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# Branching diffusion processes and spectral properties of Feynman-Kac semigroup

Pierre Collet\*, Sylvie Méléard†, Jaime San Martín‡

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## Abstract

In this article we study the long time behavior of linear functionals of branching diffusion processes as well as the time reversal of the spinal process by means of spectral properties of the Feynman-Kac semigroup. We generalize for this non Markovian semigroup the theory of quasi-stationary distribution (q.s.d.) and  $Q$ -process. The most amazing result is the identification of the law of the reversal time spinal process issued from q.s.d. with the  $Q$ -process of the Feynman-Kac semigroup.

Keywords: Stochastic differential equation - Eigenvalues asymptotics - quasistationary distribution -  $Q$ -process - spinal process - time reversal.

## 1 Introduction and model

We consider a population dynamics where individuals are characterized by a trait  $x \in \mathbb{R}$  (or a position) and give birth and die in continuous time. During their life the individual trait (or position) variations are modelled by a diffusion operator. More precisely we consider a branching-diffusion process  $(Z_t)$  defined as follows: an individual with trait  $x$  gives birth to a (unique) new individual with the same trait at rate  $b(x)$ . and dies with rate  $d(x)$ . Changes in the trait during the individuals life are driven by independent diffusion processes, accounting for infinitesimal changes of the phenotypes (or of the positions). The underlying diffusion process is given by

$$dX_t = dB_t - a(X_t)dt. \quad (1.1)$$

Let us denote by

$$V(x) = b(x) - d(x)$$

the growth rate for an individual with trait  $x$ .

Denote by  $\mathcal{V}_t$  the set of individuals alive at time  $t$ . Then the process  $(Z_t, t \geq 0)$  is the point measure valued process on the trait space  $\mathbb{R}$ , defined for any  $t$  by

$$Z_t = \sum_{i \in \mathcal{V}_t} \delta_{X_t^i},$$

where  $X_t^i$  is the trait of individual  $i$  at time  $t$ . We denote the total mass of the branching diffusion process by

$$N_t = \#\mathcal{V}_t = \langle Z_t, 1 \rangle.$$

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The empirical measure  $Z_t$  at time  $t$  on the individual traits satisfies a semi-martingale decomposition, and is the unique solution of a stochastic equation driven by independent Poisson point measures and Brownian motions (see for example Champagnat-Méléard [8]).

The main object of this article is to study the long time behavior for the linear functionals of this process, that is  $\langle Z_t, f \rangle$  for suitable functions  $f$  and its relation with the theory of quasi-stationary distributions. This is done through the well known relation between the branching diffusion process  $Z$  and the following Feynman-Kac semigroup  $(P_t)$ : for any  $f \in C_b(\mathbb{R})$ ,  $t \geq 0$ ,  $x \in \mathbb{R}$ ,

$$P_t f(x) = \mathbb{E}_{\delta_x}(\langle Z_t, f \rangle) = \mathbb{E}_x \left( \exp \left( \int_0^t V(X_s) ds \right) f(X_t) \right). \quad (1.2)$$

where  $X$  is the drifted Brownian motion defined in (1.1). So a main tool in this article is the study of the long time behavior of this semigroup. Ideas are coming from the theory of quasi-stationary distributions and  $Q$ -processes cf. Cattiaux et al. [7] (see also [3]). A main difference with these two cited articles is that here we do not have a strong drift from infinity, but a strong potential at  $\pm\infty$ , and because of that, another important source of inspiration are the ideas developed by Kac in [19].

Note that (1.2) can be generalized to trajectories if we introduce the historical process  $(L_t, t \geq 0)$  (cf. [13], [26] or [25]), i.e. the point measure valued process on the trajectories, defined for any  $t$  by

$$L_t = \sum_{i \in \mathcal{V}_t} \delta_{X_t^i}. \quad (1.3)$$

Indeed, for any  $T > 0$  and  $F : C([0, T], \mathbb{R}) \rightarrow \mathbb{R}$  a bounded and continuous function and  $x \in \mathbb{R}$ ,

$$\mathbb{E}_{\delta_x}(\langle L_T, F \rangle) = \mathbb{E}_x \left( \exp \left( \int_0^T V(X_s) ds \right) F(X^T) \right), \quad (1.4)$$

where  $X^T = (X_{t \wedge T})$ . These formulas/theory come from [21, 22] further developed for instance in [4, 10, 16, 24]. Different and simpler proofs are given in [6].

We denote by  $\mathcal{G}$  the generator of  $(X_t)_t$ , that is,  $\mathcal{G}(u) = \frac{1}{2}u'' - au'$ , and by

$$\mathcal{L} = \frac{1}{2}u'' - au' + Vu,$$

the generator of  $(P_t)$ . In what follows, we also introduce the following function which will play a main role in the estimates,

$$\ell(x) = \int_0^x a(z) dz. \quad (1.5)$$

We make the following assumptions on  $a$ ,  $\ell$  and  $V$ :

H0.1)  $a \in C^1(\mathbb{R})$ .

H0.2)  $|\ell|$  has at most a linear growth at  $\pm\infty$ , namely there exists constants  $\beta, \gamma$  so that for all  $x$  it holds  $|\ell(x)| \leq \gamma + \beta|x|$ .

H0.3) the function  $a' - a^2$  is bounded from above.

H1)  $V$  is continuous.

H2)  $V$  is bounded from above, that is  $A(V) := \sup_{z \in \mathbb{R}} V(z) < \infty$ .

H3)  $\lim_{|x| \rightarrow \infty} V(x) = -\infty$ .

H4)  $V$  has at least a linear decay at  $\pm\infty$ , that is, there exist constants  $E > 0, x_0 \geq 0$  so that for all  $|x| \geq x_0$  it holds

$$V(x) \leq -E|x|.$$

We point out that  $H1$  and  $H4$  imply  $H2$  and  $H3$ , but it will be convenient for further references to keep this redundancy.

This article is structured as follows. In Section 2 we shall state the main results, whose proofs given in Section 4, are based on the spectral properties of  $(P_t)$  developed in Section 3. Subsection 2.1 is devoted to the long time behaviour for linear functional of  $(Z_t)$ , while Section 2.2 studies the spinal process for the branching diffusion. Section 5 is devoted to the specific example developed in [6], article that motivated the present work. Note that we arrive more rapidly to this result than in [6]. This example shows the efficiency of our result.

Our results were obtained with a drifted Brownian motion but it is not a real constraint and the diffusion (1.1) can be chosen more general. Indeed, it is well known (cf. Karatzas-Shreve [20]) that a diffusion

$$dY_t = \sigma(Y_t)dB_t - a_0(Y_t)dt,$$

where  $\sigma$  is assumed to be non vanishing and  $C^1$ , can be reduced to the case  $\sigma = 1$  and drift  $a$  by the image-measure transformation on the trait space (leading to a change of measure on the measure space), with the diffeomorphism  $y \rightarrow G(y) = \int_0^y \frac{1}{\sigma(u)} du$  with  $a = (\frac{a_0}{\sigma} + \frac{1}{2}\sigma') \circ G^{-1}$ , and a new potential  $W = V \circ G^{-1}$ . The transformed diffusion  $X_t = G(Y_t)$  and its Feynman-Kac semigroup satisfies  $P_t^Y(\phi)(y) = P_t^X(\phi \circ G^{-1})(G(y))$ , which permits to study the long time behaviour of the original semigroup.

In the last years, numerous works have been developed, studying the long time behavior of non Markovian semigroups. Some of them use the Harris approach, introducing either Doeblin's condition (cf. Champagnat-Villemonais [9] and all references of the authors issued from this seminal paper) or exploiting the existence of a Lyapounov function (cf. [23], [11], [1], [2]). Our work gives the complete spectrum of the Feynman-Kac semigroup but the main point of the paper is to translate the longtime properties of this semigroup as properties for the branching-diffusion process. In a subsequent paper, we will explore more deeply these results in the case of critical branching -diffusion processes to obtain the time scale at which the process goes to extinction.

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## 2 Main results

In this section we state the main results of this article, whose proofs are based on the spectral properties of the Feynman-Kac semigroup  $(P_t)_t$ . A main tool is provided by Theorem 2.7 in Section 2.3.

Since  $(P_t)$  is, in general, not symmetric with respect to the Lebesgue measure, it is classical (see for example [7]) to introduce a new reference measure which makes it symmetric, the so called speed measure

$$\rho(dy) = e^{-2\ell(y)}dy,$$

where  $\ell$  has been defined in (1.5).

## 2.1 Long time behaviour for linear functionals of $(Z_t)$

We will prove in Section 3 the following result giving the long time behavior for linear functionals of the branching diffusion process.

**Theorem 2.1.** *Under the assumptions H0-H4,*

(1) *There exists  $\lambda_0 \in \mathbb{R}$  such that for any  $x \in \mathbb{R}$ ,*

$$\lim_{t \rightarrow \infty} -\frac{\log(\mathbb{E}_x(N_t))}{t} = \lim_{t \rightarrow \infty} -\frac{\log\left(\mathbb{E}_x\left(e^{\int_0^t V(X_s)}\right)\right)}{t} = \lambda_0.$$

(2) *There exists a positive function  $\Theta_0 \in L^2(d\rho)$  defined on  $\mathbb{R}$  such that for all  $\phi \in C(\mathbb{R})$  that has at most exponential linear growth at  $\pm\infty$  and for any  $x \in \mathbb{R}$ , it holds*

$$\lim_{t \rightarrow \infty} e^{\lambda_0 t} \mathbb{E}_x(\langle Z_t, \phi \rangle) = \lim_{t \rightarrow \infty} e^{\lambda_0 t} \mathbb{E}_x\left(e^{\int_0^t V(X_s)} \phi(X_t)\right) = \Theta_0(x) \int \Theta_0(z) \phi(z) e^{-2\ell(z)} dz.$$

*In particular, we have the following asymptotic for the total mass*

$$\lim_{t \rightarrow \infty} e^{\lambda_0 t} \mathbb{E}_x(N_t) = \lim_{t \rightarrow \infty} e^{\lambda_0 t} \mathbb{E}_x\left(e^{\int_0^t V(X_s)}\right) = \Theta_0(x) \int \Theta_0(z) e^{-2\ell(z)} dz,$$

*and the following ratio limit:*

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}_x(\langle Z_t, \phi \rangle, N_t > 0)}{\mathbb{E}_x(N_t)} = \frac{\int \phi(z) \Theta_0(z) e^{-2\ell(z)} dz}{\int \Theta_0(z) e^{-2\ell(z)} dz}.$$

In what follows we denote by  $\nu$  the probability measure on  $\mathbb{R}$ , appearing in (3), given by

$$\nu(dx) = \frac{\Theta_0(x) e^{-2\ell(x)}}{\int \Theta_0(y) e^{-2\ell(y)} dy} dx,$$

which it turns out to be the unique quasi-stationary distribution for  $(P_t)$  (see Theorem 2.2 and Corollary 2.4). We notice that  $\nu$  is well defined because  $\Theta_0$  has exponential linear decay, of any order, at  $\pm\infty$  and  $\ell$  has at most a fixed linear growth at  $\pm\infty$  (see Corollary 3.4 and formula (3.6)).

