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# Title : Building up a "flat" wormhole by isometric embedding

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## Abstract

Wormholes have drawn the attention of science fiction writers, but do wormholes exist in the real world ? It is a relevant question, but unfortunately, general relativity answers negatively. It is indeed admitted that these entities are unstable and collapse instantly once they form. Significant efforts have been made these last years to remedy this situation, especially by using various extensions of general relativity; however, the results are not very convincing. Yet, a positive response could have important applications in the domain of astrophysics, especially in the field of galaxies and stars. We invoke here some novel perspectives for studying these elusive entities. In this situation a wormhole is seen as an entity with non-existing curvature, i.e. a kind of "flat" manifold.

Keywords: isometric embedding, wormhole, galaxy, star,

## 1 Introduction

In 1935, Albert Einstein and Nathan Rosen, both at the Institute for Advanced Study in Princeton, were investigating the possibility of a mixed theory of matter, gravity and electromagnetism deprived of punctual singularities (Einstein & Rosen, 1935). They started with the equations for a spherically symmetric mass distribution, leading to the Schwarzschild solution. By performing a coordinate transformation to remove the region containing the curvature singularity, they obtain a new mathematical representation, composed of two asymptotically flat sheets connected by a throat (Fig. 1).

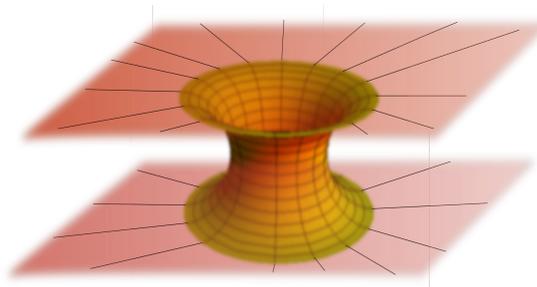


Figure 1 A stylized wormhole in space-time

More generally, this study would seem to suggest the possible existence of topological features bringing two distant points close together through the space-time fabric. These "shortcuts" came to be called Einstein-Rosen bridges, or wormholes. Unfortunately, it has been shown that the Einstein-Rosen bridge metric is not a solution of the vacuum Einstein equations, but it requires the presence of a nonzero energy-momentum tensor source that is divergent and violates the energy conditions at the throat of the wormhole.

At this stage, the wormholes are definitively unstable objects. Several pathways have been studied for many years to circumvent this deadlock imposed by the principles of general relativity. The basic idea was to add new ingredients to the recipe in order to equilibrate the gravitational field. Initially, these new ingredients

were a couple of scalar and vector fields (Ellis, 1973; Bronnikov, 1973), or the presence of some form of exotic matter (Morris & Thorne, 1988). Thereafter, the sophistication of the models significantly increased with various proposals : higher dimensional extensions of general relativity (Baruah & Deshamukhya, 2019; Sengupta et al, 2022), hybrid metric-Palatini gravity (Rosa, 2021), physics beyond the Standard Model (Maldacena & Milekhin, 2021), etc.

On the other hand, claims for the reality of stable wormholes also exist in the observational domain (Kirillov & Savelova, 2011; Dzhunushaliev et al, 2011; Dai & Stojkovic, 2020). The model, developed by Kirillov & Savelova (2011), is based on the fact that in its very early (quantum) stage, the Universe should have a foam-like topological structure (Hawking, 1978). The inflationary stage (Linde, 1982) should enormously stretch the characteristic scales of the relic foam. Its relics might well survive the cosmological expansion, thus creating a gas of wormholes randomly distributed in the Universe. Kirillov & Savelova then assume that this gas could offer a new picture of DM phenomena. Another very different application of wormholes has been suggested by Dzhunushaliev et al (2011). In this case a new configuration for the stars is considered, whose main ingredient is a macroscopic wormhole filled with a perfect fluid. Instead of a Universe composed of individual stars, we observe couples of stars, connected by a wormhole, one of the two mouths of this wormhole being located at the center of each star forming the couple. A third exploitation for wormholes in the Universe is related to the galaxy centers, where a very massive black hole is assumed to reside. Dai and Stojkovic (2020) assume that if a traversable wormhole smoothly connects two separate regions of spacetime, then the flux cannot be separately conserved in any of these regions, individually. Namely, by studying the orbits of stars around the black hole at the center of the Milky Way, we could know if this black hole harbors a traversable wormhole. However a clear link between wormholes and astrophysical observations has to be researched. There is also another thorny problem raised by all of these theories above mentioned. We need to ensure that stable wormholes indeed exist, that is far from being achieved.

