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# Anisotropic damage modeling of concrete materials

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## Abstract

An anisotropic damage model is proposed for concrete materials. As required by thermodynamics a single damage variable, tensorial, is considered for any loading: as a state variable it represents the micro-cracking pattern whatever the loading sign. Damage anisotropy is used to model the strong dissymmetry tension/compression. Ladevèze damage variable  $\mathbf{H} = (\mathbf{1} - \mathbf{D})^{-1/2}$  is introduced within a deviatoric/hydrostatic split. An original shear-bulk coupling is derived, in accordance with numerical Discrete Element computations. The sought property of gradual stress softening, with a tail in stress-strain diagram, is obtained. Stress triaxiality is used to enhance the performance of Mazars criterion therefore of the full anisotropic damage model in bicompression.

## Keywords

anisotropic damage, concrete, shear-bulk coupling, triaxiality, dissipation, nonlocal

## Introduction

Damage anisotropy is loading induced. From a micro-mechanics point of view it is due to an oriented micro-cracking pattern. From Continuum Damage Mechanics point of view anisotropic damage is represented by a tensorial damage variable, either a fourth order tensor  $\mathbf{D}$  of components  $D_{ijkl}$  (Chaboche 1979; Leckie and Onat 1981; Chaboche 1984; Lemaitre and Chaboche 1985; Ju 1989; Maire and Chaboche 1997; Chaboche and Maire 2000) or a symmetric second order damage tensor  $\mathbf{D}$  of components  $D_{ij}$  (Cordebois and Sidoroff 1982; Ladevèze 1983; Murakami 1988) such as  $D_{ijkl} = D_{ij} = 0$  for virgin material and such as rupture at vanishing stress corresponds to maximum principal damage equal to 1.

Second order anisotropic damage representation is restrictive compared to fourth order tensorial formulation but since its interpretation is quite simple it has been widely and successfully used for either metallic or quasi-brittle materials (Kattan and Voyiadjis 1990; Ramtani et al 1992; Papa and Taliercio 1996; Halm and Dragon 1998; Steinmann and Carol 1998; Lemaitre et al 2000; Carol et al 2001; Menzel and

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Steinmann 2001; Menzel et al 2002; Brünig 2003; Billardon and Pétry 2005; Desmorat et al 2007; Badel et al 2007; Desmorat and Otin 2008; Gatuingt 2008; Ragueneau et al 2008). The three principal values  $D_i$  correspond to 3 orthogonal families of microcracks. Damage evolution law takes then the tensorial rate form  $\dot{\mathbf{D}} = \dots \geq 0$  with then the need to properly bound the principal damages to 1 for Finite Element computations. In the context of quasi-brittle failure, alternative theories are the microplane damage models usually introducing several scalar damage variables  $d_k$  defined for fixed damage directions  $k$  all over unit sphere (Bazant and Gambarova 1984; Bazant and Prat 1988a,b; Bazant and Ozbolt 1990; Fichant et al 1997; Kuhl and Ramm 1998).

Other tensorial damage representations exist in literature such as the use as thermodynamics damage variable of integrity tensor  $\Phi = \mathbf{1} - \mathbf{D}$ , its invert  $\Phi^{-1}$  or even their squareroot  $\Phi^{\frac{1}{2}} = (\mathbf{1} - \mathbf{D})^{\frac{1}{2}}$ ,  $\Phi^{-\frac{1}{2}} = (\mathbf{1} - \mathbf{D})^{-\frac{1}{2}} = \mathbf{H}$  (Ladevèze 1983, 1995; Ladevèze et al 1994; Steinmann and Carol 1998; Menzel and Steinmann 2001; Badel et al 2007; Souid et al 2009; Desmorat et al 2010a,b; Chen et al 2011; Chambart et al 2014) as introduced in pioneering work of Cordebois and Sidoroff (1982). The tensorial damage evolution laws take different forms according to the authors, from  $\dot{\Phi} = \dots \leq 0$  to  $\dot{\mathbf{H}} = \dots \geq 0$ . Note that in a similar spirit a compliance version of microplane damage modeling (Jirasek 1999) performs unit sphere averaging process with damage evolution law applied to inverse of integrity tensor in order to ensure gradual softening and numerical efficiency. As the principle of energy equivalence is used the tensor thus defined is in fact tensor  $\mathbf{H}$ . Carol et al (2001) and Pröchtel and Häußler-Combe (2008) for instance use Ladevèze general framework but define damage evolution as  $\dot{\mathbf{L}} = 2 \Phi^{\frac{1}{2}} \cdot \frac{d}{dt} \Phi^{-1} \cdot \Phi^{\frac{1}{2}} = 2 \mathbf{H} \cdot \dot{\mathbf{D}} \cdot \mathbf{H} = 2(\dot{\mathbf{H}} \cdot \mathbf{H}^{-1} + \mathbf{H}^{-1} \cdot \dot{\mathbf{H}}) = \dots \geq 0$  from the concept of pseudo-logarithmic damage tensor rate. Last, some authors instead straightforwardly consider the effective (damaged) elasticity tensor as damage variable (Govindjee et al 1995; Meschke et al 1998).

One aims in present work to model monotonic multiaxial concrete materials behavior and loading induced anisotropic damage. It has been shown that a proper consideration of damage anisotropy allows to diminish the number of material parameters needed to represent concrete response in not too confined state of stresses (Desmorat 2004; Desmorat et al 2007): the strong dissymmetry of tension/compression behaviors can be obtained by means of 5 material parameters, including elasticity parameters, *i.e.* less than in reference Mazars (1984, 1986) damage model. This property has been gained in so-called next "initial anisotropic damage model":

- within Ladevèze second order damage thermodynamics framework,
- as required by thermodynamics (Lemaitre and Desmorat 2005) by considering a single thermodynamics variable, standard damage variable  $\mathbf{D}$ : as a state variable it represents the micro-cracking pattern whatever the loading sign,
- by means of a deviatoric/hydrostatic splitting within effective stress (Papa and Taliercio 1996; Lemaitre et al 1999, 2000),
- from a full hydrostatic stiffness recovery in compressive stress states,
- with damage intensity and anisotropy both governed by the extensions (Mazars 1986).

The constitutive equations are recalled in next section with an emphasis on the shear-bulk coupling. The biaxial responses of the initial model are derived next from a closed form polar representation in principal stresses plane. The model advantages but also drawbacks are discussed so that a novel shear-bulk coupling is then proposed, in accordance with numerical Discrete Element results of Delaplace and Desmorat (2007).

Main features of finally proposed anisotropic damage model are *i*) the use of unbounded second order tensor  $\mathbf{H}$  as damage variable instead of  $\mathbf{D}$ , *ii*) that both damage level and damage anisotropy are assumed to be governed by the extensions (Mazars 1984) and *iii*) that an improved multiaxial behavior in confined stress states is gained from an original stress triaxiality enhancement of Mazars criterion function for anisotropic damage. In present work stress triaxiality is classically defined as the ratio hydrostatic stress  $\sigma_H$  / von Mises equivalent stress  $\sigma_{eq}$ ,

$$T_X = \frac{1}{3} \frac{\text{tr } \boldsymbol{\sigma}}{\sigma_{eq}} = \frac{\sigma_H}{\sigma_{eq}} \quad (1)$$

if  $\boldsymbol{\sigma}$  is the stress tensor.

## Initial anisotropic damage model

For concrete, the microcracks due to tension are mainly orthogonal to the loading direction, when the microcracks due to compression are mainly parallel to the loading direction. The damage state is then chosen to be represented by tensorial variable  $\mathbf{D}$ , bounded to unit tensor  $\mathbf{1}$ . As mentioned in the introduction the use of a second order damage tensor is convenient for practical applications but also for the material parameters identification. The damage anisotropy induced by either tension or compression is simply modeled by the consideration of damage evolution laws ensuring a damage rate proportional to the positive part of the strain tensor, i.e. damage – and its anisotropy – governed by the extensions  $\langle \varepsilon_i \rangle = \max(0, \varepsilon_i)$  if  $\varepsilon_i$  stand for the eigenstrains (Mazars 1984, 1986).

## Thermodynamics framework

In initial anisotropic damage model (Desmorat 2004), Gibbs free enthalpy is assumed to be a function of the stress tensor and of the second order damage tensor, as follows

$$\rho\psi^*(\boldsymbol{\sigma}, \mathbf{D}) = \frac{1}{4G} \text{tr} \left( (\mathbf{1} - \mathbf{D})^{-\frac{1}{2}} \cdot \boldsymbol{\sigma}' \cdot (\mathbf{1} - \mathbf{D})^{-\frac{1}{2}} \cdot \boldsymbol{\sigma}' \right) + \frac{1}{18K} \left[ \frac{\langle \text{tr } \boldsymbol{\sigma} \rangle^2}{1 - \frac{1}{3}\eta \text{tr } \mathbf{D}} + \langle -\text{tr } \boldsymbol{\sigma} \rangle^2 \right] \quad (2)$$

with  $E$  Young's modulus,  $\nu$  Poisson's ratio,  $G = E/2(1 + \nu)$  and  $K = E/3(1 - 2\nu)$  the shear and bulk moduli.  $\langle x \rangle = \max(x, 0)$  denotes the positive part of a scalar and  $(\cdot)' = (\cdot) - \frac{1}{3} \text{tr}(\cdot) \mathbf{1}$  stands for the deviatoric part of a tensor.

A splitting between deviatoric and hydrostatic contributions has been made and the hydrostatic term has itself been split into two parts, the part at positive hydrostatic stresses  $\sigma_H = \frac{1}{3} \text{tr } \boldsymbol{\sigma}$  being affected by damage (through its mean / hydrostatic damage value  $D_H = \frac{1}{3} \text{tr } \mathbf{D}$ , not an additional variable), the negative hydrostatic stresses remaining unaffected by damage.

Corresponding state laws are

$$\boldsymbol{\epsilon} = \rho \frac{\partial \psi^*}{\partial \boldsymbol{\sigma}} \quad \mathbf{Y} = \rho \frac{\partial \psi^*}{\partial \mathbf{D}} \quad (3)$$

Elasticity law coupled with anisotropic damage can be recast as Eq. (4)-(5). See the works of Lemaitre and Desmorat (2005) and Lemaitre et al (2009) for the derivation of the thermodynamics force  $\mathbf{Y}$  associated with anisotropic damage  $\mathbf{D}$ .

The model is completed by a criterion function  $f \leq 0$  defining both the elasticity domain and the mean damage growth and by a tensorial non standard damage evolution law  $\dot{\mathbf{D}} = \dots \geq 0$  ensuring positivity of the damage rate tensor and of the intrinsic dissipation.

### Constitutive equations

The full set of constitutive equations for the initial anisotropic damage model reads (Desmorat et al 2007; Chambart 2009)

– *Initially isotropic elasticity coupled with damage,*

$$\boldsymbol{\epsilon} = \mathbb{E}^{-1} : \tilde{\boldsymbol{\sigma}} = \frac{1+\nu}{E} \tilde{\boldsymbol{\sigma}} - \frac{\nu}{E} \text{tr} \tilde{\boldsymbol{\sigma}} \mathbf{1} \quad \text{i.e.} \quad \begin{cases} \boldsymbol{\epsilon}' = \frac{\tilde{\boldsymbol{\sigma}}'}{2G} \\ \text{tr} \boldsymbol{\epsilon} = \frac{\text{tr} \tilde{\boldsymbol{\sigma}}}{3K} = \frac{\tilde{\sigma}_H}{K} \end{cases} \quad (4)$$

$\tilde{\sigma}_H = \frac{1}{3} \text{tr} \tilde{\boldsymbol{\sigma}}$  is the effective hydrostatic stress and  $\mathbb{E}$  is isotropic Hooke's tensor for virgin (undamaged) material.

– *Effective stress,*

$$\tilde{\boldsymbol{\sigma}} = \left( (\mathbf{1} - \mathbf{D})^{-\frac{1}{2}} \cdot \boldsymbol{\sigma}' \cdot (\mathbf{1} - \mathbf{D})^{-\frac{1}{2}} \right)' + \frac{1}{3} \left[ \frac{\langle \text{tr} \boldsymbol{\sigma} \rangle}{1 - \frac{1}{3} \eta \text{tr} \mathbf{D}} - \langle -\text{tr} \boldsymbol{\sigma} \rangle \right] \mathbf{1} \quad (5)$$

Material constant  $\eta \geq 1$  is the hydrostatic sensitivity parameter (Lemaitre et al 2000).

– *Damage criterion,*

$$f = \hat{\epsilon} - \kappa \leq 0 \quad (6)$$

with  $\hat{\epsilon} = \sqrt{\sum \langle \epsilon_i \rangle^2}$  Mazars (1984) equivalent strain. Consolidation function  $\kappa$  is set as a function of the trace of the damage tensor,

$$\kappa = \kappa(\text{tr} \mathbf{D}) = a \cdot \tan \left[ \frac{\text{tr} \mathbf{D}}{aA} + \arctan \left( \frac{\kappa_0}{a} \right) \right] \quad (7)$$

– *Induced damage anisotropy* governed by the positive – in terms of principal values – effective strain tensor,

$$\dot{\mathbf{D}} = \dot{\lambda} \langle \tilde{\boldsymbol{\epsilon}} \rangle^+ \quad \tilde{\boldsymbol{\epsilon}} = \mathbb{E}^{-1} : \boldsymbol{\sigma} \quad (8)$$

In such a rate independent formulation, the damage multiplier  $\dot{\lambda}$  satisfies Kuhn-Tucker loading-unloading conditions  $f \leq 0, \dot{\lambda} \geq 0, \dot{\lambda} f = 0$ .

There are 5 material parameters introduced if  $\eta = 3$  is set:  $E, \nu$  for elasticity,  $\kappa_0$  as damage threshold and  $A$  and  $a$  as damage parameters.