On the other hand,  $\Theta_0$  is a positive eigenfunction for the semigroup  $(P_t)$ , acting in  $L^2(\rho(dx))$  (see Corollary 3.4) associated to  $-\lambda_0$ , that is, for all  $t > 0$  it holds

$$P_t \Theta_0 = e^{-\lambda_0 t} \Theta_0.$$

**Main remark:** The sign of  $\lambda_0$  can be positive or negative. We will say that the process is supercritical (resp. subcritical or critical) if  $\lambda_0 < 0$  (resp.  $\lambda_0 > 0$  or  $\lambda_0 = 0$ ). In Section 4, it will be proved that

$$\lambda_0 > -\sup_x (V(x) + 1/2(a' - a^2)(x)) = -\sup_x \tilde{V}(x).$$

Therefore if there exists a region (which can be small) such that  $\tilde{V}$  is positive, then  $\lambda_0$  could be negative and the process supercritical with a mass exploding as  $e^{-\lambda_0 t}$ .

On the other hand, if the potential  $V$  is positive in some region, the process can be either critical, subcritical or supercritical depending on the drift of the underlying diffusion. For example, if  $a = 0$  and  $V$  such that  $\lambda_0 < 0$ , the process is supercritical. Nevertheless adding a suitable drift (for example a constant large enough) can make the process subcritical. The drift will push the individuals to have traits larger and larger and less and less adapted. In the long run, we have extinction of the population.

## 2.2 Time reversal of the spinal process

In this section we will establish a result on the time reversal for the spinal process. Previous to state it and using the spectral properties that we will developed, we will go deeper in the quasi-stationary (q.s.d.) properties of the semigroup  $(P_t)$ . In particular, we shall prove the existence and uniqueness of a q.s.d. distribution and the characterization of the  $Q$ -process. The latter will be related to the distribution of the reverse in time for the spinal process done in Theorem 2.6.

**Theorem 2.2.** *Assume that H0-H4 holds, then  $\nu$  is a q.s.d. for the semigroup  $(P_t)$ , that is, for all  $t > 0$  and all  $\phi$  bounded and measurable functions, it holds*

$$\int \mathbb{E}_x \left( e^{\int_0^t V(X_s) ds} \phi(X_t) \right) \nu(dx) = e^{-\lambda_0 t} \int \phi(x) \nu(dx).$$

In particular, we have

$$\frac{\int \mathbb{E}_x \left( e^{\int_0^t V(X_s) ds} \phi(X_t) \right) \nu(dx)}{\int \mathbb{E}_x \left( e^{\int_0^t V(X_s) ds} \right) \nu(dx)} = \int \phi(x) \nu(dx).$$

In the following result we study the domain of attraction of  $\nu$ . In particular, we prove that for any initial probability measure  $\mu$  supported on  $\mathbb{R}$ , the conditional evolution starting in  $\mu$  converges to  $\nu$ .

**Theorem 2.3.** *Assume that H0-H4 holds and assume  $\mu$  is a measure on  $\mathbb{R}$ , so that  $\int e^{-R|x|} \mu(dx) < \infty$ , for some  $R < \infty$ . The “conditional” evolution  $(\mu_t)_{t>0}$  given by: for all bounded and measurable  $\phi$ , and  $t > 0$*

$$\int \phi(y) \mu_t(dy) = \frac{\int \mathbb{E}_x \left( e^{\int_0^t V(X_s) ds} \phi(X_t) \right) \mu(dx)}{\int \mathbb{E}_x \left( e^{\int_0^t V(X_s) ds} \right) \mu(dx)},$$

is a well defined probability measure on  $\mathbb{R}$ , and

$$\lim_{t \rightarrow \infty} \int \phi(y) \mu_t(dy) = \int \phi(y) \nu(dy).$$

**Corollary 2.4.**  *$\nu$  is the unique q.s.d. for the semigroup  $(P_t)$ .*

The next result gives the existence of the  $Q$ -process.

**Theorem 2.5.** *Assume that H0-H4 holds, then for all  $s > 0$  fixed, for all  $\phi$  bounded measurable function and for all  $x$*

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}_x \left( e^{\int_0^t V(X_u) du} \phi(X_s) \right)}{\mathbb{E}_x \left( e^{\int_0^t V(X_u) du} \right)} = \mathbb{E}_x \left( e^{\int_0^s V(X_u) du} \frac{e^{\lambda_0 s} \Theta_0(X_s)}{\Theta_0(x)} \phi(X_s) \right)$$

We denote by  $(Q_t)$  here the semigroup associated with the  $Q$ -process, that is,

$$Q_t(\phi)(x) = \mathbb{E}_x \left( e^{\int_0^t V(X_u) du} \frac{e^{\lambda_0 t} \Theta_0(X_t)}{\Theta_0(x)} \phi(X_t) \right),$$

In subsection 4.3, we prove Theorem 2.5 and provide the generator of this semigroup, which is given by

$$\mathcal{L}^Q(u) = \frac{1}{2} u'' + \frac{\Psi'_0}{\Psi_0} u',$$

which corresponds to the stochastic differential equation

$$dY_t = dB_t + \frac{\Psi'_0(Y_t)}{\Psi_0(Y_t)} dt,$$

where  $\Psi_0 = \Theta_0 e^{-\ell}$ . This process has a unique invariant measure given by  $\Psi_0^2(x)dx$ .

For every  $T > 0$ , the spinal process  $Y^T = (Y_t^T)_{t \leq T}$ , associated to the branching diffusion process, is the non-homogeneous Markov process defined as, for any  $\Phi : C([0, T], \mathbb{R}) \rightarrow \mathbb{R}$  a bounded continuous function

$$\mathbb{E}_x(\langle L_T, \Phi \rangle) = m_T(x) \mathbb{E}_x(\Phi(Y^T)),$$

where  $L_T$  is the historical process at time  $T$  defined in (1.3) and  $m_T(x) = \mathbb{E}_x(\langle L_T, 1 \rangle) = \mathbb{E}_x(\langle Z_T, 1 \rangle)$  is the total mass at time  $T$ . This spinal process represents the typical trajectory explaining the history of the trait of an individual picked up at random in the population at time  $T$ . This spinal process has been the object of numerous studies (cf. for example [16]) but our motivation comes from [6] (see also [17]) where the aim is to better understand the ancestors in the past explaining the distribution of traits in the population at time  $T$ . Then we are interested in capturing the dynamics of the time reversal process  $\overleftarrow{Y}^T$  for the spinal process given by

$$\overleftarrow{Y}_s^T = Y_{T-s}^T : 0 \leq s \leq T.$$

**Theorem 2.6.** *For every  $T > 0$ , the time reversed spinal process  $\overleftarrow{Y}^T$ , with initial distribution  $\nu$ , that is,  $\overleftarrow{Y}_0^T \stackrel{\mathcal{L}}{=} \nu$ , is an homogeneous Markov process with transition probability operators: for  $0 \leq t \leq T$  and  $\phi$  continuous and bounded*

$$\mathbb{E}_\nu \left( \phi \left( \overleftarrow{Y}_t^T \right) \right) = \langle \nu, Q_t(\phi) \rangle,$$

where the semigroup  $(Q_t)$  is given in Theorem 2.5 and its initial distribution is  $\nu$ .

### 2.3 Main tool, spectral gap of $(P_t)$

This subsection provides the main tool we need in this article, which is a spectral gap for the semigroup  $(P_t)$ . In particular this will prove that the semigroup is ultracontractive in the sense of [12].

We denote by  $\Pi$  the projection onto the space generated by  $\Theta_0$ , namely

$$\Pi(g)(x) = \Theta_0(x) \int \Theta_0(y) g(y) e^{-2\ell(y)} dy = \Theta_0(x) \frac{\int \Theta_0(y) g(y) e^{-2\ell(y)} dy}{\int \Theta_0^2(y) e^{-2\ell(y)} dy},$$

where we will use the normalization  $\int \Theta_0^2(y) e^{-2\ell(y)} dy = 1$ . This projection is well defined for all functions  $g$  for which  $\Theta_0 g$  is  $\rho$ -integrable. As we will see, this includes the bounded functions, all  $L^p(dx)$  and  $L^p(\rho(dx))$ , and functions with at most linear exponential growth at  $\pm\infty$ . We will also see that  $\Theta_0$  is a bounded  $C^2$ -function, so  $\Pi(g)$  fulfills the same properties (see Section 3 for these and other properties). Note also that  $P_t \Pi = e^{-\lambda_0 t} \Pi$ .

In the next result we provide bounds in  $L^q(dx)$  and in  $L^q(\rho(dx))$ . Then, in Theorem 2.7, we denote by  $\|\cdot\|_q$  the norm in either  $L^q(dx)$  or  $L^q(\rho(dx))$  for  $q \in [1, \infty]$ , and the constant  $K$  below depends on the current measure, either  $dx$  or  $\rho(dx)$ . We denote by  $\mathbb{I}$  the identity function.

**Theorem 2.7.** *Assume H0-H4 holds*

- (1) *Consider  $r, q \in [1, \infty]$ . There exists a finite constant  $K$  independent of  $t, r, q$ , so that, if  $t_0 = 4(\beta + 1)/E$  and  $t \geq 3t_0$ , it holds*

$$\|P_t(\mathbb{I} - \Pi)\|_{q,r} = \sup \{ \|P_t(\mathbb{I} - \Pi)(g)\|_r : \|g\|_q = 1 \} \leq K e^{-\lambda_1 t}.$$

*In particular, for all  $g \in L^q$  it holds*

$$\|e^{\lambda_0 t} P_t(g) - \Pi(g)\|_\infty \leq K \|g\|_q e^{-(\lambda_1 - \lambda_0)t}$$

- (2) If  $g$  has at most an exponential linear growth at  $\pm\infty$ , that is, for all  $x$  it holds  $|g(x)| \leq Ae^{\kappa|x|}$ , for some positive constants  $A$  and  $\kappa$ , then there exists a constant  $D = D(\kappa)$ , so that, if  $t_1 = 4(\kappa + \beta + 1)/E$ ,  $t_0 = 4(\beta + 1)/E$  and  $t \geq 3(t_0 + t_1)$ , it holds

$$\|e^{\lambda_0 t} P_t(g) - \Pi(g)\|_\infty \leq AD e^{-(\lambda_1 - \lambda_0)t}.$$

The proof of Theorem 2.1 follows immediately from Theorem 2.7 (2).