Thus, in search of a stable wormhole, the aim of the current paper is to propose a very different approach of the topic. It is well known that a geometric 2D-torus has a variable non-null Ricci curvature. However a few years ago a group of mathematicians (Borrelli et al, 2012) succeeded to graphically depicting an object that sounds like a kind of "flat" geometric 2D-torus. We raise here the question of whether we can apply a similar procedure to a wormhole. In this case, we could possibly build a kind of "flat" wormhole.

## 2 The zero-curvature 2-torus

For a geometric 2D-torus, the line element  $ds^2$  is

$$ds^2 = (c + a \cos v)^2 du^2 + a^2 dv^2 \tag{1}$$

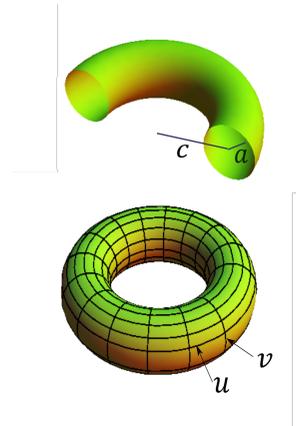


Figure 2 a 2D-torus

Then, from the metric, the calculation of the non-null components of the Riemann tensor is straightforward. We have

$$R_{vvv}^u = \frac{a \cos v}{c + a \cos v} \quad (2)$$

$$R_{uvv}^v = -\frac{1}{a} \cos v (c + a \cos v) \quad (3)$$

the other components being identically null. Then the Ricci curvature<sup>1</sup> is

$$R = \frac{2}{a} \frac{\cos v}{c + a \cos v} \quad (4)$$

We can see that the curvature is positive on the outside of the torus ( $v \in [0, \pi[$ ), and negative on the inside ( $v \in ]\pi, 2\pi[$ ). Thus, the geometric 2D-torus has a variable non-null curvature, contrarily to the cylinder characterized by a zero-curvature.

On the other hand, a topological 2D-torus is defined as being homeomorphic to the product of two circles. A circle is itself homeomorphic to the unitary group of complex numbers having a modulus 1, labeled  $U(1)$ . Then a topological 2D-torus is homeomorphic to the connected and compact topological group  $U(1) \times U(1)$ . In a more graphic way, a topological torus is still obtained by identifying the opposite faces of a square, without twisting. A topological 2D-torus is deprived of metric and thus deprived of curvature because curvature is a geometric property, and not a topological property. However in the following we are dealing with the geometric characteristics of the 2D-torus, and not with the topological one.

Figure 3 displays how a geometric 2D-torus acquires its non-null Ricci curvature. Even though the Ricci curvature is an intrinsic property, we can always embed a 2D-surface in the "ambient" 3D-space (in the operation the Ricci curvature is left unchanged). This embedding permits direct visualization of the different steps of the building process of a geometric 2D-torus. We start with a simple rectangle. The rectangle is a piece of the Euclidean plane with an identically null Ricci curvature. Starting from a rectangle, we can easily form a cylinder by rolling it up (step 1). Again, the cylinder has an identically null Ricci curvature. It is under the second operation (step 2) that the curvature appears, when we try to join the two extremities of the cylinder. In this case, the generating lines are stretched on one side, and simultaneously, the generating lines are shrunk on the opposite side. Then by an embedding with  $C^2$  regularity, we can form a 2D-torus but, unfortunately, the isometry is lost. In other words, we cannot build a geometric 2D-torus without distorting the lengths or angles drawn on the cylinder.

