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On variable Lebesgue spaces and generalized nonlinear heat equations

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Abstract

In this work we address some questions concerning the Cauchy problem for a generalized nonlinear heat equations considering as functional framework the variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. More precisely, by mixing some structural properties of these spaces with decay estimates of the fractional heat kernel, we were able to prove two well-posedness results for these equations. In a first theorem, we prove the existence and uniqueness of global-in-time mild solutions in the mixed-space $\mathcal{L}^{\frac{p(\cdot)}{2\alpha-(1)\gamma}}(\mathbb{R}^n, L^\infty([0, T]))$. On the other hand, by introducing a new class of variable exponents, we demonstrate the existence of an unique local-in-time mild solution in the space $L^{p(\cdot)}([0, T], L^q(\mathbb{R}^3))$.

1 Introduction

1.1 General setting

In this paper we study the Cauchy problem for the generalized nonlinear heat equations

$$\begin{cases} \partial_t \mathbf{u} + (-\Delta)^\alpha \mathbf{u} = F(\mathbf{u}) + f, & (t, \mathbf{x}) \in]0, +\infty[\times \mathbb{R}^n \\ \mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n, \end{cases} \quad (1.1)$$

where $\mathbf{u} : [0, +\infty[\times \mathbb{R}^n \rightarrow \mathbb{R}$, and $f : [0, +\infty[\times \mathbb{R}^n \rightarrow \mathbb{R}$ is a given external force. Equation (1.1) possess two main features, on the one hand we have involved a nonlinear term given by

$$F(\mathbf{u}) = |\mathbf{u}|^b \mathbf{u} \quad \text{or} \quad F(\mathbf{u}) = \vec{\Gamma} \cdot \vec{\nabla} (|\mathbf{u}|^b \mathbf{u}), \quad (1.2)$$

with $b \in \mathbb{N} \setminus \{0\}$ and $\vec{\Gamma} := (1, 1, \dots, 1) \in \mathbb{R}^n$, and on the other hand we considered the fractional Laplacian operator $(-\Delta)^\alpha$ in the diffusion term. Recall that this operator is defined at the Fourier level by the symbol $|\xi|^{2\alpha}$, whereas in the spatial variable we have

$$(-\Delta)^\alpha \mathbf{u}(t, \mathbf{x}) = C_\alpha \text{p.v.} \int_{\mathbb{R}^n} \frac{\mathbf{u}(t, \mathbf{x}) - \mathbf{u}(t, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{3+2\alpha}} d\mathbf{y},$$

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where $C_\alpha > 0$ is a constant depending on α , and $\mathbf{p.v.}$ denotes the principal value.

The evolution equation (1.1) models two of the classical equations in the literature; the generalized semi-linear power dissipative equation and the generalized convection-diffusion equation. The case $\alpha = 1$ corresponds to the well-known classical semi-linear heat equation and has been widely studied by many authors, where we highlight the papers [6, 10, 15, 17, 19] and the classical book [16]. In particular, the case with $\alpha = 1$, nonlinear term $\vec{1} \cdot \vec{\nabla}(|\mathbf{u}|^b \mathbf{u})$ and $\mathbf{b} = 1$, is of special interest as it can be interpreted as a scalar toy model for the Navier-Stokes equations [1, 2, 9, 18]. Regarding more general cases, in [11, 12] the authors proved for $\alpha \in \mathbb{N}$ the existence and uniqueness of strong solutions of (1.1) considering as functional setting the classical Lebesgue spaces. On the other hand, in [13] the authors provide a global well-posedness result in the case of $\alpha \geq 1$ considering small initial data in pseudomeasure spaces.

In this paper we address questions concerning the existence and uniqueness of solutions for equations (1.1) considering as functional framework the Lebesgue spaces of variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$. To the best of our knowledge, this kind of functional spaces have not been considered previously in the analysis of semi-linear power dissipative type equations nor convection-diffusion type equations. Roughly speaking, the spaces $L^{p(\cdot)}(\mathbb{R}^n)$ can be interpreted as a natural generalization of the classical Lebesgue spaces L^p , in the sense that, the usual constant parameter $p \in [1, +\infty[$ is replaced now by an appropriate function $p(\cdot) : \mathbb{R}^n \rightarrow [1, +\infty[$. However, as we will see in the following lines, in the rigorous definition of the variable Lebesgue spaces there are subtle issues that make them very different of the classical ones. To define the spaces $L^{p(\cdot)}$, we consider a measurable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, and the *modular function* $m_{p(\cdot)}$ associated to $p(\cdot)$, which is defined by

$$m_{p(\cdot)}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx. \quad (1.3)$$

At this stage of the construction, is illustrative to see that, if we consider $p(\cdot) \equiv p \in [1, +\infty[$, we can define the classical L^p -norm, and thus the L^p -space, by considering the expression

$$\|f\|_{L^p(\mathbb{R}^n)} = (m_p(f))^{\frac{1}{p}}.$$

In this point appears an important issue; if we consider a measurable function $p(\cdot)$ non constant, the previous formula does not have sense (because the exponent outside the integral). Then, the classical strategy to avoid this difficulty is to equip to the space $L^{p(\cdot)}$ with the following *Luxemburg norm*:

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \lambda > 0 : m_{p(\cdot)} \left(\frac{f}{\lambda} \right) = \int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}. \quad (1.4)$$

Thus, we define the variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$, as the set of measurable functions such that quantity $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ is finite. For a complete presentation of the theory of variable Lebesgue spaces we refer to the interested reader to the books [4], [5] and [8].

With this information at hand, our objective in this paper is to provide a first application of the variable Lebesgue spaces to the analysis of the generalized semi-linear power dissipative equation and the generalized convection-diffusion equation. For doing this, we present an unified approach which deals both cases. More precisely, we will construct solutions for the Cauchy problem (1.1) provided with initial data and external forces in appropriate variable Lebesgue spaces. The presentation of our main results motivate the next subsection.

1.2 Presentation of the results

In this subsection we state our main results about the existence and uniqueness of mild solution of the Cauchy problem for the generalized nonlinear heat equation (1.1) on variable Lebesgue spaces. Thus, in order to distinguish appropriately the different cases involved in (1.2), in the following we introduce some useful notations. Given a constant $s \in \mathbb{R}$, we denote by $\langle s \rangle_\gamma$ to the quantity given by

$$\langle s \rangle_\gamma = \begin{cases} 0 & \text{if } \gamma = 0, \\ s & \text{if } \gamma = 1. \end{cases} \quad (1.5)$$

Similarly, given the operators $\text{Id}(\cdot)$ and $\vec{1} \cdot \vec{\nabla}(\cdot)$, we denote by $\vec{\nabla}^\gamma(\cdot)$ to the operator defined by the expression

$$\vec{\nabla}^\gamma(\cdot) = \begin{cases} \text{Id}(\cdot) & \text{if } \gamma = 0, \\ \vec{1} \cdot \vec{\nabla}(\cdot) & \text{if } \gamma = 1. \end{cases} \quad (1.6)$$

With these notations fixed, let us proceed with the presentation of our first main result, which state the existence and uniqueness of a global mild solution of the Cauchy problem for the equations (1.1) in the framework of variable Lebesgue spaces as long the external force and the initial data are sufficiently small. This first theorem reads as follows.

Theorem 1 *Let $\gamma \in \{0, 1\}$ fixed. Consider $\alpha \in]\frac{1}{2}, 1]$, a variable exponent $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ such that $p^- > 1$, an initial data $\mathbf{u}_0 \in \mathcal{L}^{\frac{nb}{2\alpha - \langle 1 \rangle_\gamma}}(\mathbb{R}^n)$, and let f be an external force such that $f = \vec{\nabla}^\gamma(\mathcal{F})$ where \mathcal{F} is a function in $\mathcal{L}^{\frac{p(\cdot)}{b+1}}_{\frac{nb}{(b+1)(2\alpha - \langle 1 \rangle_\gamma)}}(\mathbb{R}^n, L^\infty([0, T]))$. If $\|\mathbf{u}_0\|_{\mathcal{L}^{\frac{nb}{2\alpha - \langle 1 \rangle_\gamma}}(\mathbb{R}^n)} + \|\mathcal{F}\|_{\mathcal{L}^{\frac{p(\cdot)}{b+1}}_{\frac{nb}{(b+1)(2\alpha - \langle 1 \rangle_\gamma)}}(\mathbb{R}^n, L^\infty([0, T]))}$ is small enough, then the equation (1.1) admits an unique global mild solution in the space $\mathcal{L}^{\frac{p(\cdot)}{b+1}}_{\frac{nb}{2\alpha - \langle 1 \rangle_\gamma}}(\mathbb{R}^n, L^\infty([0, T]))$.*

We must remark the fact that, we have studied the behaviour of the mild solution in the time variable by considering the L^∞ space, and thus we have analyzed the information in the space variable by considering the *mixed variable Lebesgue spaces* $\mathcal{L}_T^{p(\cdot)}$ (see Subsection 2.1 for the definition of these spaces). Now, note that the mixed variable Lebesgue spaces considered here by merely technical reasons, and it is motivated by the lack of flexibility in the parameters that intervene in the boundedness of the Riesz transforms involved. In Subsection 2.1 and Remark 3.1 below we will provide precise details on this particular issue.

In our second main result we will state the existence and uniqueness of a local mild solution for the equations (1.1) by considering a Lebesgue space of variable exponent in the time variable, and by setting a classical L^q -space in the space variable. This theorem reads as follows.

Theorem 2 *Let $\gamma \in \{0, 1\}$ fixed. Consider $\alpha \in]\frac{1}{2}, 1]$, $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $b + 1 < p^- \leq p^+ < +\infty$, fix a parameter $q > \frac{nb}{2\alpha - \langle 1 \rangle_\gamma}$ by the relationship $\frac{\alpha b}{p(\cdot)} + \frac{nb}{2q} < \alpha - \langle \frac{1}{2} \rangle_\gamma$ and $\bar{q}(\cdot) \in \mathcal{P}_q^{emb}(\mathbb{R}^n)$. If $f \in L^1([0, +\infty[, L^{\bar{q}(\cdot)}(\mathbb{R}^n))$ is an external force and if the initial data $\mathbf{u}_0 \in L^{\bar{q}(\cdot)}(\mathbb{R}^n)$, then there exist a time $0 < T < +\infty$ and an unique mild solution of the equation (1.1) in the space $L^{p(\cdot)}([0, T], L^q(\mathbb{R}^n))$.*

Note that, the variable exponent $\bar{q}(\cdot)$ involved in this theorem belongs to the class $\mathcal{P}_q^{\text{emb}}(\mathbb{R}^n)$, which characterize the variable Lebesgue spaces embedded in the space $L^q(\mathbb{R}^n)$. We will give a precise notion of this class of exponents in Definition 2.2 and Lemma 2.3 below.

