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# STRENGTHENED INJECTIVITY RADIUS BOUNDS FOR MANIFOLDS WITH POSITIVE SCALAR CURVATURE

THOMAS RICHARD

ABSTRACT. Green's inequality shows that a compact Riemannian manifold with scalar curvature at least  $n(n-1)$  has injectivity radius at most  $\pi$ , and that equality is achieved only for the radius 1 sphere. In this work we show how extra topological assumptions can lead to stronger upper bounds. The topologies we consider are  $\mathbb{S}^2 \times \mathbb{T}^{n-k-2} \times \mathbb{R}^k$  for  $n \leq 7$  and  $0 \leq k \leq 2$  and 3-manifolds with positive scalar curvature except lens spaces  $L(p, q)$  with  $p$  odd. We also prove a strengthened inequality for 3-manifolds with positive scalar curvature and large diameter. Our proof uses previous results of Gromov and Zhu.

In stark contrast with positive lower bounds on the sectional or Ricci curvature, positive scalar curvature doesn't provide any control on the volume or diameter of a Riemannian manifold  $(M^n, g)$  of dimension at least 3 as is already clear by considering product metrics on  $\mathbb{S}^2 \times \mathbb{T}^1$ . However in 1963, Leon Green proved the following result:

**Theorem 0.1.** [Gre63] *Let  $(M^n, g)$  be a complete manifold with finite volume and Ricci curvature bounded from below. Assume*

$$\bar{s} = \frac{1}{\text{vol}(M^n, g)} \int_M \text{scal}_g \, dv_g > 0.$$

*Then the conjugacy radius of  $(M^n, g)$  satisfies*

$$\text{conj}(M^n, g) \leq \pi \sqrt{\frac{n(n-1)}{\bar{s}}}.$$

*Moreover, if equality is achieved then  $(M^n, g)$  has constant sectional curvature metric.*

Since the injectivity radius is smaller than the conjugate radius and the round sphere is the only spherical space form with injectivity radius  $\pi$ , a straightforward corollary is:

**Corollary 0.2.** *If a compact manifold  $(M^n, g)$  has  $\text{scal}_g \geq n(n-1)$ , then  $\text{inj}_g \leq \pi$ . Moreover if equality is achieved then  $(M^n, g)$  is isometric to the round sphere.*

As with other sharp geometric inequalities, it is natural to ask if some form of stability occurs:

**Question 0.3.** *If a compact  $(M^n, g)$  has  $\text{scal}_g \geq n(n-1)$  and injectivity radius close to  $\pi$ , is it close to the round sphere in some sense ?*

A more precise (and less ambitious) version of this question is to look for a purely topological answer:

**Question 0.4.** *If a compact  $(M^n, g)$  has  $\text{scal}_g \geq n(n-1)$  and injectivity radius close to  $\pi$ , is it homeomorphic to the sphere ?*

Some progress has been made in this direction by Tuschmann and Wiemeler in [TW19] under additional Ricci curvature and volume bounds.

To get a better feel of the problem we compile here the scalar curvature and injectivity radius of some classical examples built from rank one symmetric spaces and products in Table 1, together with what the injectivity radius would be after rescaling to have the same scalar curvature as the sphere.

Manifold $(M^n, g)$	$\text{scal}_g$	$\text{inj}_g$	$\sqrt{\frac{\text{scal}_g}{n(n-1)}} \text{inj}_g$
$\mathbb{S}^n$	$n(n-1)$	$\pi$	$\pi$
$\mathbb{S}^{n-1} \times \mathbb{T}^1$	$(n-1)(n-2)$	$\pi$	$\sqrt{\frac{n-2}{n}} \pi$
$\mathbb{S}^{n-2} \times \mathbb{S}^2, n \geq 4$	$n^2 - 5n + 10$	$\pi$	$\sqrt{\frac{n^2-5n+10}{n(n-1)}} \pi$
$\mathbb{RP}^n$	$n(n-1)$	$\pi/2$	$\frac{\pi}{2}$
$\mathbb{S}^2 \times \mathbb{T}^{n-2}$	2	$\pi$	$\sqrt{\frac{2}{n(n-1)}} \pi$
$(\mathbb{S}^2)^{n/2}, n$ even	$2n$	$\pi$	$\sqrt{\frac{2}{n-1}} \pi$
$\mathbb{CP}^{n/2}, n$ even	$n(n+2)$	$\pi/2$	$\sqrt{\frac{n+2}{n-1}} \frac{\pi}{2}$
$\mathbb{HP}^{n/4}, n \equiv 0 [4]$	$n(n+8)$	$\pi/2$	$\sqrt{\frac{n+8}{n-1}} \frac{\pi}{2}$
$\mathbb{OP}^2, n = 16$	576	$\pi/2$	$\sqrt{\frac{12}{5}} \frac{\pi}{2}$

TABLE 1. Injectivity radius of some classical examples of  $n$ -manifolds.

Of all the examples we have tested,  $\mathbb{S}^{n-1} \times \mathbb{T}^1$  always achieves the biggest injectivity radius for a given scalar curvature bound for a given  $n$ , though in dimension 4 it is tied with the Fubini-Study metric on  $\mathbb{CP}^2$ . For instance in dimension 3, this gives an example of manifold non homeomorphic to  $\mathbb{S}^2 \times \mathbb{T}^1$  with a product metric of scalar curvature 6 has injectivity radius  $\pi/\sqrt{3}$ .

Another way to look at this problem is to ask to ask for a strengthening of Green's inequality in a given non trivial topological context, namely:

**Question 0.5.** *Given a smooth  $n$ -manifold  $M^n$  which is not homeomorphic to  $\mathbb{S}^n$  and a metric  $g$  on  $M^n$  with  $\text{scal}_g \geq n(n-1)$ , can one find  $\iota = \iota(M^n) < \pi$  such that  $\text{inj}_g \leq \iota$  ?*

To the knowledge of the author, the only known result of this kind follows from [BBEN10], though the authors didn't phrase it in this way, where it is shown that if  $(\mathbb{RP}^3, g)$  has  $\text{scal}_g \geq 6$  then its 1-systole is at most  $\pi$  and hence its injectivity radius is at most  $\pi/2$ . Gromov's [Gro23b] also contains related conjecture and partial results. We here prove some further results in this direction.

In dimension 3 we get:

**Theorem 0.6.** *Let  $(M^3, g)$  be a compact 3-manifold with  $\text{scal}_g \geq 6$  and  $\text{inj}_g > \frac{2\pi}{3}$ , then  $M^3$  is diffeomorphic to a lens space  $L(p, q)$  with  $p$  odd.*

Recall that the lens space  $L(p, q)$  (defined for coprime integers  $p$  and  $q$ ) is the quotient of  $\mathbb{S}^3 \subset \mathbb{C}^2$  by the action of  $\mathbb{Z}/p\mathbb{Z}$  generated by the diffeomorphism:

$$(0.1) \quad (z, w) \mapsto \left( e^{i\frac{2\pi}{p}} z, e^{i\frac{2\pi q}{p}} w \right).$$

The occurrence of the odd lens spaces in the conclusion of the theorem is probably not necessary and one could hope to conclude that  $M^n$  is actually diffeomorphic to  $\mathbb{S}^3$ .

Compactness of  $M$  is actually not necessary since we will also show:

**Theorem 0.7.** *Let  $(M^3, g)$  be a complete non-compact 3-manifold with  $\text{scal}_g \geq 6$ , then  $\text{inj}_g \leq \frac{2\pi}{3}$ .*

Note that since no volume assumption is made, Green's upper bound of  $\pi$  does not apply here. However Gromov indicates in [Gro23b] that a (possibly huge) upper bound can be derived for complete  $n$ -manifolds with uniformly positive scalar curvature with  $n \leq 5$  by quantifying the contractibility argument in [CL23].

The proof of Theorem 0.7 can be adapted to yield an improvement of Green's bound for manifolds with big diameter. For convenience we set:

$$\begin{aligned} \bar{D} : (2\pi/3, \pi) &\rightarrow \mathbb{R} \\ r &\mapsto 2r + \frac{2\pi}{3\sqrt{1 - \frac{4\pi^2}{9r^2}}}. \end{aligned}$$

and include a plot of  $\bar{D}$  as Figure 1.

