$$

We define D'_A as the metric on X'_A defined by translating $D_{|X_A}$ by $z_A \coloneqq \varepsilon \sum_{i\in A} x_i \mathrm{e}_i$, i.e. $D'_A \coloneqq \mathrm{Tr}_{z_A}(D_{|X_A})$ (see Lemmas 2.14 and 2.16). We define D' as the metric on $(1+\varepsilon)X$ constructed as in Lemma 2.15, with the family of subsets $(X'_A)_A$ and the family of metrics $(D'_A)_A$. Finally, we define $\hat{D} \coloneqq \mathrm{Sc}_{\frac{1}{1+\varepsilon}}(D')$ (see Lemma 2.13). Thanks to the equality of gradients provided by these four lemmas,

$$\begin{aligned} I^+_X(\hat{D}) &= \frac{I^+_{(1+\varepsilon)X}(D')}{(1+\varepsilon)^d} \\ &= \frac{1}{(1+\varepsilon)^d}\left[\sum_{A \subseteq [\![1\,,d]\!]} I^+_{X'_A}(D'_A) + \mathrm{Leb}\left((1+\varepsilon)X \setminus \bigcup_{A\subseteq[\![1\,,d]\!]} X'_A \right) I^+_{[0\,,1]^d}(b\|\cdot\|) \right] \\ &= \frac{1}{(1+\varepsilon)^d}\Big[I^+_X(D) + \Big((1+\varepsilon)^d - 1\Big)\,\mathrm{Leb}(X) I^+_{[0\,,1]^d}(b\|\cdot\|) \Big]. \end{aligned} \tag{6.30}$$

We claim that

$$\hat{D}(0,x) \geq \frac{\zeta + \varepsilon b\|x\|}{1+\varepsilon} > \zeta. \tag{6.31}$$

Indeed let $0 \overset{\gamma}{\rightsquigarrow} (1+\varepsilon)x$ be a D'-geodesic. Since the gradient of D' is $b\|\cdot\|$ outside the tiles X'_A (see (2.34)), $D'(\gamma)$ does not increase when replacing every excursion of γ outside the tiles by the concatenation of d (possibly degenerate) segments, such that the i^{th} one is colinear to e_i. In particular we may assume that γ is of the form

$$0 \overset{\gamma_1^{\text{int}}}{\rightsquigarrow} y(1) \overset{\gamma_1^{\text{ext}}}{\rightsquigarrow} z(1) \overset{\gamma_2^{\text{int}}}{\rightsquigarrow} y(2) \overset{\gamma_2^{\text{ext}}}{\rightsquigarrow} \dots \overset{\gamma_r^{\text{ext}}}{\rightsquigarrow} z(r) \overset{\gamma_{r+1}^{\text{int}}}{\rightsquigarrow} x, \tag{6.32}$$

where

(i) Every γ_j^{int} is included in some tile $X'_{A(j)}$.

(ii) For all $j \in [\![1\,,r]\!]$, $\gamma_j^{\text{ext}} = [y(j)\,,z(j)]$.

(iii) For all $j \in [\![1\,,r]\!]$, $y(j) - z(j)$ is colinear to some $\mathrm{e}_{i(j)}$ and $]y(j)\,,z(j)[\subseteq [(1+\varepsilon)X] \setminus \left(\bigcup_{A\subseteq[\![1\,,d]\!]} X'_A\right)$.

The concatenation of the translated paths $\left(\gamma_j^{\text{int}} - z_{A(j)}\right)_{j\in[\![1\,,r+1]\!]}$ is a path from 0 to x (see Figure 7), thus

$$\sum_{j=1}^{r+1} D'\left(\gamma_j^{\text{int}}\right) \geq D(0,x) \geq \zeta. \tag{6.33}$$

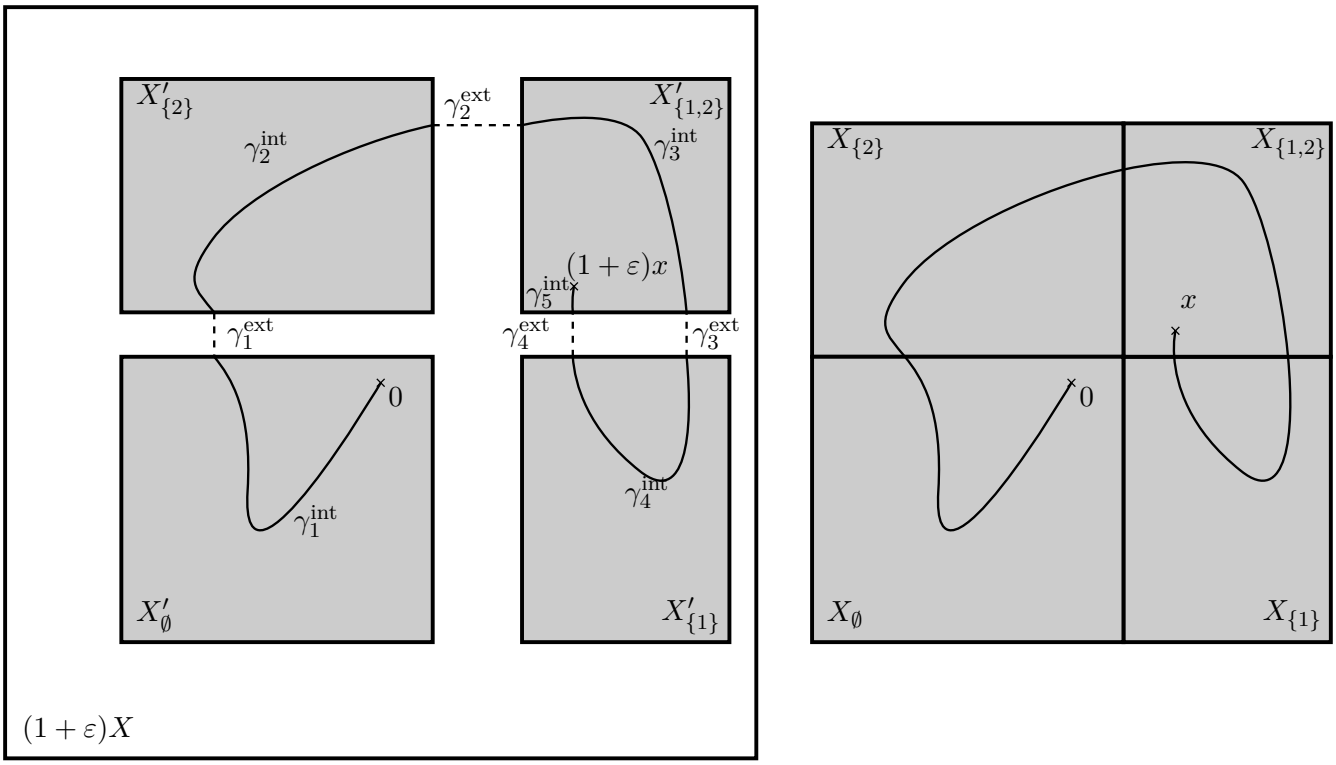


Figure 7: Up to translations, the paths γ_j^{int} may be concatenated into a path from 0 to x.

Moreover,

$$\begin{aligned}\sum_{j=1}^{r} D'\left(\gamma_j^{\text{ext}}\right) &= \sum_{j=1}^{r} b\|y(j) - z(j)\| \\ &= b\varepsilon \sum_{j=1}^{r} x_{i(j)}.\end{aligned}$$

Since $(1+\varepsilon)x \in X'_{[\![1\,,d]\!]}$, every coordinate of x appear at least once in the sum above, hence

$$\sum_{j=1}^{r} D'\left(\gamma_j^{\text{ext}}\right) \geq b\varepsilon\|x\|. \tag{6.34}$$

Inequalities (6.33) and (6.34) give

$$D'(0,(1+\varepsilon)x) \geq \zeta + b\varepsilon\|x\|,$$

i.e. (6.31). By (6.17), (6.30), and (6.31),

$$I_{\text{pp}(x)}\left(\frac{\zeta + \varepsilon b\|x\|}{1+\varepsilon}\right) \leq \frac{1}{(1+\varepsilon)^d}\left[I_X^+(D) + \left((1+\varepsilon)^d - 1\right)\text{Leb}(X) I^+_{[0\,,1]^d}(b\|\cdot\|)\right].$$

Besides, (4.20) and (6.9) imply $I^+_{[0\,,1]^d}(b\|\cdot\|) < \infty$. Letting $\varepsilon \to 0$, we obtain (6.29).