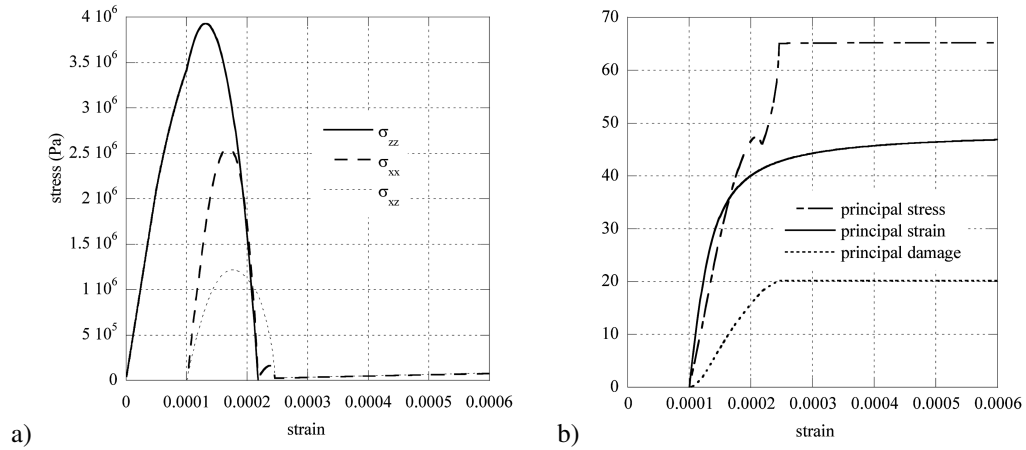
### Mathematical and thermodynamics features

Let us first recall that thermodynamics potential  $\rho\psi^*$  (Eq. 2) can be continuously differentiated (Ladevèze 1983; Ladevèze and Lemaitre 1984; Lemaitre and Desmorat 2005).

This mathematical property ensures the continuity of the model response in the non proportional loading cases. As a counterexample for example, if  $\langle \boldsymbol{\sigma} \rangle^+$  is the positive part of stress tensor in terms of principal

values and  $\tilde{\mathbf{E}}$  damaged elasticity tensor, a term of the form  $\langle \boldsymbol{\sigma} \rangle^+ : \tilde{\mathbf{E}}^{-1} : \langle \boldsymbol{\sigma} \rangle^+$  cannot be continuously differentiated and leads to discontinuities in multiaxial stress-strain response.

The mathematical differentiability feature can be illustrated from non proportional numerical tests (with rotating principal directions). The result to Willam et al (1987) test are given in Fig. 1. The loading is applied in two steps: *i*) a first uniaxial tension is applied in direction  $z$  till the peak in the stress-strain diagram, *ii*) the second loading consists in a rotation of the strain tensor principal axis by applying a non proportional loading using biaxial-tension and shear ( $\Delta \epsilon_{xz}$ ). The applied increments for step *ii*) are as follows,  $\Delta \epsilon_{zz} = \frac{1}{2} \Delta \epsilon_{xx}$ ,  $\Delta \epsilon_{xx} = \frac{3}{4} \Delta \epsilon_{zz}$ . The major principal directions angles for second order tensors  $\boldsymbol{\sigma}$ ,  $\boldsymbol{\epsilon}$  and  $\mathbf{D}$  are denoted respectively  $\varphi_\sigma$ ,  $\varphi_\epsilon$  and  $\varphi_D$ , with  $\varphi_\sigma = \varphi_\epsilon = \varphi_D = 0$  up to the end of step *i*). Continuity of the stress components (Fig. 1a) as well as of the principal directions (Fig. 1b) is obtained.



**Figure 1.** Willam's test result for initial anisotropic damage model (Ragueneau et al 2008) a) continuous stresses, b) continuous principal directions angles  $\varphi_\sigma$ ,  $\varphi_\epsilon$ ,  $\varphi_D$  (angles in degree) for stress, strain and damage tensors.

Last the damage evolution law is non standard, *i.e.* it does not derive from a convex evolution (dissipation) potential with respect to thermodynamics force  $\mathbf{Y}$ . The proof of the positivity of the intrinsic dissipation for the initial anisotropic damage model has therefore to be derived. It is given in (Desmorat 2006; Desmorat et al 2007; Lemaitre et al 2009). Numerical schemes for the computation of intrinsic dissipation due to anisotropic damage can be found in (Chambart et al 2014).

### Effective Hooke's tensor

The elasticity law can be inverted in a closed form expression  $\boldsymbol{\sigma} = \tilde{\mathbf{E}} : \boldsymbol{\epsilon}$  so that the effective (damaged) elasticity tensor is

$$\begin{aligned} \tilde{\mathbf{E}} = 2\tilde{\mathbf{G}} + \tilde{K} \mathbf{1} \otimes \mathbf{1} = 2G \left[ (\mathbf{1} - \mathbf{D})^{1/2} \underline{\otimes} (\mathbf{1} - \mathbf{D})^{1/2} - \frac{(\mathbf{1} - \mathbf{D}) \otimes (\mathbf{1} - \mathbf{D})}{3 - \text{tr } \mathbf{D}} \right] \\ + K \left( 1 - \frac{1}{3} \eta \text{tr } \mathbf{D} \mathcal{H}(\text{tr } \boldsymbol{\epsilon}) \right) \mathbf{1} \otimes \mathbf{1} \end{aligned} \quad (9)$$

where  $\mathcal{H}(x)$  is the Heaviside function, and tensorial product  $\overline{\otimes}$  is defined by the identity  $(\mathbf{A} \overline{\otimes} \mathbf{B}) : \mathbf{C} = \mathbf{A} \cdot \mathbf{C} \cdot \mathbf{B}$  for all symmetric tensors  $\mathbf{A}, \mathbf{B}, \mathbf{C}$ .

### Bulk modulus

One sees from Eq. (9) that effective bulk modulus is

$$\tilde{K} = K \left( 1 - \frac{1}{3} \eta \operatorname{tr} \mathbf{D} \mathcal{H}(\operatorname{tr} \epsilon) \right) = K (1 - \eta D_H \mathcal{H}(\operatorname{tr} \epsilon)) \quad (10)$$

It is constant equal to virgin value  $K$  in compressive loadings at  $\operatorname{tr} \sigma = 3K \operatorname{tr} \epsilon < 0$  (negative stress triaxiality). This physically corresponds to a state of closed micro-cracks. At  $\operatorname{tr} \sigma = 3\tilde{K} \operatorname{tr} \epsilon > 0$  (positive stress triaxiality) it is the same linear function  $\tilde{K} = K(1 - \eta D_H)$  of the hydrostatic damage  $D_H = \frac{1}{3} \operatorname{tr} \mathbf{D} = \frac{1}{3}(D_1 + D_2 + D_3)$  whatever the stress triaxiality. It is therefore not a function of quadratic norm  $\|D\| = \sqrt{D_1^2 + D_2^2 + D_3^2}$  nor of infinite norm  $\|D\|_\infty = \max D_i$ , and this for any positive triaxiality.

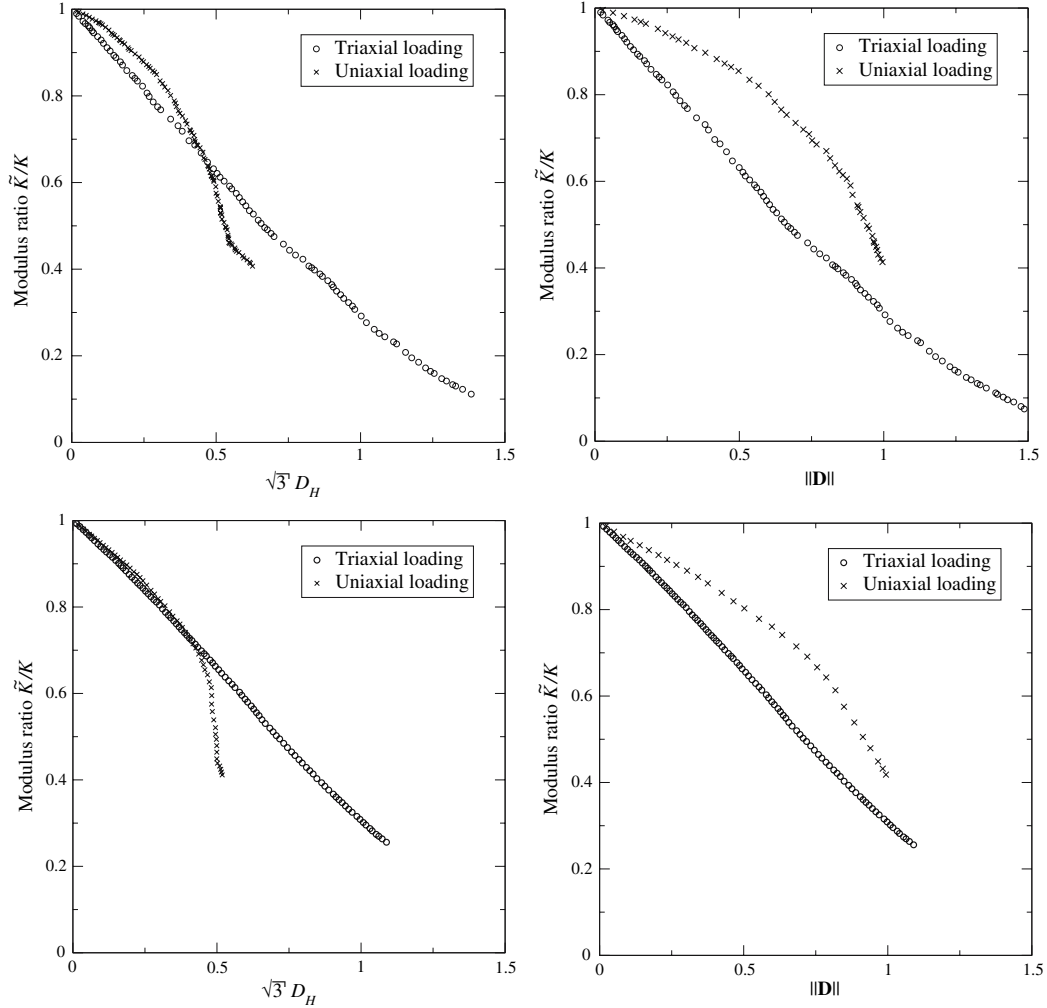
The linearity of effective bulk modulus with respect to  $D_H$  has been checked by Delaplace and Desmorat (2007) – at least at low damage – for two states of micro-cracking:

- left curves of Figure 2 show that Eq. (10) is valid, perfectly for the  $16 \times 16 \times 16 \text{ mm}^3$  cube, up to  $D_H = 0.3$ ,
- right curves Figure 2 show that  $\tilde{K}$  cannot be a function of  $\|D\|$ .

Uniaxial state of micro-cracking – not represented – is due to uniaxial tension loading applied on the cubes of Fig. 3 (considered as a Representative Volume Element (RVE)), triaxial state of micro-cracking – represented in Fig. 3 – is due to equi-triaxial tension loading applied on the cubes. Both monotonic loadings are applied numerically as Discrete Element computations on the lattice medium (the cubes made of Voronoi cells and considered as the RVE). In such numerical tests the material is described as a particles assembly representative of the material heterogeneity, the particles being here linked by elastic-brittle beams (Herrmann and Roux 1990; Schlangen and Garboczi 1997; Van Mier et al 2002). Two sizes  $8 \times 8 \times 8 \text{ mm}^3$  and  $16 \times 16 \times 16 \text{ mm}^3$  of RVE of a micro-concrete were considered (Fig. 3), the increase in size corresponding to an increase in the number of particles and in the number of degrees of freedom (dof): 512 particles and 4,096 dof for the 8-cube, 3,072 particles and 24,576 dof for the 16-cube. The crack patterns obtained at the end of the triaxial loading are those of Fig. 3 for the two samples. Note that the number of beams to break before failure varies from 1,500 beams for the 8-cube sample to 8,000 for the 16 one. The components of damage tensor have been measured (for each mark of Fig. 2) by means of repeated numerical elastic loading-unloading sequences performed in uniaxial tension (even for the triaxial loading), using then the coupling of elasticity with anisotropic damage given by Eq. (11)-(12) with one non zero principal stress  $\sigma_i = \sigma$ , the two others  $\sigma_{j \neq i} = 0$ .

The negative slope  $-\eta$  in the  $\tilde{K}/K$  vs  $D_H$  diagram is found close to  $-1$  ( $\approx -1.2$ ). The precise value is subjected to caution as it is obtained from a numerical modeling (with no aggregates for instance) but it can nevertheless be noticed that it is quite different from the slope  $-\eta = -3$  obtained from first set of material parameters identified for initial anisotropic damage model (Desmorat 2004) or for metals (Lemaitre et al 2000).

Note that considering  $\eta = 1$  in such an initial damage model means that bulk modulus  $\tilde{K}$  fully vanishes at  $\operatorname{tr} \mathbf{D} = 3$ , *i.e.* at maximum principal damage  $\max D_i$  larger than 1 in uniaxial and equi-biaxial tension ( $\tilde{K}$

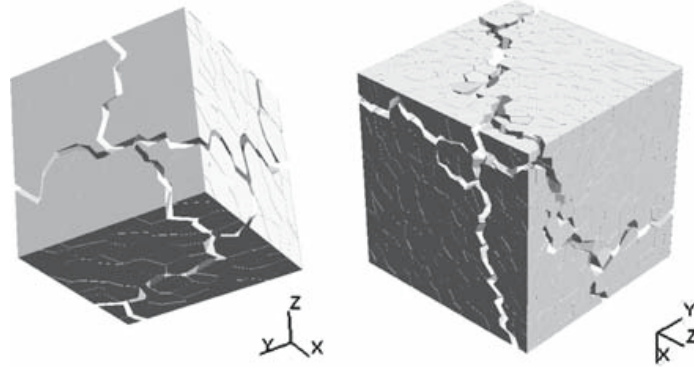


**Figure 2.** Effective bulk modulus  $\tilde{K}$  from Discrete Element computations as a function of hydrostatic damage (of  $\sqrt{3}D_H$  for comparison) and of quadratic norm  $\|\mathbf{D}\|$ . Top:  $8 \times 8 \times 8 \text{ mm}^3$  cube, bottom:  $16 \times 16 \times 16 \text{ mm}^3$  cube (from (Delaplace and Desmorat 2007)).

cannot vanish then as principal damages are bounded by 1 !). This corresponds to a quite high (spurious) elastic stiffness which is kept at rupture. On the other hand the therefore preferred case  $\eta = 3$  leads to  $\tilde{K} = 0$  at  $\text{tr } \mathbf{D} = 1$ , *i.e.* at maximum principal damage  $\max D_i$  equal to  $1/2$  in equi-biaxial tension, equal to  $1/3$  in equi-triaxial tension: at values of shear moduli far to be zero. Enforcing then  $\tilde{K} = 0$  but allowing still damage tensor  $\mathbf{D}$  to evolve up to unit tensor  $\mathbf{1}$  in an adequate procedure for the numerical control of rupture is a solution which leads to numerical difficulties in Finite Element computations (Badel et al 2007; Desmorat et al 2007; Ragueneau et al 2008; Chambart 2009; Leroux 2012).