To finish this subsection we must mention that the theorems recently presented here seems to be, to the best of our knowledge, are the first applications of the variable Lebesgue spaces in the analysis of the the generalized nonlinear heat equations (1.1). We hope that these results will inform future studies about applications of variable Lebesgue spaces to various evolution PDEs.

Organization of the paper

The next subsections are structured as follows. In Section 2 we will present a review of the main definitions and properties of Variable Lebesgue spaces $L^{p(\cdot)}$ and some decay estimates of fractional heat kernels. Section 3 is dedicated to the proof of Theorems 1 and 2.

2 Preliminaries

With the aim of keeping this article reasonably self-contained, in this subsection we will present the key results and definitions about variable Lebesgue spaces and fractional heat kernel involved in the proofs of our main results.

2.1 Variable Lebesgue spaces

We begin this subsection by specifying some of the basic concept involved in the theory of variable Lebesgue spaces; the notions of variable exponent, set of variable exponents and limit exponents.

Definition 2.1 *Let $n \in \mathbb{Z}_+$ and $p : \mathbb{R}^n \rightarrow [1, +\infty[$ a function. We say that $p(\cdot)$ is a variable exponent if it is a measurable function. We denote by $\mathcal{P}(\mathbb{R}^n)$ the set of variable exponents, and we define the limit exponents p^- and p^+ as*

$$p^- = \inf \text{ess}_{x \in \mathbb{R}^n} \{p(x)\} \quad \text{and} \quad p^+ = \sup \text{ess}_{x \in \mathbb{R}^n} \{p(x)\}. \quad (2.1)$$

Given a variable exponent $p(\cdot)$, in the rest of the article we will consider that

$$1 < p^- \leq p^+ < +\infty.$$

In this point we must emphasise the fact that the spaces $L^{p(\cdot)}(\mathbb{R}^n)$ are Banach function spaces, and thus they possess interesting and natural properties. In the next, we present the generalization of the Hölder inequalities to this functional setting.

Lemma 2.1 *Let consider the variable exponents $p_1(\cdot), p_2(\cdot), p_3(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that the following pointwise relationship follows $\frac{1}{p_1(x)} = \frac{1}{p_2(x)} + \frac{1}{p_3(x)}$, $x \in \mathbb{R}^n$. Then, given $f \in L^{p_2(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p_3(\cdot)}(\mathbb{R}^n)$, the pointwise product fg belongs to the space $L^{p_1(\cdot)}(\mathbb{R}^n)$, and there exists a numerical constant $C > 0$ such that*

$$\|fg\|_{L^{p_1(\cdot)}} \leq C \|f\|_{L^{p_2(\cdot)}} \|g\|_{L^{p_3(\cdot)}}. \quad (2.2)$$

Note that, given two vector fields $\vec{f}, \vec{g} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the estimate (2.2) can be generalized to the product $\vec{f} \cdot \vec{g}$. The proof of this result can be consulted in [4, Section 2.4] or [5, Section 3.2].

To continue we remark that the Luxemburg norm $\|\cdot\|_{L^{p(\cdot)}}$ defined in (1.4) satisfies the *norm conjugate formula* [5, Corollary 3.2.14]. This result reads as follow.

Proposition 2.1 *Let consider two variable exponents $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1 = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$, for all $x \in \mathbb{R}^n$. Then, given $f \in L^{p_1(\cdot)}$, there exists a numerical constant $C > 0$ such that*

$$\|f\|_{L^{p_1(\cdot)}} \leq 2 \sup_{\|g\|_{L^{p_2(\cdot)}} \leq 1} \int_{\mathbb{R}^n} |f(x)| |g(x)| dx. \quad (2.3)$$

Remark 2.1 *As is natural the notions and results recently presented in the setting of the whole space \mathbb{R}^n can be adapted to any subset $\Omega \subset \mathbb{R}^n$.*

In the next we present some embedding results in the setting of variable Lebesgue spaces.

Lemma 2.2 *Let $n \geq 1$, a bounded domain $\Omega \subset \mathbb{R}^n$ and $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\Omega)$ such that $1 < p_1^+, p_2^+ < +\infty$. Then, $L^{p_2(\cdot)}(\Omega) \subset L^{p_1(\cdot)}(\Omega)$ if and only if $p_1(x) \leq p_2(x)$ almost everywhere. Moreover, in the case we have the estimate*

$$\|f\|_{L^{p_1(\cdot)}(\Omega)} \leq (1 + |\Omega|) \|f\|_{L^{p_2(\cdot)}(\Omega)}.$$

The proof of this lemma can be consulted in [4, Corollary 2.48].

An interesting fact in the setting of variable Lebesgue spaces is given by the extension of the result presented in Lemma 2.2 to unbounded domains. In particular, we will be interested in the case when $p_1(x)$ is a constant function. In order to characterize it in a concise manner we introduce the following class of exponents.

Definition 2.2 *Given a constant exponent $q \in (1, \infty)$, we define the class of variable exponents $\mathcal{P}_q^{emb}(\mathbb{R}^n)$, as the set*

$$\mathcal{P}_q^{emb}(\mathbb{R}^n) = \left\{ \bar{q}(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n) : q \leq (\bar{q})^- \leq (\bar{q})^+ < +\infty \quad \text{and} \quad \frac{q\bar{q}(x)}{\bar{q}(x) - q} \rightarrow +\infty \text{ as } |x| \rightarrow +\infty \right\}. \quad (2.4)$$

A nice and natural consequence of considering a variable exponent in the class $\mathcal{P}_q^{emb}(\mathbb{R}^n)$ is given in the following result, which follows by considering Theorem 2.45 and Remark 2.46 in [4].

Lemma 2.3 *Let $q \in (1, \infty)$ and $\bar{q}(\cdot) \in \mathcal{P}_q^{emb}(\mathbb{R}^n)$. Then, $L^{\bar{q}(\cdot)}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$, and there exists a numerical constant $C > 0$ such that*

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^{\bar{q}(\cdot)}(\mathbb{R}^n)}.$$

In this point we must emphasize that the variable Lebesgue spaces are not translation invariant, and thus, the Young's inequality for convolution cannot be generalized to $L^{p(\cdot)}$ spaces for non-constant exponents (see [5, Section 3.6]). In consequence, new ideas are needed to deal with many of the classical operators and techniques that appear in the analysis of PDEs.

One of the classical techniques to study the boundedness of such operators on $L^{p(\cdot)}$ spaces, is to consider some constrains on the variable exponents. The most important condition on these exponents is given by the *log-Hölder continuity condition*, which we present in the following.

Definition 2.3 *Let consider a variable exponent $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that there exists the limit value $\frac{1}{p_\infty} = \lim_{|x| \rightarrow +\infty} \frac{1}{p(x)}$.*

1. *We say that $p(\cdot)$ is locally log-Hölder continuous if for each $x, y \in \mathbb{R}^n$, there exists a constant $C > 0$ such that*

$$\left| \frac{1}{p(x)} - \frac{1}{p(y)} \right| \leq \frac{C}{\log(e + 1/|x - y|)}.$$

2. *We say that $p(\cdot)$ satisfies the log-Hölder decay condition, if for each $x \in \mathbb{R}^n$, there exists a constant $C > 0$ such that*

$$\left| \frac{1}{p(x)} - \frac{1}{p_\infty} \right| \leq \frac{C}{\log(e + |x|)}.$$

3. *We say that $p(\cdot)$ is globally log-Hölder continuous in \mathbb{R}^n if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition.*

4. *We define the class as the following class of variable exponents*

$$\mathcal{P}^{\log}(\mathbb{R}^n) = \{p(\cdot) \in \mathcal{P}(\mathbb{R}^n) : p(\cdot) \text{ is globally log-Hölder continuous in } \mathbb{R}^n\}. \quad (2.5)$$

To continue, we recall the definition of the Hardy-Littlewood maximal function.

Definition 2.4 *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ a locally integrable function. The Hardy-Littlewood maximal function \mathcal{M} is defined by*

$$\mathcal{M}(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy$$

where B is an open ball of \mathbb{R}^n .

With the notions of globally log-Hölder continuous exponents we can present the following key result (see [5, Section 4.3]).

Theorem 3 *Let consider a variable exponent $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ satisfying $p^- > 1$. Then, given $f \in L^{p(\cdot)}$, there exist a constant $C > 0$ such that*

$$\|\mathcal{M}(f)\|_{L^{p(\cdot)}} \leq C \|f\|_{L^{p(\cdot)}}. \quad (2.6)$$

In the next we recall a classical Lemma about the Hardy-Littlewood maximal function (see [7, Section 2.1]):

Lemma 2.4 *If φ is a radially decreasing function on \mathbb{R}^n and f is a locally integrable function, then*

$$|(\varphi * f)(x)| \leq \|\varphi\|_{L^1} \mathcal{M}(f)(x),$$

where \mathcal{M} is the Hardy-Littlewood maximal function.

Let consider the Riesz transforms $(\mathcal{R}_j)_{1 \leq j \leq n}$, which are defined at the Fourier level by the expression

$$\widehat{\mathcal{R}_j(f)}(\xi) = -\frac{i\xi_j}{|\xi|} \widehat{f}(\xi).$$

The next result state that the Riesz transform of a measurable function are also bounded in Lebesgue spaces of variable exponent. A proof of this fact can be consulted in [5, Sections 6.3 and 12.4].