Case 2: $\nu(\{b\}) = 0$. For all $z \in X$, we write $g_z \coloneqq (\text{grad}\, D)_z$. We follow the proof of Proposition 1.4 in Basu, Ganguly and Sly [3], in a deterministic setting: rather than slightly increasing every edge

passage times in an configuration satisfying $\left\{\widetilde{\mathbf{T}}_n(0,x) \geq \zeta\right\}$, we slightly increase g_z to build $\hat{D}$. Let $0 < \varepsilon \leq \delta \leq \frac{b\|x\|-\zeta}{10\|x\|}$. Let ω_δ be a modulus of continuity for the restriction of $I^+_{[0,1]^d}$ on the set of norms g such that $\alpha_0\|\cdot\| \leq g \leq (b-\delta)\|\cdot\|$, which is compact (see Lemma 6.3). Define the sets

$$X_1(\delta) \coloneqq \left\{ z \in X \;\middle|\; \|g_z\|_{\mathrm{Hom}} \leq (b-2\delta) \right\} \text{ and } X_2(\delta) \coloneqq X \setminus X_1(\delta). \tag{6.35}$$

For all $z \in X$, define

$$\hat{g}_z \coloneqq \begin{cases} g_z + \varepsilon\|\cdot\| & \text{if } z \in X_1(\delta), \\ [(b-8\delta)\|\cdot\|] \vee g_z & \text{if } z \in X_2(\delta). \end{cases}$$

Let $\hat{D}$ denote the metric defined by (2.19), with the family $(\hat{g}_z)_{z\in X}$. Note that by (2.20), for all $z \in X$, $(\operatorname{grad}\hat{D})_z \leq \hat{g}_z$. Consequently,

$$I^+_X(\hat{D}) \leq \int_{X_1(\delta)} I^+_{[0,1]^d}(g_z + \varepsilon\|\cdot\|)\mathrm{d}z + \int_{X_2(\delta)} I^+_{[0,1]^d}([(b-8\delta)\|\cdot\|] \vee g_z)\mathrm{d}z. \tag{6.36}$$

The first term is upper bounded by

$$\int_{X_1(\delta)} I^+_{[0,1]^d}(g_z + \varepsilon\|\cdot\|)\mathrm{d}z \leq \int_{X_1(\delta)} I^+_{[0,1]^d}(g_z)\mathrm{d}z + \mathrm{Leb}(X_1(\delta))\omega_\delta(\varepsilon) \leq I^+_X(D) + \mathrm{Leb}(X)\omega_\delta(\varepsilon). \tag{6.37}$$

Let us bound the second. Let $z \in X_2(\delta)$. First, note that for all $\alpha > 0$, by (4.20) and (4.21),

$$\begin{aligned} I^{(\alpha)}_{[0,1]^d}((b-8\delta)\|\cdot\|) &\leq -d\log\nu^{(\alpha)}([b-8\delta\,,b]) \\ &\leq 2dI^{(\alpha)}_{[0,1]^d}(g_z) + 2d\log 2. \end{aligned}$$

Letting $\alpha \to 0$, we get

$$I^+_{[0,1]^d}((b-8\delta)\|\cdot\|) \leq 2dI^+_{[0,1]^d}(g_z) + 2d\log 2. \tag{6.38}$$

Moreover, by the FKG inequality,

$$\begin{aligned} I^+_{[0,1]^d}([(b-8\delta)\|\cdot\|] \vee g_z) &\leq I^+_{[0,1]^d}((b-8\delta)\|\cdot\|) + I^+_{[0,1]^d}(g_z) \\ &\leq (2d+1)I^+_{[0,1]^d}(g_z) + 2d\log 2. \end{aligned}$$

This implies

$$\int_{X_2(\delta)} I^+_{[0,1]^d}([(b-8\delta)\|\cdot\|] \vee g_z)\mathrm{d}z \leq (2d+1)\int_{X_2(\delta)} I^+_{[0,1]^d}(g_z)\mathrm{d}z + 2d\,\mathrm{Leb}(X_2(\delta))\log 2. \tag{6.39}$$

Combining (6.37) and (6.39), we obtain

$$I^+_X(\hat{D}) \leq I^+_X(D) + \mathrm{Leb}(X)\omega_\delta(\varepsilon) + (2d+1)\int_{X_2(\delta)} I^+_{[0,1]^d}(g_z)\mathrm{d}z + 2d\,\mathrm{Leb}(X_2(\delta))\log 2. \tag{6.40}$$

We claim that $\hat{D}(0,x) > \zeta$. Let $0 \overset{\gamma}{\underset{X}{\rightsquigarrow}} x$ be a Lipschitz path. If

$$\int_0^{T_\gamma} \|\gamma'(t)\| \mathbb{1}_{X_2(\delta)}(\gamma(t))\mathrm{d}t \geq \frac{\zeta + \delta\|x\|}{b-8\delta},$$

then

$$\begin{aligned}\int_0^{T_\gamma} \hat{g}_{\gamma(t)}(\gamma'(t))\mathrm{d}t &\geq (b-8\delta)\int_0^{T_\gamma} \|\gamma'(t)\|\mathbb{1}_{X_2(\delta)}(\gamma(t))\mathrm{d}t \\ &\geq \zeta + \delta\|x\|. \end{aligned} \tag{6.41}$$

Otherwise,

$$\begin{aligned}\int_0^{T_\gamma} \|\gamma'(t)\|\mathbb{1}_{X_1(\delta)}(\gamma(t))\mathrm{d}t &= \int_0^{T_\gamma} \|\gamma'(t)\|\mathrm{d}t - \int_0^{T_\gamma} \|\gamma'(t)\|\mathbb{1}_{X_2(\delta)}(\gamma(t))\mathrm{d}t \\ &\geq \|x\| - \frac{\zeta+\delta\|x\|}{b-8\delta} \\ &= \frac{b\|x\| - \zeta - 9\delta\|x\|}{b-8\delta} \\ &\geq \frac{\delta\|x\|}{b-8\delta}.\end{aligned}$$

Consequently,

$$\begin{aligned}\int_0^{T_\gamma} \hat{g}_{\gamma(t)}(\gamma'(t))\mathrm{d}t &\geq \int_0^{T_\gamma} g_{\gamma(t)}(\gamma'(t))\mathrm{d}t + \int_0^{T_\gamma} \Big(\hat{g}_{\gamma(t)}(\gamma'(t)) - g_{\gamma(t)}(\gamma'(t))\Big)\mathbb{1}_{X_1(\delta)}(\gamma(t))\mathrm{d}t \\ &\geq D(\gamma) + \varepsilon\int_0^{T_\gamma} \|\gamma'(t)\|\mathbb{1}_{X_1(\delta)}(\gamma(t))\mathrm{d}t \\ &\geq \zeta + \frac{\varepsilon\delta}{b-8\delta}\|x\|. \end{aligned} \tag{6.42}$$

Combining inequalities (6.41) and (6.42) with the definition of $\hat{D}$, we get $\hat{D}(0,x) > \zeta$. Thus, by (6.28) and (6.40),

$$I_{\mathrm{pp}(x)}(\zeta^+) \leq I_{\mathrm{pp}(x)}(\zeta) + \mathrm{Leb}(X)\omega_\delta(\varepsilon) + (2d+1)\int_{X_2(\delta)} I^+_{[0,1]^d}(g_z)\mathrm{d}z + 2d\,\mathrm{Leb}(X_2(\delta))\log 2. \tag{6.43}$$

Besides, since $\zeta < b\|x\|$, $I^+_X(D) = I_{\mathrm{pp}(x)}(\zeta) \leq I^+_X\Big(\frac{\zeta\|\cdot\|}{\|x\|}\Big) < \infty$, therefore by (6.38), $\mathrm{Leb}(X_2(\delta))$ converges to 0 as $\delta \to 0$. Thus, letting $\varepsilon \to 0$ then $\delta \to 0$ in (6.43), we get (6.29). □

6.3 Crossing time: proof of Corollary 1.7

In this subsection, we only assume that ν has a bounded support, i.e. $0 \leq a < b < \infty$, and $X = [0,1]^d$. For all $\alpha > 0$, Lemma 6.1 implies that $\Big(\widetilde{\mathbf{T}}^{(\alpha)}_{\mathrm{cross}}(n)\Big)_{n\geq 1}$ satisfies the LDP at speed n^d with the rate function

$$\begin{aligned} I^{(\alpha)}_{\mathrm{cross}} : [\alpha,b]^d &\longrightarrow [0,\infty] \\ \zeta &\longrightarrow \min_{\substack{D\in\mathcal{D}^{(\alpha)}_{[0,1]^d}: \\ \forall i, D(H_i,H_i')=\zeta_i}} I^{(\alpha)}_{[0,1]^d}(D) \end{aligned} \tag{6.44}$$

Corollary 1.7 is a consequence of Lemmas 6.6, 6.7 and 6.8.