**Theorem 0.8.** *Let  $(M^3, g)$  be a compact Riemannian 3-manifold with  $\text{scal}_g \geq 6$  and let  $r \in (\frac{2\pi}{3}, \pi)$ . If:*

$$\text{diam}(M^3, g) > \bar{D}(r),$$

then:

$$\text{inj}_g \leq r.$$

Since

$$\bar{D}(r) \xrightarrow{r \rightarrow 2\pi/3} +\infty,$$

one consequence of this result is that a manifold  $(M^3, g)$  with  $\text{scal}_g \geq 6$  and very big diameter cannot have injectivity radius much bigger than  $\frac{2\pi}{3}$ . On the other hand the smallest diameter for which the theorem applies can be found by minimizing  $\bar{D}$  on  $(2\pi/3, \pi)$ . Unfortunately studying the sign of  $\bar{D}'$  leads to a cubic equation in  $r$  whose solutions are not pretty. However this minimum of  $\bar{D}$  is close to  $r = \frac{5\pi}{6}$  for which we get  $\bar{D}(r) = \frac{25\pi}{9}$ . Hence we have that if  $(M^3, g)$  has  $\text{scal}_g \geq 6$  and  $\text{diam}(M^3, g) > \frac{25\pi}{9} \simeq 2.777\pi$  then  $\text{inj}_g \leq \frac{5\pi}{6} \simeq 0.833\pi$ . Numerically  $\bar{D}$  attains its minimum of approximately  $2.775\pi$  at  $r \simeq 0.851\pi$ .

Note that given our previous result, this last theorem is only interesting if  $M^3$  is an odd-order lens space, otherwise Theorem 0.6 gives a better estimate.

In dimension less than 7 we obtain:

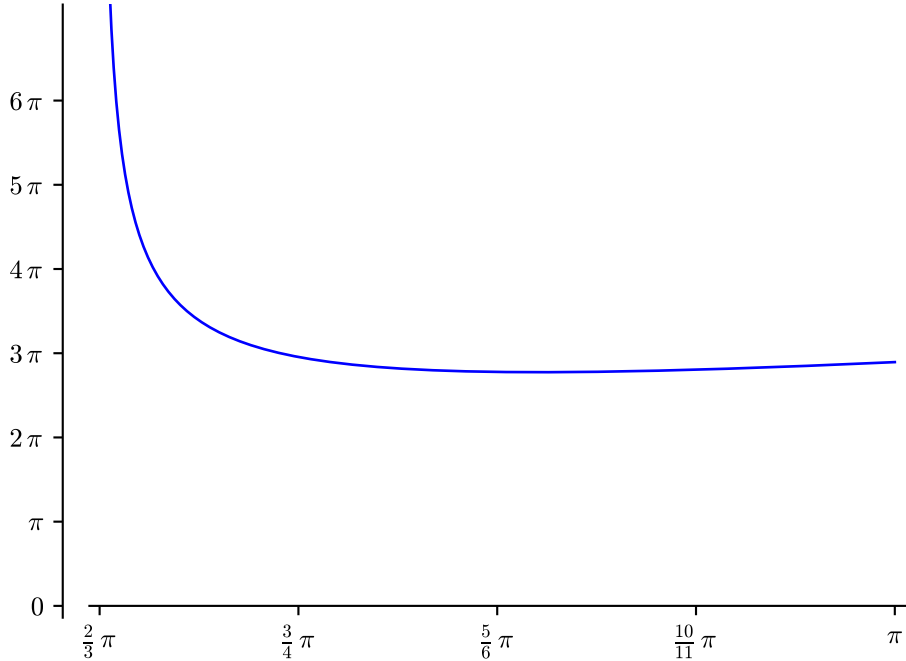


FIGURE 1. Plot of  $\bar{D}$  on  $(2\pi/3, \pi)$ .

**Theorem 0.9.** *Let  $3 \leq n \leq 7$ ,  $k \in \{0, 1, 2\}$  and let  $M^n$  be a manifold such that there exists a proper map  $F : M^n \rightarrow \mathbb{S}^2 \times \mathbb{T}^{n-k-2} \times \mathbb{R}^k$ , of non-vanishing degree. Let  $g$  be a complete metric on  $M^n$  with  $\text{scal}_g \geq n(n-1)$ , then  $\text{inj}_g \leq \frac{2\pi}{n}$ .*

For  $k = 0$ , this result can be found in [Gro23b, 1.1 (b)] at least if  $M^n$  is assumed to be homeomorphic to a product  $\mathbb{S}^2 \times \mathbb{T}^{n-2}$ , however the proof is only sketched so we include it here.

When  $k$  is not 0, we say that  $F : M^n \rightarrow \mathbb{S}^2 \times \mathbb{T}^{n-k-2} \times \mathbb{R}^k$  has non-vanishing degree if the induced map on compactly supported  $n$ -dimensional de Rham cohomology  $F^* : H_c^n(\mathbb{S}^2 \times \mathbb{T}^{n-k-2} \times \mathbb{R}^k) \rightarrow H_c^n(M^n)$  is not zero.

Both proofs use a Bonnet-Myers type inequality by Gromov for  $\mathbb{T}^{n-2}$ -invariant positive scalar curvature metrics on  $\mathbb{S}^2 \times \mathbb{T}^{n-2}$ . In dimension 3, this inequality is used to give a diameter estimate for stable immersed minimal spheres. In higher dimension, we use a construction by Zhu [Zhu20]. Our quantitative result (Theorem 0.8) and our results for complete non compact manifolds (Theorem 0.7 and Theorem 0.9 for  $k \neq 0$ ) rely on the use of Gromov's  $\mu$ -bubbles (see Theorem 1.2).

It is at present unclear if this  $\frac{2\pi}{n}$  bound is optimal, it obtained by the diameter estimate from Theorem 2.2. One could believe that suitably rescaled product metrics on  $\mathbb{S}^2 \times \mathbb{T}^{n-2}$  are optimal: these have injectivity radius  $\sqrt{\frac{2}{n(n-1)}}\pi$ . However we show in our appendix that product metrics are not optimal for Gromov's Bonnet-Myers type inequality, which shows that our method of proof cannot be used to yield optimality of products if it holds, we

also explain in the appendix why this doesn't rule out optimality of products for Theorem 0.9.

We now outline the rest of the paper. In section 1 we state the symmetrization results by Gromov and Zhu that we will need, in section 2 we spend some time discussing Gromov's toroidal band inequality and Bonnet-Myers type inequality, in section 3 we prove Theorem 0.9 for  $k = 0$ , in section 4 we prove Theorem 0.6. We then treat the case of 3-manifolds with big diameter in section 5 and the case of open 3-manifolds in section 5 section 6. Theorem 0.9 for  $k \neq 0$  is treated in section 7. The non optimality of products for Gromov's Theorem 2.2 is proved in Appendix A.

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## 1. GROMOV'S AND ZHU'S SYMMETRIZATIONS

We state two symmetrization results based on the stable minimal hypersurface method. The first is Zhu's construction, which is an elegant application of Fischer-Colbrie-Schoen symmetrization to  $\mathbb{S}^2 \times \mathbb{T}^{n-2}$ .

**Theorem 1.1.** [Zhu20, Prop. 2.2] *Let  $3 \leq n \leq 7$  and let  $M^n$  such that there exists a non-zero degree map  $F : M^n \rightarrow \mathbb{S}^2 \times \mathbb{T}^{n-2}$ . Let  $g$  be a riemannian metric on  $M^n$  such that  $\text{scal}_g \geq n(n-1)$ , then there exists a genus 0 surface  $\Sigma^2 \subset M^n$  such that:*

- $\int_{\Sigma} F^* \sigma \neq 0$  where  $\sigma$  is the fundamental cohomology class of  $\mathbb{S}^2 \subset \mathbb{S}^2 \times \mathbb{T}^{n-2}$ .
- there are  $n-2$  smooth positive functions  $f_1, \dots, f_{n-2} : \Sigma^2 \rightarrow \mathbb{R}$  such that  $\Sigma^2 \times \mathbb{T}^{n-2}$  endowed with the metric  $\tilde{g} = g|_{\Sigma} + \sum_{i=1}^{n-2} f_i^2 d\theta_i^2$  satisfies  $\text{scal}_{\tilde{g}} \geq n(n-1)$ .