Lemma 2.5 *Let consider a variable exponent $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ satisfying $1 < p^- \leq p^+ < +\infty$. Then, given $f \in L^{p(\cdot)}$, there exist a constant $C > 0$ such that*

$$\|\mathcal{R}_j(f)\|_{L^{p(\cdot)}} \leq C \|f\|_{L^{p(\cdot)}}, \quad (2.7)$$

In the following we recall the notion of Riesz potential operator.

Definition 2.5 *Let consider a parameter $\beta \in (0, n)$ and a measurable function f . We define the Riesz potential operator $\mathcal{J}_\beta(f) : \mathbb{R}^n \rightarrow [0, +\infty]$ by the expression*

$$\mathcal{J}_\beta(f)(x) := \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\beta}} dy. \quad (2.8)$$

This operator is bounded in variable Lebesgue spaces if we consider appropriate globally log-Hölder continuous variable exponents. A precise statement of this important is presented in the next.

Theorem 4 *Let consider a variable exponent $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and a parameter $\beta \in (0, n/p^+)$. Then, there exist a numerical constant $C > 0$ such that the following estimate follows*

$$\|\mathcal{J}_\beta(f)\|_{L^{q(\cdot)}} \leq C \|f\|_{L^{p(\cdot)}}, \quad \text{with} \quad \frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\beta}{n}. \quad (2.9)$$

A proof of this result can be consulted in [5, Section 6.1].

Remark 2.2 *Note that the estimate (2.9) introduces a very strong relationship between the variable exponents $p(\cdot)$ and $q(\cdot)$. For instance, if $p(\cdot)$ is a constant function, this forces to the exponent $q(\cdot)$ to be constant function as well.*

In order to provide more flexibility on these parameters (see Remark 3.1 below) we will consider the following functional spaces (see in [3] for more details).

Definition 2.6 *Let consider a variable exponent $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $q \in (1, +\infty)$. Then, we define the mixed Lebesgue space $\mathcal{L}_q^{p(\cdot)}(\mathbb{R}^n)$ as*

$$\mathcal{L}_q^{p(\cdot)}(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n),$$

which can be normed by the expression

$$\|\cdot\|_{\mathcal{L}_q^{p(\cdot)}} = \max\{\|\cdot\|_{L^{p(\cdot)}}, \|\cdot\|_{L^q}\}. \quad (2.10)$$

As we mentioned above, provided with these functional spaces we have the following result (see [3]).

Proposition 2.2 *Let consider $q \in (1, +\infty)$, a variable exponent $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and a parameter $0 < \beta < \min\{n/p^+, n/q\}$. Given a function $f \in \mathcal{L}_q^{p(\cdot)}(\mathbb{R}^n)$ and a variable exponent $r(\cdot)$ defined by the following condition*

$$r(\cdot) = \frac{np(\cdot)}{n - \beta q}, \quad (2.11)$$

then, there exist a numerical constant $C > 0$ such that the following estimate follows

$$\|J_\beta(f)\|_{L^{r(\cdot)}} \leq C \|f\|_{\mathcal{L}_q^{p(\cdot)}}. \quad (2.12)$$

Remark 2.3 *Note that the index q is not related to the limit exponents p^- or p^+ nor to $p(\cdot)$.*

Remark 2.4 *Note that the functional spaces $\mathcal{L}_q^{p(\cdot)}$ inherit the properties of the spaces $L^{p(\cdot)}$ and L^q . Thus, given variable exponents $p_1(\cdot), p_2(\cdot), p_3(\cdot)$, and constant exponents q, q_2, q_3 , such that $\frac{1}{p_1(\cdot)} = \frac{1}{p_2(\cdot)} + \frac{1}{p_3(\cdot)}$ and $\frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3}$ the following Hölder-type inequality follows*

$$\|fg\|_{\mathcal{L}_{q_1}^{p_1(\cdot)}} \leq \|f\|_{\mathcal{L}_{q_2}^{p_2(\cdot)}} \|g\|_{\mathcal{L}_{q_3}^{p_3(\cdot)}}$$

Furthermore, the Riesz transforms are also bounded in these spaces.

For more details about the theory of Variable Lebesgue spaces, we recommend to the interested reader to see the books [4], [5] and [8].

2.2 Some key estimates on the fractional heat kernel

In this subsection we state some estimates related to the fractional heat kernel $\mathfrak{g}_t^\alpha(x)$ involved in the mild formulation of the system (1.1). Let us begin by recalling that this kernel is defined by

$$\mathfrak{g}_t^\alpha(x) = \mathcal{F}^{-1}(e^{-t|\xi|^{2\alpha}}) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{-t|\xi|^{2\alpha}} d\xi.$$

Remark 2.5 *Let consider $x \in \mathbb{R}^n$ and the fractional heat kernel $\mathfrak{g}_t^\alpha(x)$. Then, there exists a numerical constant $C > 0$ such that the following pointwise estimates follow*

$$|\mathfrak{g}_t^\alpha(x)| \leq C \frac{t}{(t^{\frac{1}{2\alpha}} + |x|)^{n+2\alpha}} \quad \text{and} \quad |\nabla \mathfrak{g}_t^\alpha(x)| \leq C \frac{1}{(t^{\frac{1}{2\alpha}} + |x|)^{n+1}}. \quad (2.13)$$

For a precise statement of this remark see [14, Lemma 2.1 and Remark 2.1].

Lemma 2.6 *Let consider $x \in \mathbb{R}^n$, the fractional heat kernel $\mathfrak{g}_t^\alpha(x)$, and the parameters $1 \leq p \leq q \leq +\infty$. Then, given $\alpha, \nu > 0$, there exists a numerical constant $C > 0$ such that following estimates follow*

1. $\|\mathfrak{g}_t^\alpha * f(x)\|_{L^q} \leq Ct^{-\frac{n}{2\alpha}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p},$
2. $\|(-\Delta)^{\nu/2} \mathfrak{g}_t^\alpha * f(x)\|_{L^q} \leq Ct^{-\frac{\nu}{2\alpha} - \frac{n}{2\alpha}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p}.$

We refer to the interested reader to [14, Lemma 3.1] for a proof of these results.

3 Mild solutions in variable Lebesgue spaces

In this section we study mild solutions for the nonlinear heat equation (1.1) in the setting of variable Lebesgue spaces. These mild solutions are obtained via the classical contraction mapping principle. Note that, by considering notations introduced in (1.5) and (1.6), the equations stated in (1.1) can be recast as

$$\begin{cases} \partial_t \mathbf{u} = -(-\Delta)^\alpha \mathbf{u} + \vec{\nabla}^\gamma (|\mathbf{u}|^b \mathbf{u}) + f, & (t, \mathbf{x}) \in]0, +\infty[\times \mathbb{R}^n, \\ \mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n, \end{cases} \quad (3.1)$$

where $\gamma \in \{0, 1\}$. Now, due to the Duhamel formula, we can recover equation (3.1) in the following equivalent form

$$\mathbf{u}(t, \mathbf{x}) = \mathbf{g}_t^\alpha * \mathbf{u}_0(\mathbf{x}) + \int_0^t \mathbf{g}_{t-s}^\alpha * f(s, \mathbf{x}) ds - \int_0^t \mathbf{g}_{t-s}^\alpha * \vec{\nabla}^\gamma (|\mathbf{u}|^b \mathbf{u})(s, \mathbf{x}) ds, \quad (3.2)$$

where \mathbf{g}_t^α denotes the fractional heat kernel. Any function satisfying the integral equation (3.2) is called a *mild solution* of (3.1) (or (1.1)). In the next subsections we will apply the contraction mapping principle on this integral equation.

3.1 Proof of Theorem 1

In this subsection we consider a variable exponent $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^+)$ such that $1 < p^- \leq p(\cdot) \leq p^+ < +\infty$, and the functional space

$$\mathcal{E} = \mathcal{L}^{\frac{p(\cdot)}{2\alpha - (1)\gamma}}(\mathbb{R}^n, L^\infty([0, T])),$$

which is dotted with the following norm

$$\|\cdot\|_{\mathcal{E}} = \max \left\{ \|\cdot\|_{L_x^{p(\cdot)}(L_t^\infty)}, \|\cdot\|_{L_x^{\frac{nb}{2\alpha - (1)\gamma}}(L_t^\infty)} \right\}. \quad (3.3)$$

As mentioned above, our strategy will be construct mild solutions for the integral equation (3.2) by considering mapping contraction principle on thus functional setting. More precisely, the proof of Theorem 1 is based in the next 3 propositions.

Let us begin by considering the following result regarding a control of the initial data.

Proposition 3.1 *Let $\gamma \in \{0, 1\}$ fixed. Consider $\alpha \in]\frac{1}{2}, 1]$, a variable exponent $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ such that $p^- > 1$, and a function $\mathbf{u}_0 \in \mathcal{L}^{\frac{p(\cdot)}{2\alpha - (1)\gamma}}(\mathbb{R}^n)$. Then, there exist a numerical constant $C > 0$ such that*

$$\|\mathbf{g}_t^\alpha * \mathbf{u}_0\|_{\mathcal{E}} \leq C \|\mathbf{u}_0\|_{\mathcal{L}^{\frac{p(\cdot)}{2\alpha - (1)\gamma}}}. \quad (3.4)$$

Proof. Note that \mathbf{u}_0 is a locally integrable function, since $\mathbf{u}_0 \in L^{\frac{nb}{2\alpha - (1)\gamma}} \subset L_{loc}^1$. Then, as the fractional heat kernel \mathbf{g}_t^α is a radially decreasing function, we can consider Lemma 2.4 to obtain

$$\|\mathbf{g}_t^\alpha * \mathbf{u}_0(\mathbf{x})\|_{L_t^\infty} \leq CM(\mathbf{u}_0)(\mathbf{x}).$$

Recalling the norm defined in (3.3), we can write the estimate

$$\|\mathbf{g}_t^\alpha * \mathbf{u}_0\|_\varepsilon \leq C \max \left\{ \|\mathcal{M}(\mathbf{u}_0)\|_{L^{p(\cdot)}}, \|\mathcal{M}(\mathbf{u}_0)\|_{L^{\frac{nb}{2\alpha-(1)\gamma}}} \right\}.$$