Lemma 6.6. *For all $\alpha > 0$, $\zeta \in [\alpha,b]^d$,*

$$I^{(\alpha)}_{\mathrm{cross}}(\zeta) = I^{+,(\alpha)}_{[0,1]^d}\big(g^\zeta\big). \tag{6.45}$$

Lemma 6.7. *The process* $\left(\widetilde{\mathbf{T}}_{\mathrm{cross}}(n)\right)_{n\geq 1}$ *satisfies the LDP at speed* n^d *with the rate function*

$$I_{\mathrm{cross}}(\zeta) \coloneqq \lim_{\alpha\to 0}\uparrow I^{(\alpha)}_{\mathrm{cross}}(\zeta) = I^{+}_{[0\,,1]^d}(g^{\zeta}). \tag{6.46}$$

Lemma 6.8.

(i) I_{cross} *is continuous on* $[a\,,b]^d$.

(ii) I_{cross} *is nondecreasing on* $[a\,,b]^d$ *for the componentwise order, i.e. for all* $\zeta = (\zeta_1,\dots,\zeta_d)$ *and* $\zeta' = (\zeta'_1,\dots,\zeta'_d)$ *in* $[a\,,b]^d$, *if* $\zeta_i \leq \zeta'_i$ *for all* i, *then* $I_{\mathrm{cross}}(\zeta) \leq I_{\mathrm{cross}}(\zeta')$.

(iii) I_{cross} *is separately convex on* $[a\,,b]^d$, *i.e. for all* $1\leq i\leq d$ *and* $\zeta_1,\dots,\zeta_{i-1},\zeta_{i+1},\dots,\zeta_d \in [a\,,b]$, *the function*

$$t \mapsto I_{\mathrm{cross}}(\zeta_1,\dots,\zeta_{i-1},t,\zeta_{i+1},\dots,\zeta_d)$$

is convex on $[a\,,b]$.

(iv) $I_{\mathrm{cross}}(b,\dots,b) < \infty$ *if and only if* $\nu(\{b\}) > 0$.

Proof of Lemma 6.6 . Fix $\alpha > 0$ and $\zeta \in [\alpha\,,b]^d$. The inequality

$$I^{(\alpha)}_{\mathrm{cross}}(\zeta) \leq I^{(\alpha)}_{[0\,,1]^d}\Big(g^{\zeta} \vee [\alpha\|\cdot\|]\Big) = I^{+,(\alpha)}_{[0\,,1]^d}\Big(g^{\zeta}\Big) \tag{6.47}$$

is straightforward. Let us prove the converse inequality.

For all $A \subseteq [\![1\,,d]\!]$, let

$$\begin{aligned} h_A : \left[0\,,\frac{1}{2}\right]^d + \frac{1}{2}\sum_{i\in A}\mathrm{e}_i &\longrightarrow \left[0\,,\frac{1}{2}\right]^d \\ z &\longmapsto z + \sum_{i\in A}(1-2z_i)\mathrm{e}_i \end{aligned} \tag{6.48}$$

denote the orthogonal symmetry with respect to the affine subspace $\Big\{z\in\mathbb{R}^d \mid \forall i\in[\![1\,,d]\!], z_i = 1/2\Big\}$. For all $D \in \mathcal{D}^{(\alpha)}_X$, let $s(D)$ denote the metric on $[0\,,1]^d$ defined as in Lemma 2.15 with the family of sets $\left(\left[0\,,\frac{1}{2}\right]^d + \frac{1}{2}\sum_{i\in A}\mathrm{e}_i\right)_{A\subseteq[\![1\,,d]\!]}$ and the family of metrics $(D_A)_{A\subseteq[\![1\,,d]\!]}$ defined by

$$D_A(x,y) \coloneqq \mathrm{Sc}_{1/2}(D)(h_A(x),h_A(y)),$$

(see (2.27) and Figure 8). We postpone the proof of Claims 6.9 and 6.10 to the end of the subsection.

Claim 6.9. *Let* $D \in \mathcal{D}^{(\alpha)}_{[0\,,1]^d}$. *For all* $1\leq i\leq d$,

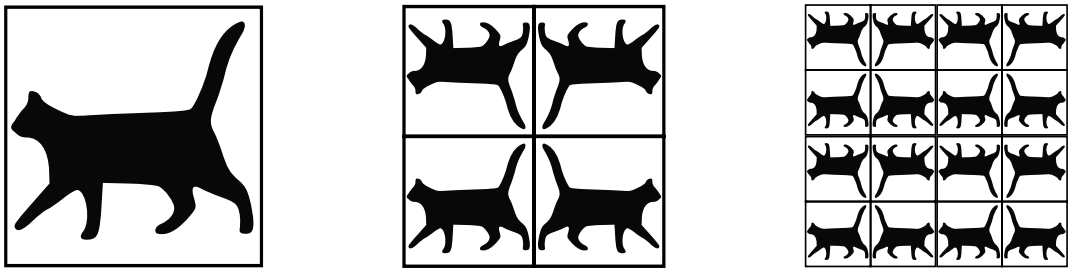
$$s(D)(H_i,H'_i) = D(H_i,H'_i). \tag{6.49}$$

Moreover,

$$I^{(\alpha)}_{[0\,,1]^d}(s(D)) = I^{(\alpha)}_{[0\,,1]^d}(D). \tag{6.50}$$

Claim 6.10. *Let* $D \in \mathcal{D}^{(\alpha)}_{[0\,,1]^d}$ *and* $n\geq 1$. *For all* $x,y\in[0\,,1]^d$ *and* $z\in 2^{-n+1}\mathbb{Z}^d$ *such that* $x+z, y+z \in [0\,,1]^d$,

$$s^n(D)(x,y) = s^n(D)(x+z,y+z). \tag{6.51}$$



D $s(D)$ $s^2(D)$

Figure 8: Illustration of the first two iterations of s in the case $d = 2$.

Let K denote the set of minimizers in (6.44). Let $D \in K$. A straightforward induction argument using Claim 6.9 implies that for all $n \geq 1$, $s^n(D) \in K$. Moreover $I^{(\alpha)}_{[0,1]^d}$ is lower semicontinuous, therefore K is compact, therefore $(s^n(D))_{n\geq 1}$ has an adherence value D' in K. Letting $n \to \infty$ in (6.51), we obtain that for all $x, y \in [0\,,1]^d$ and $z \in \mathbb{R}^d$ with dyadic coordinates such that $x + z, y + z \in [0\,,1]^d$,

$$D'(x, y) = D'(x + z, y + z).$$

Since D' is continuous, this equality is true for any z. Consequently, $z \mapsto (\operatorname{grad} D')_z$ is constant on $]0\,,1[^d$. Proposition 2.10 further implies that for all $z \in]0\,,1[^d$, $(\operatorname{grad} D')_z$ is equal to a norm $g \in \mathcal{N}^{(\alpha)}$. Consequently,

$$I^{(\alpha)}_{\text{cross}}(\zeta) = I^{(\alpha)}_{[0,1]^d}(g). \tag{6.52}$$

Moreover, g is invariant by orthogonal reflexions with respect to the hyperplanes $\Big\{x \in \mathbb{R}^d \mid x_i = 0\Big\}$, and for all $1 \leq i \leq d$, $g(\mathrm{e}_i) \geq \zeta_i$.