$\Sigma$  is such that there exists nested  $k$ -dimensional submanifolds  $M_k$  for  $2 \leq k \leq n$  with:

- $\Sigma^2 = M_2 \subset M_3 \subset \dots \subset M_n = M$ ,
- $M_{k-1} \times \mathbb{T}^{n-k}$  is a stable minimal surface in  $(M_k \times \mathbb{T}^{n-k}, g|_{M_k} + \sum_{i=1}^{n-k} f_{i,k}^2 d\theta_i^2)$  for some positive smooth functions  $f_{i,k} : M_k \rightarrow \mathbb{R}$ .
- If  $\pi_k$  denotes the projection  $\mathbb{S}^2 \times \mathbb{T}^{n-2} \rightarrow \mathbb{S}^2 \times \mathbb{T}^{k-2}$  then  $F_k = \pi_k \circ F|_{M_k} : M_k \rightarrow \mathbb{S}^2 \times \mathbb{T}^{k-2}$  has non zero degree.
- $M_{k-1} \subset M_k$  is Poincaré dual to  $F_k^* d\theta_k$  where  $\theta_k$  is a coordinate on the last factor of  $\mathbb{S}^2 \times \mathbb{T}^{k-2}$

Moreover, each metric  $g|_{M_k} + \sum_{i=1}^{n-k} f_{i,k}^2 d\theta_i^2$  has scalar curvature at least  $n(n-1)$ .

We now state a symmetrization theorem obtained by Gromov using  $\mu$ -bubbles for manifolds with positive scalar curvature with two well separated boundary components.

**Theorem 1.2.** [Gro23a, Section 3.7] *Let  $(M^n, \partial_{\pm}, g)$  ( $n \leq 7$ ) be a Riemannian band. Assume that*

$$\text{scal}_g \geq \frac{4(n-1)\pi^2}{n d_g(\partial_-, \partial_+)^2} + \delta$$

for some  $\delta > 0$ . Then there exists

- an hypersurface  $\Sigma$  which separates  $\partial_-$  and  $\partial_+$ ,
- a positive function  $u : \Sigma \rightarrow \mathbb{R}$ ,

such that the metric  $h = g|_{\Sigma} + u^2 dt^2$  on  $\Sigma \times \mathbb{R}$  has  $\text{scal}_h \geq \delta$ .

## 2. GROMOV'S TOROIDAL BAND INEQUALITY AND STABILIZED BONNET-MYERS THEOREM

We first present Gromov's inequality on positive scalar curvature metrics on  $[-1, 1] \times \mathbb{T}^{n-1}$  :

**Theorem 2.1.** *Let  $n \geq 2$  and  $g$  be a metric on  $[-1, 1] \times \mathbb{T}^{n-1}$  with  $\text{scal}_g \geq n(n-1)$  then:*

$$d_g(\{-1\} \times \mathbb{T}^{n-1}, \{+1\} \times \mathbb{T}^{n-1}) \leq \frac{2\pi}{n}$$

Note that if  $dx^2$  is a flat metric on  $\mathbb{T}^{n-1}$  then the metric  $dt^2 + (\cos \frac{nt}{2})^{4/n} dx^2$  on  $(-\frac{\pi}{n}, \frac{\pi}{n}) \times \mathbb{T}^{n-1}$  has constant scalar curvature equal to  $n(n-1)$  while its boundary components are  $\frac{2\pi}{n}$  apart, hence the inequality is optimal.

This inequality was first proved in dimension  $n \leq 7$  using the stable hypersurface method and Fischer-Colbrie-Schoen symmetrization in [Gro18], and was subsequently expanded to higher dimension by Cecchini [Cec20] using Dirac operators methods. It can also be proved using  $\mu$ -bubbles, which was Gromov's original use of Theorem 1.2. For an in-depth discussion of this inequality and its various generalizations, see sections 3.6 to 3.8 of [Gro23a].

We now state (and prove for convenience of the reader) Gromov's Bonnet-Myers theorem on  $\mathbb{T}^{n-2}$ -invariant positive scalar curvature metrics on  $\mathbb{S}^2 \times \mathbb{T}^{n-2}$  and make some remarks on its optimality.

**Theorem 2.2** ([Gro23a, 2.8]). *Let  $g$  be a metric on  $\mathbb{S}^2$  and  $f_1, \dots, f_{n-2} : \mathbb{S}^2 \rightarrow \mathbb{R}$  be smooth positive functions such that the metric  $\tilde{g} = g + \sum_{i=1}^{n-2} f_i^2 d\theta_i^2$  on  $\mathbb{S}^2 \times \mathbb{T}^{n-2}$  has  $\text{scal}_{\tilde{g}} \geq n(n-1)$ . Then:*

$$\text{diam}(\mathbb{S}^2, g) \leq \frac{2\pi}{n}.$$

*Proof.* Consider two points  $p_-, p_+ \in \mathbb{S}^2$  such that  $d_g(p_-, p_+) = \text{diam}(\mathbb{S}^2, g)$  and set  $B_+ = B_g(p_+, \varepsilon)$  and  $B_- = B_g(p_-, \varepsilon)$  for some positive  $\varepsilon$  less than half the diameter.

Set  $M^n = \mathbb{S}^2 \setminus (B_+ \cup B_-) \times \mathbb{T}^{n-2}$ . Since  $\mathbb{S}^2 \setminus (B_+ \cup B_-)$  is diffeomorphic to  $[-1, 1] \times \mathbb{T}^1$ ,  $M^n$  is diffeomorphic to  $[-1, 1] \times \mathbb{T}^{n-1}$ , where the boundary component  $\{\pm 1\} \times \mathbb{T}^{n-1}$  corresponds to  $C_{\pm} \times \mathbb{T}^{n-2}$  (where  $C_{\pm} = \partial B_{\pm}$ ). By assumption  $(M^n, \tilde{g})$  has  $\text{scal}_{\tilde{g}} \geq n(n-1)$  hence Theorem 2.1 gives that:

$$d_{\tilde{g}}(C_- \times \mathbb{T}^{n-2}, C_+ \times \mathbb{T}^{n-2}) \leq \frac{2\pi}{n}$$

We have that  $d_{\tilde{g}}(C_- \times \mathbb{T}^{n-2}, C_+ \times \mathbb{T}^{n-2}) = d_g(C_-, C_+)$ . To see this consider points  $x_{\pm} \in C_{\pm} \times \mathbb{T}^{n-2}$  and any curve  $c : [a, b] \rightarrow M^n$  from  $x_-$  to  $x_+$  and write it as  $c = (\gamma, \theta_1, \dots, \theta_{n-2})$  where  $\gamma : [a, b] \rightarrow \mathbb{S}^2 \setminus (B_+ \cup B_-)$ ,

then estimate the length of  $c$  by:

$$\begin{aligned}\mathcal{L}_{\tilde{g}}(c) &= \int_a^b \sqrt{g(\dot{\gamma}, \dot{\gamma}) + \sum_{i=1}^{n-2} (f_i \circ \gamma)^2 \dot{\theta}_i^2} dt \\ &\geq \int_a^b \sqrt{g(\dot{\gamma}, \dot{\gamma})} dt = \mathcal{L}_g(\gamma)\end{aligned}$$

which shows after taking infimums that

$$d_g(C_-, C_+) \leq d_{\tilde{g}}(C_- \times \mathbb{T}^{n-2}, C_- \times \mathbb{T}^{n-2}).$$

Equality follows from taking  $c$  with  $\mathbb{S}^2$  component a minimizing geodesic and constant  $\mathbb{T}^{n-2}$  component.