### 3 Basic estimates on the spectrum of $(P_t)$

As we mentioned previously, the Feynman-Kac semigroup  $(P_t)_t$  is not symmetric with respect to the Lebesgue measure and to make it symmetric we consider the measure  $\rho(dy) = e^{-2\ell(y)} dy$ . We notice that under the hypotheses we have made on  $V$ , for any  $t \geq 0$ , the operator  $P_t$  is well defined for all  $f$  bounded measurable, it is positive preserving, and bounded in  $L^\infty$ . In Corollary 3.4, we will prove that the semigroup  $(P_t)_t$  has a discrete spectrum on  $L^2(d\rho)$ :  $-\lambda_0 > -\lambda_1 > \dots > \lambda_k > \dots$  which converges to  $-\infty$ , and the associated set of eigenvectors  $(\Theta_k)_{k \geq 0}$  is a complete set on  $L^2(d\rho)$ . In this section we will provide some basic estimates on the eigenvalues  $\lambda_k$  and eigenvectors  $\Theta_k$ . In particular we will prove that  $\Theta_0$  can be chosen positive.

We start the analysis of the semigroup with a Girsanov's transformation to obtain that

$$P_t(\phi)(x) = \mathbb{E}_x \left( e^{\int_0^t V(B_s) ds} e^{-\int_0^t a(B_s) dB_s - \frac{1}{2} \int_0^t a^2(B_s) ds} \phi(B_t) \right),$$

where  $(B_t)$  is a Brownian motion, and from Itô's formula, we get

$$\ell(B_t) = \ell(x) + \int_0^t a(B_s) dB_s + \frac{1}{2} \int_0^t a'(B_s) ds,$$

which gives that

$$\begin{aligned} P_t(\phi)(x) &= \mathbb{E}_x \left( e^{\int_0^t V(B_s) ds} e^{\ell(x) - \ell(B_t) + \frac{1}{2} \int_0^t (a'(B_s) - a^2(B_s)) ds} \phi(B_t) \right) \\ &= e^{\ell(x)} \mathbb{E}_x \left( e^{\int_0^t \tilde{V}(B_s) ds} e^{-\ell(B_t)} \phi(B_t) \right) = e^{\ell(x)} \tilde{P}_t(e^{-\ell} \phi)(x). \end{aligned} \quad (3.1)$$

Computations show that  $(\tilde{P}_t)$  is also a semigroup, corresponding to the Brownian motion with a potential  $\tilde{V} = V(x) + \frac{1}{2}(a'(x) - a^2(x))$ , given by

$$\tilde{P}_t(g)(x) = \mathbb{E}_x \left( e^{\int_0^t \tilde{V}(B_s) ds} g(B_t) \right).$$

It is clear that  $\tilde{V}$  also satisfies Assumptions H1-H4. We denote by  $\tilde{A}, \tilde{E}, \tilde{x}_0$  the corresponding quantities in H2-H4. If  $a' - a^2 \leq M$

Notice that, we can take  $\tilde{E} = E/2$ , by choosing a large  $\tilde{x}_0$ .

The semigroup  $\tilde{P}_t$  acts naturally on  $L^2(dx)$  and if  $u$  is a  $C^2$  function, satisfying mild conditions at  $\pm\infty$ , then its generator is given by  $\tilde{\mathcal{L}}(u) = \frac{1}{2}u'' + \tilde{V}(x)u$ .

It is well known (see Chapter 2 Theorem 3.1 and Corollary 1 in [5]), that under H0-H4 the operator  $\tilde{\mathcal{L}} = \frac{1}{2}\partial_x^2 + \tilde{V}$  has a discrete spectrum in  $L^2(dx)$  of simple eigenvalues  $-\lambda_0 > -\lambda_1 > \dots > -\lambda_k > \dots$ , which converge to  $-\infty$ , and the corresponding orthonormal complete set of eigenfunctions  $(\Psi_k)_{k \geq 0}$ , are  $C^2$  functions satisfying the extra decaying property: for all  $k \geq 0$  and all  $R > 0$  there exists  $C = C(R, k)$  such that, for all  $|x| \geq C$

$$|\Psi_k(x)| \leq e^{-R|x|}. \quad (3.2)$$

In particular these eigenfunctions also belong to  $L^1(dx)$ . We will also see in Corollary 3.5 that  $\Psi_0$  can be chosen to be positive.

**Remark.** We notice that  $\lambda_0 = \lambda_0(\tilde{V})$  could be positive 0 or negative. It is enough to consider  $\hat{V} = \tilde{V} + \lambda_0$  to have a potential that satisfies H0-H4 and for which  $\lambda_0(\hat{V}) = 0$ . When  $\tilde{V}$  is negative, then it holds that  $\lambda_0 > 0$ . In general, we have  $\lambda_0 > -\sup_x \tilde{V}(x)$ . Indeed, from (3.5) below, we have

$$\lambda_0 \geq -\sup_x \tilde{V}(x) + \frac{\pi}{e} \sup_x |\Psi_0(x)|^4.$$

Recall that  $\mathcal{L}(u) = \mathcal{G}(u) + V(u) = \frac{1}{2}u'' - au' + Vu$  is the generator of  $P = (P_t)$ . We first give a rough estimate on the growth rate of the sequence  $(\lambda_k)_k$ .

**Proposition 3.1.** *Assume that H0-H4 holds. Then, for all  $k$  so that  $\lambda_k > S_0 =: \max\{|\tilde{V}(x)| : |x| \leq \tilde{x}_0\}$  it holds*

$$\lambda_k \geq C(\tilde{x}_0, \tilde{E}) k^{1/3}, \quad (3.3)$$

where  $C(\tilde{x}_0, \tilde{E}) = \left(\frac{\tilde{E}^2}{8\tilde{x}_0+2}\right)^{1/3}$ .

*Proof.* A basic estimate for the distribution of eigenvalues for the Schrödinger operator  $\tilde{\mathcal{L}}(u) = \frac{1}{2}u'' + \tilde{V}(x)u$  is obtained as follows (see Theorem 5.2 in [27]). Using the formula developed for  $\Gamma(u) = -u'' + Wu$ , with  $W = -2\tilde{V}$  and whose eigenvalues are  $\tau_k = 2\lambda_k$ , we get from [27] that

$$N_W(\tau) := \#\{j : \tau_j \leq \tau\} \leq \int |x|(\tau - W(x))_+ dx + 1.$$

So, in the rest of the proof we will give an upper estimate of the integral in the right hand side. For all  $|x| \geq \tilde{x}_0$  it holds  $\tilde{V}(x) \leq -\tilde{E}|x|$ , and we choose  $k$  so that  $\lambda_k > S_0$ . Take now,  $x_k \in \mathbb{R}$  the largest, in absolute value, of the solutions to the equation

$$2\tilde{V}(x) = -\tau_k = -2\lambda_k.$$

Clearly it holds  $|x_k| > \tilde{x}_0$  and then  $-\lambda_k = \tilde{V}(x_k) \leq -\tilde{E}|x_k|$ . We deduce that  $|x_k| \leq \frac{1}{\tilde{E}}\lambda_k$ . Also, for  $|x| > |x_k|$ , we have  $2\tilde{V}(x) < -2\lambda_k$ . Then, for  $\tau = 2\lambda_k$ , we have

$$\begin{aligned} k+1 &= N_W(2\lambda_k) \leq \int_{-|x_k|}^{|x_k|} |x|(2\lambda_k + 2\tilde{V}(x))_+ dx + 1 \leq \frac{2}{\tilde{E}}\lambda_k \int_{-|x_k|}^{|x_k|} (\lambda_k + \tilde{V}(x))_+ dx + 1 \\ &= \frac{2}{\tilde{E}}\lambda_k \int_{|x| \leq \tilde{x}_0} (\lambda_k + \tilde{V}(x))_+ dx + \frac{2}{\tilde{E}}\lambda_k \int_{\tilde{x}_0 \leq |x| \leq |x_k|} (\lambda_k + \tilde{V}(x))_+ dx + 1 \\ &\leq \frac{4\tilde{x}_0}{\tilde{E}}\lambda_k(\lambda_k + S_0) + \frac{2}{\tilde{E}}\lambda_k \int_{\tilde{x}_0 \leq |x| \leq |x_k|} (\lambda_k + \tilde{V}(x))_+ dx + 1 \end{aligned}$$

To estimate the last integral we use, for  $\tilde{x}_0 \leq |x| \leq |x_k|$

$$\lambda_k + \tilde{V}(x) \leq \lambda_k - \tilde{E}|x|.$$

Since the right hand side is positive, we get  $(\lambda_k + \tilde{V}(x))_+ \leq \lambda_k - E|x|$  on the range  $\tilde{x}_0 \leq |x| \leq |x_k|$ . We conclude that

$$k \leq \frac{8\tilde{x}_0}{\tilde{E}}\lambda_k^2 + 4\lambda_k \int_0^{\lambda_k/\tilde{E}} \frac{1}{\tilde{E}}\lambda_k - x dx \leq \frac{8\tilde{x}_0}{\tilde{E}}\lambda_k^2 + \frac{2}{\tilde{E}^2}\lambda_k^3 \leq \frac{8\tilde{x}_0+2}{\tilde{E}^2}\lambda_k^3,$$

and the result follows.  $\square$

**Corollary 3.2.** *For all  $t > 0$  it holds  $\sum_{k \geq 0} e^{-\lambda_k t} < \infty$ .*

*Proof.* For large  $k$  we have  $e^{-\lambda_k t} \leq e^{-Ct k^{1/3}} \leq k^{-2}$ .  $\square$

The first result is the existence of densities for both  $(P_t)$  and  $(\tilde{P}_t)$ . We also provide some a priori estimates on these eigenfunctions.