Let $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $1 \leq i \leq d$. We have

$$x_i \mathrm{e}_i = 2^{-d+1} \sum_{\substack{\lambda \in \{-1,1\}^d \\ \lambda_i = 1}} \sum_{j=1}^{d} \lambda_j x_j \mathrm{e}_j,$$

thus

$$|x_i| g(\mathrm{e}_i) \leq 2^{-d+1} \sum_{\substack{\lambda \in \{-1,1\}^d : \\ \lambda_i = 1}} g\left(\sum_{j=1}^{d} \lambda_j x_j \mathrm{e}_j\right).$$

Besides, since g is invariant by orthogonal reflexions with respect to the hyperplanes $\Big\{x \in \mathbb{R}^d \mid x_j = 0\Big\}$, every term the right-hand side is equal to $g(x)$. Applying $g(\mathrm{e}_i) \geq \zeta_i$ leads to

$$\zeta_i |x_i| \leq g(x),$$

hence $g \geq g^\zeta$. Proposition 3.1 and Equation (6.52) yield (6.45). □

Proof of Lemma 6.7. For all $\zeta \in [a\,,b]^d$, we define

$$\overline{I}_{\mathrm{cross}}(\zeta) \coloneqq \lim_{\varepsilon\to 0}\uparrow \overline{\lim_{n\to\infty}} -\frac{1}{n^d}\log \mathbb{P}\Big(\Big\|\widetilde{\mathbf{T}}_{\mathrm{cross}}(n)-\zeta\Big\|_\infty \le \varepsilon\Big), \tag{6.53}$$

$$\underline{I}_{\mathrm{cross}}(\zeta) \coloneqq \lim_{\varepsilon\to 0}\uparrow \underline{\lim_{n\to\infty}} -\frac{1}{n^d}\log \mathbb{P}\Big(\Big\|\widetilde{\mathbf{T}}_{\mathrm{cross}}(n)-\zeta\Big\|_\infty \le \varepsilon\Big), \tag{6.54}$$

$$\overline{I}^{+}_{\mathrm{cross}}(\zeta) \coloneqq \lim_{\varepsilon\to 0}\uparrow \overline{\lim_{n\to\infty}} -\frac{1}{n^d}\log \mathbb{P}\left(\bigcap_{i=1}^{d}\Big\{\widetilde{\mathbf{T}}_{n,[0\,,1]^d}(H_i,H_i')\ge \zeta_i-\varepsilon\Big\}\right), \tag{6.55}$$

$$\underline{I}^{+}_{\mathrm{cross}}(\zeta) \coloneqq \lim_{\varepsilon\to 0}\uparrow \underline{\lim_{n\to\infty}} -\frac{1}{n^d}\log \mathbb{P}\left(\bigcap_{i=1}^{d}\Big\{\widetilde{\mathbf{T}}_{n,[0\,,1]^d}(H_i,H_i')\ge \zeta_i-\varepsilon\Big\}\right). \tag{6.56}$$

The same arguments as the ones used in the proof of Lemmas 6.2 and 6.3 lead to

$$I^{+,(\alpha)}_{\mathrm{cross}}(\zeta) \coloneqq \overline{I}^{+,(\alpha)}_{\mathrm{cross}}(\zeta) = \underline{I}^{+,(\alpha)}_{\mathrm{cross}}(\zeta) = I^{+,(\alpha)}_{[0\,,1]^d}\Big(g^\zeta\Big), \tag{6.57}$$

and

$$I^{+}_{\mathrm{cross}}(\zeta) \coloneqq \overline{I}^{+}_{\mathrm{cross}}(\zeta) = \underline{I}^{+}_{\mathrm{cross}}(\zeta) = \lim_{\alpha\to 0}\uparrow I^{+,(\alpha)}_{\mathrm{cross}}(\zeta) = I^{+}_{[0\,,1]^d}\Big(g^\zeta\Big). \tag{6.58}$$

Following the proof of (6.21), we now show by a modification argument that

$$I_{\mathrm{cross}}(\zeta) \coloneqq \overline{I}_{\mathrm{cross}}(\zeta) = \underline{I}_{\mathrm{cross}}(\zeta) = I^{+}_{\mathrm{cross}}(\zeta). \tag{6.59}$$

The inequality

$$\underline{I}_{\mathrm{cross}}(\zeta) \ge I^{+}_{\mathrm{cross}}(\zeta) \tag{6.60}$$

is straightforward. Let us prove the converse inequality. Consider a configuration τ' with the same distribution as τ, and (X_e) a family of independent Bernoulli variables with parameter $1/2$, indexed by $\bigcup_{i=1}^{d}\{(r\mathrm{e}_i,(r+1)\mathrm{e}_i), 0\le r\le n-1\}$. We assume τ,τ' and (X_e) to be independent. The configuration defined by

$$\tau^*_e \coloneqq \begin{cases} X_e\tau'_e + (1-X_e)\tau_e & \text{if } e\in \bigcup_{i=1}^{d}\{(r\mathrm{e}_i,(r+1)\mathrm{e}_i), 0\le r\le n-1\},\\ \tau_e & \text{otherwise,}\end{cases} \tag{6.61}$$

has the same distribution as τ. As for Lemma 3.6 and (6.21), the idea is to construct an event $\mathrm{Fav}^\diamond$ whose probability has the same order as $\mathbb{P}\Big(\bigcap_{i=1}^{d}\Big\{\widetilde{\mathbf{T}}_{n,[0\,,1]^d}(H_i,H_i')\ge\zeta_i-\varepsilon\Big\}\Big)$, such that

$$\mathrm{Fav}^\diamond \subseteq \Big\{\Big\|\widetilde{\mathbf{T}}^*_{\mathrm{cross}}(n)-\zeta\Big\|_\infty\le\varepsilon\Big\}. \tag{6.62}$$

Let $n\ge 1$ and $1\le i\le d$. For all $0\le R\le n$, consider the modified configuration defined by

$$\tau^{[i,R]}_e \coloneqq \begin{cases} a & \text{if } e\in\{(r\mathrm{e}_i,(r+1)\mathrm{e}_i), 0\le r\le R-1\}\\ \tau_e & \text{otherwise.}\end{cases}$$

Define the random variable

$$R_i \coloneqq \min\left\{0\le R\le n \;\middle|\; \widetilde{\mathbf{T}}^{[i,R]}_{n,[0\,,1]^d}(H_i,H_i')\le\zeta_i\right\}.$$

It is clear that $\widetilde{\mathbf{T}}^{[i,n]}_{n,[0\,,1]^d}(H_i,H_i') = a\le\zeta_i$, thus R_i is well-defined. Note that

$$\zeta_i - \frac{b}{n} \le \widetilde{\mathbf{T}}^{[i,R_i]}_{n,[0\,,1]^d}(H_i,H_i')\le\zeta_i. \tag{6.63}$$

Let

$$E_{\mathrm{fast}} \coloneqq \bigcup_{i=1}^{d} \{(r\mathrm{e}_i, (r+1)\mathrm{e}_i), 0 \le r \le R_i - 1\}. \tag{6.64}$$

We can now define $\mathrm{Fav}^\diamond$ as

$$\begin{aligned} \mathrm{Fav}^\diamond \coloneqq &\left(\bigcap_{i=1}^{d} \left\{ \widetilde{\mathbf{T}}_{n,[0\,,1]^d}(H_i, H_i') \ge \zeta_i - \varepsilon \right\} \right) \cap \left(\bigcap_{i=1}^{d} \bigcap_{r=0}^{n-1} \left\{ X_{(r\mathrm{e}_i,(r+1)\mathrm{e}_i)} = \mathbb{1}_{E_{\mathrm{fast}}}(e) \right\} \right) \\ &\cap \left(\bigcap_{i=1}^{d} \bigcap_{r=0}^{n-1} \{ \tau'(r\mathrm{e}_i, (r+1)\mathrm{e}_i) \le a + \varepsilon \} \right). \end{aligned} \tag{6.65}$$

By independence of τ, τ' and (X_e),

$$\mathbb{P}(\mathrm{Fav}^\diamond) = \mathbb{P}\left(\bigcap_{i=1}^{d} \left\{ \widetilde{\mathbf{T}}_{n,[0\,,1]^d}(H_i, H_i') \ge \zeta_i - \varepsilon \right\} \right) \cdot \left(\frac{1}{2}\right)^{nd} \cdot \nu([a\,, a+\varepsilon])^{nd}. \tag{6.66}$$

For all $1 \le i \le d$, there exists a $\mathbf{T}_{[0\,,n]^d}$-geodesic $nH_i \overset{\gamma}{\rightsquigarrow} nH_i'$ which does not use any edge in nH_i, therefore $\widetilde{\mathbf{T}}^*_{n,[0\,,1]^d}(H_i, H_i') = \widetilde{\mathbf{T}}^{[i,R_i]}_{n,[0\,,1]^d}(H_i, H_i')$. Consequently, applying (6.63), we get (6.62) for large enough n. The inequality (6.66) then implies

$$\begin{aligned} &-\frac{1}{n^d} \log \mathbb{P}\Big(\Big\| \widetilde{\mathbf{T}}_{\mathrm{cross}}(n) - \zeta \Big\|_\infty \le \varepsilon \Big) \\ &\quad \le -\frac{1}{n^d} \log \mathbb{P}\left(\bigcap_{i=1}^{d} \left\{ \widetilde{\mathbf{T}}_{n,[0\,,1]^d}(H_i, H_i') \ge \zeta_i - \varepsilon \right\} \right) - \frac{d}{n^{d-1}} \log\Big(\frac{\nu([a\,, a+\varepsilon])}{2} \Big). \end{aligned}$$

Taking the superior limit as $n \to \infty$ then $\varepsilon \to 0$,

$$\overline{I}_{\mathrm{cross}}(\zeta) \le I^+_{\mathrm{cross}}(\zeta), \tag{6.67}$$

thus (6.59). Consequently, by Lemma 1.2 and the remark below it, $(\widetilde{\mathbf{T}}_{\mathrm{cross}}(n))_{n\ge1}$ satisfies the LDP with the rate function I_{cross}. Besides, by (6.9) and (6.58),

$$I_{\mathrm{cross}}(\zeta) = I^+_{[0\,,1]^d}(g^\zeta).$$

□

Proof of Lemma 6.8. We prove the different parts in order of difficulty.