One will propose next an adequate shear-bulk coupling that makes coincide the full loss of both bulk and shear stiffnesses  $\tilde{K} = \tilde{G} = 0$  with no need of a procedure for the numerical control of rupture to bound the damage eigenvalues to 1.





**Figure 3.** Discrete Element samples considered as Representative Volume Element (left  $8 \times 8 \times 8 \text{ mm}^3$  cube, right:  $16 \times 16 \times 16 \text{ mm}^3$  cube) and crack patterns at the end of triaxial loading (high damage level) for the two samples.

### Uniaxial and biaxial responses of initial model

In proportional loading, radial loading at constant stress triaxiality  $T_X = \text{tr } \boldsymbol{\sigma} / 3\sigma_{eq} = \sigma_H / \sigma_{eq}$ , the stress, strain and damage tensors are colinear and closed form expressions can be derived.

In the principal framework the different tensors are all diagonal (of principal components  $\sigma_i, \varepsilon_i, D_i$ ). The effective stress  $\tilde{\boldsymbol{\sigma}}$  is related to the stress and damage state by Eq. (5) and is also diagonal. In plane stress condition ( $\sigma_3 = 0$ ) its deviatoric part is given by

$$\begin{aligned}\tilde{\sigma}'_1 &= \frac{\sigma_1}{9} \left( \frac{4}{1-D_1} + \frac{1}{1-D_2} + \frac{1}{1-D_3} \right) - \frac{\sigma_2}{9} \left( \frac{4}{1-D_1} - \frac{2}{1-D_2} + \frac{1}{1-D_3} \right) \\ \tilde{\sigma}'_2 &= -\frac{\sigma_1}{9} \left( -\frac{2}{1-D_1} + \frac{4}{1-D_2} + \frac{1}{1-D_3} \right) + \frac{\sigma_2}{9} \left( \frac{1}{1-D_1} + \frac{4}{1-D_2} + \frac{1}{1-D_3} \right) \\ \tilde{\sigma}'_3 &= -\frac{\sigma_1}{9} \left( -\frac{1}{1-D_1} + \frac{2}{1-D_2} + \frac{2}{1-D_3} \right) - \frac{\sigma_2}{9} \left( -\frac{1}{1-D_1} + \frac{2}{1-D_2} + \frac{2}{1-D_3} \right)\end{aligned}\quad (11)$$

when its hydrostatic part by

$$\tilde{\sigma}_H = \frac{\sigma_1 + \sigma_2}{3(1 - \eta D_H)} \quad (12)$$

The elasticity law coupled with damage  $\boldsymbol{\epsilon}' = \tilde{\boldsymbol{\sigma}}' / 2G$ ,  $\text{tr } \boldsymbol{\epsilon} = \tilde{\sigma}_H / K$  gives for an in-plane stress loading  $\sigma_1 = \rho \cos \theta$ ,  $\sigma_2 = \rho \sin \theta$ , interpreting  $\rho$  as an equivalent polar stress,

$$\varepsilon_i = \frac{\rho}{E} B_i(\theta, D_i) \quad (13)$$

with  $B_i$  a function of the parametric angle in the stress plane  $\theta$  and of the principal damages  $D_i$  given in Appendix B.

Similar but simpler calculations give the effective strain principal components as

$$\tilde{\varepsilon}_i = \frac{\rho}{E} \tilde{B}_i(\theta) \quad \tilde{B}_i(\theta) = B_i(\theta, D_i = 0) \quad (14)$$

The damage evolution law  $\dot{\mathbf{D}} = \dot{\lambda} \langle \tilde{\boldsymbol{\varepsilon}} \rangle^+$  becomes

$$\dot{D}_i = \dot{\lambda} \langle \tilde{\varepsilon}_i \rangle = \frac{\dot{\lambda} \rho}{E} \langle \tilde{B}_i \rangle \quad (15)$$

and can be integrated at constant  $\theta$  in

$$D_i = \frac{\langle \tilde{\varepsilon}_i \rangle}{\langle \tilde{\varepsilon}_1 \rangle + \langle \tilde{\varepsilon}_2 \rangle + \langle \tilde{\varepsilon}_3 \rangle} \text{tr } \mathbf{D} = \frac{\langle \tilde{B}_i \rangle}{\langle \tilde{B}_1 \rangle + \langle \tilde{B}_2 \rangle + \langle \tilde{B}_3 \rangle} \text{tr } \mathbf{D} \quad (16)$$

Consistency condition

$$f = \hat{\varepsilon} - \kappa = \frac{\rho}{E} \sqrt{\langle B_1 \rangle^2 + \langle B_2 \rangle^2 + \langle B_3 \rangle^2} - \kappa(\text{tr } \mathbf{D}) = 0 \quad (17)$$

allows to determine equivalent polar stress  $\rho = \rho(\theta, D_i)$  as a function of angle  $\theta$  and principal damages  $D_i$ .

Previous derivations give analytically the model response in proportional biaxial loading by proceeding as follows:

1. Consider a loading biaxiality through angle  $\theta$  (constant for each proportional loading calculations) and any given value for the trace of the damage tensor  $\text{tr } \mathbf{D}$ , starting from  $\text{tr } \mathbf{D} = 0$ .
2. Calculate the principal damage components,

$$D_i = \frac{\langle \tilde{B}_i \rangle}{\langle \tilde{B}_1 \rangle + \langle \tilde{B}_2 \rangle + \langle \tilde{B}_3 \rangle} \text{tr } \mathbf{D} \quad (18)$$

as  $\tilde{B}_i$  only depends on  $\theta$ .

3. Calculate the equivalent polar stress,

$$\rho = \frac{E \kappa(\text{tr } \mathbf{D})}{\sqrt{\langle B_1 \rangle^2 + \langle B_2 \rangle^2 + \langle B_3 \rangle^2}} \quad (19)$$

with  $B_i = B_i(\theta, D_i)$ .

4. Calculate the stress components

$$\sigma_1 = \rho \cos \theta \quad \sigma_2 = \rho \sin \theta \quad (20)$$

5. Calculate the strain components

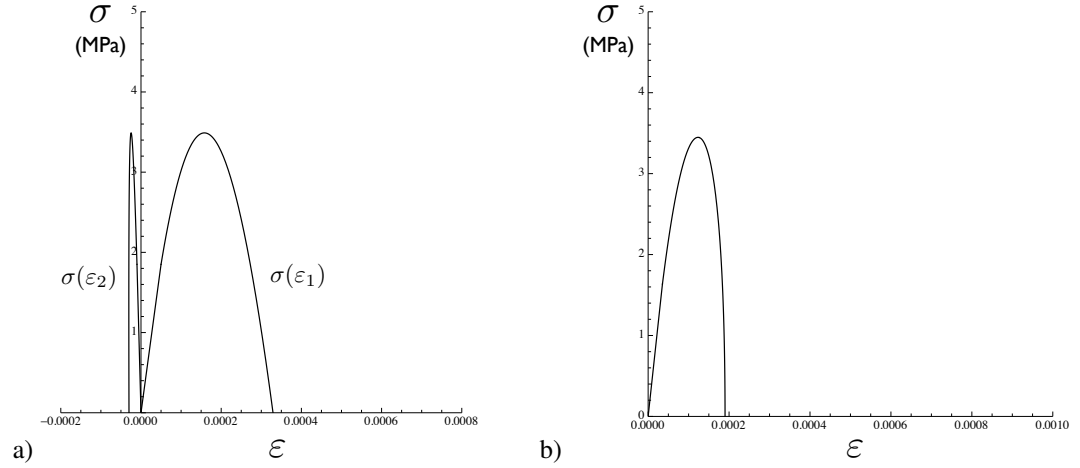
$$\varepsilon_i = \frac{\rho}{E} B_i \quad (21)$$

The model responses are plotted next. The material parameters are representative of a concrete (peak stress in tension  $\sigma_u^t = 3.5$  MPa, peak stress in compression  $\sigma_u^c = 30.5$  MPa).

$$E = 37000 \text{ MPa}, \nu = 0.2, \eta = 3, \kappa_0 = 5 \cdot 10^{-5}, A = 5000, a = 3 \cdot 10^{-3}.$$

### Uniaxial tension

The case of tension – performed in direction 1 – corresponds to  $\theta = 0$  (stress triaxiality  $T_X = 1/3$ ). The stress-strain response is given in Fig. 4a. The different curves  $\sigma(\varepsilon_1)$ ,  $\sigma(\varepsilon_2)$  correspond to the tensile strain  $\varepsilon_1$  and to transverse strain  $\varepsilon_2 = \varepsilon_3$  abscissa. The damage state is simply  $D_1 \geq 0$ ,  $D_2 = D_3 = 0$ .



**Figure 4.** Response of initial anisotropic damage model a) in tension, b) in equi-biaxial tension.

### Equi-biaxial tension

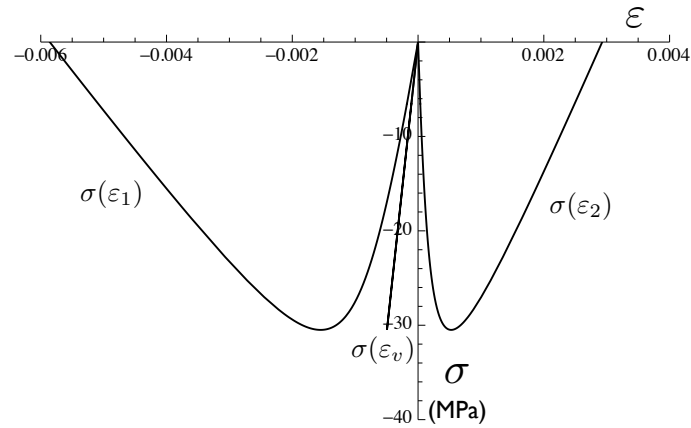
Equi-biaxial tension corresponds to  $\theta = \pi/4$  (stress triaxiality  $T_X = 2/3$ ). The stress  $\sigma = \sigma_1 = \sigma_2$  vs strain  $\varepsilon = \varepsilon_1 = \varepsilon_2$  curve (Fig. 4b) exhibits a peak a bit lower than in tension (3.45 instead of 3.5 MPa) and a more brittle behavior, as expected for a quasi-brittle material. The damage state is  $D_1 = D_2 \geq 0$ ,  $D_3 = 0$ .

### Uniaxial compression

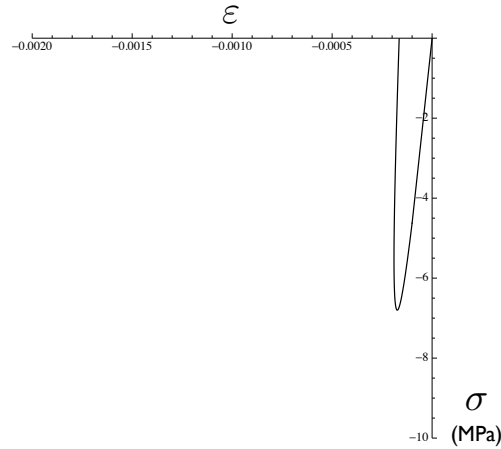
The stress strain responses  $\sigma(\varepsilon_i)$  and  $\sigma(\varepsilon_v)$  for compression in direction 1 ( $\theta = \pi$ , stress triaxiality  $T_X = -1/3$ ) are given in Fig. 5, denoting  $\varepsilon_v = \text{tr } \boldsymbol{\varepsilon} = \varepsilon_1 + 2\varepsilon_2$  the volumetric strain. As expected, the  $\sigma(\varepsilon_v)$  response is found linear ( $\sigma = \text{tr } \boldsymbol{\sigma} = K\varepsilon_v$ ) due to the deviatoric/hydrostatic splitting with no effect of the damage on bulk modulus in compressive states. The factor of around 10 between the peak stresses in tension and in compression, usual for concrete, is obtained. The tension/compression dissymmetric behavior is mainly due to damage anisotropy: micro-cracks perpendicular to loading direction in tension ( $D_1 > 0$ ,  $D_2 = D_3 = 0$ ), micro-cracks parallel to loading direction in compression  $D_1 = 0$ ,  $D_2 = D_3 > 0$ .

### Equi-biaxial compression

The response in equi-biaxial compression ( $\theta = -\frac{3\pi}{4}$ , stress triaxiality  $T_X = -2/3$ , Fig. 6) is found much too brittle (a snapback is exhibited) with a much too low peak stress (6.5 MPa) as for concrete it is usually larger than compression peak stress  $\sigma_u^t$  (Kupfer et al 1969). This feature has been pointed out in earlier works. Modeling improvements, not fully satisfactory due to their complexity, have been proposed in (Ragueneau et al 2008; Leroux 2012). The damage state is  $D_1 = D_2 = 0$ , out of plane damage  $D_3 \geq 0$ .