Now, by hypothesis we know that $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p^- > 1$, then, by Theorem 3 we have that the maximal function \mathcal{M} is bounded on the Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$. Considering this fact, and since \mathcal{M} is also bounded on the space $L^{\frac{nb}{2\alpha-(1)\gamma}}$, we conclude

$$\|\mathbf{g}_t^\alpha * \mathbf{u}_0\|_\varepsilon \leq C \max \left\{ \|\mathbf{u}_0\|_{L^{p(\cdot)}}, \|\mathbf{u}_0\|_{L^{\frac{nb}{2\alpha-(1)\gamma}}} \right\} \leq C \|\mathbf{u}_0\|_{\mathcal{L}^{\frac{nb}{2\alpha-(1)\gamma}}(\mathbb{R}^n)}.$$

With this we conclude the proof. \square

Proposition 3.2 *Let $\gamma \in \{0, 1\}$ fixed. Consider $\alpha \in [\frac{1}{2}, 1]$, a variable exponent $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p^- > 1$, and a function $f = \vec{\nabla}^\gamma(\mathcal{F})$, where $\mathcal{F} \in \mathcal{L}^{\frac{p(\cdot)}{b+1}, \frac{nb}{(b+1)(2\alpha-(1)\gamma)}}(\mathbb{R}^n, L^\infty([0, T]))$. Then, there exist a numerical constant $C > 0$ such that*

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * f(\cdot, \cdot) ds \right\|_\varepsilon \leq C \|\mathcal{F}\|_{\mathcal{L}^{\frac{p(\cdot)}{b+1}, \frac{nb}{(b+1)(2\alpha-(1)\gamma)}, x}(L_t^\infty)}. \quad (3.5)$$

Proof. To conclude the estimate (3.5), in the following we will study separately the cases when $\gamma = 1$ and $\gamma = 0$.

- **Case $\gamma = 1$.** We begin the analysis of this case by noticing that since $f = \vec{\nabla}^\gamma(\mathcal{F})$ we can write

$$\left| \int_0^t \mathbf{g}_{t-s}^\alpha * f(s, x) ds \right| \leq C \int_0^t \int_{\mathbb{R}^n} |\vec{\nabla} \mathbf{g}_{t-s}^\alpha(x-y)| |\mathcal{F}(s, y)| dy ds.$$

Then, by Fubini's theorem and the decay properties of the fractional heat kernel in Remark 2.5, we get

$$\left| \int_0^t \mathbf{g}_{t-s}^\alpha * f(s, x) ds \right| \leq C \int_{\mathbb{R}^n} \int_0^t \frac{1}{(|t-s|^{\frac{1}{2\alpha}} + |x-y|)^{n+1}} |\mathcal{F}(s, y)| ds dy.$$

Now, by considering the L_t^∞ norm on \mathcal{F} , we obtain

$$\left| \int_0^t \mathbf{g}_{t-s}^\alpha * f(s, x) ds \right| \leq C \int_{\mathbb{R}^n} \int_0^t \frac{1}{(|t-s|^{\frac{1}{2\alpha}} + |x-y|)^{n+1}} ds \|\mathcal{F}(\cdot, y)\|_{L_t^\infty} dy.$$

Then, as $\alpha \in [\frac{1}{2}, 1]$, we can consider the Riesz potential (see definition (2.5)) provided of the estimate

$$\begin{aligned} \int_0^t \frac{ds}{\left(|t-s|^{\frac{1}{2\alpha}} + |x-y|\right)^{n+1}} &\leq \int_0^{+\infty} \frac{ds}{\left(s^{\frac{1}{2\alpha}} + |x-y|\right)^{n+1}} \\ &= \int_0^{+\infty} \frac{|x-y|^{2\alpha} d\beta}{\left(\left(|x-y|^{2\alpha} \beta\right)^{\frac{1}{2\alpha}} + |x-y|\right)^{n+1}} \\ &= \frac{1}{|x-y|^{n+1-2\alpha}} \int_0^{+\infty} \frac{d\beta}{\left(1 + \beta^{\frac{1}{2\alpha}}\right)^{n+1}}, \end{aligned} \quad (3.6)$$

in order to obtain

$$\left| \int_0^t \mathbf{g}_{t-s}^\alpha * f(s, \mathbf{x}) ds \right| \leq C \int_{\mathbb{R}^n} \frac{1}{|\mathbf{x} - \mathbf{y}|^{n+1-2\alpha}} \|\mathcal{F}(\cdot, \mathbf{y})\|_{L_t^\infty} d\mathbf{y} = C \mathcal{J}_{2\alpha-1}(\|\mathcal{F}(\cdot, \cdot)\|_{L_t^\infty})(\mathbf{x}),$$

and then, we can write

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * f(s, \mathbf{x}) ds \right\|_{L_t^\infty} \leq C \mathcal{J}_{2\alpha-1}(\|\mathcal{F}(\cdot, \cdot)\|_{L_t^\infty})(\mathbf{x}). \quad (3.7)$$

Now, to reconstruct the $\mathcal{L}_{\frac{nb}{2\alpha-1}}^{p(\cdot)}$ -norm given in (3.3), from the estimate (3.7) we write

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * f(s, \mathbf{x}) ds \right\|_{\mathcal{L}_x^{p(\cdot)}(L_t^\infty)} \leq C \|\mathcal{J}_{2\alpha-1}(\|\mathcal{F}(\cdot, \cdot)\|_{L_t^\infty})(\cdot)\|_{L_x^{p(\cdot)}},$$

and

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * f(s, \mathbf{x}) ds \right\|_{\mathcal{L}_x^{\frac{nb}{2\alpha-1}}(L_t^\infty)} \leq C \|\mathcal{J}_{2\alpha-1}(\|\mathcal{F}(\cdot, \cdot)\|_{L_t^\infty})(\cdot)\|_{L_x^{\frac{nb}{2\alpha-1}}}.$$

Thus, by Proposition 2.2 we get

$$\|\mathcal{J}_{2\alpha-1}(\|\mathcal{F}(\cdot, \cdot)\|_{L_t^\infty})(\cdot)\|_{L_x^{p(\cdot)}} \leq C \|\|\mathcal{F}(\cdot, \cdot)\|_{L_t^\infty}\|_{\mathcal{L}_x^{\frac{nb}{(b+1)(2\alpha-1)}, x}} \stackrel{p(\cdot)}{=} \|\mathcal{F}\|_{\mathcal{L}_x^{\frac{nb}{(b+1)(2\alpha-1)}, x}(L_t^\infty)}. \quad (3.8)$$

Note that, since the Riesz potentials are bounded on space $L^{\frac{nb}{2\alpha-1}}$, we obtain

$$\|\mathcal{J}_{2\alpha-1}(\|\mathcal{F}(\cdot, \cdot)\|_{L_t^\infty})(\cdot)\|_{L_x^{\frac{nb}{2\alpha-1}}} \leq C \|\|\mathcal{F}\|_{L_t^\infty}\|_{L_x^{\frac{nb}{(b+1)(2\alpha-1)}}} = \|\mathcal{F}\|_{L_x^{\frac{nb}{(b+1)(2\alpha-1)}}(L_t^\infty)}. \quad (3.9)$$

Then, gathering together the definition of the norm $\|\cdot\|_\mathcal{E}$ given in (3.3) with the estimates (3.8) and (3.9), we conclude

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * f(s, \mathbf{x}) ds \right\|_{\mathcal{L}_x^{p(\cdot)}(L_t^\infty)} \leq C \|\mathcal{F}\|_{\mathcal{L}_x^{\frac{nb}{(b+1)(2\alpha-1)}, x}(L_t^\infty)}. \quad (3.10)$$

- **Case $\gamma = 0$.** Similarly to the previous case, if we consider Fubini's theorem and the decay properties of the fractional heat kernel in Remark 2.5, we can write

$$\left| \int_0^t \mathbf{g}_{t-s}^\alpha * \mathcal{F}(s, \mathbf{x}) ds \right| \leq C \int_{\mathbb{R}^n} \int_0^t \frac{|t-s|}{(|t-s|^{\frac{1}{2\alpha}} + |\mathbf{x} - \mathbf{y}|)^{n+2\alpha}} |\mathcal{F}(s, \mathbf{y})| ds d\mathbf{y},$$

and then, involving the L_t^∞ -norm on the previous estimate, we conclude

$$\left| \int_0^t \mathbf{g}_{t-s}^\alpha * \mathcal{F}(s, \mathbf{x}) ds \right| \leq C \int_{\mathbb{R}^n} \left(\int_0^t \frac{|t-s|}{(|t-s|^{\frac{1}{2\alpha}} + |\mathbf{x} - \mathbf{y}|)^{n+2\alpha}} ds \right) \|\mathcal{F}(\cdot, \mathbf{y})\|_{L_t^\infty} d\mathbf{y}.$$

Now, considering the estimate

$$\begin{aligned}
\int_0^t \frac{|t-s|ds}{\left(|t-s|^{\frac{1}{2\alpha}} + |x-y|\right)^{n+2\alpha}} &\leq \int_0^{+\infty} \frac{sds}{\left(s^{\frac{1}{2\alpha}} + |x-y|\right)^{n+1}} \\
&= \int_0^{+\infty} \frac{|x-y|^{2\alpha}\beta|x-y|^{2\alpha}d\beta}{\left(\left(|x-y|^{2\alpha}\beta\right)^{\frac{1}{2\alpha}} + |x-y|\right)^{n+2\alpha}} \\
&= \frac{1}{|x-y|^{n-2\alpha}} \int_0^{+\infty} \frac{\beta d\beta}{\left(1 + \beta^{\frac{1}{2\alpha}}\right)^{n+2\alpha}},
\end{aligned} \tag{3.11}$$

by the definition of the Riesz potential (see definition (2.5)), we obtain

$$\begin{aligned}
\left| \int_0^t \mathbf{g}_{t-s}^\alpha * \mathcal{F}(s, \mathbf{x}) ds \right| &\leq C \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2\alpha}} \|\mathcal{F}(\cdot, \mathbf{y})\|_{L_t^\infty} d\mathbf{y} \\
&= C J_{2\alpha}(\|\mathcal{F}(\cdot, \mathbf{y})\|_{L_t^\infty})(\mathbf{x}).
\end{aligned}$$