**Proposition 3.3.** *The following properties hold under H0-H4.*

(i) *for every  $k$ , the function  $\Psi_k$  is an eigenfunction of  $\tilde{P}$ , that is, for all  $x, k, s > 0$  it holds*

$$\Psi_k(x) = e^{\lambda_k t} \mathbb{E}_x \left( e^{\int_0^t \tilde{V}(B_s) ds} \Psi_k(B_s) \right). \quad (3.4)$$

(ii) *For every  $k \geq 0$  and all  $x$  it holds*

$$|\Psi_k(x)| \leq \left( \frac{e}{\pi} \right)^{1/4} (\lambda_k + \tilde{A})^{1/4}. \quad (3.5)$$

*On the other hand, for all  $R > 0$ , there exist a constant  $C_1 = C_1(R)$  (see 3.12 below) and  $t_0 = t_0(R) = 2R/\tilde{E}$  such that, for all  $k \geq 0$  and all  $x$  it holds*

$$e^{R|x|} |\Psi_k(x)| \leq C_1(R) e^{\lambda_k t_0}. \quad (3.6)$$

*Moreover, for any  $t > 0$  there exists a constant  $C_2(R, t)$  so that, for all  $x \in \mathbb{R}, k \geq 0, 0 < s \leq t$  it holds*

$$e^{R|x|} |\Psi_k(x)| \leq C_2(R, t) \frac{e^{\lambda_k s}}{s^{1/4}}. \quad (3.7)$$

(iii) *The semigroup  $\tilde{P}$  has a transition density  $\tilde{p}(t, x, y)$  given by*

$$\tilde{p}(t, x, y) = \sum_{k \geq 0} e^{-\lambda_k t} \Psi_k(x) \Psi_k(y), \quad (3.8)$$

*where this series converges absolutely for all  $t > 0, x, y$ , uniformly on compact sets of  $(0, \infty) \times \mathbb{R}^2$ . Moreover, for all  $t > 0, x, y$ , we have the following bound*

$$\tilde{p}(t, x, y) \leq \sum_{k \geq 0} e^{-\lambda_k t} |\Psi_k(x) \Psi_k(y)| \leq \frac{e^{\tilde{A}t}}{\sqrt{2\pi t}}. \quad (3.9)$$

(iv) *the density  $\tilde{p}$  is continuous in the three variables on  $(0, \infty) \times \mathbb{R}^2$ .*

**Remark.** *In Section 5 we shall prove that  $\tilde{p}$  is  $C^{\infty,2}$  on  $(0, \infty) \times \mathbb{R}^2$ , improving (iv) in this Proposition.*

*Proof.* (i) From Itô's formula and using the stopping time  $T_n = \inf\{s : |B_s| > n\}$  we have,

$$\mathbb{E}_x \left( e^{\int_0^{t \wedge T_n} \tilde{V}(B_s) ds} e^{\lambda_k(t \wedge T_n)} \Psi_k(B_{t \wedge T_n}) \right) = \Psi_k(x),$$

where the stochastic integral part vanishes because its integrand is bounded, and the finite variation part is 0, because  $\Psi_k$  is an eigenfunction of  $\tilde{\mathcal{L}}$ . Using the dominated convergence theorem (recall that  $\Psi_k$  is continuous and bounded), we get

$$\mathbb{E}_x \left( e^{\int_0^t \tilde{V}(B_s) ds} \Psi_k(B_t) \right) = e^{-\lambda_k t} \Psi_k(x). \quad (3.10)$$

(ii) This follows from the eigenfunction equation given in (i). We first consider  $R = 0$ . Then, we have

$$\begin{aligned} |\Psi_k(x)| &\leq e^{(\lambda_k + \tilde{A})t} \mathbb{E}_x (|\Psi_k(B_t)|) = e^{(\lambda_k + \tilde{A})t} \int |\Psi_k(z)| \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(z-x)^2} dz \\ &\leq e^{(\lambda_k + \tilde{A})t} \left( \int |\Psi_k(z)|^2 dz \right)^{1/2} \left( \frac{1}{2\pi t} \int e^{-\frac{1}{t}(z-x)^2} dz \right)^{1/2} = \frac{e^{(\lambda_k + \tilde{A})t}}{(4\pi t)^{1/4}}. \end{aligned}$$

Notice that  $\lambda_k + \tilde{A} > 0$ , otherwise we conclude that  $\Psi_k \equiv 0$ , by taking  $t \rightarrow \infty$ . So, optimizing over  $t$  we get  $t^* = t^*(k) = \frac{1}{4(\lambda_k + \tilde{A})}$ , which gives

$$|\Psi_k(x)| \leq \left(\frac{e}{\pi}\right)^{1/4} (\lambda_k + \tilde{A})^{1/4},$$

proving the desired property when  $R = 0$ .

Now, assume  $R > 0$ . We first assume  $x > 0$ . Consider  $t > 0$  to be defined latter, which satisfies  $t\tilde{E} \geq 2R$ . For  $z \geq x/2 \geq \tilde{x}_0$  one has

$$\tilde{V}(z) \leq -\tilde{E}z \leq -\tilde{E}x/2.$$

Now, we decompose the expectation in two sets  $D, D^c$ , where

$$D = \left\{ \min_{0 \leq u \leq t} W_u \geq -x/2 \right\},$$

with  $B_s = W_s + x$ , that is,  $W$  is a standard BM. We have

$$\begin{aligned} |h^D(x)| &= e^{\lambda_k t} e^{Rx} \left| \mathbb{E}_x \left( e^{\int_0^t \tilde{V}(B_u) du} \Psi_k(B_t) 1_D \right) \right| \leq e^{\lambda_k t} \mathbb{E}_0 (|\Psi_k(x + W_t)|) \\ &= e^{\lambda_k t} \frac{1}{\sqrt{2\pi t}} \int |\Psi_k(z)| e^{-\frac{1}{2t}(z-x)^2} dz \leq e^{\lambda_k t} \frac{1}{\sqrt{2\pi t}} \left( \int \Psi_k^2(z) dz \right)^{1/2} \left( \int e^{-\frac{1}{t}(z-x)^2} dz \right)^{1/2} \\ &= e^{\lambda_k t} \frac{1}{\sqrt{2\pi t}} (\pi t)^{1/4} = \frac{e^{\lambda_k t}}{(4\pi t)^{1/4}}. \end{aligned}$$

Recall that  $\lambda_0 + \tilde{A} > 0$ . Then, we get

$$|h^D(x)| \leq \frac{1}{(4\pi)^{1/4}} \frac{e^{(\lambda_k + \tilde{A})t}}{t^{1/4}}.$$

Now, we consider the expectation over  $D^c$

$$\begin{aligned} |h^{D^c}(x)| &= e^{\lambda_k t} e^{Rx} \left| \mathbb{E}_x \left( e^{\int_0^t \tilde{V}(B_u) du} \Psi_k(B_t) 1_{D^c} \right) \right| \leq e^{(\lambda_k + \tilde{A})t} e^{Rx} \mathbb{E}_x (|\Psi_k(B_t)| 1_{D^c}) \\ &\leq e^{(\lambda_k + \tilde{A})t} e^{Rx} (\mathbb{P}_0(D^c))^{1/2} \left( \frac{1}{\sqrt{2\pi t}} \int |\Psi(z)|^2 dz \right)^{1/2} = \frac{e^{(\lambda_k + \tilde{A})t}}{(2\pi t)^{1/4}} e^{Rx} (\mathbb{P}_0(D^c))^{1/2}. \end{aligned}$$

By the reflection principle (see for example [20])

$$\begin{aligned} \mathbb{P}_0(D^c) &= \mathbb{P}_0 \left( \max_{s \leq t} W_u \geq x/2 \right) = \frac{2}{\sqrt{2\pi t}} \int_{\frac{x}{2}}^{\infty} e^{-\frac{1}{2t}z^2} dz = \frac{2}{\sqrt{2\pi}} \int_{\frac{x}{2\sqrt{t}}}^{\infty} e^{-\frac{1}{2}u^2} du \\ &\leq \frac{2}{\sqrt{2\pi}} \int_{\frac{x}{2\sqrt{t}}}^{\infty} \frac{u}{2\sqrt{t}} e^{-\frac{1}{2}u^2} du = \frac{2}{\sqrt{2\pi}} \frac{1}{2\sqrt{t}} e^{-\frac{1}{8t}x^2}. \end{aligned}$$

This implies that, for  $x > 8Rt + \sqrt{8t}$

$$e^{2Rx} \mathbb{P}_0(D^c) \leq \frac{2e^{8tR^2}}{\sqrt{2\pi}} \frac{1}{2\sqrt{t}} e^{-\frac{1}{8t}(x-8tR)^2} \leq \frac{e^{8tR^2}}{\sqrt{\pi}} \frac{1}{\frac{x-8tR}{\sqrt{8t}}} e^{-\frac{1}{8t}(x-8tR)^2} \leq \frac{e^{8tR^2}}{\sqrt{\pi}},$$

which gives

$$|h^{D^c}(x)| \leq \frac{1}{(2\pi^3)^{1/4}} \frac{e^{(\lambda_k + \tilde{A} + 4R^2)t}}{t^{1/4}}.$$

Therefore, we get the bound, for  $x \geq x_1 = x_1(R, t) = \max\{2\tilde{x}_0, 8Rt + \sqrt{8t}\}$ ,

$$e^{Rx} |\Psi_k(x)| \leq \left( \frac{1}{(4\pi)^{1/4}} + \frac{1}{(2\pi^3)^{1/4}} \right) \frac{e^{(\lambda_k + \tilde{A} + 4R^2)t}}{t^{1/4}}.$$

A similar inequality holds for  $x < -x_1(R, t)$ . For  $|x| \leq x_1$ , we have, using the first part, that

$$e^{R|x|} |\Psi_k(x)| \leq \left(\frac{e}{\pi}\right)^{1/4} (\lambda_k + \tilde{A})^{1/4} e^{Rx_1}.$$

Notice that  $z^{1/4} \leq e^z$  is valid for all  $z \geq 0$ , from which we deduce

$$e^{R|x|} |\Psi_k(x)| \leq \left[ \left( \frac{1}{(4\pi)^{1/4}} + \frac{1}{(2\pi^2)^{1/4}} \right) e^{(\tilde{A}+4R^2)t} + \left(\frac{e}{\pi}\right)^{1/4} e^{Rx_1} \right] \frac{e^{\lambda_k t}}{t^{1/4}} \leq \Gamma(R, t) \frac{e^{\lambda_k t}}{t^{1/4}}, \quad (3.11)$$

where

$$\Gamma(R, t) = \left[ \left( \frac{1}{(4\pi)^{1/4}} + \frac{1}{(2\pi^2)^{1/4}} \right) e^{|\tilde{A}+4R^2|t} + \left(\frac{e}{\pi}\right)^{1/4} e^{Rx_1(R, t)} \right].$$

Now, we choose  $t = t_0(R) = 2R/\tilde{E}$  to get

$$e^{R|x|} |\Psi_k(x)| \leq C_1(R) e^{\lambda_k t_0},$$

where

$$C_1(R) = \frac{\Gamma(R, t_0)}{t_0^{1/4}} = \frac{1}{(2R/\tilde{E})^{1/4}} \left( \frac{1}{(4\pi)^{1/4}} + \frac{1}{(2\pi^2)^{1/4}} \right) e^{|\tilde{A}+4R^2|2R/\tilde{E}} + \left(\frac{e}{\pi}\right)^{1/4} e^{Rx_1} \quad (3.12)$$

with  $x_1 = x_1(R, 2R/\tilde{E})$ .