Item (ii) is a consequence of Lemma 6.7 and the monotonicity of $I^+_{[0\,,1]^d}$.

Item (iv) is a consequence of Lemma 4.2.

To prove Item (iii), it is sufficient by Lemma 6.7 to prove that for all $\alpha > 0$, $I^{(\alpha)}_{\mathrm{cross}}$ is separately convex on $[\alpha\,, b]^d$. Let $\alpha > 0$. Without loss of generality, we only prove that, given $\alpha \le \zeta_2, \dots, \zeta_d \le b$, $I^{(\alpha)}_{\mathrm{cross}}(\cdot, \zeta_2, \dots, \zeta_d)$ is convex. Let $0 \le \theta \le 1$ and $\alpha \le \zeta_1, \zeta_1' \le b$. Consider the metric D defined on $[0\,,1]^d$ by prescribing its gradient, as in Lemma 2.11, (2.19), with

$$g_z \coloneqq \begin{cases} g^{(\zeta_1, \zeta_2, \dots, \zeta_d)} \vee \alpha\|\cdot\| & \text{if } 0 \le z_1 \le \theta, \\ g^{(\zeta_1', \zeta_2, \dots, \zeta_d)} \vee \alpha\|\cdot\| & \text{if } \theta \le z_1 \le 1. \end{cases}$$

Then

$$D(H_1, H_1') = \theta\zeta_1 + (1-\theta)\zeta_1'$$

and for all $2 \le i \le d$,

$$D(H_i, H_i') = \zeta_i.$$

Moreover, by (1.20),

$$I_{[0\,,1]^d}(D) = \theta I^{(\alpha)}_{[0\,,1]^d}\Big(g^{(\zeta_1,\zeta_2,\dots,\zeta_d)} \vee \alpha\|\cdot\|\Big) + (1-\theta) I^{(\alpha)}_{[0\,,1]^d}\Big(g^{(\zeta_1',\zeta_2,\dots,\zeta_d)} \vee \alpha\|\cdot\|\Big).$$

Equations (6.44) and (6.45) yield

$$I^{(\alpha)}_{\mathrm{cross}}(\theta\zeta_1 + (1-\theta)\zeta_1', \zeta_2, \dots, \zeta_d) \le \theta I^{(\alpha)}_{\mathrm{cross}}(\zeta_1, \zeta_2, \dots, \zeta_d) + (1-\theta) I^{(\alpha)}_{\mathrm{cross}}(\zeta_1', \zeta_2, \dots, \zeta_d), \tag{6.68}$$

thus Item (iii) is proven.

To prove Item (i), we proceed differently whether ν has an atom at b or not.

Case 1: $\nu(\{b\}) > 0$. Let $\zeta \in [a\,,b]^d$, $\varepsilon > 0$ and $\alpha > 0$. Consider the metric D on $[0\,,1]^d$ defined by prescribing its gradient, as in Lemma 2.11, (2.19), with

$$g_z \coloneqq \begin{cases} g^\zeta \vee \alpha\|\cdot\| & \text{if } z \in [0\,,1-\varepsilon]^d, \\ b\|\cdot\| & \text{otherwise.} \end{cases}$$

Then for all $1 \le i \le d$,

$$D(H_i, H_i') \ge \varepsilon b + (1-\varepsilon)\zeta_i =: \zeta_i'.$$

Note that $U \coloneqq \Big\{\hat\zeta \in [a\,,b]^d \mid \forall 1 \le i \le d, \hat\zeta_i \le \zeta_i'\Big\}$ is a neighbourhood of ζ in $[a\,,b]^d$, regardless of whether $\zeta_i = b$ for some i or not. Besides,

$$I^{(\alpha)}_{[0\,,1]^d}(D) = (1-\varepsilon)^d I^{+,(\alpha)}_{[0\,,1]^d}\Big(g^\zeta\Big) + \Big[1-(1-\varepsilon)^d\Big] I^{+,(\alpha)}_{[0\,,1]^d}(b\|\cdot\|),$$

therefore

$$I^{+,(\alpha)}_{\mathrm{cross}}(\zeta') \le (1-\varepsilon)^d I^{+,(\alpha)}_{[0\,,1]^d}\Big(g^\zeta\Big) + \Big[1-(1-\varepsilon)^d\Big] I^{+,(\alpha)}_{[0\,,1]^d}(b\|\cdot\|).$$

Letting $\alpha \to 0$, applying (6.46) and Item (ii), we deduce that for all $\hat\zeta \in U$,

$$\begin{aligned} I_{\mathrm{cross}}(\hat\zeta) &\le (1-\varepsilon)^d I^{+}_{[0\,,1]^d}\Big(g^\zeta\Big) + \Big[1-(1-\varepsilon)^d\Big] I^{+}_{[0\,,1]^d}(b\|\cdot\|) \\ &\le I^{+}_{[0\,,1]^d}\Big(g^\zeta\Big) + \Big[1-(1-\varepsilon)^d\Big] I^{+}_{[0\,,1]^d}(b\|\cdot\|) \\ &= I_{\mathrm{cross}}(\zeta) + \Big[1-(1-\varepsilon)^d\Big] I^{+}_{[0\,,1]^d}(b\|\cdot\|) \end{aligned}$$

therefore I_{cross} is upper semicontinuous on $[a\,,b]^d$. Since I_{cross} is lower semicontinuous, it is continuous on $[a\,,b]^d$.

Case 2: $\nu(\{b\}) = 0$. Lemma 6.3 and Equation (6.46) imply that I_{cross} is continuous on $[a\,,b[^d$. By Proposition 1.10, $I_{\mathrm{cross}}(\zeta) = \infty$ if $\zeta_i = b$ for some $1 \le i \le d$. Since I_{cross} is lower semicontinuous, it is continuous at such ζ, thus (i). □

This concludes the proof of Corollary 1.7, up to the proof of Claims 6.9 and 6.10.

Proof of Claim 6.9. Let $\alpha > 0$ and D as in the lemma. Without loss of generality, we only prove (6.49) for $i = 1$. By rescaling and concatenating a D-geodesic between H_1 and H_1' with its symmetric, one shows the inequality

$$s(D)(H_1, H_1') \leq D(H_1, H_1').$$

For all $z \in [0\,,1]^d$ and $u \in \mathbb{R}^d$, let

$$g_z(u) \coloneqq \min\left\{(\operatorname{grad} D_A)_z(u), A \subseteq [\![1\,,d]\!] \text{ s.t. } z \in \left[0\,,\frac{1}{2}\right]^d + \frac{1}{2}\sum_{i\in A} \mathrm{e}_i\right\}$$

be the map involved in the definition of $s(D)$ (see Lemma 2.15). Note that if z belongs the interior of $[0\,,1/2]^d + 1/2\sum_{i\in A}\mathrm{e}_i$ for some $A \subseteq [\![1\,,d]\!]$, then A in the definition of $g_z(u)$ is unique. Moreover, for all pairs (A_1, A_2), h_{A_1} and h_{A_2} agree on the intersection

$$\left(\left[0\,,\frac{1}{2}\right]^d + \frac{1}{2}\sum_{i\in A_1}\mathrm{e}_i\right) \cap \left(\left[0\,,\frac{1}{2}\right]^d + \frac{1}{2}\sum_{i\in A_2}\mathrm{e}_i\right).$$