This last estimate implies

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * f(s, \mathbf{x}) ds \right\|_{L_t^\infty} \leq C J_{2\alpha}(\|\mathcal{F}(\cdot, \cdot)\|_{L_t^\infty})(\mathbf{x}). \tag{3.12}$$

In order to reconstruct the $\mathcal{L}_{\frac{nb}{2\alpha}}^{p(\cdot)}$ -norm given in (3.3), from the estimate (3.12) we write

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * f(s, \mathbf{x}) ds \right\|_{L_x^{p(\cdot)}(L_t^\infty)} \leq C \|J_{2\alpha}(\|\mathcal{F}(\cdot, \cdot)\|_{L_t^\infty})(\cdot)\|_{L_x^{p(\cdot)}},$$

and

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * f(s, \mathbf{x}) ds \right\|_{L_x^{\frac{nb}{2\alpha}}(L_t^\infty)} \leq C \|J_{2\alpha}(\|\mathcal{F}(\cdot, \cdot)\|_{L_t^\infty})(\cdot)\|_{L_x^{\frac{nb}{2\alpha}}}.$$

Now, considering Proposition 2.2, we get the estimate

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * f(s, \mathbf{x}) ds \right\|_{L_x^{p(\cdot)}(L_t^\infty)} \leq C \|\mathcal{F}\|_{L_t^\infty} \left\| \mathcal{L}_{\frac{nb}{(b+1)(2\alpha)}}^{p(\cdot)} \right\|_{\mathcal{L}_{\frac{nb}{(b+1)(2\alpha)}, x}^{p(\cdot)}(L_x^\infty)}, \tag{3.13}$$

and, since the Riesz potential $J_{2\alpha}$ satisfies $\|J_{2\alpha}(\varphi)\|_{L^{\frac{nb}{2\alpha}}} \leq C\|\varphi\|_{L^{\frac{nb}{(b+1)(2\alpha)}}$, we obtain

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * f(s, \mathbf{x}) ds \right\|_{L_x^{\frac{nb}{2\alpha}}(L_t^\infty)} \leq C \|\mathcal{F}\|_{L_t^\infty} \left\| \mathcal{L}_{\frac{nb}{(b+1)(2\alpha)}} \right\|_{L_x^{\frac{nb}{(b+1)(2\alpha)}}(L_t^\infty)} = C \|\mathcal{F}\|_{L_x^{\frac{nb}{(b+1)(2\alpha)}}(L_t^\infty)}. \tag{3.14}$$

Then, gathering together the norm $\|\cdot\|_{\mathcal{L}}$ given in (3.3) with the estimates (3.13) and (3.14), we get

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * f(s, \mathbf{x}) ds \right\|_{\mathcal{L}_{\frac{nb}{(2\alpha)}, x}^{p(\cdot)}(L_t^\infty)} \leq C \|\mathcal{F}\|_{\mathcal{L}_{\frac{nb}{(b+1)(2\alpha)}, x}^{p(\cdot)}(L_t^\infty)}. \tag{3.15}$$

Considering the estimates (3.10) and (3.15) we conclude (3.5) and thus the proof. \square

Proposition 3.3 Consider $\alpha \in [\frac{1}{2}, 1]$ and a variable exponent $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ such that $p^- > 1$. Then, there exist a constant $C_{\mathcal{B}} > 0$ such that

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * \vec{\nabla}^\gamma (|u|^b u)(s, x) ds \right\|_{\mathcal{E}} \leq C_{\mathcal{B}} \|u\|_{\mathcal{E}}^b \|u\|_{\mathcal{E}}. \quad (3.16)$$

Proof. In order to conclude (3.16) in the next we will study separately the cases when $\gamma = 1$ and $\gamma = 0$.

• **Case $\gamma = 1$.**

We start the study of this case by noticing that, due the Minkowski's integral inequality and decay properties of the fractional heat kernel (see Remark 2.5), we can write

$$\begin{aligned} \left| \int_0^t \mathbf{g}_{t-s}^\alpha * \vec{\mathbf{I}} \cdot \vec{\nabla} (|u|^b u)(s, x) ds \right| &\leq C \int_0^t \int_{\mathbb{R}^n} |\vec{\nabla} \mathbf{g}_{t-s}^\alpha(x-y)| |u(s, y)|^b |u(s, y)| dy ds \\ &\leq C \int_{\mathbb{R}^n} \int_0^t \frac{1}{(|t-s|^{\frac{1}{2\alpha}} + |x-y|)^{n+1}} |u(s, y)|^b |u(s, y)| ds dy. \end{aligned}$$

Then, considering the L_t^∞ -norm on the last inequality we obtain

$$\begin{aligned} \left| \int_0^t \mathbf{g}_{t-s}^\alpha * \vec{\mathbf{I}} \cdot \vec{\nabla} (|u|^b u)(s, x) ds \right| &\leq C \int_{\mathbb{R}^n} \left(\int_0^t \frac{1}{(|t-s|^{\frac{1}{2\alpha}} + |x-y|)^{n+1}} ds \right) \|u(\cdot, y)\|_{L_t^\infty}^b \|u(\cdot, y)\|_{L_t^\infty} dy. \end{aligned}$$

Inspired by the proof of Proposition 3.2 (see (3.6)), we can write

$$\int_0^t \frac{ds}{(|t-s|^{\frac{1}{2\alpha}} + |x-y|)^{n+1}} \leq C \frac{1}{|x-y|^{n+1-2\alpha}} \quad (3.17)$$

and then, we conclude the estimate

$$\left| \int_0^t \mathbf{g}_{t-s}^\alpha * \vec{\mathbf{I}} \cdot \vec{\nabla} (|u|^b u)(s, x) ds \right| \leq C \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n+1-2\alpha}} \|u(\cdot, y)\|_{L_t^\infty}^b \|u(\cdot, y)\|_{L_t^\infty} dy. \quad (3.18)$$

Considering the definition of the Riesz Potential (see 2.5), the last inequality can be recast as

$$\left| \int_0^t \mathbf{g}_{t-s}^\alpha * \vec{\mathbf{I}} \cdot \vec{\nabla} (|u|^b u)(s, x) ds \right| \leq C \mathcal{J}_{2\alpha-1} (\|u\|_{L_t^\infty}^b \|u\|_{L_t^\infty})(x).$$

To reconstruct the norm $\|\cdot\|_{\mathcal{E}}$ (see (3.3)) we write

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * \vec{\mathbf{I}} \cdot \vec{\nabla} (|u|^b u)(s, x) ds \right\|_{L_x^{p(\cdot)}(L_t^\infty)} \leq C \|\mathcal{J}_{2\alpha-1} (\|u\|_{L_t^\infty}^b \|u\|_{L_t^\infty})\|_{L_x^{p(\cdot)}},$$

and

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * \vec{\Gamma} \cdot \vec{\nabla}(|\mathbf{u}|^b \mathbf{u})(s, \mathbf{x}) ds \right\|_{L_x^{p(\cdot)}(L_t^\infty)} \leq C \|J_{2\alpha-1}(\|\mathbf{u}\|_{L_t^\infty}^b \|\mathbf{u}\|_{L_t^\infty})\|_{L_x^{\frac{nb}{2\alpha-1}\gamma}}.$$

Then, considering that $\|J_{2\alpha-1}(\varphi)\|_{L^{\frac{nb}{2\alpha-1}\gamma}} \leq C \|\varphi\|_{L^{\frac{nb}{(b+1)(2\alpha-1)}}$, an Hölder inequality (see Remark 2.4) and Proposition 2.2, we conclude

$$\begin{aligned} \left\| \int_0^t \mathbf{g}_{t-s}^\alpha * \vec{\Gamma} \cdot \vec{\nabla}(|\mathbf{u}|^b \mathbf{u})(s, \mathbf{x}) ds \right\|_{L_x^{p(\cdot)}(L_t^\infty)} &\leq C \left\| \|\mathbf{u}\|_{L_t^\infty}^b \|\mathbf{u}\|_{L_t^\infty} \right\|_{\mathcal{L}^{\frac{p(\cdot)}{b+1}} \frac{nb}{(b+1)(2\alpha-1)}} \\ &\leq C \|\mathbf{u}\|_{\mathcal{L}^{\frac{nb}{2\alpha-1}\gamma, x}(L_x^\infty)}^b \|\mathbf{u}\|_{\mathcal{L}^{\frac{nb}{2\alpha-1}\gamma, x}(L_x^\infty)}, \end{aligned} \quad (3.19)$$

and

$$\begin{aligned} \left\| \int_0^t \mathbf{g}_{t-s}^\alpha * \vec{\Gamma} \cdot \vec{\nabla}(|\mathbf{u}|^b \mathbf{u})(s, \mathbf{x}) ds \right\|_{L_x^{\frac{nb}{2\alpha-1}}(L_t^\infty)} &\leq C \left\| \|\mathbf{u}\|_{L_t^\infty}^b \|\mathbf{u}\|_{L_t^\infty} \right\|_{L_x^{\frac{nb}{(b+1)(2\alpha-1)}}} \\ &\leq C \|\mathbf{u}\|_{L_x^{\frac{nb}{2\alpha-1}\gamma}(L_t^\infty)}^b \|\mathbf{u}\|_{L_x^{\frac{nb}{2\alpha-1}\gamma}(L_t^\infty)}. \end{aligned} \quad (3.20)$$

Remark 3.1 Note that, in the case that we had considered Theorem 4 instead of Proposition 2.2, we had obtained an estimate of the form $\|J_{2\alpha-1}(\varphi)\|_{L^{p(\cdot)}} \leq \|\varphi\|_{L^{\frac{p(\cdot)}{b+1}}}$, which, due the strong relationship between the the variable exponents involved, yields the constant exponent $p(\cdot) \equiv \frac{nb}{2\alpha-1}$.