The bound (3.7) is obtained in the same way with

$$C_2(R, t) = \Gamma(R, t).$$

The result is shown (recall that  $\tilde{E} = E/2$ ).

(iii) Take  $\phi$  a continuous non negative function with compact support, in particular  $\phi \in L^2(dx)$ , therefore

$$\phi(x) = \sum_{k \geq 0} \Psi_k(x) \int \Psi_k(y) \phi(y) dy,$$

where the series converges in  $L^2(dx)$ . Using that the exponential part is bounded and the dominated convergence theorem, we conclude that

$$\int \phi(x) \mathbb{E}_x \left( e^{\int_0^t \tilde{V}(B_s) ds} \phi(B_t) \right) dx = \sum_{k \geq 0} e^{-\lambda_k t} \int \Psi_k(x) \phi(x) dx \int \Psi_k(y) \phi(y) dy.$$

In particular we have, for all  $n \geq 1$

$$\begin{aligned} \sum_{k=1}^n e^{-\lambda_k t} \left( \int \Psi_k(x) \phi(x) dx \right)^2 &\leq \int \phi(x) \mathbb{E}_x \left( e^{\int_0^t \tilde{V}(B_s) ds} \phi(B_t) \right) dx \\ &\leq e^{\tilde{A}t} \int_{\mathbb{R}^2} \phi(x) \phi(y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} dx dy \leq \frac{e^{\tilde{A}t}}{\sqrt{2\pi t}} \int_{\mathbb{R}^2} \phi(x) \phi(y) dx dy. \end{aligned}$$

Using the integrability and continuity of the eigenfunctions, and taking a sequence  $\phi$  converging to  $\delta_x$  we obtain the bound, for all  $n, x, t > 0$

$$\sum_{k=0}^n e^{-\lambda_k t} \Psi_k^2(x) \leq \frac{e^{\tilde{A}t}}{\sqrt{2\pi t}},$$

from where follows the point-wise convergence of the series

$$\sum_{k \geq 0} e^{-\lambda_k t} \Psi_k^2(x) \leq \frac{e^{\tilde{A}t}}{\sqrt{2\pi t}}.$$

Then, Cauchy-Schwarz inequality shows the absolute convergence and domination for

$$\sum_{k \geq 0} e^{-\lambda_k t} |\Psi_k(x)| |\Psi(y)| \leq \frac{e^{\tilde{A}t}}{\sqrt{2\pi t}}.$$

This implies that  $\tilde{p}$  is well defined and bounded on  $(t_0, \infty) \times \mathbb{R}^2$ , for all  $t_0 > 0$ .

Consider  $\phi_2 \in L^2(dx)$  a finite combination of the eigenfunctions:  $\phi_2(y) = \sum_{k=0}^m a_k \Psi_k(y)$ . Notice that  $\phi_2$  is continuous and bounded. As before, for all  $x, t$  it holds for all  $n \geq m$  (here we use the continuity of each  $\Psi_k$ )

$$\begin{aligned} \mathbb{E}_x \left( e^{\int_0^t \tilde{V}(B_s) ds} \phi_2(B_t) \right) &= \sum_{k=0}^m e^{-\lambda_k t} \Psi_k(x) \int \Psi_k(y) \phi_2(y) dy = \\ &= \int \sum_{k=0}^n e^{-\lambda_k t} \Psi_k(x) \Psi_k(y) \phi_2(y) dy = \int \tilde{p}(t, x, y) \phi_2(y) dy, \end{aligned}$$

where the last equality follows from dominated convergence theorem. Using the density of such functions  $\phi_2$  in  $L^2(dx)$ , we get that  $\tilde{p}(t, x, \bullet)$  is a density for  $\tilde{P}_t(\bullet)(x)$ .

(iv) Assume that  $(t_n, x_n, y_n) \rightarrow (t, x, y) \in (0, \infty) \times \mathbb{R}$ . Consider  $r = \inf\{t_n; n \in \mathbb{N}\} > 0$  and  $k_0 = \inf\{k; \lambda_k > 0\}$ , then for all  $k \geq k_0$  we have (see (3.5))

$$\sup_n e^{-\lambda_k t_n} |\Psi_k(x_n)| |\Psi_k(y_n)| \leq e^{-\lambda_k r} \left( \frac{e}{\pi} \right)^{1/2} (\lambda_k + \tilde{A})^{1/2}$$

which according to Lemma 3.2 is summable. Then we can use the dominated convergence theorem (for series) to get that

$$\lim_{n \rightarrow \infty} \sum_{k \geq 0} e^{-\lambda_k t_n} \Psi_k(x_n) \Psi_k(y_n) = \sum_{k \geq 0} e^{-\lambda_k t} \Psi_k(x) \Psi_k(y),$$

holds uniformly on compact sets of  $(0, \infty) \times \mathbb{R}^2$ . □

The following result gives a representation of the density for the semigroup  $(P_t)$ . Recall that  $|\ell(x)| \leq \gamma + \beta|x|$ , for all  $x$  (see H0.2).

**Corollary 3.4.** *Assume H0-H4, then*

- (1) *The set  $(\Theta_k = \Psi_k e^{\ell})_k$  is an orthonormal basis of  $L^2(\rho(dy))$ .*
- (2) *The semigroup  $(P_t)$  can be extended to  $L^2(\rho(dx))$  and  $L^2(dx)$ . Each  $\Theta_k = \Psi_k e^{\ell} \in L^2(\rho(dx)) \cap L^2(dx)$  is an eigenvector of  $P_t$ , with eigenvalue  $e^{-\lambda_k t}$ . This semigroup has a symmetric density with respect to  $\rho(dx)$  given by, for  $t > 0$*

$$r(t, x, y) = e^{\ell(x) + \ell(y)} \sum_{k \geq 0} e^{-\lambda_k t} \Psi_k(x) \Psi_k(y) = \sum_{k \geq 0} e^{-\lambda_k t} \Theta_k(x) \Theta_k(y),$$

and similarly a density with respect to  $dx$

$$p(t, x, y) = e^{\ell(x) - \ell(y)} \sum_{k \geq 0} e^{-\lambda_k t} \Psi_k(x) \Psi_k(y). \tag{3.13}$$

Also we have the bound

$$p(t, x, y) \leq e^{\ell(x) - \ell(y)} \sum_{k \geq 0} e^{-\lambda_k t} |\Psi_k(x) \Psi_k(y)| \leq e^{\ell(x) - \ell(y)} \frac{e^{\tilde{A}t}}{\sqrt{2\pi t}}, \tag{3.14}$$

and a similar bound for  $r$ .

(3) Let  $R' \geq 0$  and  $s_0 = 2(R' + \beta)/\tilde{E}$ . Then, there exists  $F = F(R') < \infty$  so that for  $t > 3s_0$  it holds

$$r(t, x, y) \vee p(t, x, y) \leq F e^{-\lambda_0(t-2s_0)} e^{-R'(|x|+|y|)}. \quad (3.15)$$

Moreover, for every compact set  $[t_0, t_1] \subset (0, \infty)$ , both series  $r, p$ , converge absolutely and they are uniformly bounded on  $[t_0, t_1] \times \mathbb{R}^2$ . If  $\lambda_0 \geq 0$ , the same conclusion holds on  $[t_0, \infty) \times \mathbb{R}^2$ . Also  $r, p$  are continuous on  $(0, \infty) \times \mathbb{R}^2$ .

(4)  $(P_t)_{t>0}$  can be extended to the class of functions that has at most an exponential linear growth at  $\pm\infty$ , and for every  $\phi$  in this class and  $t > 0$  we have that

$$P_t(\phi)(x) = \int r(t, x, y)\phi(y)e^{-2\ell(y)}dy = \int p(t, x, y)\phi(y)dy,$$

is bounded in  $x$ , and has an exponential decay at  $x = \pm\infty$ , uniformly on  $t \geq t_0$ , for all  $t_0 > 0$ .

*Proof.* Only part (3) needs some extra argument. Notice that, by Proposition 3.3 with  $R = R' + \beta$  (see (3.6))

$$\begin{aligned} r(t, x, y) \vee p(t, x, y) &\leq e^{2\gamma+\beta(|x|+|y|)} \sum_{k \geq 0} e^{-\lambda_k t} |\psi_k(x)| |\psi_k(y)| \leq e^{2\gamma} (C_1(R'))^2 e^{-R'(|x|+|y|)} \sum_{k \geq 0} e^{-\lambda_k(t-2s_0)} \\ &\leq e^{2\gamma} (C_1(R'))^2 e^{-\lambda_0(t-2s_0)} \sum_{k \geq 0} e^{-(\lambda_k - \lambda_0)s_0} e^{-R'(|x|+|y|)} = F(R') e^{-\lambda_0(t-2s_0)} e^{-R'(|x|+|y|)}, \end{aligned}$$

where  $F(R') = e^{2\gamma} (C_1(R'))^2 \sum_{k \geq 0} e^{-(\lambda_k - \lambda_0)s_0}$  and the bound is shown. The second part is obtained by using the same proposition with the bound provided by (3.7).  $\square$

The next Theorem is a Yaglom-type of result

**Corollary 3.5.** *Under H0-H4, for all  $x, y$  it holds*

$$\lim_{t \rightarrow \infty} e^{\lambda_0 t} p(t, x, y) = e^{\ell(x) - \ell(y)} \Psi_0(x) \Psi_0(y) = e^{-2\ell(y)} \Theta_0(x) \Theta_0(y)$$

and  $\Psi_0, \Theta_0$  have constant sign, that is, we can take them to be positive. A similar result holds for  $r$ .

*Proof.* The assertion follows directly from the representation of  $p(t, x, y)$ .  $\square$

## 4 Proof of the results

### 4.1 Proof of Theorem 2.7

In the proof below we do the estimates in  $L^q(dx)$ . Similar estimates can be done in  $L^q(\rho(dx))$ .