This shows that  $\text{diam}(\mathbb{S}^2, g) - 2\varepsilon = d_g(C_-, C_+) \leq \frac{2\pi}{n}$  for any small enough  $\varepsilon$ , hence the Theorem is proved.  $\square$

The optimality of this inequality is an open problem. Note that the product of a round metric on  $\mathbb{S}^2$  and a flat metric on  $\mathbb{T}^{n-2}$  gives a diameter for  $\mathbb{S}$  of  $\frac{\pi}{\sqrt{n(n-1)}}$ . If one looks at the optimal example for Theorem 2.1, a candidate for an optimal  $g$  on  $\mathbb{S}^2$  would be  $g = dr^2 + f(r)^2 d\phi^2$  where  $(r, \phi)$  denote latitude and longitude on the sphere and  $f(r) = (\cos \frac{nr}{2})^{2/n}$ , which is badly singular at the north and south pole: for instance while  $g$  has positive Gauss curvature  $K_g$  outside of the poles, it can be seen that  $K_g$  blows up in a non integrable fashion at the poles, hence the underlying distance cannot be approximated uniformly by positive Gauss curvature metrics. Moreover, the corresponding metric on  $\mathbb{S}^2 \times \mathbb{T}^{n-2}$  would be  $\tilde{g} = g + f(r)^2 dx^2$  for some flat metric on  $\mathbb{T}^{n-2}$ , which completely shrinks the  $\mathbb{T}^{n-2}$  fibers above the poles of  $\mathbb{S}^2$ .

To get a genuine near optimal example from this, one should desingularize this metric  $dr^2 + f(r)^2 d\phi^2 + f(r)^2 dx^2$  into some metric  $dr^2 + a(r)^2 d\phi^2 + b(r)^2 dx^2$  with:

- $a, b : [-\frac{\pi}{n} + \delta, \frac{\pi}{n} - \delta] \rightarrow \mathbb{R}$ ,
- $b > 0$ ,
- $b'(-\frac{\pi}{n} + \delta) = b'(\frac{\pi}{n} - \delta) = 0$
- $a > 0$  on  $(-\frac{\pi}{n} + \delta, \frac{\pi}{n} - \delta)$ ,
- $a(-\frac{\pi}{n} + \delta) = a(\frac{\pi}{n} - \delta) = 0$ ,
- $a'(-\frac{\pi}{n} + \delta) = 1, a'(\frac{\pi}{n} - \delta) = -1$ ,
- $\text{scal} \geq n(n-1)$ .

My attempts to do so were inconclusive, however I managed to show that product metrics are not optimal, the details are included in the appendix.

### 3. INJECTIVITY RADIUS BOUND FOR $\mathbb{S}^2 \times \mathbb{T}^{n-2}$

We now prove Theorem 0.9 for  $k = 0$ .

*Proof.* Let  $M^n$  be an  $n$ -manifold with a nonvanishing degree map  $M^n \rightarrow \mathbb{S}^2 \times \mathbb{T}^{n-2}$  endowed with a metric  $g$  with  $\text{scal}_g \geq n(n-1)$ . We use Theorem 1.1 to get an homologically non-trivial  $\Sigma^2 \subset M^n$  and positive smooth functions  $f_1, f_2, \dots, f_{n-2}$  such that  $\Sigma^2 \times \mathbb{T}^{n-2}$  endowed with the metric  $\tilde{g} = g|_{\Sigma} + \sum_{i=1}^{n-2} f_i^2 d\theta_i^2$  satisfies  $\text{scal}_{\tilde{g}} \geq n(n-1)$ . Note that since  $(\Sigma^2, g|_{\Sigma^2})$



is isometrically immersed in  $(M^n)$ , the extrinsic diameter of  $\Sigma^2$  as a subset of  $(M^n, g)$  denoted by  $\text{diam}_g(\Sigma^2)$  is smaller than its intrinsic diameter  $\text{diam}_{g|_{\Sigma^2}}(\Sigma^2)$ . In short:

$$(3.1) \quad \text{diam}_g(\Sigma^2) \leq \text{diam}_{g|_{\Sigma^2}}(\Sigma^2).$$

Applying Theorem 2.2 to  $(\Sigma^2 \times \mathbb{T}^{n-2}, \tilde{g})$  gives that

$$(3.2) \quad \text{diam}_{g|_{\Sigma^2}}(\Sigma^2) \leq \frac{2\pi}{n}.$$

Hence by (3.1) and (3.2) we get:

$$(3.3) \quad \text{diam}_g(\Sigma^2) \leq \frac{2\pi}{n}.$$

Now assume  $(M^n, g)$  has injectivity radius bigger than  $\frac{2\pi}{n}$ . Then for any  $p \in \Sigma^2$ ,  $B_M(p, \frac{2\pi}{n})$  is contractible. The diameter estimate (3.3) implies that  $\Sigma^2 \subset B_M(p, \frac{2\pi}{n})$  thus  $\Sigma^2$  is trivial in homology which contradicts the definition of  $\Sigma^2$ .  $\square$

#### 4. INJECTIVITY RADIUS BOUNDS FOR 3-MANIFOLDS WITH RICH TOPOLOGY

Before proving Theorem 0.6, we first state a diameter estimate for stable minimal immersions of 2-spheres in manifolds with positive scalar curvature initially observed by Schoen and Yau (see Lemma 16 in [CL23]). We include here a proof to show how this can be obtained directly from Gromov's Theorem 2.2.

**Lemma 4.1.** *Let  $(M^3, g)$  be a 3-manifold with  $\text{scal}_g \geq 6$  and let  $\iota : \mathbb{S}^2 \rightarrow M^3$  be a stable minimal immersion, then:*

$$\text{diam}(\mathbb{S}^2, \iota^*g) \leq \frac{2\pi}{3}.$$

*Proof.* As in Gromov's first proof of Theorem 2.1 in [Gro18], we will use Fischer-Colbrie–Schoen symmetrization. The stability of  $\iota$  means that the Jacobi operator  $J = -\Delta_{\iota^*g} - \frac{1}{2}(\text{scal}_g - \text{scal}_{\iota^*g} + |A|^2)$  is nonnegative.

Let  $f$  be the first eigenfunction of  $J$  (which doesn't vanish since  $J$  is a Schrödinger operator) and consider the metric  $\tilde{g} = \iota^*g + f^2 d\theta^2$  on  $\mathbb{S}^2 \times \mathbb{T}^1$ . A classical computation shows that:

$$\text{scal}_{\tilde{g}} = \text{scal}_{\iota^*g} - \frac{2\Delta_{\iota^*g}f}{f} \geq 6.$$

Now, as in the proof of Theorem 0.9, we can apply Gromov's Theorem 2.2 to  $(\mathbb{S}^2 \times \mathbb{T}^1, \tilde{g})$  to show that  $(\mathbb{S}^2, \iota^*g)$  has diameter at most  $\frac{2\pi}{3}$ .  $\square$

We are now ready to prove Theorem 0.6.

*Proof of Theorem 0.6.* First, we note that by going to the orientation cover if needed we can assume that  $M^3$  is orientable.

Let  $(M^3, g)$  be a compact 3-manifold with  $\text{scal}_g \geq 6$  and:

$$(4.1) \quad \text{inj}_g > \frac{2\pi}{3}.$$

Since  $(M^3, g)$  has positive scalar curvature, it follows from Perelman's solution to the Geometrization conjecture ([Per03]) that  $M^3$  is diffeomorphic to a connected sum:

$$M^3 \simeq P_1 \# \cdots \# P_k$$

where the  $P_i$  are either non simply connected spherical 3-manifolds or  $\mathbb{S}^2 \times \mathbb{T}^1$ , note that we can have  $k = 0$  in which case  $M^3$  is a 3-sphere and there is nothing to prove.

If one of the  $P_i$ 's is  $\mathbb{S}^2 \times \mathbb{T}^1$ , then by collapsing all the other factors we can build a non zero degree map  $M^3 \rightarrow \mathbb{S}^2 \times \mathbb{T}^1$ . Thus Theorem 0.9 applies and  $\text{inj}_g \leq \frac{2\pi}{3}$ . This contradicts (4.1).

Hence all the  $P_i$ 's are non simply connected spherical manifolds.