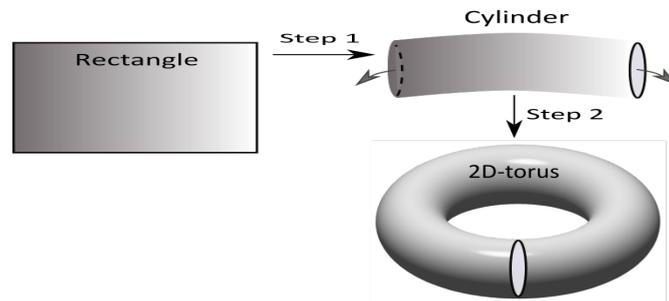


Figure 3 Building up a 2D-torus

However, is there a possibility of building, in a practical manner, a geometric 2D-torus with "zero Ricci curvature" at every point? In light of what we have just said, the question is obviously preposterous, but nevertheless, in some sense, we can give it a positive answer. However a price has to be paid: we must admit that the isometric embedding is no longer with  $C^2$  regularity, but restricted just to  $C^1$  regularity. Let us note that the property "zero Ricci curvature" is a manner of speaking. Because *stricto sensu* the curvature no longer exists for such embedding with  $C^1$  regularity. In the following, quotation marks have to be understood in this sense. The basic idea of the methodology comes from the Hevea Project. The aim of the Hevea Project is to provide an implementation of the Convex Integration Theory in order to visualize

<sup>1</sup>The Ricci curvature is an intrinsic property of a manifold. In 2D the curvature tensor eventually reduces to a single number (the Ricci scalar  $R$ ). The latter quantity is then identical to the extrinsic Gaussian curvature  $K$ , by analogy with the theory of surfaces developed by Carl Friedrich Gauss in the early nineteenth century (more exactly  $K = \frac{1}{2}R$ ).

isometric embeddings of "flat" tori in the three-dimensional Euclidean space (Borrelli et al, 2012). This project follows the works of John Nash and Nicolaas Kuiper developed in the mid-1950s. While examining the isometric embedding problem in four dimensional space, Nash and Kuiper found a very strange result: the obstruction to the existence of such embeddings (i.e. the curvature) can be bypassed (Nash, 1954; Kuiper, 1955). In the 1970s and 1980s, following this way and the results on immersions by Morris Hirsch and Stephen Smale, Misha Gromov puts into light the concept of homotopy principle in various formulations and, at the same time, develops a technique, called convex integration, which dramatically systematizes and generalizes the Nash and Kuiper construction process of isometric embeddings (for a list of references, see for instance, Gromov, 2017). Noticing that convex integration is basically algorithmic in nature, a practical step has been eventually achieved by Borrelli et al (2012), which has led to the production of very amazing pictures of isometric embeddings of "flat" tori in tridimensional space. The pictures reveal some kind of self-similarity in an infinite succession of corrugations (Fig. 3). However, this very interesting method has been insufficiently exploited, and it seems that the procedure has not been extended to other bidimensional or three-dimensional manifolds.