Gathering together the estimates (3.19), (3.20), and using the definition of the norm $\|\cdot\|_{\mathcal{E}}$ given in (3.3) we get the estimate

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * \vec{\Gamma} \cdot \vec{\nabla}(|\mathbf{u}|^b \mathbf{u})(s, \mathbf{x}) ds \right\|_{\mathcal{L}^{\frac{nb}{(2\alpha-1)}, x}(L_t^\infty)} \leq C \|\mathbf{u}\|_{\mathcal{L}^{\frac{nb}{(2\alpha-1)}, x}(L_t^\infty)}^b \|\mathbf{u}\|_{\mathcal{L}^{\frac{nb}{(2\alpha-1)}, x}(L_t^\infty)}. \quad (3.21)$$

- **Case $\gamma = 0$.** By considering the Minkowski's integral inequality and the decay properties of the fractional heat kernel (see Remark 2.5), we get the estimates

$$\left| \int_0^t \mathbf{g}_{t-s}^\alpha * (|\mathbf{u}|^b \mathbf{u})(s, \mathbf{x}) ds \right| \leq C \int_{\mathbb{R}^n} \int_0^t \frac{|t-s|}{(|t-s|^{\frac{1}{2\alpha}} + |\mathbf{x}-\mathbf{y}|)^{n+2\alpha}} |\mathbf{u}(s, \mathbf{y})|^b |\mathbf{u}(s, \mathbf{y})| ds d\mathbf{y}.$$

Considering L_t^∞ -norm we conclude

$$\begin{aligned} \left| \int_0^t \mathbf{g}_{t-s}^\alpha * (|\mathbf{u}|^b \mathbf{u})(s, \mathbf{x}) ds \right| &\leq C \int_{\mathbb{R}^n} \left(\int_0^t \frac{|t-s|}{(|t-s|^{\frac{1}{2\alpha}} + |\mathbf{x}-\mathbf{y}|)^{n+2\alpha}} ds \right) \|\mathbf{u}(\cdot, \mathbf{y})\|_{L_t^\infty}^b \|\mathbf{u}(\cdot, \mathbf{y})\|_{L_t^\infty} d\mathbf{y}. \end{aligned}$$

Now, considering the definition of the Riesz potential, and the estimate (see (3.11))

$$\int_0^t \frac{|t-s|ds}{\left(|t-s|^{\frac{1}{2\alpha}} + |x-y|\right)^{n+2\alpha}} \leq \frac{1}{|x-y|^{n-2\alpha}} \int_0^{+\infty} \frac{\beta d\beta}{\left(1 + \beta^{\frac{1}{2\alpha}}\right)^{n+2\alpha}}, \quad (3.22)$$

we obtain the estimate

$$\begin{aligned} \left| \int_0^t \mathbf{g}_{t-s}^\alpha * (|u|^b u)(s, x) ds \right| &\leq C \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2\alpha}} \|u(\cdot, y)\|_{L_t^\infty}^b \|u(\cdot, y)\|_{L_t^\infty} dy \\ &= C \mathcal{J}_{2\alpha}(\|u(\cdot, y)\|_{L_t^\infty}^b \|u(\cdot, y)\|_{L_t^\infty})(x). \end{aligned}$$

In order to reconstruct the norm $\|\cdot\|_\varepsilon$ given in (3.3) we can write

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * (|u|^b u)(s, x) ds \right\|_{L_x^{p(\cdot)}(L_t^\infty)} \leq C \|\mathcal{J}_{2\alpha}(\|u\|_{L_t^\infty}^b \|u\|_{L_t^\infty})\|_{L_x^{p(\cdot)}},$$

and

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * (|u|^b u)(s, x) ds \right\|_{L_x^{\frac{nb}{2\alpha}}(L_t^\infty)} \leq C \|\mathcal{J}_{2\alpha}(\|u\|_{L_t^\infty}^b \|u\|_{L_t^\infty})\|_{L_x^{\frac{nb}{2\alpha}}}.$$

If we consider an Hölder inequality (see Remark 2.4), Proposition 2.2 and the fact that the Riesz potential is bounded in $L^{\frac{nb}{(b+1)(2\alpha)}}$, we get the estimates

$$\begin{aligned} \left\| \int_0^t \mathbf{g}_{t-s}^\alpha * (|u|^b u)(s, x) ds \right\|_{L_x^{p(\cdot)}(L_t^\infty)} &\leq C \left\| \|u\|_{L_t^\infty}^b \|u\|_{L_t^\infty} \right\|_{\mathcal{L}^{\frac{p(\cdot)}{b+1}} \frac{nb}{(b+1)(2\alpha)}} \\ &\leq C \|u\|_{\mathcal{L}^{\frac{p(\cdot)}{2\alpha}, x} \frac{nb}{2\alpha}}^b \|u\|_{\mathcal{L}^{\frac{p(\cdot)}{2\alpha}, x} \frac{nb}{2\alpha}}, \end{aligned} \quad (3.23)$$

and

$$\begin{aligned} \left\| \int_0^t \mathbf{g}_{t-s}^\alpha * (|u|^b u)(s, x) ds \right\|_{L_x^{\frac{nb}{2\alpha}}(L_t^\infty)} &\leq C \left\| \|u\|_{L_t^\infty}^b \|u\|_{L_t^\infty} \right\|_{L_x^{\frac{nb}{(b+1)(2\alpha)}}} \\ &\leq C \|u\|_{L_x^{\frac{nb}{2\alpha}}(L_t^\infty)}^b \|u\|_{L_x^{\frac{nb}{2\alpha}}(L_t^\infty)}. \end{aligned} \quad (3.24)$$

Thus, gathering (3.23), (3.24), and using the definition of the norm $\|\cdot\|_\varepsilon$ given in (3.3), we obtain

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * (|u|^b u)(s, x) ds \right\|_{\mathcal{L}^{\frac{p(\cdot)}{(2\alpha)}, x} \frac{nb}{2\alpha}} \leq C \|u\|_{\mathcal{L}^{\frac{p(\cdot)}{2\alpha}, x} \frac{nb}{2\alpha}}^b \|u\|_{\mathcal{L}^{\frac{p(\cdot)}{2\alpha}, x} \frac{nb}{2\alpha}}. \quad (3.25)$$

Considering the estimates (3.21) and (3.25) we conclude (3.16), and then Proposition 3.6 is proven. \square

End of the proof of Theorem 1

Given an element $\mathbf{u} \in \mathcal{E}$, we consider the operator Ψ

$$\Psi(\mathbf{u}) := \mathbf{g}_t^\alpha * \mathbf{u}_0(\mathbf{x}) + \int_0^t \mathbf{g}_{t-s}^\alpha * f(s, \mathbf{x}) ds - \int_0^t \mathbf{g}_{t-s}^\alpha * \vec{\nabla}^\gamma(|\mathbf{u}|^b \mathbf{u})(s, \mathbf{x}) ds.$$

Thus, by fixing a parameter $\gamma \in \{0, 1\}$ and considering the estimates obtained in Propositions 3.1, 3.2 and 3.3, we can conclude

$$\|\Psi(\mathbf{u})\|_{\mathcal{E}} \leq C \|\mathbf{u}_0\|_{\mathcal{L}^{\frac{nb}{2\alpha-(1)\gamma}}(\cdot)} + C \|\mathcal{F}\|_{\mathcal{L}^{\frac{p(\cdot)}{b+1}}(\cdot)} \frac{nb}{(b+1)(2\alpha-(1)\gamma), x} (\mathbb{L}_t^\infty) + C \|\mathbf{u}\|_{\mathcal{E}}^{b+1},$$

and

$$\|\Psi(\mathbf{u}) - \Psi(\mathbf{v})\|_{\mathcal{E}} \leq C \|\mathbf{u} - \mathbf{v}\|_{\mathcal{E}} (\|\mathbf{u}\|_{\mathcal{E}}^b + \|\mathbf{v}\|_{\mathcal{E}}^b).$$

Let us denote by

$$B = C \|\mathbf{u}_0\|_{\mathcal{L}^{\frac{nb}{2\alpha-(1)\gamma}}(\cdot)} + C \|\mathcal{F}\|_{\mathcal{L}^{\frac{p(\cdot)}{b+1}}(\cdot)} \frac{nb}{(b+1)(2\alpha-(1)\gamma), x} (\mathbb{L}_t^\infty),$$

and set $R = (2B)^{\frac{1}{b}}$. Now, considering the ball

$$B_R = \{\mathbf{u} \in \mathcal{E} : \|\mathbf{u}\|_{\mathcal{E}} \leq R\},$$

we can write

$$\|\Psi(0)\|_{\mathcal{E}} \leq C \|\mathbf{u}_0\|_{\mathcal{L}^{\frac{nb}{2\alpha-(1)\gamma}}(\cdot)} + C \|\mathcal{F}\|_{\mathcal{L}^{\frac{p(\cdot)}{b+1}}(\cdot)} \frac{nb}{(b+1)(2\alpha-(1)\gamma), x} (\mathbb{L}_t^\infty) = B,$$

and then we obtain

$$\|\Psi(\mathbf{u})\|_{\mathcal{E}} \leq \|\Psi(\mathbf{u}) - \Psi(0) + \Psi(0)\|_{\mathcal{E}} \leq \|\Psi(\mathbf{u}) - \Psi(0)\|_{\mathcal{E}} + \|\Psi(0)\|_{\mathcal{E}} \leq CR^b \|\mathbf{u}\|_{\mathcal{E}} + B.$$

Now, if $2CB \leq \frac{1}{2}$, then $CR^b \leq \frac{1}{2}$, and we conclude that

$$\|\Psi(\mathbf{u}) - \Psi(\mathbf{v})\|_{\mathcal{E}} \leq \frac{1}{2} \quad \text{and} \quad \|\Psi(\mathbf{u})\|_{\mathcal{E}} \leq \frac{1}{2}.$$

Thus, it follows from the contraction mapping principle that there exists an unique mild solution $\mathbf{u} \in \mathcal{E}$ of the equation (1.1). With this we conclude the proof of Theorem 1.

3.2 Proof of Theorem 2

In this subsection we consider a variable exponent $p(\cdot) \in \mathcal{P}^{\text{log}}([0, +\infty[)$ such that $1 < p^- \leq p^+ < +\infty$, and the functional space

$$E_T = L^{p(\cdot)}([0, T], L^q(\mathbb{R}^n)),$$

with $T \in]0, +\infty[$ to be precised later. The space E_T is dotted with the following Luxemburg-type norm:

$$\|\vec{\varphi}\|_{E_T} = \inf \left\{ \lambda > 0 : \int_0^T \left| \frac{\|\vec{\varphi}(t, \cdot)\|_{L^q}}{\lambda} \right|^{p(t)} dt \leq 1 \right\}. \quad (3.26)$$

In this subsection we attack the proof following the same strategy that in Theorem 1, we will construct mild solutions for the integral equation (3.2) by considering the mapping contraction principle on the space E_T .