We will now prove that there is only one summand in the connected sum (4). To see this note that if  $k \geq 2$  then inside a neck  $\mathbb{S}^2 \times [-1, 1]$  used to perform the connected sum one can find an embedded 2-sphere which does not bound a 3-ball. Now by Theorem 4.1 of [HS88], we can minimize area among all such embedded spheres to obtain a stable minimal immersion  $\iota : \mathbb{S}^2 \rightarrow (M^3, g)$  (though it is not useful for our purpose, Hass and Scott actually prove that  $\iota$  is either an embedding or a double cover of a 1-sided projective plane).

We can now apply Lemma 4.1 to show that  $\text{diam}(\mathbb{S}^2, \iota^*g) \leq \frac{2\pi}{3}$ . Thus the image of  $\iota : \mathbb{S}^2 \rightarrow M^3$  is contained in a ball of radius  $\frac{2\pi}{3}$ . Since  $\iota$  is homotopically non trivial this implies that the injectivity radius of  $(M, g)$  is less than  $\frac{2\pi}{3}$ . This contradicts (4.1) and proves that  $k = 1$ .

We now have that  $M^3$  is a spherical 3-manifold, and we can write it as a quotient  $\mathbb{M}^3 = \mathbb{S}^3/\Gamma$  for some finite subgroup  $\Gamma \subset SO(4)$  of fixed point free isometries of the round  $\mathbb{S}^3$ .

We will now show that  $\Gamma$  has odd order. Assume to the contrary that  $\Gamma$  has even order, then a classical exercise in group theory shows that  $\Gamma$  contains an element  $\sigma$  of order 2. Since in  $SO(4)$  we have  $\sigma^2 = I_4$ , the eigenvalues of  $\sigma$  are all equal to  $\pm 1$ . Since  $\sigma$  acts on  $\mathbb{S}^3$  without fixed points, 1 cannot be an eigenvalue of  $\sigma$ . Hence  $\sigma = -I_4$ .

This shows that if  $\Gamma$  has even order then  $\{\pm I_4\} \subset \Gamma$ . This gives rise to a covering  $\Pi : \mathbb{RP}^3 \rightarrow M^3$ . Thus  $\tilde{g} = \Pi^*g$  is a metric on  $\mathbb{RP}^3$  with scalar curvature at least 6. By [BBEN10, Theorem 1], this implies that the 1-systole of  $(\mathbb{RP}^3, \tilde{g})$  is at most  $\pi$ , and thus that its injectivity radius is at most  $\pi/2$ . Since the injectivity radius can only increase by coverings,  $\text{inj}_g \leq \pi/2$  which contradicts (4.1). Thus  $\Gamma$  has odd order.

All that needs to be shown now is that  $M^3 = \mathbb{S}^3/\Gamma$  is a lens space, this follows from the classification of spherical 3-manifolds. Inspecting section 7.5 of [Wol11] we see that all spherical 3-manifolds which are not lens spaces have fundamental group which contain binary dihedral, tetrahedral, octahedral or icosahedral groups as subgroups, thus all those non lens space spherical 3-manifolds will have even order fundamental group. Thus the only possible spherical 3-manifolds with odd order fundamental group are lens spaces.  $\square$

5. INJECTIVITY RADIUS BOUNDS FOR 3-MANIFOLDS WITH LARGE DIAMETER

In this section we prove Theorem 0.8.

We first state an elementary topological lemma, certainly well known, whose simple proof is only included because of our inability to locate it somewhere in the literature:

**Lemma 5.1.** *Let  $(M^n, g)$  be a compact manifold and  $p, q \in M^n$ . Let  $r < \text{inj}_g$ . Then  $[S(p, r)] \neq 0 \in H_2(M \setminus \{p, q\})$ .*

*Proof.* Since  $r < \text{inj}_g$ , all balls of radius  $r$  are diffeomorphic to euclidean ball and all spheres of radius  $r$  are diffeomorphic to standard spheres.

Let  $\hat{M} = M \setminus \{p, q\}$  et  $U = B(p, r) \cup B(q, r)$ . We will show that  $[\Sigma] \neq 0$  in  $H_{n-1}(\hat{M})$ . The Mayer Vietoris exact sequence for the decomposition  $M = \hat{M} \cup U$  gives:

$$\cdots \rightarrow H_n(M) \rightarrow H_{n-1}(\hat{M} \cap U) \rightarrow H_{n-1}(\hat{M}) \oplus H_{n-1}(U) \rightarrow \cdots$$

Note that  $M \cap U$  is homotopy equivalent to  $\Sigma \times \{-1, 1\}$  and that the map  $H_n(M) \rightarrow H_{n-1}(\hat{M} \cap U)$  sends the fundamental class  $[M^n]$  to  $[(\Sigma, 1)] - [(\Sigma, -1)]$ .

Since the sequence is exact this implies that the image of  $[(\Sigma, 1)] + [(\Sigma, -1)]$  by  $H_{n-1}(\hat{M} \cap U) \rightarrow H_{n-1}(\hat{M}) \oplus H_{n-1}(U)$  is not zero. Since  $H_{n-1}(U) = 0$  this implies that  $[\Sigma]$  is not zero in  $H_{n-1}(\hat{M})$ .  $\square$

We can now move to the proof of Theorem 0.8.

*Proof of Theorem 0.8.* Let  $D = \text{diam}(M^3, g)$  and let  $p, q \in M^3$  be such that  $d_g(p, q) = D$ . For  $r < \text{inj}_g$ , we set  $\hat{M} = M \setminus \{p, q\}$  and  $\tilde{M} = M \setminus \{B(p, r) \cup B(q, r)\}$ .

$\tilde{M}$  has two spherical boundary components  $\partial_+ \tilde{M} = S(p, r)$  and  $\partial_- \tilde{M} = S(q, r)$  which satisfy:  $d_g(\partial_- \tilde{M}, \partial_+ \tilde{M}) = D - 2r$ .

Assume that  $D - 2r > 2\pi/3$ , then  $\delta = 6 - \frac{8\pi^2}{3(D-2r)^2} > 0$  and we can apply Theorem 1.2 to  $(\tilde{M}, g)$  to get a compact surface  $\Sigma \subset \tilde{M}$  homologous to  $S(p, r)$  in  $\hat{M}$  and a positive function  $f : \Sigma \rightarrow \mathbb{R}$  such that  $\tilde{g} = g|_\Sigma + f^2 d\theta^2$  has  $\text{scal}_{\tilde{g}} \geq \delta$  and  $\Sigma \in [\partial_- \tilde{M}]$ .

We can now apply Theorem 2.1 (after proper rescaling) to show that:

$$(5.1) \quad \text{diam}(\Sigma, g) \leq \frac{2\pi}{3\sqrt{\delta/6}} = \frac{2\pi}{3\sqrt{1 - \frac{4\pi^2}{9(D-2r)^2}}}$$

Since  $\Sigma \subset \tilde{M}$ , for any  $x \in \Sigma$ ,  $d_g(x, \{p, q\}) > r$  and thus  $B(x, r) \subset \hat{M}$ .

Assume  $r \geq \text{diam}(\Sigma, g)$  then for any  $x \in \Sigma$ ,  $\Sigma$  is included in  $B(x, r)$  which is a topological ball contained in  $\hat{M}$  since  $r$  is smaller than the injectivity radius. Hence  $\Sigma$  is contractible in  $\hat{M}$ , which contradicts Lemma 5.1. Thus :

$$r < \text{diam}(\Sigma, g) \leq \frac{2\pi}{3\sqrt{1 - \frac{4\pi^2}{9(D-2r)^2}}}$$

Solving for  $D$  we get the inequality:

$$(5.2) \quad D < 2r + \frac{2\pi}{3\sqrt{1 - \frac{4\pi^2}{9r^2}}}.$$

We have proved that if  $r < \text{inj}_g$  and  $D > 2r + \frac{2\pi}{3}$  then (5.2) holds.

Hence by contraposition if  $D \geq 2r + \frac{2\pi}{3\sqrt{1 - \frac{4\pi^2}{9r^2}}}$  then  $r \geq \text{inj}_g$  or  $D \leq 2r + \frac{2\pi}{3}$ .