*Proof.* (1). Consider  $g \in L^q(dx)$ , then we have

$$\begin{aligned} P_t(g - \Pi(g))(x) &= e^{\ell(x)} \sum_{k \geq 0} e^{-\lambda_k t} \Psi_k(x) \int \Psi_k(y) (g(y) - \Pi(g)(y)) e^{-\ell(y)} dy \\ &= e^{\ell(x)} \sum_{k \geq 1} e^{-\lambda_k t} \Psi_k(x) \int \Psi_k(y) g(y) e^{-\ell(y)} dy. \end{aligned}$$

Then, according to Proposition 3.3 for  $R = \beta + 1, t_0 = 2R/\tilde{E} = 4R/E$ , there exists a constant  $C = e^\gamma C_1(R)$ , where  $C_1(R)$  is given in (3.12), so that for all  $k, x$  it holds

$$|\Psi_k(x)| e^{\pm\ell(x)} \leq |\Psi_k(x)| e^{\gamma + \beta|x|} \leq C e^{\lambda_k t_0} e^{-|x|},$$

and, for  $p$  the conjugate of  $q$

$$\left| \int \Psi_k(y) e^{-\ell(y)} g(y) dy \right| \leq \|\Psi_k e^{-\ell}\|_p \|g\|_q \leq C e^{\lambda_k t_0} \left( \int e^{-p|x|} dx \right)^{1/p} \|g\|_q \leq 2C e^{\lambda_k t_0} \|g\|_q.$$

Hence, we obtain that

$$\begin{aligned} \|P_t(g - \Pi(g))\|_r &\leq 2C \|g\|_q \sum_{k \geq 1} e^{-\lambda_k(t-t_0)} \|\Psi_k e^\ell\|_r \leq 4C^2 \|g\|_q \sum_{k \geq 1} e^{-\lambda_k(t-2t_0)} \\ &= e^{-\lambda_1(t-2t_0)} 4C^2 \sum_{k \geq 1} e^{-(\lambda_k - \lambda_1)(t-2t_0)} \|g\|_q \leq K e^{-\lambda_1 t} \|g\|_q \end{aligned}$$

where  $K = 4C^2 e^{2\lambda_1 t_0} \sum_{k \geq 1} e^{-(\lambda_k - \lambda_1)t_0} < \infty$ . For the second part we notice that  $P_t(\Pi(g)) = e^{-\lambda_0 t} \Pi(g)$ , then we get

$$\|e^{\lambda_0 t} P_t(g) - \Pi(g)\|_r \leq K \|g\|_r e^{-(\lambda_1 - \lambda_0)t},$$

proving the result in this case.

(2). The proof is similar, where this time we take  $R = \kappa + \beta + 1$ ,  $C = e^\gamma C_1(R)$ ,  $t_1 = 2R/\tilde{E}$  to bound

$$\left| \int \Psi_k(y) e^{-\ell(y)} g(y) dy \right| \leq A C e^{\lambda_k t_1} \int e^{-|y|} dy = 2A C e^{\lambda_k t_1},$$

and we bound  $\|\Psi_k e^\ell\|_\infty \leq e^\gamma C_1(\beta + 1) e^{\lambda_k t_0}$ , to get

$$\|P_t(g - \Pi(g))\|_\infty \leq A 2e^{2\gamma} C_1(\kappa + \beta + 1) C_1(\beta + 1) \sum_{k \geq 1} e^{-\lambda_k(t-t_1-t_0)} \leq A D e^{-\lambda_1 t},$$

where

$$D = D(\kappa) = 2e^\gamma C_1(\kappa + \beta + 1) C_1(\beta + 1) e^{\lambda_1(t_1+t_0)} \sum_{k \geq 1} e^{-(\lambda_k - \lambda_1)(t_1+t_0)}. \quad (4.1)$$

The last part of (2) is shown as above.  $\square$

## 4.2 Proof of Theorem 2.2 and Theorem 2.3

*Proof.* (Theorem 2.2) Assume that  $\phi$  is a nonnegative function and with compact support. Then,

$$\begin{aligned} \int \mathbb{E}_x \left( e^{\int_0^t V(X_s) ds} \phi(X_t) \right) \Theta_0(x) e^{-2\ell(x)} dx &= \int_{\mathbb{R}^2} \Theta_0(x) \phi(y) p(t, x, y) \phi(y) e^{-2\ell(y)} dy e^{-2\ell(x)} dx \\ &= \int_{\mathbb{R}^2} \Theta_0(x) \phi(y) \sum_{k \geq 0} e^{-\lambda_k t} \Theta_k(x) \Theta_k(y) e^{-2\ell(y)} dy e^{-2\ell(x)} dx \\ &= \int \Theta_0(x) \sum_{k \geq 0} e^{-\lambda_k t} \Theta_k(x) \left( \int \Theta_k(y) \phi(y) e^{-2\ell(y)} dy \right) e^{-2\ell(x)} dx \\ &= e^{-\lambda_0 t} \int \Theta_0(y) \phi(y) e^{-2\ell(y)} dy, \end{aligned}$$

where the last three equalities follow from Fubini's Theorem, the absolute convergence of the series and the bound given in (3.14), the fact that  $\phi$  is in  $L^2(\rho(dy))$  and the orthonormality of  $(\Theta_k)_k$ . Hence, the nonnegative measures  $\mu, \xi$  defined by

$$\int \phi(y) \mu(dy) := \int \mathbb{E}_x \left( e^{\int_0^t V(X_s) ds} \phi(X_t) \right) \Theta_0(x) e^{-2\ell(x)} dx$$

and

$$\int \phi(y)\xi(dy) := e^{-\lambda_0 t} \int \phi(y)\Theta_0(y)e^{-2\ell(y)} dy,$$

coincide for all  $\phi$  nonnegative measurable and with compact support. Since  $\xi$  is a finite measure (thanks to (3.2)), we conclude  $\mu = \xi$ . In particular, for all  $\phi$  bounded and measurable it holds

$$\int \mathbb{E}_x \left( e^{\int_0^t V(X_s) ds} \phi(X_t) \right) \Theta_0(x) e^{-2\ell(x)} dx = e^{-\lambda_0 t} \int \phi(x)\Theta_0(x)e^{-2\ell(x)} dx$$

If we take  $\phi \equiv 1$  we get

$$\int \mathbb{E}_x \left( e^{\int_0^t V(X_s) ds} \right) \Theta_0(x) e^{-2\ell(x)} dx = e^{-\lambda_0 t} \int \Theta_0(x) e^{-2\ell(x)} dx,$$

from which the result follows.  $\square$

*Proof.* (Theorem 2.3) For every  $t_0 > 0$ , there exists a constant  $K(R, t_0)$  so that (see (3.15)), for every  $t > 2t_0$  and  $\phi$  bounded, we have

$$\left| e^{\lambda_0 t} \mathbb{E}_x \left( e^{\int_0^t V(X_s) ds} \phi(X_t) \right) \right| \leq K(R, t_0) \|\phi\|_\infty e^{-R|x|}.$$

So, the dominated convergence theorem allows us to conclude

$$\lim_{t \rightarrow \infty} e^{\lambda_0 t} \int \mathbb{E}_x \left( e^{\int_0^t V(X_s) ds} \phi(X_s) \right) \mu(dx) = \int \Psi_0(z) e^{-\ell(z)} \phi(z) dz \int \Psi_0(x) e^{\ell(x)} \mu(dx).$$

A similar relation holds for  $\phi \equiv 1$ , from where the result follows.  $\square$

### 4.3 Proof of Theorem 2.5: The generator of the $Q$ -process

*Proof.* We first notice that for  $t > s \geq 0$  it holds

$$\mathbb{E}_x \left( e^{\int_0^t V(X_u) du} \phi(X_s) \right) = \mathbb{E}_x \left( e^{\int_0^s V(X_u) du} \phi(X_s) \mathbb{E}_{X_s} \left( e^{\int_0^{t-s} V(X_u) du} \right) \right)$$

From what we have proved, for all  $y$  it holds (see Theorem 2.1)

$$\lim_{t \rightarrow \infty} e^{\lambda_0(t-s)} \mathbb{E}_y \left( e^{\int_0^{t-s} V(X_u) du} \right) = e^{\ell(y)} \Psi_0(y) \int \Psi_0(z) e^{-\ell(z)} dz.$$

Using Theorem 2.7 part (2) (see the definition of  $t_0, t_1$  and  $D = D(1)$ ) and for all  $(t-s) \geq 3(t_0 + t_1), y \in \mathbb{R}$ , it holds

$$e^{\lambda_0(t-s)} \mathbb{E}_y \left( e^{\int_0^{t-s} V(X_u) du} \right) = e^{\lambda_0(t-s)} P_{t-s}(1)(y) \leq C,$$

where  $C = D + \|\Pi(1)\|_\infty$ . Therefore, using the dominated convergence theorem, we conclude the proof.  $\square$

Recall that  $(Q_t)$  is given by

$$Q_t(\phi)(x) = \mathbb{E}_x \left( e^{\int_0^t V(X_u) du} \frac{e^{\lambda_0 t} e^{\ell(X_t)} \Psi_0(X_t)}{e^{\ell(x)} \Psi_0(x)} \phi(X_t) \right),$$

whose transition kernel is then given by

$$q(t, x, y) = e^{\lambda_0 t} \frac{\Psi_0(y)}{\Psi_0(x)} e^{-\ell(x) + \ell(y)} p(t, x, y) = e^{\lambda_0 t} \frac{\Theta_0(y)}{\Theta_0(x)} p(t, x, y) = e^{\lambda_0 t} \frac{\Psi_0(y)}{\Psi_0(x)} \tilde{p}(t, x, y). \quad (4.2)$$

It follows from the results in Section 5 that  $q$  satisfies the following equation

$$\partial_t q(t, x, y) = (\mathcal{L}^Q)_x^* q(t, x, y),$$

where

$$(\mathcal{L}^Q)^*(u) = \frac{1}{2}u'' - (\Psi'_0/\Psi_0)u' + 2\left(\lambda_0 + \tilde{V}\Psi_0 + \frac{1}{2}(\Psi'_0/\Psi_0)^2\right)u.$$

We notice that  $\eta(dx) = \Psi_0^2(x)dx$  is an invariant probability measure for  $(Q_t)$ , which can be checked by proving that  $(\mathcal{L}^Q)^*(\Psi_0) = 0$  or by using 3.4. Indeed, it holds

$$\int q(t, x, y)\Psi_0^2(x) dx = \Psi_0(y)e^{\lambda_0 t} \int \tilde{p}(t, x, y)\Psi_0(x) dx = \Psi_0(y)e^{\lambda_0 t} \int \tilde{p}(t, y, x)\Psi_0(x) dx = \Psi_0^2(y).$$