**Figure 5.** Response of initial anisotropic damage model in compression.



**Figure 6.** a) Response of initial anisotropic damage model in equi-biaxial compression.

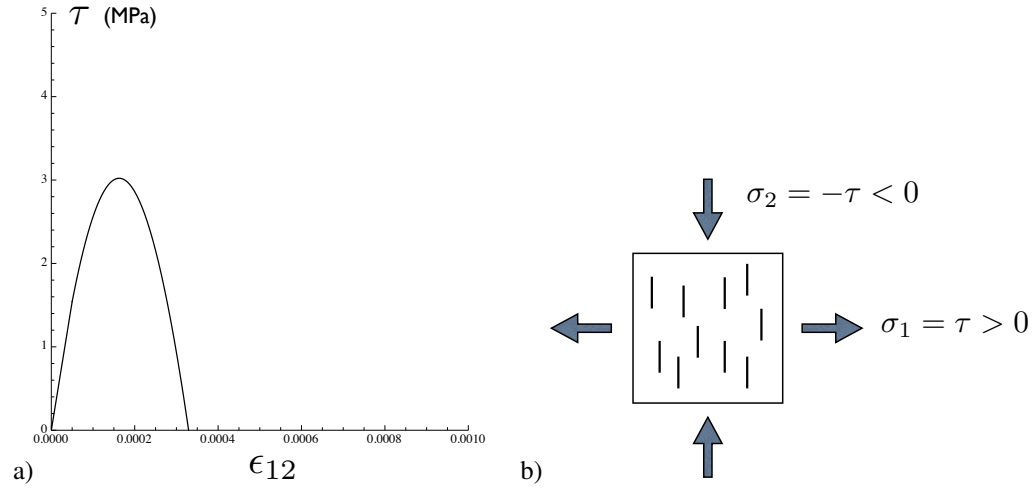
### Shear

The shear stress  $\tau = \sigma_1$  vs shear strain  $\epsilon_{12} = \epsilon_1$  curve is plotted in Fig. 7 (case  $\theta = -\pi/4$ , stress triaxiality  $T_X = 0$ ). It exhibits a peak stress of the same order of magnitude than the one in tension as for well established Mazars (1984, 1986) isotropic damage model. But present model does not give any ductility at all in shear (one would have expected a plateau or a slow softening more representative of friction-wear behavior).

### Discussion

This initial anisotropic damage model, dedicated to initially isotropic quasi-brittle materials such as concrete, has some nice properties:

- It represents the damage state encountered in tension (micro-cracking  $D_1, D_2 = D_3 = 0$ , parallel perpendicular to loading direction 1), in compression (micro-cracking  $D_2 = D_3, D_1 = 0$ , parallel to loading direction 1), in bicompression (out of plane micro-cracking  $D_3, D_1 = D_2 = 0$ ), in shear.



**Figure 7.** Response of initial anisotropic damage model in shear, b) Microcracking pattern corresponding to damage state  $D_1 \geq 0$ ,  $D_2 = 0$  in shear.

- The damage anisotropy is itself responsible for the dissymmetry of tension and compression responses, as observed experimentally.
- There is only one damage variable, tensorial, for all loadings including tension and compression, as requested by the status of a state variable of thermodynamics:  $\mathbf{D}$  represents the micro-cracking pattern present in the material whatever the loading sign.
- As a consequence the number of material parameters is quite low: 5 including elasticity parameters if  $\eta = 3$  is reasonably set.
- It can be proven that dissipation due to damage is positive in any case Desmorat (2006), this thanks to the deviatoric/hydrostatic splitting in the definition of the effective stress (5).
- Total softening of all stress components is obtained in 3D at positive stress triaxiality (property due again to the deviatoric/hydrostatic splitting in the definition of effective stress). At negative stress triaxiality bulk modulus is unchanged,  $\text{tr } \boldsymbol{\sigma} = 3K \text{tr } \boldsymbol{\epsilon} < 0$ , and only (all) deviatoric stress components  $\sigma'_{ij}$  softens to zero.

There are of course drawbacks. The main one is the response in equi-bicompression which is way too brittle. One can argue that the behavior in confined states has not to be modeled by elasticity coupled with damage only and that plasticity and permanent strains has to take over (Govindjee et al 1995; Meschke et al 1998; Burlion et al 2002; Gatingt et al 2002; Grassl and Jirasek 2006). Nevertheless an elasticity coupled with damage model that includes acceptable monotonic responses in confined stress states would be appreciated (see for instance the work of Leroux (2012)). A second drawback is that the post-peak response has no tail. Maximum principal damage reaches 1 at finite rupture strain, whose value is a bit small in tensile cases. Brittleness is physical but it leads to costly numerical difficulties. One needs a specific numerical control of rupture (Badel et al 2007; Desmorat et al 2007; Chambart 2009) in order to enforce the principal damages to remain bounded to 1. In case of plain concrete applications this works well but difficulties arise at concrete/bars sheared interfaces in reinforced concrete structures (Leroux 2012)). Related to this control

of rupture procedure, there is the fact that full softening at stress triaxiality larger than 1/3, such as in equibiaxial tension case, occurs at principal damages  $D_i < 1$ , strictly smaller than 1. This is due to shear-bulk coupling considered as damage  $\mathbf{D}$  acts on both shear modulus (in a tensorial manner as  $(\mathbf{1} - \mathbf{D})^{\frac{1}{2}}$  terms, Eq. (9)) and bulk modulus as  $\tilde{K} = K(1 - \text{tr } \mathbf{D})$ .

Next section is dedicated to the proposal of an anisotropic damage model that attempts to keep the advantages of initial model and to correct its drawbacks. The validity domain sought still consists in monotonic applications at not too high triaxial confining pressures, so that one allows us not to model the irreversible strains, neither the volumetric dilatancy in simple compression, nor all complex cyclic effects, see for instance (Bazant and Prat 1988a,b; Ragueneau et al 2000; Goidescu et al 2015) and (Desmorat 2004; Ramtani et al 1992; Halm and Dragon 1998; Souid et al 2009; Lebon 2011; Richard and Ragueneau 2013) in case of tensorial damage).

### Model with modified shear-bulk coupling

The change of second order tensorial damage variable  $\mathbf{H} = (\mathbf{1} - \mathbf{D})^{-\frac{1}{2}}$  is operated and the damage dependency of deviatoric and of hydrostatic parts of novel effective stress (23) are both made homogeneous to  $\mathbf{H}^2$ . This allows to consider an unbounded damage evolution up to  $\mathbf{H} \rightarrow \infty$  for any loading case. The same damage variable  $\mathbf{H}$  is chosen to act on both shear and bulk parts, representative of so-called shear-bulk coupling. There is no need to refer anymore to standard damage variable  $\mathbf{D}$  but it is nevertheless convenient to still calculate it (as  $\mathbf{D} = \mathbf{1} - \mathbf{H}^{-2}$ ) as principal damages  $D_i \in [0, 1]$  are more easily interpreted than unbounded principal values  $H_i \in [1, \infty[$  of tensor  $\mathbf{H}$ . The "no damage" case corresponds to  $\mathbf{H} = \mathbf{1} \Leftrightarrow \mathbf{D} = \mathbf{0}$  or  $H_{ij} = \delta_{ij} \Leftrightarrow D_{ij} = 0$  in terms of components.

### Constitutive equations

The proposed anisotropic damage constitutive equations for quasi-brittle materials, such as concrete, are as follows.

- *Initially isotropic elasticity coupled with damage,*

$$\boldsymbol{\epsilon} = \frac{\tilde{\boldsymbol{\sigma}}'}{2G} + \frac{1}{9K} \text{tr } \tilde{\boldsymbol{\sigma}} \mathbf{1} \quad (22)$$

- *Effective stress,*

$$\tilde{\boldsymbol{\sigma}} = (\mathbf{H} \cdot \boldsymbol{\sigma}' \cdot \mathbf{H})' + \frac{1}{3} \left[ \frac{1}{3} \text{tr } \mathbf{H}^2 \langle \text{tr } \boldsymbol{\sigma} \rangle - \langle -\text{tr } \boldsymbol{\sigma} \rangle \right] \mathbf{1} \quad (23)$$

- *Damage criterion,*

$$f = \hat{\epsilon} - \kappa \leq 0 \quad (24)$$

with  $\hat{\epsilon}$  Mazars equivalent strain.

Consolidation function is taken linear in  $\text{tr } \mathbf{H}$  as,

$$\kappa = \kappa_0 + SR_\nu^s (\text{tr } \mathbf{H} - 3) \quad (25)$$

with as material parameters the damage threshold  $\kappa_0$ , the triaxiality exponent  $s$  and the damage strength  $S$ , enhanced by (negative) stress triaxiality  $T_X$  by means of triaxiality function  $R_\nu$  (Lemaitre

and Chaboche (1985), see Appendix D), here normed to unity in shear and possibly bounded to material constant  $B > R_\nu^c = 3/2(1 + \nu)$  in order to properly model biaxial compression,

$$R_\nu = \min \left[ 1 + \frac{9}{2} \frac{1 - 2\nu}{1 + \nu} \langle -T_X \rangle^2, B \right] \quad T_X = \frac{\sigma_H}{\sigma_{eq}} \quad (26)$$

– *Induced damage anisotropy* governed by the positive – in terms of principal values – effective strain tensor  $\langle \tilde{\epsilon} \rangle^+$  but written in terms of rate of damage tensor  $\mathbf{H}$ ,

$$\dot{\mathbf{H}} = \dot{\lambda} \langle \tilde{\epsilon} \rangle^+ \quad \tilde{\epsilon} = \mathbf{E}^{-1} : \boldsymbol{\sigma} \quad (27)$$

The damage multiplier  $\dot{\lambda}$  satisfies Kuhn-Tucker loading-unloading conditions  $f \leq 0$ ,  $\dot{\lambda} \geq 0$ ,  $\dot{\lambda} f = 0$ .

The stress triaxiality is  $T_X = 1/3$  in tension leading to the value  $R_\nu^t = 1$  for triaxiality function, it is  $T_X = 2/3$  in equibiaxial tension so that  $R_\nu = 1$ , it is  $T_X = 0$  in shear with then  $R_\nu = 1$ ,  $T_X = -1/3$  in compression with  $R_\nu = R_\nu^c = 3/2(1 + \nu) > 1$ , and  $T_X = -2/3$  in equibiaxial compression in which case  $R_\nu = 3(1 - \nu)/(1 + \nu) > R_\nu^c > 1$ . Standard definition for stress triaxiality function is here preferred to the definition of equivalent strain reducing factor  $\gamma = \sqrt{\langle -\boldsymbol{\sigma} \rangle^+ : \langle -\boldsymbol{\sigma} \rangle^+} / \langle -\text{tr } \boldsymbol{\sigma} \rangle$  introduced by Mazars and coworkers in criterion function  $f$  (Mazars 1984; Pontiroli 1995; La Borderie 2003).

There are 6 or 5 material parameters introduced, depending on whether bounding value  $B$  is introduced or not:  $E$ ,  $\nu$  for elasticity,  $\kappa_0$  as damage threshold, damage strength  $S$ , damage triaxiality exponent  $s$ , bicompression parameter  $B$ . Their number is quite low for constitutive equations attempting to properly model concrete multiaxial behavior with dissymmetry tension/compression. Again the key point of modeling is the fact that such a dissymmetry is due to damage anisotropy with a tensorial damage state represented by single variable  $\mathbf{H} = (\mathbf{1} - \mathbf{D})^{-\frac{1}{2}}$ . There is no – thermodynamically inconsistent – use of a first damage variable for ”tension” and of a second damage variable for ”compression”.

### Effective Hooke's tensor

The elasticity law can be inverted in a closed form expression  $\boldsymbol{\sigma} = \tilde{\mathbf{E}} : \boldsymbol{\epsilon}$  so that the effective (damaged) elasticity tensor is at positive stress triaxiality  $T_X$  (positive  $\text{tr } \boldsymbol{\sigma}$  and  $\text{tr } \boldsymbol{\epsilon}$ ):

$$\tilde{\mathbf{E}} = 2G \left[ \mathbf{H}^{-1} \underline{\otimes} \mathbf{H}^{-1} - \frac{\mathbf{H}^{-2} \otimes \mathbf{H}^{-2}}{\text{tr } \mathbf{H}^{-2}} \right] + \frac{3K}{\text{tr } \mathbf{H}^2} \mathbf{1} \otimes \mathbf{1} \quad (28)$$

It tends toward  $\mathbf{0}$  in limiting case  $\mathbf{H} \rightarrow \infty$ .

At negative  $T_X$  (negative  $\text{tr } \boldsymbol{\sigma}$  and  $\text{tr } \boldsymbol{\epsilon}$ ):

$$\tilde{\mathbf{E}} = 2G \left[ \mathbf{H}^{-1} \underline{\otimes} \mathbf{H}^{-1} - \frac{\mathbf{H}^{-2} \otimes \mathbf{H}^{-2}}{\text{tr } \mathbf{H}^{-2}} \right] + K \mathbf{1} \otimes \mathbf{1} \quad (29)$$

It tends toward  $K \mathbf{1} \otimes \mathbf{1}$  in limiting case  $\mathbf{H} \rightarrow \infty$ .

### Bulk modulus

Bulk modulus  $\tilde{K} = K$  remains unchanged at negative stress triaxiality  $T_X$  ( $\sigma_H = \frac{1}{3} \text{tr } \boldsymbol{\sigma} < 0$ ,  $\varepsilon_v = \text{tr } \boldsymbol{\epsilon} < 0$ ).