Since for  $r \in (2\pi/3, \pi)$ ,  $\frac{2\pi}{3\sqrt{1 - \frac{4\pi^2}{9r^2}}} > \frac{2\pi}{3}$ , we get that  $\text{inj}_g \leq r$ .  $\square$

## 6. INJECTIVITY RADIUS OF OPEN 3-MANIFOLDS

In this section we will prove Theorem 0.7. Like in the previous section we start with a topological lemma:

**Lemma 6.1.** *Let  $(M^n, g)$  be an open manifold,  $p \in M^n$  and  $r < \text{inj}_g$ . The the  $[S(p, r)] \neq 0 \in H_2(M \setminus \{p, q\})$ .*

*Proof.* The proof is very similar to the proof of Lemma 5.1. Since  $r < \text{inj}_g$ , all balls of radius  $r$  are diffeomorphic to euclidean ball and all spheres of radius  $r$  are diffeomorphic to standard spheres. We set  $\hat{M} = M \setminus \{p\}$  and  $U = B(p, r)$ . We will show that  $[S(p, r)] \neq 0$  in  $H_{n-1}(\hat{M})$ . Once again the Mayer Vietoris exact sequence for the decomposition  $M = \hat{M} \cup U$  gives:

$$\cdots \rightarrow H_n(M) \rightarrow H_{n-1}(\hat{M} \cap U) \rightarrow H_{n-1}(\hat{M}) \oplus H_{n-1}(U) \rightarrow \cdots$$

Now since  $M$  is not compact,  $H_n(M) = 0$  and thus  $H_{n-1}(\hat{M} \cap U) \rightarrow H_{n-1}(\hat{M}) \oplus H_{n-1}(U) \simeq H_{n-1}(\hat{M})$  is an injection. Now  $M \cap U$  is homotopy equivalent to  $S(p, r)$  and thus  $[S(p, r)]$  generates  $H_{n-1}(\hat{M} \cap U)$ . This shows that  $[S(p, r)]$  is not zero in  $H_{n-1}(\hat{M})$ .  $\square$

We can now prove Theorem 0.7.

*Proof of Theorem 0.7.* Fix a point  $p$  in our open  $(M^3, g)$  with  $\text{scal}_g \geq 6$ . Consider a smooth proper 1-Lipschitz function  $\rho$  such that  $|\rho - d(p, \cdot)| < 1$ . For  $R > 2 + \frac{2\pi}{3}$  such that  $R$  and  $2R$  are regular values of  $\rho$  we set:

$$\tilde{M} = \rho^{-1}([R, 2R]).$$

Then  $\partial\tilde{M} = \rho^{-1}(R) \cup \rho^{-1}(2R)$  and  $[\rho^{-1}(R)] = [S(p, r)]$  in  $H_2(M \setminus \{p\})$  for  $r$  small enough. Moreover we have:

$$d_g(\rho^{-1}(2R), \rho^{-1}(R)) \geq R - 2 > 2\pi/3.$$

Thus we can apply Theorem 1.2 to  $(\tilde{M}, g)$  with  $\delta = 6 - \frac{8\pi^2}{3(R-2)^2}$  to get  $\Sigma^2 \subset \tilde{M}$  in the homology class of  $\rho^{-1}(\{R\})$  (and thus of  $S(p, r)$ ) and a function  $f : \Sigma^2 \rightarrow \mathbb{R}$  such that  $g|_{\Sigma} + f^2 d\theta^2$  has scalar curvature at least  $\delta$ . As in section 5, we get that:

$$\text{diam}(\Sigma^2, g) \leq \frac{2\pi}{3\sqrt{1 - \frac{4\pi^2}{9(R-2)^2}}}.$$

We now argue that  $\text{inj}_g \leq \text{diam}(\Sigma^2, g)$ . Let  $x \in \Sigma^2 \subset \tilde{M}$ . Then  $\Sigma^2 \subset \overline{B}(x, \text{diam}(\Sigma^2, g)) \subset M \setminus \{p\}$  and, since  $\Sigma^2$  is not contractible in  $M \setminus \{p\}$ ,  $\overline{B}(x, \text{diam}(\Sigma^2, g))$  is not contractible, hence:

$$\text{inj}_g \leq \text{diam}(\Sigma^2, g) \leq \frac{2\pi}{3\sqrt{1 - \frac{4\pi^2}{9(R-2)^2}}}.$$

Picking  $R$  as big as we want we get that:

$$\text{inj}_g \leq \frac{2\pi}{3}.$$

□

## 7. INJECTIVITY RADIUS OF $\mathbb{S}^2 \times \mathbb{T}^{n-3} \times \mathbb{R}$ AND $\mathbb{S}^2 \times \mathbb{T}^{n-4} \times \mathbb{R}^2$

Once again we start with a topological lemma:

**Lemma 7.1.** *Let  $M$  be a smooth  $n$ -manifold,  $X$  be a compact  $(n-1)$ -manifold and  $F : M \rightarrow X \times \mathbb{R}$  be smooth proper non zero-degree map. For any  $p \in M$ , set  $F(p) = (x(p), r(p)) \in X \times \mathbb{R}$ . Let  $r_0$  be a regular value of  $r : M \rightarrow \mathbb{R}$ , then for any  $\Sigma \in [r^{-1}(r_0)]$ ,  $x|_{\Sigma} : \Sigma \rightarrow X$  has non-zero degree.*

*Proof.* Since  $F$  has non-zero degree there exists  $k \neq 0$  such for any compactly supported  $n$ -form  $\omega$  on  $X \times \mathbb{R}$ :

$$(7.1) \quad \int_M F^* \omega = k \int_{X \times \mathbb{R}} \omega.$$

Let  $\Sigma_0 = r^{-1}(r_0)$ ,  $\Sigma_0$  is a compact smooth hypersurface. Since  $r_0$  is a regular value of  $r$ ,  $F$  is a submersion from  $M_\varepsilon = r^{-1}((r_0 - \varepsilon, r_0 + \varepsilon))$  for some positive  $\varepsilon$ , hence we can assume that  $M_\varepsilon$  is diffeomorphic to  $\Sigma_0 \times (r_0 - \varepsilon, r_0 + \varepsilon)$  and that in these coordinates,  $F$  can be written as:

$$F : M_\varepsilon \simeq \Sigma_0 \times (r_0 - \varepsilon, r_0 + \varepsilon) \rightarrow X \times \mathbb{R} \\ (s, \rho) \mapsto (x(s, \rho), \rho).$$

Now let  $\chi : (r_0 - \varepsilon, r_0 + \varepsilon) \rightarrow \mathbb{R}$  be a compactly supported function. Let  $\xi$  be an  $n-1$ -form on  $X$  which is not zero in  $H^{n-1}(X)$ . Then, then writing  $t$  for the coordinate on the  $\mathbb{R}$  factor we set  $\omega = \chi(t)(\xi \wedge dt)$ . Then  $\int_M F^* \omega = k \int_{X \times \mathbb{R}} \omega$  by (7.1). We can compute:

$$\int_{X \times \mathbb{R}} \omega = \int_{X \times \mathbb{R}} \chi(t)(\xi \wedge dt) = \int_{\mathbb{R}} \chi(t) dt \int_X \xi$$

and:

$$\int_M F^* \omega = \int_{M_\varepsilon} F^*(\chi(t)(\xi \wedge dt)) = \int_{(r_0 - \varepsilon, r_0 + \varepsilon)} \chi(\rho) \left( \int_{\Sigma_0} x^* \xi(s, \rho) \right) d\rho.$$

This shows that for any  $\chi \in C_0^\infty((r_0 - \varepsilon, r_0 + \varepsilon))$ :

$$\int_{(r_0 - \varepsilon, r_0 + \varepsilon)} \chi(t) \left( k \int_X \xi - \int_{\Sigma_0} x^* \xi(s, t) \right) dt$$

and thus for any  $t \in (r_0 - \varepsilon, r_0 + \varepsilon)$ :

$$\int_{\Sigma_0} x^* \xi(s, t) = k \int_X \xi \neq 0.$$

For  $t = r_0$ , we get that  $\int_{\Sigma_0} x^* \xi = k \int_X \xi \neq 0$  which proves that  $x|_{\Sigma_0} : \Sigma_0 \rightarrow X$  has nonzero degree.