Notice that the generator of the  $Q$ -process is

$$\mathcal{L}^Q(u) = \frac{1}{2}u'' + \frac{\Psi'_0}{\Psi_0}u',$$

which corresponds to the stochastic differential equation

$$dY_t = dB_t + \frac{\Psi'_0(Y_t)}{\Psi_0(Y_t)} dt.$$

#### 4.4 Proof of Theorem 2.6

*Proof.* The first thing to compute is the joint density for  $Y_s^T, Y_{t+s}^T$ , where  $0 \leq s \leq t+s \leq T$ . We consider  $\phi, \psi$  two bounded continuous functions and

$$\begin{aligned} m_T(x)\mathbb{E}_x(\phi(Y_s^T)\Psi(Y_{t+s}^T)) &= \mathbb{E}_x(e^{\int_0^T V(X_u)du}\phi(X_s)\psi(X_{t+s})) \\ &= \mathbb{E}_x\left(e^{\int_0^s V(X_u)du}\phi(X_s)\mathbb{E}_{X_s}\left(e^{\int_0^t V(X_u)du}\psi(X_t)\mathbb{E}_{X_t}\left(e^{\int_0^{T-(t+s)} V(X_u)du}\right)\right)\right). \end{aligned}$$

From here we deduce that the joint density, with respect to  $dy dz$  is

$$\mathbb{P}_x(Y_s^T \in dy, Y_{t+s}^T \in dz) = p(s, x, y)p(t, y, z)\frac{m_{T-(t+s)}(z)}{m_T(x)} dy dz \quad (4.3)$$

In particular, if the initial distribution of  $Y_0^T$  is  $e^{\lambda_0 T}m_T(x)\nu(dx)$ , we get the distribution of  $Y_s^T$  has density with respect to Lebesgue measure  $dy$  given by (take  $T = t+s$  in the previous equality)

$$\begin{aligned} e^{\lambda_0 T} \int \Theta_0(x)e^{-2\ell(x)}p(s, x, y)p(T-s, y, z)dx dz &= e^{\lambda_0(T-s)}\Theta_0(y)e^{-2\ell(y)} \int p(T-s, y, z)dz \\ &= e^{\lambda_0(T-s)}\Theta_0(y)e^{-2\ell(y)}m_{T-s}(y). \end{aligned}$$

Here we have used that  $\Theta_0 e^{-2\ell}$  is a left eigenvector for  $P_s$  with eigenvalue  $e^{-\lambda_0 s}$  (this is equivalent to say that  $\nu$  is a q.s.d. for  $(P_t)$ ). So, we get

$$\mathbb{E}_{e^{\lambda_0 T}m_T\nu}(Y_s^T \in dy) = e^{\lambda_0(T-s)}m_{T-s}(y)\nu(dy).$$

This implies that  $Y_T^T$  has distribution  $\nu$ , if the initial distribution is  $e^{\lambda_0 T}m_T(x)\nu(dx)$ . Now we compute the conditional density for the reversed process, when the initial distribution is  $\mathbb{P}\left(\overleftarrow{Y}_0^T \in dy\right) = \nu(dy)$

$$\mathbb{P}\left(\overleftarrow{Y}_{u+r}^T \in dy \mid \overleftarrow{Y}_r^T = z\right) = \mathbb{P}\left(Y_{T-(u+r)}^T \in dy \mid Y_{T-r}^T = z\right).$$

We put  $s = T - (u + r)$  and  $t + s = T - r$ , that is  $t = u$  in (4.3) and using the marginal density for  $\overleftarrow{Y}_r^T$

$$\begin{aligned}
\mathbb{P}\left(\overleftarrow{Y}_{u+r}^T \in dy \mid \overleftarrow{Y}_r^T = z\right) &= \frac{1}{e^{\lambda_0 r} m_r(z) \theta_0(z) e^{-2\ell(z)}} \int p(s, x, y) p(u, y, z) \frac{m_r(z)}{m_T(x)} m_T(x) e^{\lambda_0 T} \Theta_0(x) e^{-2\ell(x)} dx \\
&= e^{\lambda_0(T-r)} \frac{p(u, y, z)}{\Theta_0(z)} e^{2\ell(z)} \int p(s, x, y) \Theta_0(x) e^{-2\ell(x)} dx \\
&= e^{\lambda_0(T-r-s)} \frac{\Theta_0(y)}{\Theta_0(z)} e^{2\ell(z) - 2\ell(y)} p(u, y, z) = e^{\lambda_0 u} \frac{\Theta_0(y)}{\Theta_0(z)} p(u, z, y) \\
&= q(u, z, y).
\end{aligned} \tag{4.4}$$

Here, we have used again that  $\Theta_0 e^{-2\ell}$  is a left eigenvector of  $P_s$  with eigenvalue  $e^{-\lambda_0 s}$  and the symmetry relation  $e^{2\ell(z) - 2\ell(y)} p(u, y, z) = p(u, z, y)$  (see (3.13)). The last equality in (4.4) is given in (4.2). The result follows.  $\square$

## 5 Example : Case $V(x) = 1 - x^2/2$ , $dX_t = \sigma(dB_t - cdt)$

We develop here our approach in the case studied in [6], where  $b \equiv 1$ ,  $d(x) = x^2/2$  and the underlying diffusion process is a drifted Brownian motion with variance  $\sigma^2$  and drift  $c\sigma$ ,  $\sigma$  being the mutation scale parameter. Indeed in this case, we will have explicit computations involving Hermite polynomials.

We define as in the above framework the semigroup

$$P_t(\phi)(x) = \mathbb{E}_x\left(e^{\int_0^t V(X_s) ds} \phi(X_t)\right) = \mathbb{E}_0\left(e^{\int_0^t V(x + \sigma(B_s - cs)) ds} \phi(x + \sigma(B_t - ct))\right).$$

Using Girsanov's transform (see (3.1)), we obtain

$$P_t(\phi)(x) = \mathbb{E}_0\left(e^{\int_0^t (V(x + \sigma W_s) - c^2/2) ds} e^{-cW_t} \phi(x + \sigma W_t)\right),$$

where  $W$  is a new Brownian motion. Writing  $\sigma W_t = \widehat{B}_{\sigma^2 t}$  with another Brownian motion  $\widehat{B}$ , we obtain

$$P_t(\phi)(x) = e^{cx/\sigma} \mathbb{E}_0\left(\exp\left(\frac{1}{\sigma^2} \int_0^{\sigma^2 t} V(x + \widehat{B}_u) - c^2/2 du - \frac{c}{\sigma} (x + \widehat{B}_{\sigma^2 t})\right) \phi(x + \widehat{B}_{\sigma^2 t})\right).$$

Writing  $V(x) = 1 - x^2/2$ , we finally obtain that

$$P_t(\phi)(x) = e^{(1-c^2/2)t} e^{cx/\sigma} R_{\sigma^2 t}(\varphi)(x),$$

where  $\varphi(y) = e^{-\frac{c}{\sigma} y} \phi(y)$ , and

$$R_t(g)(y) = \mathbb{E}_y\left(e^{-\frac{1}{2\sigma^2} \int_0^t (\widehat{B}_u)^2 du} g(\widehat{B}_t)\right).$$

Notice that the generator of the semigroup  $(R_t)$  is given for suitable functions  $v$  by

$$\mathcal{L}_R v = \frac{1}{2} \partial_x^2 v - \frac{x^2}{2\sigma^2} v.$$

The eigenvectors and eigenvalues associated to this generator satisfy

$$\frac{1}{2} \Psi_k''(x) - \frac{1}{2\sigma^2} x^2 \Psi_k(x) = -\lambda_k \Psi_k(x).$$

Consider  $g_k(x) = \Psi_k(\sqrt{\sigma}x)$  which satisfies the equation

$$\frac{1}{2}g''(x) - \frac{1}{2}x^2g(x) = -\sigma\lambda_k g(x) = -\nu_k g(x),$$

whose solutions are the Hermite functions  $H_k(x)e^{-\frac{1}{2}x^2}$ , and associated eigenvalues are  $\nu_k = k + 1/2$ , with  $H_k$  the Hermite polynomials of degree  $k$  (see [28] page 106). So,

$$\Psi_k(x) = g_k(x/\sqrt{\sigma}) = C_k H_k(x/\sqrt{\sigma}) e^{-\frac{x^2}{2\sigma}}, \lambda_k = \frac{k + 1/2}{\sigma}$$

and  $C_k$  is a normalizing constant, in order to have  $\int \Psi_k^2(x) dx = 1$ . Since  $\int H_k^2(x) e^{-x^2} dx = 2^k k! \sqrt{\pi}$ , we conclude that

$$C_k = \sigma^{-1/4} \frac{1}{\sqrt{2^k k! \sqrt{\pi}}}$$

In particular

$$\Psi_0(x) = \left(\frac{1}{\sigma\pi}\right)^{1/4} e^{-\frac{x^2}{2\sigma}}.$$

On the other hand, Cramer inequality (see [15] page 208 and [18]) gives the following refinement of (3.5) in this context, for all  $x, k$

$$|\Psi_k(x)| \leq \left(\frac{1}{\sigma\pi}\right)^{1/4}.$$

For the moment we have the asymptotic

$$\lim_{t \rightarrow \infty} e^{\frac{1}{2\sigma}t} e^{\frac{c}{\sigma}x} R_t(e^{-\frac{c}{\sigma}\bullet} \phi(\bullet))(x) = \Psi_0(x) e^{\frac{c}{\sigma}x} \int \Psi_0(y) e^{-\frac{c}{\sigma}y} \phi(y) dy.$$

This translates into the original semigroup, with potential  $V(x) = 1 - \frac{1}{2}x^2$ ,  $dX_t = \sigma(dB_t - cdt)$ ,  $X_0 = x$  and  $\lambda_0 = 1 - \frac{c^2}{2} - \frac{\sigma}{2}$ :

$$\lim_{t \rightarrow \infty} e^{\lambda_0 t} \mathbb{E}_y \left( e^{\int_0^t V(X_s) ds} \phi(X_t) \right) = e^{-\frac{x^2}{2\sigma} + \frac{c}{\sigma}x} \int \frac{1}{\sqrt{\pi\sigma}} e^{-\frac{y^2}{2\sigma} - \frac{c}{\sigma}y} \phi(y) dy = e^{\frac{c^2}{\sigma}} e^{-\frac{1}{2\sigma}(x-c)^2} \int \frac{1}{\sqrt{\pi\sigma}} e^{-\frac{1}{2\sigma}(y+c)^2} \phi(y) dy.$$

In particular for  $\phi = 1$ , we get for  $m_t(x) = \mathbb{E}_y \left( e^{\int_0^t V(X_s) ds} \right)$

$$\lim_{t \rightarrow \infty} e^{\lambda_0 t} m_t(x) = \sqrt{2} e^{\frac{c^2}{\sigma}} e^{-\frac{1}{2\sigma}(x-c)^2}.$$

Notice that an explicit form for the function  $m_t$  has been given in [6], providing another way to calculate this limit.