To this end, we begin by proving the following result regarding a control of the initial data.

Proposition 3.4 *Let $\alpha \in]\frac{1}{2}, 1]$, $p(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ with $b + 1 < p^- \leq p^+ < +\infty$, fix an index $q > \frac{nb}{2\alpha - \langle 1 \rangle_\gamma}$ by the relationship $\frac{\alpha b}{p(\cdot)} + \frac{nb}{2q} < \alpha - \langle \frac{1}{2} \rangle_\gamma$ and $\bar{q}(\cdot) \in \mathcal{P}_q^{emb}(\mathbb{R}^n)$. Consider a function $u_0 \in L^{\bar{q}(\cdot)}(\mathbb{R}^n)$. Then, there exist a constant $C > 0$ such that*

$$\|\mathbf{g}_t^\alpha * u_0\|_{E_T} \leq C \|u_0\|_{L^{\bar{q}(\cdot)}}. \quad (3.27)$$

Proof. To deduce the inequality (3.27), we begin by considering the Young inequality to obtain

$$\|\mathbf{g}_t^\alpha * u_0\|_{L^q(\mathbb{R}^n)} \leq \|\mathbf{g}_t^\alpha\|_{L^1(\mathbb{R}^n)} \|u_0\|_{L^q(\mathbb{R}^n)} = \|u_0\|_{L^q(\mathbb{R}^n)}.$$

Now, let recall the following result in the context of Variable Lebesgue spaces (see [5, Lemma 3.2.12, Section 3.2]).

Lemma 3.1 *Let $p(\cdot) \in \mathcal{P}([0, +\infty[)$ such that $1 < p^- \leq p^+ < +\infty$. Then, there exist $C > 0$ such that*

$$\|1\|_{L^{p(\cdot)}([0, T])} \leq C \max \left\{ T^{\frac{1}{p^-}}, T^{\frac{1}{p^+}} \right\}.$$

To continue, we consider the $L_t^{p(\cdot)}$ -norm on the last inequality to get

$$\|\mathbf{g}_t^\alpha * u_0\|_{L_t^{p(\cdot)} L_x^q} \leq C \|u_0\|_{L^q(\mathbb{R}^n)} \|1\|_{L^{p(\cdot)}([0, T])} \quad (3.28)$$

$$\leq C \max \left\{ T^{\frac{1}{p^-}}, T^{\frac{1}{p^+}} \right\} \|u_0\|_{L^q(\mathbb{R}^n)}. \quad (3.29)$$

Then, by Lemma 2.3 we conclude the estimate

$$\|\mathbf{g}_t^\alpha * u_0\|_{L_t^{p(\cdot)} L_x^q} \leq C \max \left\{ T^{\frac{1}{p^-}}, T^{\frac{1}{p^+}} \right\} \|u_0\|_{L^{\bar{q}(\cdot)}(\mathbb{R}^n)}. \quad (3.30)$$

With this we conclude the proof of Proposition 3.4. \square

Proposition 3.5 *Let $\gamma \in \{0, 1\}$ fixed. Consider $\alpha \in (\frac{1}{2}, 1)$, a variable exponent $p(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ with $b + 1 < p^- \leq p^+ < +\infty$, fix an index $q > \frac{nb}{2\alpha - \langle 1 \rangle_\gamma}$ by the relationship $\frac{\alpha b}{p(\cdot)} + \frac{nb}{2q} < \alpha - \langle \frac{1}{2} \rangle_\gamma$ and $\bar{q}(\cdot) \in \mathcal{P}_q^{emb}(\mathbb{R}^n)$. Then, given a function $f \in L^{p(\cdot)}([0, +\infty[, L^{\bar{q}(\cdot)}(\mathbb{R}^3))$, there exist a numerical constant $C > 0$ such that*

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * f(s, \cdot) ds \right\|_{E_T} \leq C \|f\|_{L_t^1 L_x^{\bar{q}(\cdot)}}. \quad (3.31)$$

Proof. We begin by taking the L_x^q -norm to the term $\int_0^t \mathbf{g}_{t-s}^\alpha * f(s, \mathbf{x}) ds$ in order to obtain the estimate

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * f(s, \cdot) ds \right\|_{L^q} \leq \int_0^t \|\mathbf{g}_{t-s}^\alpha\|_{L^1} \|f(s, \cdot)\|_{L^q} ds.$$

Then, by considering Lemma 2.3 we can write

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * f(s, \cdot) ds \right\|_{L^q} \leq \int_0^t \|\mathbf{g}_{t-s}^\alpha\|_{L^1} \|f(s, \cdot)\|_{L^{\bar{q}(\cdot)}} ds \leq C \|f\|_{L_t^1 L_x^{\bar{q}(\cdot)}}.$$

Proceeding as in the previous lines, *i.e.* taking the $L^{p(\cdot)}$ -norm in the time variable and using Lemma 3.1, we get the estimates

$$\begin{aligned} \left\| \int_0^t \mathbf{g}_{t-s}^\alpha * f(s, \cdot) ds \right\|_{L_t^{p(\cdot)}(L_x^q)} &\leq C \left\| \|f\|_{L_t^1 L_x^{\bar{q}(\cdot)}} \right\|_{L^{p(\cdot)}} \\ &\leq C \|f\|_{L_t^1 L_x^{\bar{q}(\cdot)}} \|1\|_{L^{p(\cdot)}([0, T])} \\ &\leq C \max \left\{ T^{\frac{1}{p^-}}, T^{\frac{1}{p^+}} \right\} \|f\|_{L_t^1 L_x^{\bar{q}(\cdot)}}. \end{aligned}$$

Considering the last estimate, we conclude (3.31) and we finish the proof of Proposition 3.5. \square

Proposition 3.6 *Let $\gamma \in \{0, 1\}$ fixed. Consider $\alpha \in [\frac{1}{2}, 1]$, a variable exponent $p(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ with $b + 1 < p^- \leq p^+ < +\infty$, fix an index $q > \frac{nb}{2\alpha - (1)\gamma}$ by the relationship $\frac{\alpha b}{p(\cdot)} + \frac{nb}{2q} < \alpha - \langle \frac{1}{2} \rangle_\gamma$, and $\bar{q}(\cdot) \in \mathcal{P}_q^{emb}(\mathbb{R}^n)$. Then, there exist a constant $C_B > 0$ such that*

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * \vec{\nabla}^\gamma (|\mathbf{u}|^b \mathbf{u})(s, \mathbf{x}) ds \right\|_{E_T} \leq C_B \|\mathbf{u}\|_{E_T}^b \|\mathbf{u}\|_{E_T}. \quad (3.32)$$

Proof. We begin the proof by considering the L_x^q -norm to the term $\int_0^t \mathbf{g}_{t-s}^\alpha * \vec{\nabla}^\gamma (|\mathbf{u}|^b \mathbf{u})(s, \mathbf{x}) ds$ in order to obtain

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * \vec{\nabla}^\gamma (|\mathbf{u}|^b \mathbf{u})(s, \mathbf{x}) ds \right\|_{L^q} \leq C \int_0^t \left\| \mathbf{g}_{t-s}^\alpha * \vec{\nabla}^\gamma (|\mathbf{u}|^b \mathbf{u})(s, \cdot) \right\|_{L^q} ds.$$

From this inequality we can conclude

$$\left\| \int_0^t \mathbf{g}_{t-s}^\alpha * \vec{\nabla}^\gamma (|\mathbf{u}|^b \mathbf{u})(s, \mathbf{x}) ds \right\|_{L^q} \leq \begin{cases} C \int_0^t \left\| \mathbf{g}_{t-s}^\alpha * (|\mathbf{u}|^b \mathbf{u})(s, \cdot) \right\|_{L^q} ds & \text{if } \gamma = 0, \\ C \int_0^t \left\| \vec{\nabla} \mathbf{g}_{t-s}^\alpha * (|\mathbf{u}|^b \mathbf{u})(s, \cdot) \right\|_{L^q} ds & \text{if } \gamma = 1. \end{cases}$$

Note that, Lemma 2.6 yields the estimate

$$\int_0^t \left\| \vec{\nabla} \mathbf{g}_{t-s}^\alpha * \mathbf{u} \otimes \mathbf{u}(s, \cdot) \right\|_{L^q} ds \leq C \int_0^t \frac{1}{(t-s)^{\langle \frac{1}{2\alpha} \rangle_\gamma + \frac{nb}{2\alpha q}}} \|\mathbf{u}|^b \mathbf{u}(s, \cdot)\|_{L^{\frac{q}{b+1}}} ds.$$

Thus, considering an Hölder inequality we obtain

$$\int_0^t \left\| \vec{\nabla} \mathbf{g}_{t-s}^\alpha * \mathbf{u} \otimes \mathbf{u}(s, \cdot) \right\|_{L^q} ds \leq C \int_0^t \frac{1}{(t-s)^{\langle \frac{1}{2\alpha} \rangle_\gamma + \frac{nb}{2\alpha q}}} \|\mathbf{u}(s, \cdot)\|_{L^q}^b \|\mathbf{u}(s, \cdot)\|_{L^q} ds.$$

Now, considering the $L_t^{p(\cdot)}$ -norm and the norm conjugate formula (2.3), provided with the variable exponent $p'(\cdot)$ defined by $1 = \frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)}$, we get

$$\begin{aligned} \left\| \int_0^t \mathbf{g}_{t-s}^\alpha * \vec{\nabla}^\gamma (|u|^b u)(s, x) ds \right\|_{L_t^{p(\cdot)}(L_x^q)} &\leq \left\| \int_0^t \frac{1}{(t-s)^{\langle \frac{1}{2\alpha} \rangle_\gamma + \frac{nb}{2\alpha q}}} \|u(s, \cdot)\|_{L^q}^{b+1} ds \right\|_{L_t^{p(\cdot)}([0, T])} \\ &\leq \sup_{\|\psi\|_{L^{p'(\cdot)}} \leq 1} \int_0^T \int_0^t \frac{|\psi(t)|}{|t-s|^{\langle \frac{1}{2\alpha} \rangle_\gamma + \frac{nb}{2\alpha q}}} \|u(s, \cdot)\|_{L^q}^{b+1} ds dt. \end{aligned} \quad (3.33)$$