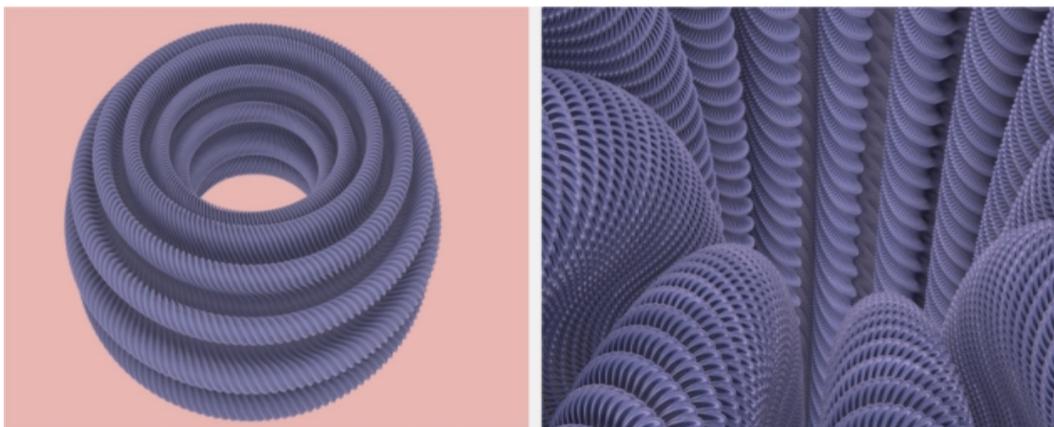


Figure 4 Exterior and interior views of a 2D-torus of revolution after four corrugations (according to Borrelli et al, 2012)

### 3 A potential 2D-wormhole with an "identically null Ricci curvature"

If "flat" tori exist, is it possible that "flat" wormholes also exist ? A smooth 2D-wormhole can easily be constructed from pieces of a 2D-torus. On Figure 4 we can see how three concatenated pieces of a 2D-torus can join together in a  $C^2$ -continuous way, at every point, to form a 2-wormhole. This operation is possible because the piecewise surfaces are very smooth. Can we make the same operation with a corrugated 2D-torus? The question is not trivial because the smoothness of the complete surface has to be realized at every point across the two boundaries.

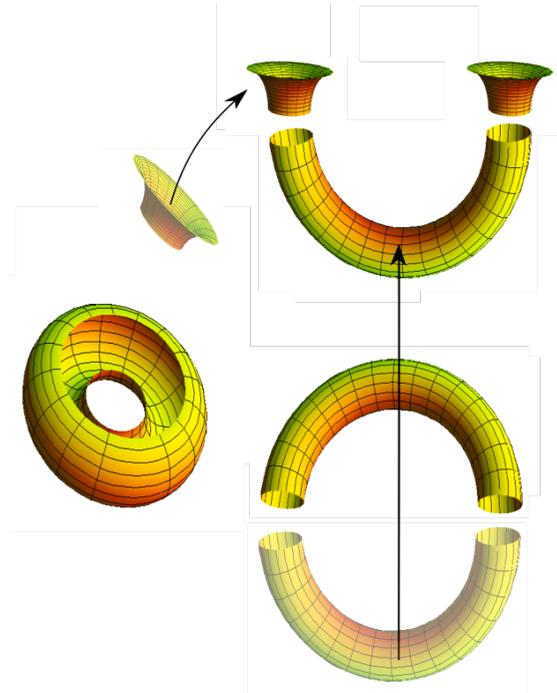


Figure 5 Decomposition of a bidimensional wormhole into three pieces of a bidimensional torus

Now, the three steps of the next (and very ambitious) study could be : i. to prove the existence of a 3D-torus with "zero Ricci curvature" at every point, ii. to prove the existence of a 3D-wormhole with "zero Ricci curvature" at every point, and eventually iii. to prove the existence of a real wormhole with "zero-Ricci curvature" in space-time. The way seems to be direct and without difficulties, but the appearance in mathematics can be deceptive and, each proof needs to be supported by a reliable demonstration.

## 4 Conclusion

This very short note aims at suggesting a new path of reflection on the possibility of the existence of real (i.e. stable) wormholes in space-time. We are still far from achieving a full proof, but the simple conjecture we have presented in this paper, may be, deserves some future consideration. Eventually, even if we can build it, there is no guarantee that a corrugated wormhole is traversable by a macroscopic object. However, may be, it is traversable by electromagnetic flux or possibly by elementary particles (electrons) and thus, very interestingly, by information.

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