One sees from Eq. (28) that effective (damaged) bulk modulus at positive stress triaxiality ( $\sigma_H > 0$ ,  $\varepsilon_v > 0$ ) is

$$\tilde{K} = \frac{3K}{\text{tr} \mathbf{H}^2} = \frac{3K}{\text{tr}(\mathbf{1} - \mathbf{D})^{-1}} \quad (30)$$

It decreases with damage as potted in Fig. 8, where hydrostatic damage  $D_H$  in terms of  $\mathbf{D}$  has been calculated from the knowledge of damage tensor  $\mathbf{H}$ ,

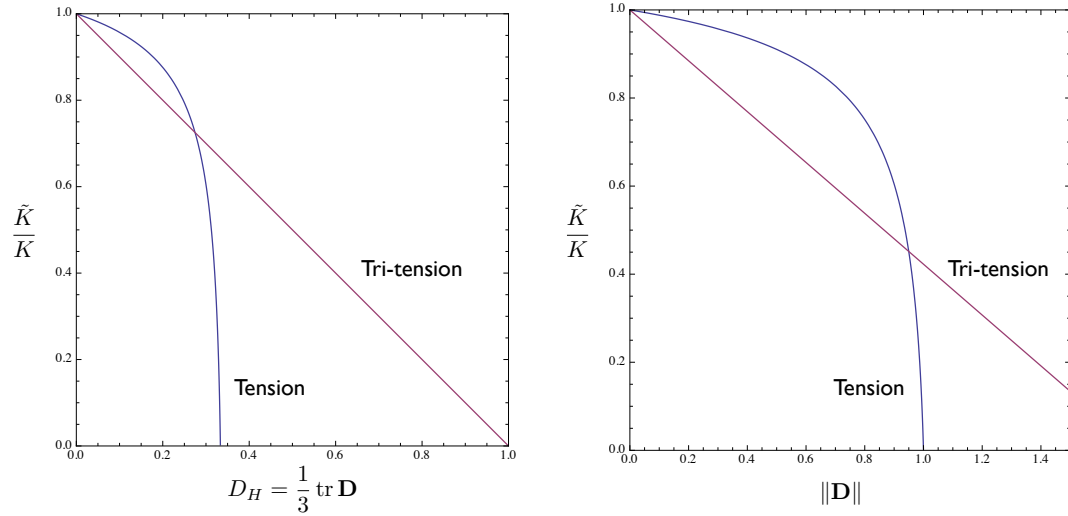
$$D_H = \frac{1}{3} \text{tr} \mathbf{D} = 1 - \frac{1}{3} \text{tr} \mathbf{H}^{-2} \quad (31)$$

This gives a nonlinear variation in most cases including uniaxial tension  $H_1 \geq 1$ ,  $H_2 = H_3 = 1$ . Equi-triaxial tension case corresponds to spherical damage tensors of principal components

$$H_i = \frac{1}{\sqrt{1 - D_i}} = \frac{1}{3} \text{tr} \mathbf{H} \quad D_i = 1 - \frac{1}{H_i^2} = 1 - \left( \frac{3}{\text{tr} \mathbf{H}} \right)^2 = \frac{1}{3} \text{tr} \mathbf{D} = D_H \quad (32)$$

so that in such a particular case one recovers a linear loss of stiffness  $\tilde{K} = K(1 - D_H)$  which corresponds to hydrostatic sensitivity parameter  $\eta = 1$  in initial anisotropic damage modeling. The loss of bulk modulus is found very similar to the one of Fig. 2 obtained from Discrete Element computations.

Altogether vanishing shear and bulk stiffnesses are gained from proposed shear-bulk coupling (at infinite strain as illustrated in the examples of next section).



**Figure 8.** Effective bulk modulus  $\tilde{K}$  from shear-bulk coupling (23)-(30) as a function of hydrostatic damage  $D_H$  and of quadratic norm  $\|\mathbf{D}\|$  with  $\mathbf{D} = \mathbf{1} - \mathbf{H}^{-2}$ .

## Responses of proposed anisotropic model

Let us consider cases of proportional loading and use again the plane stress polar representation  $\sigma_1 = \rho \cos \theta$ ,  $\sigma_2 = \rho \sin \theta$ .



Following the same formal derivation than for the initial anisotropic damage model one sets for elasticity law coupled with damage (see Appendix C for functions  $C_i$  and  $\tilde{C}_i$ ,  $C_{eq} = \sqrt{\langle C_1 \rangle^2 + \langle C_2 \rangle^2 + \langle C_3 \rangle^2}$ ),

$$\varepsilon_i = \frac{\rho}{E} C_i(\theta, H_i) \quad \hat{\varepsilon} = \frac{\rho}{E} C_{eq}(\theta, H_i) \quad (33)$$

and for effective strain  $\tilde{\varepsilon} = \mathbf{E}^{-1} : \boldsymbol{\sigma}$ ,

$$\tilde{\varepsilon}_i = \frac{\rho}{E} \tilde{C}_i(\theta) \quad \tilde{C}_i = C_i(\theta, H_i = 1) \quad (34)$$

The damage evolution law  $\dot{\mathbf{H}} = \dot{\lambda} \langle \tilde{\varepsilon} \rangle^+$  becomes in proportional loading

$$\dot{H}_i = \dot{\lambda} \langle \tilde{\varepsilon}_i \rangle = \frac{\dot{\lambda} \rho}{E} \langle \tilde{C}_i \rangle \quad (35)$$

and can be integrated in

$$H_i = 1 + \frac{\langle \tilde{\varepsilon}_i \rangle}{\langle \tilde{\varepsilon}_1 \rangle + \langle \tilde{\varepsilon}_2 \rangle + \langle \tilde{\varepsilon}_3 \rangle} (\text{tr } \mathbf{H} - 3) = 1 + \frac{\langle \tilde{C}_i \rangle}{\langle \tilde{C}_1 \rangle + \langle \tilde{C}_2 \rangle + \langle \tilde{C}_3 \rangle} (\text{tr } \mathbf{H} - 3) \quad (36)$$

The consistency condition gives then, with triaxiality function  $R_\nu = R_\nu(T_X(\theta))$

$$f = \hat{\varepsilon} - \kappa(\theta, \text{tr } \mathbf{H}) = 0 \quad \kappa(\theta, \text{tr } \mathbf{H}) = \kappa_0 + S R_\nu^s (\text{tr } \mathbf{H} - 3) = 0 \quad (37)$$

In the same manner than for the initial anisotropic damage model, proceed then as follows to calculate the model response in proportional biaxial loading:

1. Consider a loading biaxility through angle  $\theta$  and any given value for  $\text{tr } \mathbf{H}$ , starting from  $\text{tr } \mathbf{H} = 3$ .

The stress triaxiality is:

$$T_X = \frac{1}{3} \frac{\sigma_1 + \sigma_2}{\sqrt{\sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2}} = \frac{1}{3} \frac{\sin \theta + \cos \theta}{\sqrt{1 - \sin \theta \cos \theta}} = T_X(\theta) \quad (38)$$

2. Calculate the principal damage components,

$$H_i = 1 + \frac{\langle \tilde{C}_i \rangle}{\langle \tilde{C}_1 \rangle + \langle \tilde{C}_2 \rangle + \langle \tilde{C}_3 \rangle} (\text{tr } \mathbf{H} - 3) \quad (39)$$

as  $\tilde{C}_i$  only depends on  $\theta$ .

3. Calculate the equivalent polar stress,

$$\rho = \frac{E \kappa(\theta, \text{tr } \mathbf{H})}{C_{eq}} \quad (40)$$

due to  $C_{eq} = C_{eq}(\theta, H_i)$ .

4. Calculate the stress components

$$\sigma_1 = \rho \cos \theta \quad \sigma_2 = \rho \sin \theta \quad (41)$$

5. Calculate the strain components

$$\varepsilon_i = \frac{\rho}{E} C_i \quad (42)$$

One cross-identifies the proposed anisotropic model on the initial tension and compression responses of initial model, Eq. (4)-(8). As already mentioned the purpose is to avoid the sudden softening to zero stress and to gain ductility, as a tail decreasing gently to zero stress in stress-strain diagrams. The peak stresses are enforced identical for both models in tension ( $\sigma_u^t = 3.5$  MPa) and in compression ( $\sigma_u^c = 30.5$  MPa).

The material parameters for concrete are:

$$E = 37000 \text{ MPa}, \nu = 0.2, \kappa_0 = 9 \cdot 10^{-5}, S = 1.45 \cdot 10^{-4}, s = 4.9, B = \frac{5}{3}$$

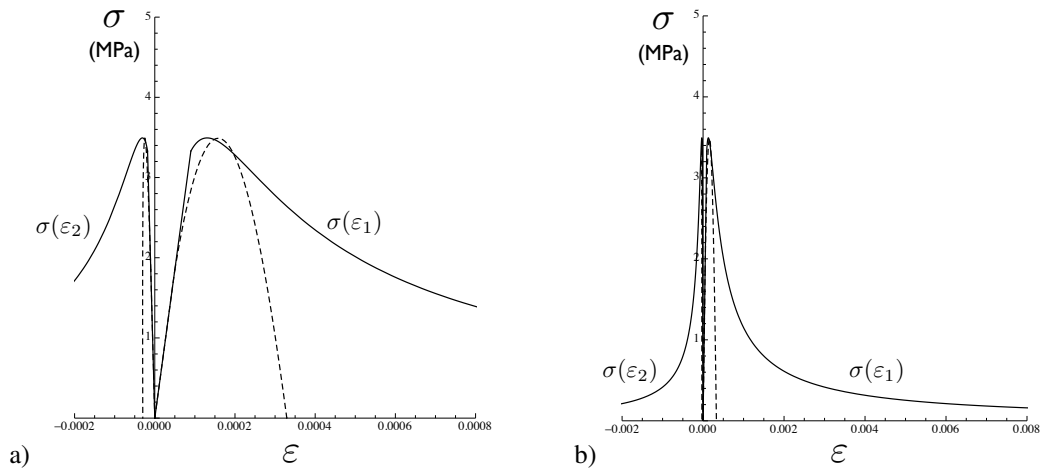
The responses of initial anisotropic damage model are reported as dashed lines in next figures.

### Uniaxial tension

The response in uniaxial tension, at polar angle  $\theta = 0$  and stress triaxiality  $T_X = 1/3$ , is given in Fig. 9a and 9b. The peak stress is well  $\sigma_u^t = 3.5$  MPa as for initial damage model. A larger strain scale is considered in Fig. 9b in order to exhibit the gain in ductility and the tail obtained at high softening (the stress nevertheless tends to zero at infinite strain).

The micro-cracking pattern characteristic of tension performed in direction 1 is recovered as  $H_2 = H_3 = 1$  (equivalent to  $D_2 = D_3 = 0$ ) and

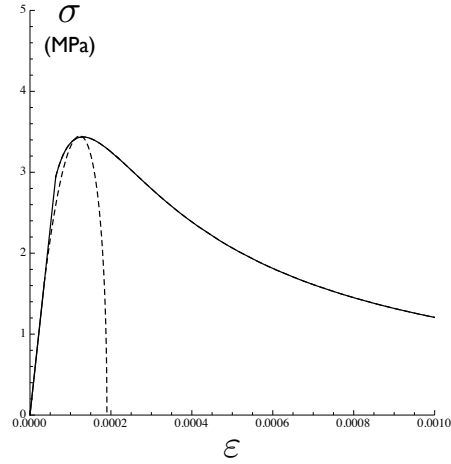
$$H_1 = \frac{1}{\sqrt{1 - D_1}} = \text{tr} \mathbf{H} - 2 > 1 \quad D_1 = 1 - \frac{1}{H_1^2} = 1 - \frac{1}{(\text{tr} \mathbf{H} - 2)^2} > 0 \quad (43)$$



**Figure 9.** Response of proposed anisotropic damage model in tension (dashed line: initial anisotropic damage model). A larger strain scale used in b).

### Biaxial tension

The same peak stress than for initial damage model (still with more ductility, Fig. 10) is obtained in equi-biaxial tension case  $\theta = \pi/4$ ,  $T_X = 2/3$ . The damage state is  $H_1 = H_2 = \frac{1}{2}(\text{tr } \mathbf{H} - 1) > 1$ ,  $H_3 = 1$  equivalent to  $D_1 = D_2 > 0$ ,  $D_3 = 0$ .



**Figure 10.** Response of proposed anisotropic damage model in equi-biaxial tension (dashed line: initial anisotropic damage model).

### Uniaxial compression

Compression in direction 1 is the case  $\theta = \pi$ , stress triaxiality  $T_X = -1/3$  (Fig. 11).

Cross-identification on compression response can be performed up to softening stage at high damage level (up to  $D_2 = D_3 \approx 0.9$ ). The fact that the stress has not softened of 90% at this damage level is due both to damage anisotropy (damages  $D_2$  and  $D_3$  are not in loading direction) and to unchanged (undamaged) bulk modulus in compression  $\tilde{K} = K$ . Such last model feature is reflected by linear response in terms of volumetric stress-strain curve  $\sigma(\varepsilon_v) = K\varepsilon_v = K \text{tr } \epsilon$  in Fig. 11. A softening tail is obtained for  $\sigma(\varepsilon_1)$  response but its role in compression (as well as in next bicompression) is much less important for most practical applications than at positive or zero stress triaxiality.

The micro-cracking pattern in compression  $D_1 = 0$ ,  $D_2 = D_3 > 0$  is gained as

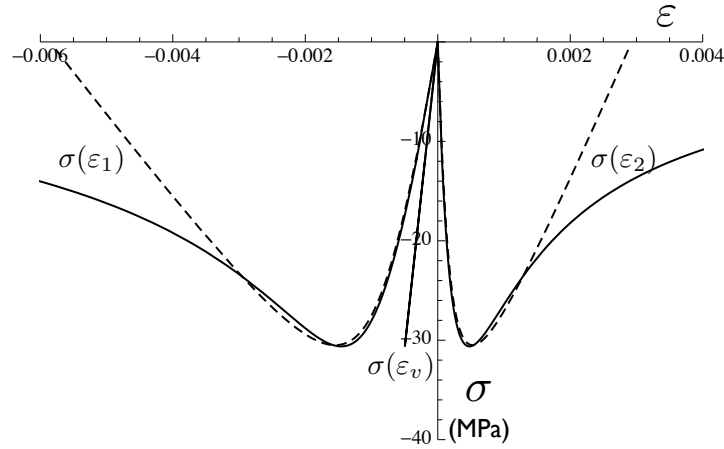
$$H_1 = 1 \quad \text{and} \quad H_2 = H_3 = \frac{1}{2}(\text{tr } \mathbf{H} - 1) > 1 \quad (44)$$

corresponding to standard damages

$$D_2 = D_3 = 1 - \frac{1}{H_2^2} = 1 - \left( \frac{2}{\text{tr } \mathbf{H} - 1} \right)^2 > 0 \quad (45)$$

### Biaxial compression

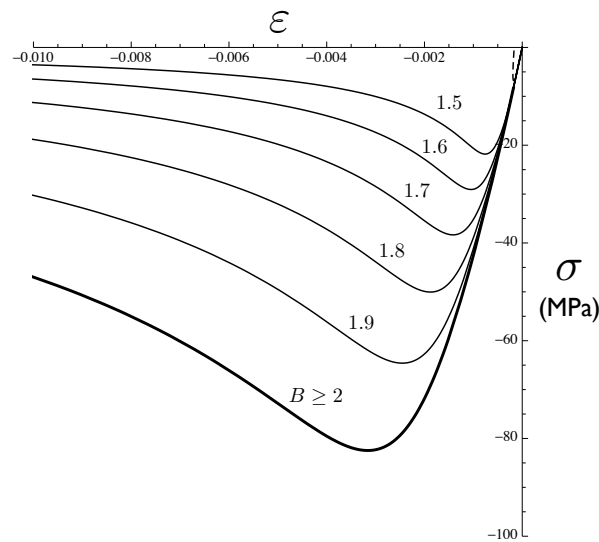
Model response in equi-biaxial compression is plotted in Fig. 12 for different values of bounding material parameter  $B$ . Poisson's ratio is  $\nu = 0.2$  so that the same response is obtained for any  $B \geq 2$  than for



**Figure 11.** Response of proposed anisotropic damage model in compression (dashed line: initial anisotropic damage model).

unbounded case  $B \rightarrow \infty$ . One sees clearly that parameter  $B$  allows to represent any physical bicompression response, depending on the concrete considered: it is named bicompression parameter. Be careful just to chose  $B > 3/2(1 + \nu) = 1.25$  so that the response in uniaxial compression remains unchanged (as well as all other model responses at larger stress triaxiality  $T_X \geq -1/3$  plotted in present work). The choice  $B = 5/3$  is found consistent with Kupfer et al (1969) multiaxial experiments exhibiting a ratio peak stress in bicompression / peak stress in tension of approximately 1.15, *i.e.* here a peak stress of 35 MPa in bicompression. The choice  $B = 1.725$  is consistent with ratio of 1.35 obtained by Yin et al (1989) (peak stress of 41 MPa in bicompression).