We can thus define without amibiguity

$$\begin{aligned} h : [0\,,1]^d &\longrightarrow \left[0\,,\frac{1}{2}\right]^d \\ z &\longmapsto h_A(z) \text{ if } z \in \left[0\,,\frac{1}{2}\right]^d + \frac{1}{2}\sum_{i\in A}\mathrm{e}_i. \end{aligned} \tag{6.69}$$

It is straightforward to check for all $z \in \left[0\,,\frac{1}{2}\right]^d + \frac{1}{2}\sum_{i\in A}\mathrm{e}_i$,

$$(\operatorname{grad} D_A)_z = (\operatorname{grad} D)_{2h_A(z)} \circ f_A, \tag{6.70}$$

where f_A denote the orthogonal symmetry with respect to $\{\forall i \in A, z_i = 0\}$. In particular, for all $z \in \left[0\,,\frac{1}{2}\right]^d + \frac{1}{2}\sum_{i\in A}\mathrm{e}_i$ and $u \in \mathbb{R}^d$ such that $z + \varepsilon u \in \left[0\,,\frac{1}{2}\right]^d + \frac{1}{2}\sum_{i\in A}\mathrm{e}_i$ for small enough $\varepsilon > 0$,

$$g_z(u) = (\operatorname{grad} D)_{2h_A(z)} \circ f_A(u). \tag{6.71}$$

Let $x \in H_1$, $y \in H_1'$ and $x \overset{\gamma}{\underset{[0\,,1]^d}{\rightsquigarrow}} y$ a Lispchitz path. For all t, let $A(t)$ denote a subset $A \subseteq [\![1\,,d]\!]$ such that $\gamma(t) \in [0\,,1/2]^d + 1/2\sum_{i\in A}\mathrm{e}_i$, chosen in a measurable way. By (6.71),

$$\begin{aligned} \int_0^{T_\gamma} g_{\gamma(t)}(\gamma'(t))\mathrm{d}t &= \int_0^{T_\gamma} (\operatorname{grad} D)_{2h\circ\gamma(t)} \circ f_{A(t)}(\gamma'(t))\mathrm{d}t \\ &= \int_0^{T_\gamma} (\operatorname{grad} D)_{2h\circ\gamma(t)}((h\circ\gamma)'(t))\mathrm{d}t \\ &= \frac{1}{2}\int_0^{T_\gamma} (\operatorname{grad} D)_{2h\circ\gamma(t)}((2h\circ\gamma)'(t))\mathrm{d}t. \end{aligned} \tag{6.72}$$

Moreover, $2h \circ \gamma$ is a Lipschitz path going from H_1 to H_1', then going back to H_1, therefore

$$\int_0^{T_\gamma} g_{\gamma(t)}(\gamma'(t))\mathrm{d}t \geq D(H_1, H_1').$$

Consequently,

$$s(D)(H_1, H_1') \geq D(H_1, H_1'). \tag{6.73}$$

This concludes the proof of (6.49).

By (2.33) and (6.71),

$$\begin{aligned} I^{(\alpha)}_{[0\,,1]^d}(s(D)) &= \int_{[0\,,1]^d} I^{(\alpha)}_{[0\,,1]^d}((\operatorname{grad} s(D))_z)\mathrm{d}z \\ &= \sum_{A\subseteq[\![1\,,d]\!]} \int_{\left[0\,,\frac{1}{2}\right]^d + \frac{1}{2}\sum_{i\in A}\mathrm{e}_i} I^{(\alpha)}_{[0\,,1]^d}\Big((\operatorname{grad} D)_{2h_A(z)} \circ f_A\Big)\mathrm{d}z. \end{aligned}$$

Since the model is invariant in distribution with respect to the orthogonal transformations of $\mathbb{Z}^d$,

$$I^{(\alpha)}_{[0\,,1]^d}(s(D)) = \sum_{A\subseteq[\![1\,,d]\!]} \int_{\left[0\,,\frac{1}{2}\right]^d + \frac{1}{2}\sum_{i\in A}\mathrm{e}_i} I^{(\alpha)}_{[0\,,1]^d}\Big((\operatorname{grad} D)_{2h_A(z)}\Big)\mathrm{d}z.$$

Equation (6.50) follows. □

Proof of Claim 6.10. Recall definition (1.35). Let x, y, z as in the lemma. We assume that $x \leq y$, the other cases being similar. Let $v_x := \lfloor 2^n x \rfloor$ and $v_y := \lceil 2^n y \rceil$. For all $v, w \in [\![0\,, 2^n - 1]\!]^d$ such that $v - w \in 2\mathbb{Z}^d$, the restriction of $s^n(D)$ on the tiles $\mathrm{Tile}(v, 2^n)$ and $\mathrm{Tile}(w, 2^n)$ are equal up to a translation (see Figure 8, right). It is thus sufficient to show that there exists a $s^n(D)$-geodesic $x \overset{\gamma}{\rightsquigarrow} y$ included in the box $B := \left\{z' \in \mathbb{R}^d \mid \frac{v_x}{2^n} \leq z' \leq \frac{v_y}{2^n}\right\}$.

We claim that for all $0 \leq k \leq \langle v_x, \mathrm{e}_1\rangle$, there exists a $s^n(D)$-geodesic $x \overset{\gamma}{\rightsquigarrow} y$ included in

$$B_k := \left\{ z' \in \mathbb{R}^d \;\middle|\; \frac{k}{2^n} \leq z_1' \leq 1 \right\}.$$

For $k = 0$, there is nothing to prove. Let $0 \leq k < \langle v_x, \mathrm{e}_1\rangle$ such that the statement is true. There exists a $s^n(D)$-geodesic $x \overset{\gamma}{\rightsquigarrow} y$ included in B_k. Applying to the points of $\gamma \cap (B_k \setminus B_{k+1})$ the orthogonal symmetry with respect to

$$\left\{ z' \in \mathbb{R}^d \;\middle|\; z_1' = \frac{k+1}{2^n} \right\},$$

we obtain a geodesic included in B_{k+1} (see Figure 9). The claim follows by induction on k.

Applying a similar construction on the other side, we obtain a geodesic included in

$$\left\{ z' \in \mathbb{R}^d \;\middle|\; \frac{\langle v_x, \mathrm{e}_1\rangle}{2^n} \leq z_1' \leq \frac{\langle v_x, \mathrm{e}_1\rangle}{2^n} \right\}.$$

Finally, repeating the procedure along other coordinates concludes the proof. □

6.4 Rescaled random ball: proof of Corollary 1.8

Let $\mathfrak{K}$ denote the set of compact subsets of $\mathbb{R}^d$ et $X := \left[-\frac{1}{a}\,, \frac{1}{a}\right]^d$. Define the map

$$\begin{aligned} \Phi : \mathcal{D}_X &\longrightarrow \mathfrak{K} \\ D &\longmapsto \{x \in X \mid D(0, x) \leq 1\}. \end{aligned} \tag{6.74}$$

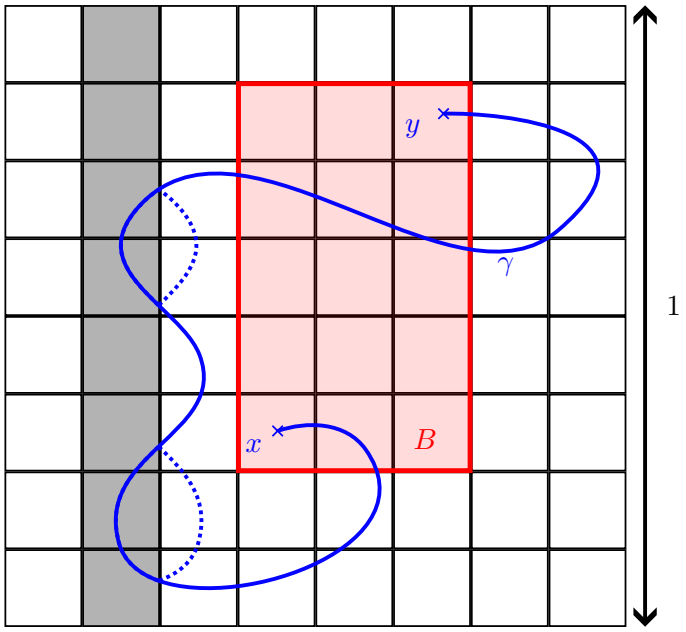


Figure 9: The symmetry argument used in Lemma 6.10, with $d = 2, n = 3, k = 1$. The set $B_1 \setminus B_2$ is represented by the shaded strip. The path γ (solid line) may be transformed into a path of same $s^n(D)$-length, included in B_2.