Notice that  $\ell(x) = \frac{c}{\sigma}$  and recall that  $\rho(dx) = e^{-2\ell(x)} dx$ . The principal right eigenvector for  $(P_t)$ , normalized to have  $L^2(\rho(dx))$ -norm equal to 1 is

$$\Theta_0(x) = e^{\frac{c^2}{2\sigma}} \frac{e^{-\frac{(x-c)^2}{2\sigma}}}{(\pi\sigma)^{1/4}}.$$

The unique q.s.d. is given by

$$d\nu(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(y+c)^2}{2\sigma}}.$$

The  $Q$  process is given by the stochastic differential equation

$$d\hat{Y}_t = -\sigma\hat{Y}_t + \sigma dB_t.$$

whose invariant measure is  $\Psi_0^2(x) = \frac{1}{\sqrt{\pi\sigma}} e^{-\frac{x^2}{\sigma}}$ .

## Appendix A: The heat equation for $p$

In this appendix we shall prove that  $p$  is a positive solution of the heat equation  $\partial_t p(t, x, y) = \mathcal{L}_y p(t, x, y)$ . To this end we first start with  $\tilde{p}$ .

We first notice that the series  $\sum e^{-\lambda_k t} \Psi_k(x) \Psi_k''(y)$  converges absolutely and uniformly on compact sets in  $(0, t) \times \mathbb{R}^2$ . Indeed, consider  $K = [t_0, t_1] \times [-c, c]^2$ , where  $0 < t_0 < t_1 < \infty, 0 < c < \infty$ . For  $(t, x, y) \in K$ , we have

$$|\Psi_k''(y)| = |2\tilde{V}(y) + 2\lambda_k| |\Psi_k(y)| \leq 2 \left( \max_{|y| \leq c} |\tilde{V}(y)| + |\lambda_k| \right) |\Psi_k(y)|,$$

which implies that for some finite constant  $C$  (see Proposition 3.3 formula (3.5))

$$\sum_{k \geq 0} e^{-\lambda_k t} |\Psi_k(x)| |\Psi_k''(y)| \leq C \sum_{k \geq 0} e^{-\lambda_k t} \left( \max_{|y| \leq c} |V(y)| + |\lambda_k| \right) (\lambda_k + \tilde{A})^{1/2}.$$

The series in the right hand side is converging. This shows the claim and moreover

$$\zeta(t, x, y) := \sum_{k \geq 0} e^{-\lambda_k t} \Psi_k(x) \Psi_k''(y),$$

is well defined and continuous in  $K$ . We now need to control the growth of  $(\Psi_k'(x^*))_k$  for some  $x^*$  fixed.

**Lemma .1.** *There exist two finite constants  $A_1 \geq 0, A_2 \geq 0$  and  $x^* \in \mathbb{R}$  such that, for all  $k$ , we have*

$$|\Psi_k'(x^*)| \leq A_1 + A_2 \lambda_k. \quad (.1)$$

*Proof.* It is well known that each  $\Psi_k$  has exactly  $k$  zeros (see Theorem 3.5 in [5]). Then,  $u = \Psi_2$  has two zeros  $x_1 < x_2$ . We conclude, there exists  $x^* \in (x_1, x_2)$  such that  $u'(x^*) = 0$ , and therefore  $|u(x^*)| > 0$ . Now, we consider  $v = \Psi_k$ , for  $k \geq 3$  and the Wronskian

$$W(x) = \frac{1}{2}(u'v - uv').$$

Since  $W'(x) = \frac{1}{2}(u''v - uv'') = (\lambda_k - \lambda_2)uv$ , we get

$$\begin{aligned} W(x^*) - W(x_1) &= \frac{1}{2}(u'(x^*)v(x^*) - u(x^*)v'(x^*)) - \frac{1}{2}(u'(x_1)v(x_1) - u(x_1)v'(x_1)) \\ &= -\frac{1}{2}u(x^*)v'(x^*) - \frac{1}{2}u'(x_1)v(x_1) = (\lambda_k - \lambda_2) \int_{x_1}^{x^*} u(z)v(z) dz, \end{aligned}$$

and therefore

$$\begin{aligned} |v'(x^*)||u(x^*)| &\leq |v(x_1)||u'(x_1)| + 2(\lambda_k - \lambda_2) \int_{-\infty}^{\infty} |u(z)v(z)| dz \\ &\leq \left(\frac{\epsilon}{\pi}\right)^{1/4} |u'(x_1)| (\lambda_k + \tilde{A})^{1/4} + 2(\lambda_k - \lambda_2) \left( \int_{-\infty}^{\infty} |u(z)|^2 dz \right)^{1/2} \left( \int_{-\infty}^{\infty} |u(z)|^2 dz \right)^{1/2} \\ &\leq \left(\frac{\epsilon}{\pi}\right)^{1/4} |u'(x_1)| (\lambda_k + \tilde{A})^{1/4} + 2(\lambda_k - \lambda_2) \\ &\leq \left(\frac{\epsilon}{\pi}\right)^{1/4} |u'(x_1)| (\lambda_k + \tilde{A})^{1/4} + 2|\lambda_2| + 2\lambda_k. \end{aligned}$$

Consider  $k_0 = \inf\{k : \lambda_k + \tilde{A} \geq 1\}$ , then for  $k \geq k_0$ , we have

$$|\Psi_k'(x^*)| \leq D_1 + A_2 \lambda_k,$$

with  $D_1 = \frac{(\frac{\epsilon}{\pi})^{1/4} |u'(x_1)| \tilde{A} + 2|\lambda_2|}{|u(x^*)|}$  and  $A_2 = \frac{(\frac{\epsilon}{\pi})^{1/4} |u'(x_1)| + 2}{|u(x^*)|}$ . The result follows by taking

$$A_1 = D_1 \vee \max\{|\Psi_j'(x^*)| + A_2|\lambda_j| : 0 \leq j \leq k_0 - 1\}.$$

□

From the previous Lemma, the series

$$S_1(t, x) := \sum_{k \geq 0} e^{-\lambda_k t} \Psi_k(x) \Psi'_k(x^*), \quad S_2(t, x, y) := \sum_{k \geq 0} e^{-\lambda_k t} \Psi_k(x) (\Psi'_k(x^*)y + \Psi_k(0))$$

are absolutely convergent, uniformly on  $K$  and continuous there.

The DCT allow us to show that

$$\begin{aligned} & \int_0^y \int_{x^*}^z \zeta(t, x, w) dw dz + \int_0^y S_1(t, x) dz + \tilde{p}(t, x, 0) \\ &= \sum_{k \geq 0} e^{-\lambda_k t} \Psi_k(x) \left( \int_0^y \int_{x^*}^z \Psi''_k(w) dw dz + \Psi'_k(x^*)y + \Psi_k(0) \right) \\ &= \tilde{p}(t, x, y). \end{aligned}$$

This shows that  $\tilde{p}(t, x, y)$  is twice continuously differentiable in  $y$  and

$$\begin{aligned} \partial_y \tilde{p}(t, x, y) &= \sum_{k \geq 0} e^{-\lambda_k t} \Psi_k(x) \Psi'_k(y), \quad \partial_y^2 \tilde{p}(t, x, y) = \sum_{k \geq 0} e^{-\lambda_k t} \Psi_k(x) \Psi''_k(y); \\ \partial_x \tilde{p}(t, x, y) &= \sum_{k \geq 0} e^{-\lambda_k t} \Psi'_k(x) \Psi_k(y), \quad \partial_x^2 \tilde{p}(t, x, y) = \sum_{k \geq 0} e^{-\lambda_k t} \Psi''_k(x) \Psi_k(y) \end{aligned}$$

The same technique shows that  $\tilde{p} \in C^{\infty, 2}((0, \infty) \times \mathbb{R}^2)$  and

$$\partial_t \tilde{p}(t, x, y) = \sum_{k \geq 0} -\lambda_k e^{-\lambda_k t} \Psi_k(x) \Psi_k(y).$$

Now, we have

$$\begin{aligned} \partial_t \tilde{p}(t, x, y) &= \sum_{k \geq 0} -\lambda_k e^{-\lambda_k t} \Psi_k(x) \Psi_k(y) = \sum_{k \geq 0} e^{-\lambda_k t} \Psi_k(x) \left( \frac{1}{2} \Psi''_k(y) + \tilde{V}(y) \Psi_k(y) \right) \\ &= \frac{1}{2} \partial_y^2 \tilde{p}(t, x, y) + \tilde{V}(y) \tilde{p}(t, x, y) = \tilde{\mathcal{L}}_y \tilde{p}(t, x, y) = \tilde{\mathcal{L}}_x \tilde{p}(t, x, y) \end{aligned}$$

Now, we can apply the relation to  $p(t, x, y) = e^{\ell(x)\ell(y)} \tilde{p}(t, x, y)$  to conclude that

$$\begin{aligned} \partial_t p(t, x, y) &= e^{\ell(x)-\ell(y)} \partial_t \tilde{p}(t, x, y), \\ \partial_x p(t, x, y) &= e^{\ell(x)-\ell(y)} [\partial_x \tilde{p}(t, x, y) + a(x) \tilde{p}(t, x, y)] \\ \partial_x^2 p(t, x, y) &= e^{\ell(x)-\ell(y)} [\partial_x^2 \tilde{p}(t, x, y) + 2a(x) \partial_x \tilde{p}(t, x, y) + a^2(x) \tilde{p}(t, x, y) + a'(x) \tilde{p}(t, x, y)]. \end{aligned}$$

So, we get

$$\partial_t p(t, x, y) = \mathcal{L}_y p(t, x, y),$$

where  $\mathcal{L}(u) = \frac{1}{2}u'' - au' + Vu$ , and  $\frac{1}{2}u'' - au'$  is the generator of  $(X_t)$ . Also, we get

$$\partial_t p(t, x, y) = \mathcal{L}_x^* p(t, x, y),$$

where  $\mathcal{L}^*(u) = \frac{1}{2}u'' + au' + a'u + Vu$ .

**Corollary .2.** *Both  $p$  and  $\tilde{p}$  satisfy Harnack's inequality*

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