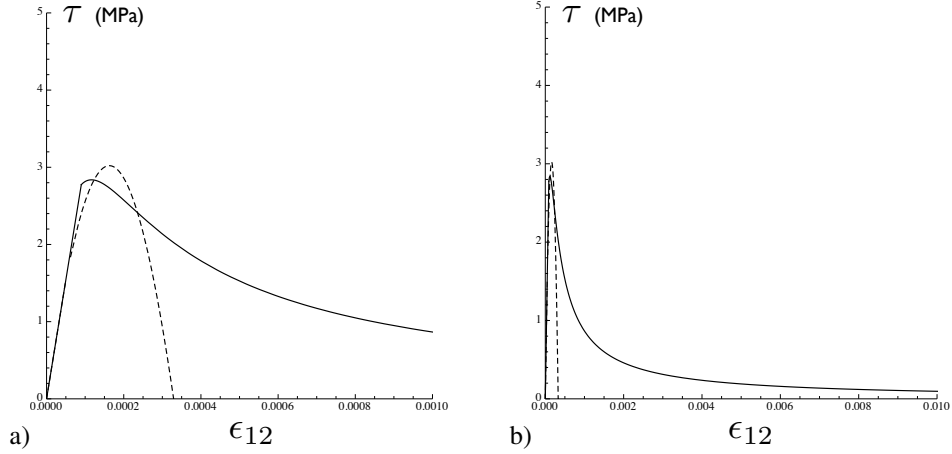
Physical damage state of initial anisotropic model  $D_3 > 0$ ,  $D_1 = D_2 = 0$  is recovered from principal out of plane damage  $H_3 = (1 - D_3)^{-\frac{1}{2}} = \text{tr } \mathbf{H} - 2 > 1$  and in-plane damages  $H_1 = H_2 = 1$ .



**Figure 12.** Response of proposed anisotropic damage model in equi-biaxial compression for different values of bicompression parameter  $B$  (dashed line: initial anisotropic damage model, thick line: unbounded  $R_\nu$  case  $B \rightarrow \infty$  or any case  $B \geq 2$ ).

## Shear

Shear response,  $\theta = -\pi/4$  and  $T_X = 0$ , is plotted in Fig. 13. The peak stress is satisfactory of the same magnitude than the peak stress in tension or than the peak stress in shear calculated from initial anisotropic damage model. The nice feature of a softening tail is obtained. Note that a non zero residual shear strain is still present (of value  $\tau = 0.1$  MPa) at quite important strain  $\varepsilon_{12} = 10^{-2}$ .



**Figure 13.** Response of proposed anisotropic damage model in shear (dashed curve: initial anisotropic damage model). A larger strain scale used in b).

## Lode angle dependency

General elasticity criterions for isotropic materials depends on the 3 stress invariants  $I_1 = \text{tr } \boldsymbol{\sigma}$ ,  $J_2 = \frac{1}{2} \text{tr } \boldsymbol{\sigma}'^2 = \sigma_{eq}^2/3$ ,  $J_3 = \frac{1}{3} \text{tr } \boldsymbol{\sigma}'^3 = \det \boldsymbol{\sigma}'$ , or in a equivalent manner of von Mises stress  $\sigma_{eq}$ , of stress triaxiality  $T_X = \frac{1}{3} \text{tr } \boldsymbol{\sigma}'/\sigma_{eq}$  and of Lode angle  $\Theta \in [0, \frac{\pi}{3}]$  defined as

$$\Theta = \frac{1}{3} \arccos \frac{27 \det \boldsymbol{\sigma}'}{2 \sigma_{eq}^3} \quad (46)$$

not to be confused with previous polar angle  $\theta$ . The principal deviatoric stresses are then  $\sigma'_i = \frac{2}{3} \sigma_{eq} \cos \Theta_i$  where  $\Theta_1 = \Theta$ ,  $\Theta_2 = \Theta - \frac{2\pi}{3}$ ,  $\Theta_3 = \Theta + \frac{2\pi}{3}$ , so that in principal basis diagonal deviatoric stress is expressed as

$$\boldsymbol{\sigma}' = \frac{2}{3} \sigma_{eq} \mathbf{n}(\Theta) \quad \mathbf{n}(\Theta) = \begin{bmatrix} \cos \Theta_1 & 0 & 0 \\ 0 & \cos \Theta_2 & 0 \\ 0 & 0 & \cos \Theta_3 \end{bmatrix} \quad (47)$$

with tensor  $\mathbf{n}(\Theta)$  deviatoric as  $\text{tr } \mathbf{n}(\Theta) = 0$ . This exhibits the feature that Mazars initial elasticity surface  $f = \hat{\varepsilon} - \kappa_0 = 0$ , built from the positive part of the principal strains within definition of equivalent strain  $\hat{\varepsilon}$ , is function of second (von Mises) stress invariant  $\sigma_{eq} = \sqrt{3J_2}$  but also on both stress triaxiality and Lode angle. Mazars elasticity surface has parametric representation

$$\sigma_i = \frac{2}{3} \frac{E \kappa_0 \cos \Theta_i}{\sqrt{\langle \hat{\varepsilon}_1(\Theta, T_X) \rangle^2 + \langle \hat{\varepsilon}_2(\Theta, T_X) \rangle^2 + \langle \hat{\varepsilon}_3(\Theta, T_X) \rangle^2}} \quad (48)$$

in principal stress space  $(\sigma_1, \sigma_2, \sigma_3)$ .

In both anisotropic damage models studied the tensorial damage evolution is governed by the effective positive strain tensor  $\langle \tilde{\epsilon} \rangle^+$  of principal components

$$\langle \tilde{\epsilon}_i \rangle = \left\langle \frac{\sigma'_i}{2G} + \frac{\text{tr } \boldsymbol{\sigma}}{9K} \right\rangle = \frac{\sigma_{eq}}{E} \langle \tilde{e}_i \rangle \quad \tilde{e}_i = \frac{2}{3}(1 + \nu) \cos \Theta_i + (1 - 2\nu)T_X \quad (49)$$

so that the rates of principal damages  $\dot{H}_i = \dot{\lambda} \langle \tilde{\epsilon}_i \rangle = \frac{\dot{\lambda} \sigma_{eq}}{E} \langle \tilde{e}_i \rangle$  and even their mean value  $\dot{H}_H = \frac{1}{3} \text{tr } \dot{\mathbf{H}}$  are Lode angle dependent. This dependency for  $H_H$  can be explicitied from  $f = \hat{\epsilon} - \kappa = 0$  as

$$H_H = \frac{1}{3} \sum H_i = 1 + \frac{\sigma_{eq} \sqrt{\sum \langle e_i(\Theta, T_X, \mathbf{H}) \rangle^2 - E \kappa_0}}{3ESR_v^s} \quad (50)$$

due to  $\hat{\epsilon} = \frac{\sigma_{eq}}{E} \sqrt{\sum \langle e_i(\Theta, T_X, \mathbf{H}) \rangle^2}$  if the  $e_i$  are the principal values of tensor

$$\mathbf{e} = \frac{2}{3}(1 + \nu)(\mathbf{H} \cdot \mathbf{n}(\Theta) \cdot \mathbf{H})' + (1 - 2\nu) \left[ \frac{1}{3} \text{tr } \mathbf{H}^2 \langle T_X \rangle - \langle -T_X \rangle \right] \mathbf{1} \quad (51)$$

Nevertheless the Lode angle dependency exhibited here is not specific to the damage models considered in present work: it is included in most anisotropic damage models, as stated in Desmorat (2012) for a  $J_2$ -plasticity model coupled with anisotropic damage.

## Thermodynamics of second order anisotropic damage

Let us now derive the thermodynamics framework for novel anisotropic damage model.

### Thermodynamics potential and state laws

The Gibbs free enthalpy density  $\rho\psi^*$ , with  $\rho$  the density, for proposed anisotropic damage model is a function of the stress and damage tensors,  $\boldsymbol{\sigma}$ ,  $\mathbf{H}$ , considered as a thermodynamics state variables,

$$\rho\psi^*(\boldsymbol{\sigma}, \mathbf{D}) = \frac{1}{4G} \text{tr}(\mathbf{H} \cdot \boldsymbol{\sigma}' \cdot \mathbf{H} \cdot \boldsymbol{\sigma}') + \frac{1}{18K} \left[ \frac{1}{3} \text{tr } \mathbf{H}^2 \langle \text{tr } \boldsymbol{\sigma} \rangle^2 + \langle -\text{tr } \boldsymbol{\sigma} \rangle^2 \right] \quad (52)$$

Thermodynamics potential  $\rho\psi^*$  can be continuously differentiated (Ladevèze and Lemaitre 1984; Lemaitre et al 2000; Desmorat 2000; Lemaitre and Desmorat 2005). This ensures continuous stress-strain response for any multiaxial non proportional loading (as exhibited in Willam et al (1987) test of Fig. 1) for initial damage model).

The state laws are:

$$\boldsymbol{\epsilon} = \rho \frac{\partial \psi^*}{\partial \boldsymbol{\sigma}} \quad \mathbf{Z} = \rho \frac{\partial \psi^*}{\partial \mathbf{H}} \quad (53)$$

They lead

- to elasticity coupled with damage ( $(\cdot)'$  denotes deviatoric part),

$$\boldsymbol{\epsilon} = \frac{1}{2G} (\mathbf{H} \cdot \boldsymbol{\sigma}' \cdot \mathbf{H})' + \frac{1}{9K} \left[ \frac{1}{3} \text{tr } \mathbf{H}^2 \langle \text{tr } \boldsymbol{\sigma} \rangle - \langle -\text{tr } \boldsymbol{\sigma} \rangle \right] \mathbf{1} \quad (54)$$

This elasticity law coupled with damage defines effective stress  $\tilde{\sigma}$  from Eq. (22) and can be recast as Eq. (22)-(23).

- to the definition of thermodynamics force associated with damage  $\mathbf{H}$ ,

$$\mathbf{Z} = \frac{1}{2G} \boldsymbol{\sigma}' \cdot \mathbf{H} \cdot \boldsymbol{\sigma}' + \frac{1}{27K} \langle \text{tr } \boldsymbol{\sigma} \rangle^2 \mathbf{H} \quad (55)$$

### Convexity with respect to stress tensor and to damage variable $\mathbf{H}$

The second derivative of thermodynamics potential with respect to stress tensor is (where  $\mathcal{H}$  is Heaviside function and tensorial product  $\overline{\otimes}$  is such as  $\mathbf{H} \overline{\otimes} \mathbf{H} : \boldsymbol{\sigma} = \mathbf{H} \cdot \boldsymbol{\sigma} \cdot \mathbf{H}$ , fourth order tensor  $\mathbf{H} \overline{\otimes} \mathbf{H}$  being symmetric),

$$\begin{aligned} \tilde{\mathbf{E}}^{-1} = \rho \frac{\partial^2 \psi^*}{\partial \boldsymbol{\sigma} \partial \boldsymbol{\sigma}} = & \frac{1}{2G} \left[ \mathbf{H} \overline{\otimes} \mathbf{H} - \frac{1}{3} (\mathbf{H} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{H}) + \frac{1}{9} (\text{tr } \mathbf{H}^2) \mathbf{1} \otimes \mathbf{1} \right] \\ & + \frac{1}{9K} \left[ \frac{1}{3} \text{tr } \mathbf{H}^2 \mathcal{H}(\text{tr } \boldsymbol{\sigma}) + \mathcal{H}(-\text{tr } \boldsymbol{\sigma}) \right] \mathbf{1} \otimes \mathbf{1} \end{aligned} \quad (56)$$

It is the invert of the effective elasticity tensor. Convexity of thermodynamics potential  $\rho\psi^*$  with respect to stress tensor is ensured – through deviatoric/hydrostatic splitting – by positivity of  $\tilde{\mathbf{E}}^{-1}(\mathbf{X})$  for any non zero symmetric second order tensor  $\mathbf{X}$ ,

$$\begin{aligned} \mathbf{X} : \tilde{\mathbf{E}}^{-1} : \mathbf{X} &= \frac{1}{2G} \mathbf{X} : (\mathbf{H} \cdot \mathbf{X}' \cdot \mathbf{H})' + \frac{1}{3} \left[ \frac{1}{3} \text{tr } \mathbf{H}^2 \langle \text{tr } \mathbf{X} \rangle - \langle -\text{tr } \mathbf{X} \rangle \right] \text{tr } \mathbf{X} \\ &= \frac{1}{2G} \text{tr} (\mathbf{X}' \cdot \mathbf{H} \cdot \mathbf{X}' \cdot \mathbf{H}) + \frac{1}{27K} \left[ \frac{1}{3} \text{tr } \mathbf{H}^2 \langle \text{tr } \mathbf{X} \rangle^2 + \langle -\text{tr } \mathbf{X} \rangle^2 \right] \\ &= \frac{1}{2G} \text{tr} (\mathbf{X}' \cdot \mathbf{H})^2 + \frac{1}{27K} \left[ \frac{1}{3} \text{tr } \mathbf{H}^2 \langle \text{tr } \mathbf{X} \rangle^2 + \langle -\text{tr } \mathbf{X} \rangle^2 \right] > 0 \end{aligned} \quad (57)$$

This proves convexity with respect to  $\boldsymbol{\sigma}$  as  $\mathbf{X} : \tilde{\mathbf{E}}^{-1} : \mathbf{X} = 2\rho\psi^*(\mathbf{X}, \mathbf{H}) > 0$  for any symmetric  $\mathbf{X} \neq 0$ .