Note that, considering Fubini's Theorem, we can write

$$\begin{aligned} \sup_{\|\psi\|_{L^{p'(\cdot)}} \leq 1} \int_0^T \int_0^t \frac{|\psi(t)|}{|t-s|^{\langle \frac{1}{2\alpha} \rangle_\gamma + \frac{nb}{2\alpha q}}} \|u(s, \cdot)\|_{L^q}^b \|u(s, \cdot)\|_{L^q} ds dt \\ = \sup_{\|\psi\|_{L^{p'(\cdot)}} \leq 1} \int_0^T \int_0^T \frac{1_{\{0 < s < t\}} |\psi(t)|}{|t-s|^{\langle \frac{1}{2\alpha} \rangle_\gamma + \frac{nb}{2\alpha q}}} dt \|u(s, \cdot)\|_{L^q}^b \|u(s, \cdot)\|_{L^q} ds, \end{aligned}$$

Now, extending the function $\psi(t)$ by zero on $\mathbb{R} \setminus [0, T]$, we obtain

$$\begin{aligned} \sup_{\|\psi\|_{L^{p'(\cdot)}} \leq 1} \int_0^T \int_0^t \frac{|\psi(t)|}{|t-s|^{\langle \frac{1}{2\alpha} \rangle_\gamma + \frac{nb}{2\alpha q}}} \|u(s, \cdot)\|_{L^q}^b \|u(s, \cdot)\|_{L^q} ds dt \\ = \sup_{\|\psi\|_{L^{p'(\cdot)}} \leq 1} \int_0^T \left(\int_{-\infty}^{+\infty} \frac{|\psi(t)|}{|t-s|^{\langle \frac{1}{2\alpha} \rangle_\gamma + \frac{nb}{2\alpha q}}} dt \right) \|u(s, \cdot)\|_{L^q}^b \|u(s, \cdot)\|_{L^q} ds \\ = \sup_{\|\psi\|_{L^{p'(\cdot)}} \leq 1} \int_0^T \mathcal{J}_\beta (|\psi|)(s) \|u(s, \cdot)\|_{L^q}^b \|u(s, \cdot)\|_{L^q} ds, \end{aligned}$$

where \mathcal{J}_β is the 1D Riesz potential with $\beta = 1 - \langle \frac{1}{2\alpha} \rangle_\gamma - \frac{nb}{2\alpha q} < 1$ (see Definition 2.5).

Remark 3.2 Note that the constraints $\frac{nb}{2\alpha - \langle 1 \rangle_\gamma} < q$ and $\alpha \in [\frac{1}{2}, 1]$ imply $0 < 1 - \langle \frac{1}{2\alpha} \rangle_\gamma - \frac{nb}{2\alpha q} < 1$, and then the Riesz potential considered is well defined.

Considering an Hölder inequality with $1 = \frac{b}{p(\cdot)} + \frac{1}{p(\cdot)} + \frac{1}{\bar{p}(\cdot)}$, we get the estimate

$$\begin{aligned} \sup_{\|\psi\|_{L^{p'(\cdot)}} \leq 1} \int_0^T \mathcal{J}_\beta (|\psi|)(s) \|u(s, \cdot)\|_{L^q}^b \|u(s, \cdot)\|_{L^q} ds \\ \leq C \sup_{\|\psi\|_{L^{p'(\cdot)}} \leq 1} \|\mathcal{J}_\beta (|\psi|)\|_{L_t^{\bar{p}(\cdot)}} \left\| \|u(\cdot, \cdot)\|_{L_x^q} \right\|_{L_t^{p(\cdot)}}^b \left\| \|u(\cdot, \cdot)\|_{L_x^q} \right\|_{L_t^{p(\cdot)}}. \end{aligned}$$

Remark 3.3 We must stress the fact that condition $p^- > b + 1$ in the statement of the proposition become from the relationship $1 = \frac{b+1}{p(\cdot)} + \frac{1}{\bar{p}(\cdot)}$.

Now, provided of Theorem 4 with indexes defined by the relationship

$$\frac{1}{\tilde{p}(\cdot)} = \frac{1}{r(\cdot)} - \left(1 - \langle \frac{1}{2\alpha} \rangle_\gamma - \frac{nb}{2\alpha q}\right), \quad (3.34)$$

we conclude the estimate

$$\begin{aligned} \sup_{\|\psi\|_{L^{p'(\cdot)}} \leq 1} \|\mathcal{J}_\beta(|\psi|)\|_{L^{\tilde{p}(\cdot)}} \|\mathbf{u}(\cdot, \cdot)\|_{L_t^{p(\cdot)}(L_x^q)}^b \|\mathbf{u}(\cdot, \cdot)\|_{L_t^{p(\cdot)}(L_x^q)} \\ \leq C \sup_{\|\psi\|_{L^{p'(\cdot)}} \leq 1} \|\psi\|_{L_t^{r(\cdot)}} \|\mathbf{u}(\cdot, \cdot)\|_{L_t^{p(\cdot)}(L_x^q)}^b \|\mathbf{u}(\cdot, \cdot)\|_{L_t^{p(\cdot)}(L_x^q)}. \end{aligned} \quad (3.35)$$

By hypothesis we know that $\frac{\alpha b}{p(\cdot)} + \frac{nb}{2q} < \alpha - \langle \frac{1}{2} \rangle_\gamma$, then from relationship (3.34) and the expressions

$$\frac{1}{\tilde{p}(\cdot)} = 1 - \frac{b+1}{p(\cdot)} \quad \text{and} \quad \frac{1}{p'(\cdot)} = 1 - \frac{1}{p(\cdot)},$$

we deduce $r(\cdot) < p'(\cdot)$. Thus, by Considering Lemma 2.2 (with $r(\cdot) < p'(\cdot)$ and $\mathcal{X} = [0, T]$), from (3.35) we obtain

$$\begin{aligned} \sup_{\|\psi\|_{L^{p'(\cdot)}} \leq 1} \|\psi\|_{L_t^{r(\cdot)}} \|\mathbf{u}(\cdot, \cdot)\|_{L_t^{p(\cdot)}(L_x^q)}^b \|\mathbf{u}(\cdot, \cdot)\|_{L_t^{p(\cdot)}(L_x^q)} \\ \leq \sup_{\|\psi\|_{L^{p'(\cdot)}} \leq 1} (1+T) \|\psi\|_{L_t^{p'(\cdot)}} \|\mathbf{u}(\cdot, \cdot)\|_{L_t^{p(\cdot)}(L_x^q)}^b \|\mathbf{u}(\cdot, \cdot)\|_{L_t^{p(\cdot)}(L_x^q)} \\ \leq (1+T) \|\mathbf{u}(\cdot, \cdot)\|_{L_t^{p(\cdot)}(L_x^q)}^b \|\mathbf{u}(\cdot, \cdot)\|_{L_t^{p(\cdot)}(L_x^q)}. \end{aligned}$$

Gathering these last estimates with (3.33), we conclude

$$\left\| \int_0^t \mathfrak{g}_{t-s}^\alpha * \vec{\nabla}^\gamma (|\mathbf{u}|^b \mathbf{u})(s, \mathbf{x}) ds \right\|_{L_t^{p(\cdot)}(L_x^q)} \leq C(1+T) \|\mathbf{u}(\cdot, \cdot)\|_{L_t^{p(\cdot)}(L_x^q)}^b \|\mathbf{u}(\cdot, \cdot)\|_{L_t^{p(\cdot)}(L_x^q)}. \quad (3.36)$$

Considering this last estimate, we deduce the inequality (3.32) and we finish the proof. \square

End of the proof of Theorem 2

Similarly to the strategy considered in Theorem 1, here we consider the operator Φ defined on E_T by the expression

$$\Phi(\mathbf{u}) := \mathfrak{g}_t^\alpha * \mathbf{u}_0(\mathbf{x}) + \int_0^t \mathfrak{g}_{t-s}^\alpha * f(s, \mathbf{x}) ds - \int_0^t \mathfrak{g}_{t-s}^\alpha * \vec{\nabla}^\gamma (|\mathbf{u}|^b \mathbf{u})(s, \mathbf{x}) ds.$$

Thus, by fixing a parameter $\gamma \in \{0, 1\}$ and considering the estimates obtained in Propositions 3.4, 3.5 and 3.6 we can conclude

$$\|\Phi(\mathbf{u})\|_{E_T} \leq C \max \left\{ T^{\frac{1}{p^-}}, T^{\frac{1}{p^+}} \right\} \left(\|\mathbf{u}_0\|_{L^q(\mathbb{R}^n)} + \|f\|_{L_t^1(L_x^q)} \right) + C(1+T) \|\mathbf{u}\|_{E_T}^{b+1}$$

and

$$\|\Phi(\mathbf{u}) - \Phi(\mathbf{v})\|_{E_T} \leq \|\mathbf{u} - \mathbf{v}\|_{E_T} (\|\mathbf{u}\|_{E_T}^b + \|\mathbf{v}\|_{E_T}^b).$$

Let us denote by

$$\mathbf{b} = C \max \left\{ T_{p^-}^{\frac{1}{p^-}}, T_{p^+}^{\frac{1}{p^+}} \right\} \left(\|u_0\|_{L^q(\mathbb{R}^n)} + \|f\|_{L_t^1(L_x^q)} \right),$$

and set $B = (2B)^{\frac{1}{b}}$. Considering the ball

$$B_R = \{u \in E_T : \|u\|_{E_T} \leq R\},$$

and similar arguments as the end of the proof Theorem 1, we conclude that if B is bounded by an appropriate constant, then Φ is a contraction map on B_R . Thus, it follows from the contraction mapping principle that there exists an unique mild solution $u \in E_T$ of the equation (1.1) and we conclude the proof of Theorem 2.

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