The lower bound in (1.16) implies that for all $n \geq 1$, $\widetilde{\mathbf{B}}(n) = \Phi(\widetilde{\mathbf{T}}_{n,X})$. By Lemma 6.1, it is thus sufficient to show that Φ is continuous for the Hausdorff distance. We actually prove that it is $\frac{1}{a}$-Lipchitz. Let $D_1, D_2 \in \mathcal{D}_X$ and $x \in \Phi(D_1)$. By definition of $\mathfrak{d}_\infty$,

$$D_2(0,x) \leq D_1(0,x) + \mathfrak{d}_\infty(D_1,D_2) \leq 1 + \mathfrak{d}_\infty(D_1,D_2).$$

Let $0 \overset{\sigma}{\rightsquigarrow} x$ be a D_2-geodesic and $y \coloneqq \sigma([D_2(0,x) - \mathfrak{d}_\infty(D_1,D_2)] \wedge 0)$. Then $y \in \Phi(D_2)$ and

$$\|x - y\| \leq \frac{1}{a} D_2(x,y) \leq \frac{1}{a} \mathfrak{d}_\infty(D_1,D_2).$$

Consequently,

$$\Phi(D_1) \subseteq \Phi(D_2) + \overline{\mathrm{B}}\left(0, \frac{1}{a}\mathfrak{d}_\infty(D_1,D_2)\right).$$

Transposing D_1 and D_2 leads to

$$\Phi(D_2) \subseteq \Phi(D_1) + \overline{\mathrm{B}}\left(0, \frac{1}{a}\mathfrak{d}_\infty(D_1,D_2)\right).$$

Hence Φ is $\frac{1}{a}$-Lipschitz. □

A Large deviations tools: proof of Lemma 1.2

Let $\mathcal{X}$, (X_n), $\overline{I}$ and $\underline{I}$ as in the lemma. We denote by $\overline{\mathrm{B}}_{\mathcal{X}}(x,r)$ the closed ball of center x and radius r in $(\mathcal{X}, d_{\mathcal{X}})$.

Proof of (i). Let $(x_k)_{k\geq 1}$ a sequence converging to x when $k \to \infty$ in $\mathcal{X}$ and $\varepsilon > 0$. Then for all $n \geq 1$,

$$\{\mathrm{d}_{\mathcal{X}}(x_k, X_n) \leq \varepsilon\} \subseteq \{\mathrm{d}_{\mathcal{X}}(x, X_n) \leq \varepsilon + \mathrm{d}_{\mathcal{X}}(x, x_k)\},$$

thus

$$-\frac{1}{n^d} \log \mathbb{P}(\mathrm{d}_{\mathcal{X}}(x_k, X_n) \leq \varepsilon) \geq -\frac{1}{n^d} \log \mathbb{P}(\mathrm{d}_{\mathcal{X}}(x, X_n) \leq \varepsilon + \mathrm{d}_{\mathcal{X}}(x, x_k)).$$

Letting $n \to \infty$, we get, for any $\varepsilon > 0$,

$$\overline{I}(x_k) \geq \overline{\lim_{n\to\infty}} -\frac{1}{n^d} \log \mathbb{P}(\mathrm{d}_{\mathcal{X}}(x, X_n) \leq \varepsilon + \mathrm{d}_{\mathcal{X}}(x, x_k)).$$

In particular, for large enough k,

$$\overline{I}(x_k) \geq \overline{\lim_{n\to\infty}} -\frac{1}{n^d} \log \mathbb{P}(\mathrm{d}_{\mathcal{X}}(x, X_n) \leq 2\varepsilon),$$

therefore

$$\underline{\lim_{k\to\infty}}\, \overline{I}(x_k) \geq \overline{\lim_{n\to\infty}} -\frac{1}{n^d} \log \mathbb{P}(\mathrm{d}_{\mathcal{X}}(x, X_n) \leq 2\varepsilon).$$

Letting $\varepsilon \to 0$ yields

$$\underline{\lim_{k\to\infty}}\, \overline{I}(x_k) \geq \overline{I}(x),$$

i.e. $\overline{I}$ is rate function. Likewise $\underline{I}$ is a rate function.

Proof of (ii). Let $U \subseteq \mathcal{X}$ be an open set and $x \in U$. There exists $\varepsilon > 0$ such that $\overline{\mathrm{B}}_{\mathcal{X}}(x, \varepsilon) \subseteq U$. In particular, for all $n \geq 1$,

$$\{\mathrm{d}_{\mathcal{X}}(x, X_n) \leq \varepsilon\} \subseteq \{X_n \in U\},$$

thus

$$\overline{\lim_{n\to\infty}} -\frac{1}{n^d} \log \mathbb{P}(X_n \in U) \leq \overline{\lim_{n\to\infty}} -\frac{1}{n^d} \log \mathbb{P}(\mathrm{d}_{\mathcal{X}}(x, X_n) \leq \varepsilon) \leq \overline{I}(x).$$

Taking the infimum over $x \in U$, we get (1.8).

Proof of (iii). Let $K \subseteq \mathcal{X}$ a compact set and $\varepsilon > 0$. There exists a finite family $(x_p)_{p=1}^P$ of elements of K such that

$$K \subseteq \bigcup_{p=1}^{P} \overline{\mathrm{B}}_{\mathcal{X}}(x_p, \varepsilon).$$

Consequently by union bound, for all $n \geq 1$,

$$\begin{aligned}
\mathbb{P}(X_n \in K) &\leq \sum_{p=1}^{P} \mathbb{P}(\mathrm{d}_{\mathcal{X}}(x_p, X_n) \leq \varepsilon) \\
&\leq P \max_{1\leq p\leq P} \mathbb{P}(\mathrm{d}_{\mathcal{X}}(x_p, X_n) \leq \varepsilon).
\end{aligned}$$

Taking the log, dividing by $-n^d$ and letting $n \to \infty$, we get

$$\begin{aligned}
\underline{\lim_{n\to\infty}} -\frac{1}{n^d} \log \mathbb{P}(X_n \in K) &\geq \underline{\lim_{n\to\infty}} -\frac{1}{n^d} \log\Big(\max_{1\leq p\leq P} \mathbb{P}(\mathrm{d}_{\mathcal{X}}(x_p, X_n) \leq \varepsilon)\Big) \\
&= \underline{\lim_{n\to\infty}} \min_{1\leq p\leq P}\Big(-\frac{1}{n^d} \log \mathbb{P}(\mathrm{d}_{\mathcal{X}}(x_p, X_n) \leq \varepsilon)\Big) \\
&= \min_{1\leq p\leq P} \underline{\lim_{n\to\infty}} -\frac{1}{n^d} \log \mathbb{P}(\mathrm{d}_{\mathcal{X}}(x_p, X_n) \leq \varepsilon). \qquad \text{(A.1)}
\end{aligned}$$

Let us denote by $\hat{x}(\varepsilon)$ an element x_p on which the minimum is attained. By compactness there exist a sequence $(\varepsilon_k)_{k\geq 1}$ converging to 0, such that $(\hat{x}(\varepsilon_k))_{k\geq 1}$ converges in K. We write $\hat{x}$ its limit. Then for all $k \geq 1$,

$$\begin{aligned}\varliminf_{n\to\infty} -\frac{1}{n^d}\log \mathbb{P}(X_n \in K) &\geq \varliminf_{n\to\infty} -\frac{1}{n^d}\log \mathbb{P}(\mathrm{d}_{\mathcal{X}}(\hat{x}(\varepsilon_k), X_n) \leq \varepsilon_k) \\ &\geq \varliminf_{n\to\infty} -\frac{1}{n^d}\log \mathbb{P}(\mathrm{d}_{\mathcal{X}}(\hat{x}, X_n) \leq \varepsilon_k + \mathrm{d}_{\mathcal{X}}(\hat{x}, \hat{x}(\varepsilon_k))).\end{aligned}$$

Letting $k \to \infty$ gives

$$\varliminf_{n\to\infty} -\frac{1}{n^d}\log \mathbb{P}(X_n \in K) \geq \underline{I}(\hat{x}),$$

thus (1.9). □

Remark A.1. The permutation of $\varliminf$ and min in Equation (A.1) is false in general with $\overline{\lim}$ instead of $\varliminf$, thus this proof may not be adapted to show

$$\overline{\lim_{n\to\infty}} -\frac{1}{n^d}\log \mathbb{P}(X_n \in K) \geq \min_{x\in K} \overline{I}(x).$$

As a consequence we have no straightforward proof of the analogue of Corollary 3.2 with $\overline{I}_X$ instead of $\underline{I}_X$.