Now for any  $\Sigma \in [\Sigma_0]$ ,  $\int_{\Sigma} x^* \xi = \int_{\Sigma_0} x^* \xi \neq 0$  hence  $x|_{\Sigma} : \Sigma \rightarrow X$  has non-zero degree.  $\square$

We can now prove Theorem 0.9 for  $k = 1$  or  $2$ .

*Proof of Theorem 0.9 for  $k=1,2$ .* For  $k = 1$ , consider a proper non zero degree map  $F : (M^n, g) \rightarrow \mathbb{S}^2 \times \mathbb{T}^{n-1} \times \mathbb{R}$  and let  $r : M^n \rightarrow \mathbb{R}$  be the last coordinate of  $f$ . Let  $L > 0$  and set  $M_L = r^{-1}([-L, L])$ . Then  $M_L$  is compact with boundary components  $\partial_+ M_L = r^{-1}(L)$  and  $\partial_- M_L = r^{-1}(-L)$ . Since  $r$  is proper and  $g$  is complete:

$$d_g(\partial_- M_L, \partial_+ M_L) \xrightarrow{L \rightarrow +\infty} +\infty.$$

By Sard's theorem, we can find an increasing sequence  $(L_k)$  diverging to  $+\infty$  such that  $L_k$  and  $-L_k$  are regular values of  $r$ .

We now apply Theorem 1.2 to  $(M_{L_k}, g)$  to find an hypersurface  $\Sigma_k \in [\partial_+ M_{L_k}]$  such that:

- By Lemma 7.1,  $\pi \circ F|_{\Sigma_k} : \Sigma_k \rightarrow \mathbb{S}^2 \times \mathbb{T}^{n-1}$  has non-zero degree where  $\pi : \mathbb{S}^2 \times \mathbb{T}^{n-1} \times \mathbb{R} \rightarrow \mathbb{S}^2 \times \mathbb{T}^{n-1}$  is the natural projection.
- there exists a smooth function  $f_k : \Sigma \rightarrow (0, +\infty)$  such that  $(\tilde{M}_k, \tilde{g}_k) = (\Sigma_k \times \mathbb{T}^1, g|_{\Sigma_k} + f_k^2 d\theta^2)$  has  $\text{scal}_{\tilde{g}_k} \geq \delta_k$  where:

$$\delta_k \xrightarrow{k \rightarrow +\infty} n(n-1).$$

Note that  $\tilde{F} = (\pi \circ F|_{\Sigma_k}, \text{id}_{\mathbb{T}^1}) : \tilde{M}_k \rightarrow \mathbb{S}^2 \times \mathbb{T}^n$  has non zero degree.

We then apply Zhu's Theorem 1.1 to  $(\tilde{M}_k, \tilde{g}_k)$  to get a 2-sphere  $S_k \subset \tilde{M}_k$  and  $n-2$  positive functions  $f_{1,k}, \dots, f_{n-2,k} : S_k \rightarrow \mathbb{R}$  such that:

- $\int_{S_k} \tilde{F}^* \sigma \neq 0$  where  $[\sigma] \in H^2(\mathbb{S}^2 \times \mathbb{T}^{n-2})$  is the fundamental class of the  $\mathbb{S}^2$  factor.
- $(\hat{M}_k, \hat{g}_k) = (S_k \times \mathbb{T}^{n-2}, \tilde{g}|_{S_k} + \sum_{i=1}^{n-2} f_{i,k}^2 d\theta_i^2)$  has  $\text{scal}_{\hat{g}_k} \geq \delta_k$ .

Moreover it follows from Zhu's proof of Theorem 1.1 that the construction of  $S_k$  respects the  $\mathbb{T}^1$  symmetry of  $(\tilde{M}_k, \tilde{g}_k)$ , which implies that  $S_k \subset \Sigma_k$  and  $\tilde{g}|_{S_k} = (g_{\Sigma_k})|_{S_k} = g|_{S_k}$ .

By Theorem 2.2 applied to  $(\hat{M}_k, \hat{g}_k)$ , we have that the diameter of  $(S_k, g|_{S_k})$  is at most  $D_k$  where  $D_k \xrightarrow{k \rightarrow +\infty} \frac{2\pi}{n}$ . Since  $S_k$  is not contractible in  $M$ , this implies that  $\text{inj}_g \leq D_k$ . Letting  $k$  go to infinity we get that  $\text{inj}_g \leq \frac{2\pi}{n}$ .

We now turn to the case of  $k = 2$  and assume that  $F : (M, g) \rightarrow \mathbb{S}^2 \times \mathbb{T}^{n-4} \times \mathbb{R}^2$  is proper and has nonzero degree. Let  $(x, y)$  denote cartesian coordinates of  $\mathbb{R}^2$  and set:

$$\begin{aligned} r : M &\rightarrow \mathbb{R} \\ p &\mapsto x(F(p))^2 + y(F(p))^2. \end{aligned}$$

$r$  is smooth and proper. Let  $\eta \in (0, 1)$  be a regular value of  $r$ . For  $L > 1$  we set  $M_L = r^{-1}((\eta, L))$ ,  $\partial_- M_L = r^{-1}(\eta)$  and  $\partial_+ M_L = r^{-1}(L)$ . Since  $r$  is proper and  $g$  is complete:

$$d_g(\partial_- M_L, \partial_+ M_L) \xrightarrow{L \rightarrow +\infty} +\infty.$$

Moreover if one set  $\dot{M} = F^{-1}(\mathbb{S}^2 \times \mathbb{T}^{n-4} \times (\mathbb{R}^2 \setminus \{0\}))$ . Then  $F|_{\dot{M}}$  has nonzero degree as a map valued in  $\mathbb{S}^2 \times \mathbb{T}^{n-4} \times (\mathbb{R}^2 \setminus \{0\}) \simeq \mathbb{S}^2 \times \mathbb{T}^{n-3} \times \mathbb{R}$ . We thus can argue as in the  $k = 1$  case.  $\square$

#### APPENDIX A. NON OPTIMALITY OF PRODUCTS FOR GROMOV'S DIAMETER ESTIMATE

We will show here that one can deform the product metric on  $\mathbb{S}^2 \times \mathbb{T}^{n-2}$  in a way which increase the scalar curvature while keeping the diameter of the  $\mathbb{S}^2$  factor, which shows that Theorem 2.1 cannot be improved to show that product metrics are optimal.

More precisely, we will build metrics  $g = g_s$  on  $\mathbb{S}^2$  and functions  $b = b_s : \mathbb{S}^2 \rightarrow (0, \infty)$  for  $s \geq 0$  such that:

- (1)  $g_0$  is the round metric on  $\mathbb{S}^2$  and  $f_0 = 1$ .
- (2)  $\text{diam}(\mathbb{S}^2, g_s) = \pi$ .
- (3) the metric  $\tilde{g} = \tilde{g}_s = g_s + b_s^2 dx^2$  on  $\mathbb{S}^2 \times \mathbb{T}^{n-2}$  (where  $dx^2$  is a flat metric on  $\mathbb{T}^{n-2}$ ) has  $\text{scal}_{\tilde{g}} > 2$  for  $s > 0$  small enough.

We will choose normal coordinates  $(r, \phi)$  on  $\mathbb{S}^2$  in which the round metric is written as  $g_0 = dr^2 + \sin^2(r)d\phi^2$  for  $r \in (0, \pi)$  and  $\phi \in (-\pi, \pi]$ . We will look at metrics of the form  $\tilde{g} = dr^2 + a^2(r)d\phi^2 + b(r)^2 dx^2$ .

We will add to our functions  $a$  and  $b$  the next Fourier mode while ensuring smoothness of  $\tilde{g}$  by setting:

- $a(r) = \frac{\sin r + \alpha \sin 3r}{1 + 3\alpha}$ ,
- $b(r) = 1 + \beta \sin^2 r$ ,

for  $\alpha, \beta$  to be determined later. Note that for  $\alpha \in (-\frac{1}{3}, 1)$  and  $\beta \in (-1, +\infty)$ ,  $\tilde{g}_s = dr^2 + a^2(r)d\theta^2 + b(r)^2 dx^2$  is a smooth metric on  $\mathbb{S}^2 \times \mathbb{T}^{n-2}$  and that since  $(\mathbb{S}^2, dr^2 + a^2(r)d\theta^2)$  has diameter at least  $\pi$ . The scalar curvature of  $\tilde{g}$  is given by:

$$\begin{aligned} \text{scal}_{\tilde{g}} = & -2 \left( \frac{a'}{a} + (n-2) \frac{b'}{b} \right)' \\ & - \left( \left( \frac{a'}{a} \right)^2 + (n-2) \left( \frac{b'}{b} \right)^2 \right) - \left( \frac{a'}{a} + (n-2) \frac{b'}{b} \right)^2. \end{aligned}$$

This can be computed for instance with Cartan's method of moving coframes applied to the orthonormal coframe  $(dr, a(r)d\phi, b(r)dx_3, \dots, b(r)dx_n)$ .