Convexity of  $\rho\psi^*$  with respect to damage tensor  $\mathbf{H}$  is also obtained as

$$\rho \frac{\partial^2 \psi^*}{\partial \mathbf{H} \partial \mathbf{H}} = \frac{1}{2G} \boldsymbol{\sigma}' \overline{\otimes} \boldsymbol{\sigma}' + \frac{1}{27K} \langle \text{tr } \boldsymbol{\sigma} \rangle^2 \mathbf{1} \quad (58)$$

and for any  $\mathbf{X} \neq 0$

$$\mathbf{X} : \rho \frac{\partial^2 \psi^*}{\partial \mathbf{H} \partial \mathbf{H}} : \mathbf{X} = \frac{1}{2G} \text{tr} (\boldsymbol{\sigma}' \cdot \mathbf{X})^2 + \frac{1}{27K} \langle \text{tr } \boldsymbol{\sigma} \rangle^2 (\text{tr } \mathbf{X})^2 > 0 \quad (59)$$

Convexity of free enthalpy with respect to damage variable is not necessary (both state potentials for initial anisotropic model and for novel anisotropic model are not convex with respect to damage  $\mathbf{D}$ ). Convexity with respect to  $\mathbf{H}$  prevents from instabilities in damage evolution driven by its associated thermodynamics force (in so-called standard generalized materials framework, Halphen and Nguyen (1975)), whatever the loading intensity. Such a mathematical property was sought for instance by Badel et al (2007) in case of anisotropic damage.

### Non standard anisotropic damage evolution law

Mazars criterion function  $f = \hat{\epsilon} - \kappa$  is used. It is not expressed as a function of  $\mathbf{Z}$ , the thermodynamics force associated with damage, and damage evolution law (27) does not derive from an evolution (pseudo-dissipation) potential whereas it does in Ladevèze (1983) pioneering work who considered damage criterion  $f_Z$  extended to anisotropy and expressed in terms of thermodynamics force  $\mathbf{Z}$

$$f_Z = \sqrt{\text{tr}(\mathbf{Z} \cdot \mathbf{H} \cdot \mathbf{Z})} + k \mathbf{H} : \mathbf{Z} - k_0 \leq 0 \quad \dot{\mathbf{H}} = \dot{\mu} \frac{\partial f_Z}{\partial \mathbf{Z}} \quad (60)$$

with  $k_0, k_1$  as material parameter and where  $\dot{\mu}$  is damage multiplier satisfying Khun-Tucker loading-unloading conditions  $f_Z \leq 0, \dot{\mu} \geq 0, \dot{\mu} f_Z = 0$ . Applications to Ceramics Matrix Composites can be found in (Ladevèze et al 1994; Ladevèze 1995).

The anisotropic damage model proposed uses evolution law  $\dot{\mathbf{H}} = \dot{\lambda} \langle \tilde{\epsilon} \rangle^+$ , it is therefore non standard and the positivity of the dissipation is not guaranteed: it has to be proven (done in further section). The damage multiplier  $\dot{\lambda} \geq 0$  is determined from consistency condition  $f = \hat{\epsilon} - \kappa = 0$  &  $\dot{f} = \dot{\hat{\epsilon}} - \dot{\kappa} = 0$ ,

$$\dot{\lambda} = \frac{\text{tr} \dot{\mathbf{H}}}{\text{tr} \langle \tilde{\epsilon} \rangle^+} = \frac{1}{S \text{tr} \langle \tilde{\epsilon} \rangle^+} \frac{d}{dt} (R_\nu^{-s} \hat{\epsilon}) \quad (61)$$

At positive stress triaxiality  $T_X$  (i.e. at  $\text{tr} \boldsymbol{\sigma} \geq 0, R_\nu = 1$ ) or in case of compressive loading at negative  $T_X$  ( $\text{tr} \boldsymbol{\sigma} \leq 0$ ) but in the second case by neglecting the triaxiality change over a time increment :

$$\dot{\lambda} = \frac{1}{SR_\nu^s} \frac{\dot{\hat{\epsilon}}}{\text{tr} \langle \tilde{\epsilon} \rangle^+} = \frac{1}{SR_\nu^s} \frac{\langle \epsilon \rangle^+ : \dot{\epsilon}}{\hat{\epsilon} \text{tr} \langle \tilde{\epsilon} \rangle^+} \quad (62)$$

so that damage law is explicitated as,

$$\dot{\mathbf{H}} = \frac{1}{SR_\nu^s} \frac{\dot{\hat{\epsilon}}}{\text{tr} \langle \tilde{\epsilon} \rangle^+} = \frac{1}{SR_\nu^s} \frac{\langle \tilde{\epsilon} \rangle^+ \langle \epsilon \rangle^+ : \dot{\epsilon}}{\hat{\epsilon} \text{tr} \langle \tilde{\epsilon} \rangle^+} \quad (63)$$

Tangent operator, such as  $\dot{\boldsymbol{\sigma}} = \mathbf{L} : \dot{\epsilon}$ , can be derived using damaged elastic tensor (28) and derivatives  $\frac{d}{dt} \mathbf{H}^{-1} = -\mathbf{H}^{-1} \cdot \dot{\mathbf{H}} \cdot \mathbf{H}^{-1}$  and  $\frac{d}{dt} \mathbf{H}^{-2} = -\mathbf{H}^{-2} \cdot [\mathbf{H} \cdot \dot{\mathbf{H}} + \dot{\mathbf{H}} \cdot \mathbf{H}] \cdot \mathbf{H}^{-2}$ ,

$$\begin{aligned} \mathbf{L} = \tilde{\mathbf{E}} - \frac{2}{SR_\nu^s} \left[ \left( \boldsymbol{\sigma}' \cdot \frac{\langle \tilde{\epsilon} \rangle^+}{\text{tr} \langle \tilde{\epsilon} \rangle^+} \cdot \mathbf{H}^{-1} \right)^{\text{Sym}} - \text{tr} \left( \boldsymbol{\sigma}' \cdot \frac{\langle \tilde{\epsilon} \rangle^+}{\text{tr} \langle \tilde{\epsilon} \rangle^+} \cdot \mathbf{H}^{-1} \right) \frac{\mathbf{H}^{-2}}{\text{tr} \mathbf{H}^{-2}} \right] \\ - \frac{2}{3S} \langle \text{tr} \boldsymbol{\sigma} \rangle \frac{\mathbf{H}}{\text{tr} \mathbf{H}^2} : \frac{\langle \tilde{\epsilon} \rangle^+}{\text{tr} \langle \tilde{\epsilon} \rangle^+} \mathbf{1} \otimes \frac{\langle \epsilon \rangle^+}{\hat{\epsilon}} \end{aligned} \quad (64)$$

showing then that  $\mathbf{L} \rightarrow 0$  at high damage  $\mathbf{H} \rightarrow \infty$  in case of positive stress triaxiality  $T_X$  ( $\text{tr} \boldsymbol{\sigma} \geq 0$ ), but not at negative stress triaxiality at which the bulk modulus remains constant in effective Hooke's tensor  $\tilde{\mathbf{E}}$ .

Tangent operator is not symmetric,  $L_{kl ij} \neq L_{ij kl}$ .



### Positivity of intrinsic dissipation

The intrinsic dissipation has for expression  $\mathcal{D} = \rho \frac{\partial \psi^*}{\partial \mathbf{H}} : \dot{\mathbf{H}} = \mathbf{Z} : \dot{\mathbf{H}}$  (Desmorat 2000, 2006; Lemaitre et al 2009) so that

$$\mathcal{D} = \left[ \frac{1}{2G} \boldsymbol{\sigma}' \cdot \mathbf{H} \cdot \boldsymbol{\sigma}' + \frac{1}{27K} \langle \text{tr } \boldsymbol{\sigma} \rangle^2 \mathbf{H} \right] : \dot{\mathbf{H}} = \mathbf{A} : \dot{\mathbf{H}} \geq 0 \quad (65)$$

The term within the brackets is the sum of two positive symmetric second order tensors  $\frac{1}{2G} \boldsymbol{\sigma}' \cdot \mathbf{H} \cdot \boldsymbol{\sigma}'$  and  $\frac{1}{27K} \langle \text{tr } \boldsymbol{\sigma} \rangle^2 \mathbf{H}$ , as  $\mathbf{H}$  is a positive symmetric tensor. The scalar product  $\mathbf{A} : \dot{\mathbf{H}}$  of two positive tensors is positive (see Appendix A) so is the dissipation, proving  $\mathcal{D} \geq 0$  and fulfilling second principle of thermodynamics, and this for any loading, possibly multiaxial, non proportional.

## Numerical implementation and nonlocal regularizations

### Numerical scheme for Finite Element implementation

The time integration procedure for the full anisotropic damage model takes place at Gauss points of a Finite Element code and solves in a coupled manner the constitutive equations of section "Model with modified shear-bulk coupling / Constitutive equations". The strain  $\boldsymbol{\epsilon}_{n+1} = \boldsymbol{\epsilon}(t_{n+1})$  at time  $t_{n+1}$ , the stress triaxiality  $T_{Xn}$ , the damages  $\mathbf{H}_n, \mathbf{D}_n$  at time  $t_n$  are the inputs of the procedure. The outputs are the stresses  $\boldsymbol{\sigma}_{n+1}$  and the damages  $\mathbf{H}_{n+1}, \mathbf{D}_{n+1}$ . In order to integrate the damage model for loading paths more complex than those of section "Responses of proposed anisotropic model" – for instance non proportional loading – proceed as follows:

1. Compute the equivalent strain  $\hat{\epsilon}_{n+1} = \sqrt{\sum \langle \epsilon_{i n+1} \rangle^2}$  from eigenstrains  $\epsilon_{i n+1}$ .
2. Make a test on the criterion function  $f = \hat{\epsilon}_{n+1} - \kappa_n, \kappa_n = \kappa_0 + SR_{\nu n}^s (\text{tr } \mathbf{H}_n - 3)$ .
  - If  $f \leq 0$ , the material behaves elastically, set then  $\mathbf{H}_{n+1} = \mathbf{H}_n, \mathbf{D}_{n+1} = \mathbf{D}_n$ .
  - If  $f > 0$ , the damage must be corrected by using the damage evolution law discretized as  $\Delta \mathbf{H} = \mathbf{H}_{n+1} - \mathbf{H}_n = \Delta \lambda \langle \tilde{\boldsymbol{\epsilon}}_n \rangle^+$  with damage direction evaluated at time step  $t_n, \tilde{\boldsymbol{\epsilon}}_n = \mathbf{E}^{-1} : \boldsymbol{\sigma}_n$  with  $\mathbf{E}$  isotropic undamaged Hooke's tensor. The expression for the damage multiplier increment  $\Delta \lambda$  is

$$\Delta \lambda = \frac{\text{tr } \mathbf{H}_{n+1} - \text{tr } \mathbf{H}_n}{\text{tr} \langle \tilde{\boldsymbol{\epsilon}}_n \rangle^+} \quad \text{where} \quad \text{tr } \mathbf{H}_{n+1} = 3 + \frac{\hat{\epsilon}_{n+1} - \kappa_0}{SR_{\nu n}^s}$$

and the actualization of the damage tensor  $\mathbf{H}$  and  $\mathbf{D}$  is

$$\mathbf{H}_{n+1} = \mathbf{H}_n + \Delta \lambda \langle \tilde{\boldsymbol{\epsilon}}_n \rangle^+ \quad \mathbf{D}_{n+1} = 1 - \mathbf{H}_{n+1}^{-2}$$

3. Compute the stresses at the end of the increment  $\boldsymbol{\sigma}_{n+1}$  using first the elasticity law  $\tilde{\boldsymbol{\sigma}}_{n+1} = \mathbf{E} : \boldsymbol{\epsilon}_{n+1}$ , then by inverting the definition (23) of effective stress,

$$\boldsymbol{\sigma}_{n+1} = \mathbf{H}_{n+1}^{-1} \cdot \tilde{\boldsymbol{\sigma}}_{n+1} \cdot \mathbf{H}_{n+1}^{-1} - \frac{\mathbf{H}_{n+1}^{-2} : \tilde{\boldsymbol{\sigma}}_{n+1}}{\text{tr } \mathbf{H}_{n+1}^{-2}} \mathbf{H}_{n+1}^{-2} + \frac{1}{3} \left[ \frac{3 \langle \text{tr } \tilde{\boldsymbol{\sigma}}_{n+1} \rangle}{\text{tr } \mathbf{H}_{n+1}^2} - \langle -\text{tr } \tilde{\boldsymbol{\sigma}}_{n+1} \rangle \right] \mathbf{1}$$

There is no need of a specific procedure for the numerical control of rupture to bound damage tensor  $\mathbf{D}$  to unity as principal damages  $D_i$  in novel model do not reach 1 at finite strain. For a nonlocal implementation,

simply replace local Mazars strain  $\hat{\varepsilon}_{n+1}$  by its nonlocal averaging  $\hat{\varepsilon}_{n+1}^{\text{nl}}$  at time  $t_{n+1}$  (see next), the test of step 2 being then made on criterion function  $f = \hat{\varepsilon}_{n+1}^{\text{nl}} - \kappa_n$  and the damage multiplier reading

$$\Delta\lambda = \frac{\text{tr} \mathbf{H}_{n+1} - \text{tr} \mathbf{H}_n}{\text{tr}(\tilde{\varepsilon}_n)^*} \quad \text{where now} \quad \text{tr} \mathbf{H}_{n+1} = 3 + \frac{\hat{\varepsilon}_{n+1}^{\text{nl}} - \kappa_0}{SR_{\nu_n}^s}$$

### *Nonlocal, nonlocal with internal time and eikonal nonlocal regularizations*

Spurious mesh dependency is obtained in computations with local damage (softening) models. In dynamics visco-damage framework allows to obtain mesh independency if the damage rate is bounded (Allix & Deü 1997; Allix 2013). See (Gatuingt 2008; Chambart 2009; Desmorat et al 2010a; Leroux 2012) for the extension to anisotropic damage including the modeling of the strain rate effect of concrete material.