B Properties of compact, convex sets with nonempty interior

In this section $X \in \mathcal{K}$ is fixed. Let $z \in \mathring{X}$. We define the function

$$\begin{aligned}\Phi_{z,X} : \mathbb{R}^d &\longrightarrow [0\,,\infty] \\ x &\longmapsto \inf\left\{t > 0 \;\middle|\; \frac{x-z}{t} + z \in X\right\}.\end{aligned} \tag{B.1}$$

Lemma B.1. *For all $x \in \mathbb{R}^d$ and $\lambda > 0$,*

$$\Phi_{z,X}(\lambda x + z) = \lambda \Phi_{z,X}(x + z). \tag{B.2}$$

Moreover, $\Phi_{z,X}$ is convex and for all $x \in X$,

$$x \in X \iff \Phi_{z,X}(x) \leq 1. \tag{B.3}$$

Proof. Equation (B.2) is straightforward.

Let $x_1, x_2 \in \mathbb{R}^d$ and $\theta_1, \theta_2 \in [0\,,1]$ such that $\theta_1 + \theta_2 = 1$. Let $t_1, t_2 > 0$ such that

$$\frac{x_1 - z}{t_1} + z \in X \text{ and } \frac{x_2 - z}{t_2} + z \in X.$$

Then

$$\frac{\theta_1 x_1 + \theta_2 x_2 - z}{\theta_1 t_1 + \theta_2 t_2} = \frac{\theta_1 t_1}{\theta_1 t_1 + \theta_2 t_2} \cdot \frac{x_1 - z}{t_1} + \frac{\theta_2 t_2}{\theta_1 t_1 + \theta_2 t_2} \cdot \frac{x_2 - z}{t_2}.$$

By convexity of X,

$$\frac{\theta_1 x_1 + \theta_2 x_2 - z}{\theta_1 t_1 + \theta_2 t_2} \in X.$$

Consequently,

$$\Phi_{z,X}(\theta_1 x_1 + \theta_2 x_2) \leq \theta_1 t_1 + \theta_2 t_2.$$

Taking the infimum over t_1 and t_2, we get

$$\Phi_{z,X}(\theta_1 x_1 + \theta_2 x_2) \leq \theta_1 \Phi_{z,X}(x_1) + \theta_2 \Phi_{z,X}(x_2),$$

thus $\Phi_{z,X}$ is convex.

We now turn to the proof of (B.3). The part $\implies$ is clear. Let $x \in \mathbb{R}^d$ such that $\Phi_{z,X}(x) \leq 1$. Then for all $n \geq 1$,

$$\frac{x - z}{1 + \frac{1}{n}} + z \in X,$$

thus by compactness,

$$x = \frac{x - z}{1} + z \in X.$$

□

Lemma B.2. *For all* $0 < \delta \leq 1$,

$$\mathrm{d}\Big((1-\delta)X + \delta z, \mathbb{R}^d \setminus X\Big) \geq \frac{\delta}{\max_{u \in \mathbb{S}} \Phi_{z,X}(u)} \tag{B.4}$$

and

$$\mathrm{d}\Big(X, \mathbb{R}^d \setminus ((1+\delta)X - \delta z)\Big) \geq \frac{\delta}{\max_{u \in \mathbb{S}} \Phi_{z,X}(u)}. \tag{B.5}$$

Proof. Let $0 < \delta \leq 1$ and $x \in (1-\delta)X + \delta z$. By Lemma B.1,

$$\Phi_{z,X}(x) \leq 1 - \delta.$$

Consequently, for all $y \in \mathrm{B}\Big(0, \frac{\delta}{\max_{u \in \mathbb{S}} \Phi_{z,X}(u)}\Big)$, since by Lemma B.1, $\Phi_{z,X}(\cdot + z)$ is convex and positively homogeneous,

$$\begin{aligned}
\Phi_{z,X}(x+y) &= \Phi_{z,X}(x - z + y + z) \\
&= 2 \cdot \Phi_{z,X}\left(\frac{x-z}{2} + \frac{y}{2} + z\right) \\
&\leq 2 \cdot \frac{1}{2} \cdot [\Phi_{z,X}(x - z + z) + \Phi_{z,X}(y + z)] \\
&= \Phi_{z,X}(x) + \Phi_{z,X}(y + z) \\
&\leq 1 - \delta + \|y\| \max_{u \in \mathbb{S}} \Phi_{z,X}(u + z) \\
&\leq 1,
\end{aligned}$$

i.e. $x + y \in X$. This concludes the proof of (B.4). Inequality (B.5) is proven analogously. □

Lemma B.3. *The boundary of* X *is Lebesgue-negligible.*

Proof. Let $\delta > 0$. By Lemma (B.2),

$$(1-\delta)X + \delta z \subseteq \partial X \subseteq (1+\delta)X - \delta z,$$

thus

$$\mathrm{Leb}(\partial X) \leq \Big[(1+\delta)^d - (1-\delta)^d\Big] \mathrm{Leb}(X).$$

Letting $\delta \to 0$, we get $\mathrm{Leb}(\partial X) = 0$, which concludes the proof. □

Lemma B.4. *Recall definitions* (1.36) *and* (1.37)*. These sets satisfy*

$$\lim_{k\to\infty} \frac{\#\mathrm{V}_k^{\mathrm{ext}}(X)}{k^d} = \lim_{k\to\infty} \frac{\#\mathrm{V}_k^{\mathrm{int}}(X)}{k^d} = \mathrm{Leb}(X). \tag{B.6}$$

Proof. Let $\delta > 0$. By Lemma B.2, for large enough k,

$$(1-\delta)X + \delta z \subseteq \bigcup_{v\in \mathrm{V}_k^{\mathrm{int}}(X)} \mathrm{Tile}(v,k) \subseteq \bigcup_{v\in \mathrm{V}_k^{\mathrm{ext}}(X)} \mathrm{Tile}(v,k) \subseteq (1+\delta)X - \delta z, \tag{B.7}$$

thus by taking the Lebesgue measure,

$$(1-\delta)^d \mathrm{Leb}(X) \leq \frac{\#\mathrm{V}_k^{\mathrm{int}}(X)}{k^d} \leq \frac{\#\mathrm{V}_k^{\mathrm{ext}}(X)}{k^d} \leq (1+\delta)^d \mathrm{Leb}(X).$$

Consequently,

$$(1-\delta)^d \mathrm{Leb}(X) \leq \underline{\lim}_{k\to\infty} \frac{\#\mathrm{V}_k^{\mathrm{int}}(X)}{k^d} \leq \overline{\lim}_{k\to\infty} \frac{\#\mathrm{V}_k^{\mathrm{ext}}(X)}{k^d} \leq (1+\delta)^d \mathrm{Leb}(X).$$

Letting $\delta \to 0$ concludes the proof. □

Lemma B.5. *The set $X^{-\delta}$ defined by* (5.3) *satisfies*

$$\lim_{\delta\to 0} \max_{x\in X} \mathrm{d}\Big(x, X^{-\delta}\Big) = 0. \tag{B.8}$$

Proof. The inequality (B.4) implies that for all $\delta > 0$,

$$\Big(1 - \delta \max_{u\in\mathbb{S}} \Phi_{z,X}(u)\Big) X + \delta \max_{u\in\mathbb{S}} (\Phi_{z,X}(u)) z \subseteq X^{-\delta}.$$

Consequently, for all $x \in X$,

$$\Big(1 - \delta \max_{u\in\mathbb{S}} \Phi_{z,X}(u)\Big) x + \delta \max_{u\in\mathbb{S}} (\Phi_{z,X}(u)) z \in X^{-\delta},$$

thus

$$\begin{aligned}
\mathrm{d}\Big(x, X^{-\delta}\Big) &\leq \Big\| x - \Big(1 - \delta \max_{u\in\mathbb{S}} \Phi_{z,X}(u)\Big) x - \delta \max_{u\in\mathbb{S}} (\Phi_{z,X}(u)) z \Big\| \\
&= \delta \max_{u\in\mathbb{S}} (\Phi_{z,X}(u)) \|x - z\| \\
&\leq \delta \max_{u\in\mathbb{S}} (\Phi_{z,X}(u)) \,\mathrm{diam}(X).
\end{aligned}$$

This concludes the proof. □

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