With the help of a computer algebra system<sup>1</sup>, we can plug our expressions  $a$  and  $b$  inside the expression above for the scalar curvature. We get a lengthy formula depending on  $r, \alpha, \beta$  and  $n$ .

In Figure 2, we plotted the scalar curvature of  $\tilde{g}$  for various choices of our parameters  $\alpha, \beta$  and  $n$ . From this plot we can guess that  $r \mapsto \text{scal}_{\tilde{g}}$  attains its minimum at either  $r = 0$  or  $r = \pi/2$ . We can then use our CAS to compute those values:

- $\text{scal}_{\tilde{g}}(0) = \frac{2-4\beta(n-2)(6\alpha+2)+54\alpha}{1+3\alpha}$
- $\text{scal}_{\tilde{g}}(\pi/2) = \frac{2(1-9\alpha-3\beta-5\alpha\beta+2n(1-\alpha)\beta)}{(1-\alpha)(1+\beta)}$

<sup>1</sup>this computation and the subsequents computation and plots have been carried on Sage-Math 10.4.

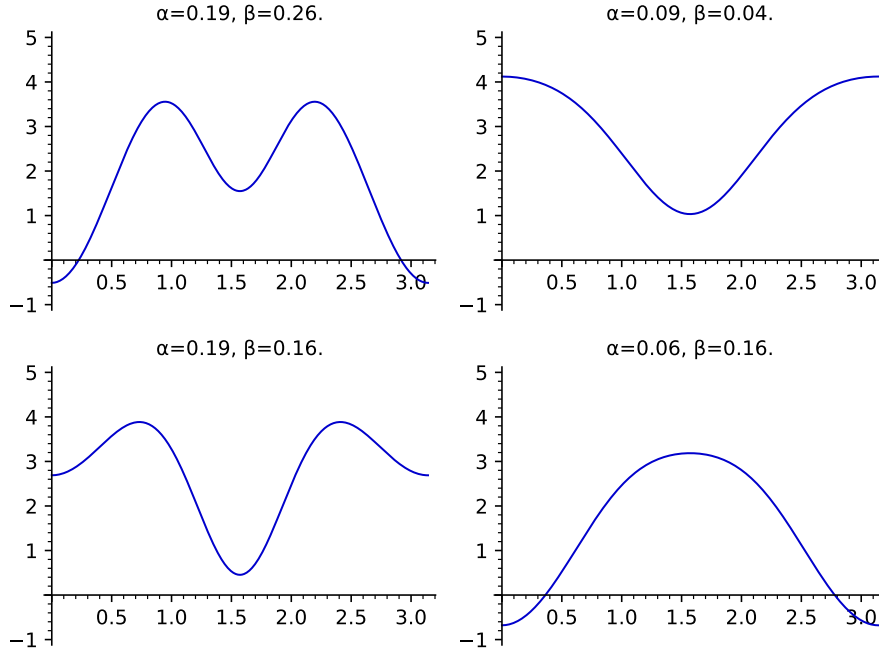


FIGURE 2. Example plots of the scalar curvature of  $\tilde{g}$  as a function of  $r$  for various choices of the  $\alpha$  and  $\beta$  parameters, for  $n = 6$ .

To find candidates for counter examples, we plotted for various values of  $n$  the regions  $\mathcal{R}_0 = \{\text{scal}_{\tilde{g}}(\pi/2) \geq 2\}$  and  $\mathcal{R}_{\pi/2} = \{\text{scal}_{\tilde{g}}(0) \geq 2\}$  in the  $(\alpha, \beta)$  plane. Luckily enough the two regions seem to always intersect in what looks like a tiny convex region of the plane as seen on Figure 3.

By solving the system:

$$\begin{cases} \text{scal}_{\tilde{g}}(\pi/2) = 2 \\ \text{scal}_{\tilde{g}}(0) = 2 \end{cases}$$

for  $\alpha$  and  $\beta$ , we get two solutions  $(0, 0)$  and  $(\frac{n-2}{3(3n-2)}, \frac{1}{2(n-1)})$ . Since we have observed that our counterexample set  $\mathcal{R}_{\pi/2} \cap \mathcal{R}_0$  should be convex in the  $(\alpha, \beta)$  plane, the whole segment between these two points should consist of counterexamples. Hence we set

$$(\alpha, \beta) = s\left(\frac{n-2}{3(3n-2)}, \frac{1}{2(n-1)}\right)$$

What remains to be shown is that at least for  $s$  small enough, we actually have  $\text{scal}_{\tilde{g}_s} > 2$ .

In order to see this we can compute

$$\left. \frac{\partial \text{scal}_{\tilde{g}_s}}{\partial s} \right|_{s=0} = \frac{(n-2)((7n-10) + (5n-14)\cos 2r)}{3(n-1)(3n-2)} \geq \frac{(n-2)(2n+4)}{3(n-1)(3n-2)}$$

which is strictly positive since  $n \geq 3$ . Hence for  $s > 0$  small enough,  $\text{scal}_{\tilde{g}_s} > \text{scal}_{\tilde{g}_0} = 2$ .



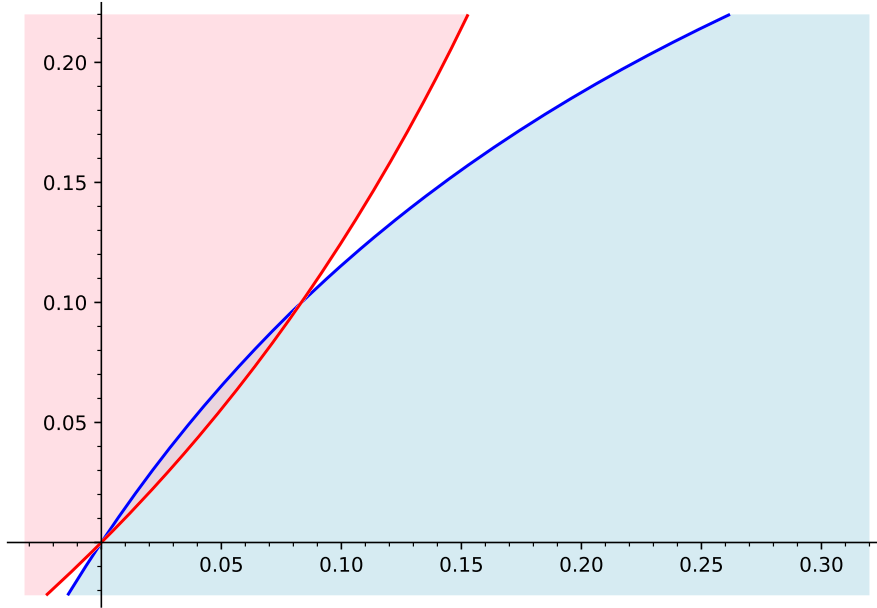


FIGURE 3. Plot of the  $\mathcal{R}_0$  (blue) and  $\mathcal{R}_{\pi/2}$  (pink) regions in the  $(\alpha, \beta)$ -plane for  $n = 4$ .

**Remark A.1.** While this shows that products are not optimal (even locally) for the diameter estimate given by Theorem 2.2, this doesn't rule out optimality of products for the injectivity radius estimate of Theorem 0.9. This follows from the fact that while the length of the meridians of  $\mathbb{S}^2$  remains fixed at  $\pi$  for our counterexamples, the equator is shrunk when  $s > 0$  which lowers the injectivity radius too fast with respect to the increase in minimal scalar curvature.

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