In present rate independent model formulation, a nonlocal averaging of the equivalent strain is needed to ensure mesh independency, the criterion function being changed into

$$f = \hat{\varepsilon}^{\text{nl}} - \kappa \leq 0 \quad (66)$$

with  $\hat{\varepsilon}^{\text{nl}}$  nonlocal equivalent strain. Different choices are possible, taking advantage or not of damage anisotropy:

- the use of standard nonlocal integral equivalent strain (Pijaudier-Cabot & Bažant 1987),

$$\hat{\varepsilon}^{\text{nl}}(\mathbf{x}) = \frac{1}{V_r(\mathbf{x})} \int_V \alpha_0 \left( \frac{\|\mathbf{x} - \boldsymbol{\xi}\|}{l_c} \right) \hat{\varepsilon}(\boldsymbol{\xi}) d\boldsymbol{\xi} \quad (67)$$

with normalizing factor

$$V_r(\mathbf{x}) = \int_V \alpha_0 \left( \frac{\|\mathbf{x} - \boldsymbol{\xi}\|}{l_c} \right) d\boldsymbol{\xi} \quad (68)$$

with the integrals performed over the whole structure  $V$ . The quantity  $\hat{\varepsilon}(\boldsymbol{\xi})$  is local value of equivalent strain at points  $\boldsymbol{\xi}$  around position  $\mathbf{x}$  and  $\|\boldsymbol{\xi} - \mathbf{x}\|$  is the geometric distance. Two frequently used weight functions are the Gaussian function  $\alpha_0(\zeta) = e^{-\frac{1}{2}\zeta^2}$  and the bell-shaped polynomial function  $\alpha_0(\zeta) = \langle 1 - \zeta^2 \rangle^2$  (Bažant & Jirásek 2002).

Such an approach considers a constant (isotropic) internal length  $l_c$  whatever the damage level and anisotropy.

- the use of gradient enhanced modeling (Aifantis 1987; Peerlings et al 1996), defining  $\hat{\varepsilon}^{\text{nl}}$  from implicit differential equation

$$\hat{\varepsilon}^{\text{nl}} - l_c^2 \nabla^2 \hat{\varepsilon}^{\text{nl}} = \hat{\varepsilon} \quad (69)$$

introducing an isotropic and constant internal length  $l_c$  with the same remark on isotropy and constancy of internal length than for standard nonlocal integral.

- the use of an internal time  $\tau_c$  as material parameter – instead of characteristic length  $l_c$  – in nonlocal integral averaging (Desmorat and Gatuingt 2007; Desmorat and Gatuingt 2010)

$$\hat{\varepsilon}^{\text{nl}}(\mathbf{x}) = \frac{1}{V_r(\mathbf{x})} \int_V \alpha_0 \left( \frac{\tau_{x\xi}}{\tau_c} \right) \hat{\varepsilon}(\boldsymbol{\xi}) d\boldsymbol{\xi} \quad V_r(\mathbf{x}) = \int_V \alpha_0 \left( \frac{\tau_{x\xi}}{\tau_c} \right) d\boldsymbol{\xi} \quad (70)$$

with  $\tau_{x\xi}$  the information time for an elastic wave to propagate from points  $\mathbf{x}$  to  $\xi$ . This leads to an evolving internal length, as suggested by Geers et al (1998), Pijaudier-Cabot et al (2004) and Simone et al (2004), but here damage induced and anisotropic. The approach makes a highly damage zone equivalent to a crack. The main drawback of the nonlocal integral with internal time framework is its prohibitive numerical cost in structural cases.

- the use of eikonal nonlocal integral framework (Desmorat et al 2015),

$$\varepsilon^{\text{nl}}(\mathbf{x}) = \frac{1}{V_r(\mathbf{x})} \int_V \alpha_0 \left( \frac{\tilde{\ell}_{x\xi}}{l_c} \right) \hat{\varepsilon}(\xi) d\xi, \quad V_r(\mathbf{x}) = \int_V \alpha_0 \left( \frac{\tilde{\ell}_{x\xi}}{l_c} \right) d\xi \quad (71)$$

where effective distances  $\tilde{\ell} = \tilde{\ell}_{x\xi}$  between points  $\mathbf{x}$  and  $\xi$  are solution of anisotropic eikonal equation

$$\nabla \tilde{\ell} \cdot \mathbf{H}^{-2} \cdot \nabla \tilde{\ell} = \nabla \tilde{\ell} \cdot (\mathbf{1} - \mathbf{D}) \cdot \nabla \tilde{\ell} = 1 \quad (72)$$

An anisotropic evolving internal length is obtained, dependent on both the damage level and anisotropy. A highly damaged zone is found equivalent to a crack. In 1D the eikonal nonlocal integral framework is equivalent to the nonlocal with internal time theory.

- the use of eikonal gradient enhancement (Desmorat et al 2015),

$$\hat{\varepsilon}^{\text{nl}} - \frac{1}{2} \frac{l_c^2}{\det \mathbf{H}} \nabla \cdot ((\det \mathbf{H}) \mathbf{H}^{-2} \cdot \nabla \hat{\varepsilon}^{\text{nl}}) = \hat{\varepsilon} \quad (73)$$

Such an approach behaves as Geers et al (1998) gradient enhancement but with an anisotropic evolving internal length, dependent on both the damage level and anisotropy. For an uniaxial state of damage  $\mathbf{H} = \text{diag}[H_1, 1, 1] = \text{diag}[1/\sqrt{1-D_1}, 1, 1]$ , setting  $x$  the normal to nucleated crack, Eq. (73) gives

$$\hat{\varepsilon}^{\text{nl}} - \frac{1}{2} l_c^2 \sqrt{1-D_1} \frac{\partial}{\partial x} \left( \sqrt{1-D_1} \frac{\partial \hat{\varepsilon}^{\text{nl}}}{\partial x} \right) = \hat{\varepsilon} \quad (74)$$

Both eikonal nonlocal integral and eikonal gradient enhancement end up

- to a local behavior normally to the nucleated crack surface at  $D_1 \rightarrow 1$ ,
- and to a non local behavior parallel to the nucleated crack, as sought by Pijaudier-Cabot et al (2007).

One does not enter here in the debate nonlocal integral *versus* gradient enhancement, both choices being most often possible. It worth it to recall the well known (non physical) feature of a highly damaged zone spreading zone in nonlocal averaging. An evolving internal length (such as from eikonal nonlocal integral, refer to Desmorat et al (2015) work) allows for present anisotropic damage modeling to localize the nucleated crack at a single point, where  $H_1 \rightarrow 0$ ,  $D_1 \rightarrow 1$ .

## Conclusion

An anisotropic damage model has been proposed for concrete materials, using single Ladevèze variable  $\mathbf{H} = (\mathbf{1} - \mathbf{D})^{-\frac{1}{2}}$  for both coupling of deviatoric and hydrostatic elastic properties with damage. The shear-bulk coupling obtained at positive stress triaxiality is found in accordance with numerical Discrete Element

results when bulk modulus is not affected by damage in compressive stress states. The model biaxial responses have been derived in a closed form parametric representation in principal stresses space and advantageously compared to initial anisotropic damage model responses. A key feature of the modeling is proposed evolution law  $\dot{\mathbf{H}} = \dot{\lambda} \langle \tilde{\varepsilon} \rangle^+$  of damage anisotropy governed by the extensions, keeping second order damage tensor  $\mathbf{H}$  unbounded and allowing for the sought property of gradual stress softening. Lemaitre stress triaxiality function has been introduced to enhance the performance of Mazars criterion therefore the performance of the full anisotropic damage model in bicompression. Related thermodynamics aspects have been discussed and both proof of convexity of thermodynamics potential with respect to stress  $\sigma$  and damage  $\mathbf{H}$  as well as proof of the positivity of the intrinsic dissipation have been given.

The anisotropic damage models have been presented first in their local form. Different nonlocal enhancements have finally been described, they simply replace the expression of local equivalent strain  $\tilde{\varepsilon}$  within Mazars criterion by its nonlocal averaging  $\tilde{\varepsilon}^{\text{nl}}$ , either integral or of gradient type, with a possibly evolving internal length, made anisotropic because of tensorial nature of damage.

The modeling of permanent strains and of dilatancy in confined stress states is left to further work; suggested improvements in considered anisotropic damage framework, not using plasticity theory for instance, can be found in (Herrmann and Kestin 1988; La Borderie 1991, 2003; Desmorat 2004; Lebon 2011). Extension to cyclic loading is straightforward from the concept of active damage (Souid et al 2009; Desmorat et al 2010a).

## Appendix A – Positive scalar product of two positive symmetric tensors

Consider two positive symmetric second order tensors  $\mathbf{A}$  and  $\dot{\mathbf{H}}$ . Spectral decomposition of  $\mathbf{A}$  gives

$$\mathbf{A} = \sum_i A_i \mathbf{a}^i \otimes \mathbf{a}^i \quad A_i \geq 0 \quad (75)$$

with eigenvalues  $A_i$  and eigenvectors  $\mathbf{a}^i$ . Scalar product

$$\mathbf{A} : \dot{\mathbf{H}} = \sum_i A_i \mathbf{a}^i \cdot \dot{\mathbf{H}} \cdot \mathbf{a}^i \geq 0 \quad (76)$$

is positive, as sum of positive terms.

## Appendix B – Functions $B_i$

$$B_i = B'_i + B_H = B_i(\theta, D_i)$$

$$\begin{aligned} B'_1 &= \frac{1+\nu}{9} \left( \frac{4}{1-D_1} + \frac{1}{1-D_2} + \frac{1}{1-D_3} \right) \cos \theta \\ &\quad - \frac{1+\nu}{9} \left( \frac{2}{1-D_1} + \frac{2}{1-D_2} - \frac{1}{1-D_3} \right) \sin \theta \\ B'_2 &= - \frac{1+\nu}{9} \left( \frac{2}{1-D_1} + \frac{2}{1-D_2} - \frac{1}{1-D_3} \right) \cos \theta \\ &\quad + \frac{1+\nu}{9} \left( \frac{1}{1-D_1} + \frac{4}{1-D_2} + \frac{1}{1-D_3} \right) \sin \theta \end{aligned}$$

$$\begin{aligned}
B'_3 &= -\frac{1+\nu}{9} \left( \frac{2}{1-D_1} - \frac{1}{1-D_2} + \frac{2}{1-D_3} \right) \cos \theta \\
&\quad + \frac{1+\nu}{9} \left( \frac{1}{1-D_1} - \frac{2}{1-D_2} - \frac{2}{1-D_3} \right) \sin \theta \\
B_H &= \frac{1-2\nu}{3-\eta(D_1+D_2+D_3)} \langle \cos \theta + \sin \theta \rangle - \frac{1-2\nu}{3} \langle -\cos \theta - \sin \theta \rangle
\end{aligned}$$

### Appendix C – Functions $C_i$

$$\begin{aligned}
C_i &= C'_i + C_H = C_i(\theta, H_i) \\
C'_1 &= \frac{1+\nu}{9} (4H_1^2 + H_2^2 + H_3^2) \cos \theta - \frac{1+\nu}{9} (2H_1^2 + 2H_2^2 - H_3^2) \sin \theta \\
C'_2 &= -\frac{1+\nu}{9} (2H_1^2 + 2H_2^2 - H_3^2) \cos \theta + \frac{1+\nu}{9} (H_1^2 + 4H_2^2 + H_3^2) \sin \theta \\
C'_3 &= -\frac{1+\nu}{9} (2H_1^2 - H_2^2 + 2H_3^2) \cos \theta + \frac{1+\nu}{9} (H_1^2 - 2H_2^2 - 2H_3^2) \sin \theta \\
C_H &= \frac{1-2\nu}{9} (H_1^{2\eta} + H_2^{2\eta} + H_3^{2\eta}) \langle \cos \theta + \sin \theta \rangle - \frac{1-2\nu}{3} \langle -\cos \theta - \sin \theta \rangle
\end{aligned}$$

### Appendix D – Triaxiality functions $R_\nu$

The strain energy density of virgin (undamaged) material is

$$w_e = \frac{1}{2} \boldsymbol{\epsilon} : \mathbf{E} : \boldsymbol{\epsilon} = \frac{\sigma_{eq}^2 R_\nu}{2E}$$

where the concept of triaxiality function  $R_\nu$  has been introduced (Lemaitre 1992), normed to 1 in tension,

$$R_\nu = \frac{2}{3}(1+\nu) + 3(1-2\nu)T_X^2 \quad T_X = \frac{\sigma_H}{\sigma_{eq}} = \frac{1}{3} \frac{\text{tr } \boldsymbol{\sigma}}{(\frac{3}{2} \boldsymbol{\sigma}' : \boldsymbol{\sigma}')^{\frac{1}{2}}}$$

In present work we use a similar definition but for negative triaxiality only and normed to 1 in pure shear ( $R_\nu = 1$  at  $T_X \geq 0$ ),

$$R_\nu = 1 + \frac{9}{2} \frac{1-2\nu}{1+\nu} \langle -T_X \rangle^2$$

and we bound it to  $B$ .

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