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# On the friable mean-value of the Erdős-Hooley Delta function

B. Martin, G. Tenenbaum, and J. Wetzer

ABSTRACT. For integer  $n$  and real  $u$ , define  $\Delta(n, u) := |\{d : d \mid n, e^u < d \leq e^{u+1}\}|$ . Then, the Erdős-Hooley Delta function is defined as  $\Delta(n) := \max_{u \in \mathbb{R}} \Delta(n, u)$ . We provide uniform upper and lower bounds for the mean-value of  $\Delta(n)$  over friable integers, i.e. integers free of large prime factors.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

For integer  $n \geq 1$  and real  $u$ , put

$$\Delta(n, u) := |\{d : d \mid n, e^u < d \leq e^{u+1}\}|, \quad \Delta(n) := \max_{u \in \mathbb{R}} \Delta(n, u).$$

The  $\Delta$ -function was introduced by Erdős in 1974 and was highlighted in 1979 by Hooley [13]. It turned out to be a key-concept in many branches of analytic number theory such as Waring type problems, circle method, Diophantine approximation, distribution of prime factors in polynomial sequences, etc.

However, the behaviour of  $\Delta(n)$  remains rather mysterious. For instance, the average order is still not known with desirable precision. Hall and Tenenbaum [8] obtained in 1982 the lower bound

$$(1.1) \quad D(x) := \sum_{n \leq x} \Delta(n) \gg x \log_2 x \quad (x \geq 3),$$

whereas Tenenbaum [16] showed in 1985 that for suitable  $c > 0$  we have

$$(1.2) \quad D(x) \ll x e^{c \sqrt{\log_2 x \log_3 x}} \quad (x \geq 16).$$

Here and in the sequel, we let  $\log_k$  denote the  $k$ -fold iterated logarithm. Recently, La Bretèche and Tenenbaum [3, th.1.1] obtained a slight improvement to (1.2) by removing the triple logarithm in the exponent and, even more recently, Koukoulopoulos and Tao [14] obtained the remarkable bound

$$D(x) \ll x (\log_2 x)^{11/4} \quad (x \geq 3).$$

A few months later, Ford, Koukoulopoulos and Tao [7] improved (1.1) by showing

$$D(x) \gg x (\log_2 x)^{1+\eta+o(1)} \quad (x \geq 3),$$

where the exponent  $\eta \approx 0.3533227$  appears in the work of Ford, Green and Koukoulopoulos [6] on the normal order of  $\Delta(n)$ . Both bounds have been recently improved by La Bretèche and Tenenbaum [4]: we have

$$(1.3) \quad x (\log_2 x)^{3/2} \ll D(x) \ll x (\log_2 x)^{5/2} \quad (x \geq 3),$$

which constitutes the current state of the art.

Let  $P^+(n)$  denote the largest prime factor of an integer  $n > 1$  and let us agree that  $P^+(1) = 1$ . Following usual notation, we define  $S(x, y)$  as the set of  $y$ -friable integers not exceeding  $x$ , and denote by  $\Psi(x, y)$  its cardinality, viz.

$$S(x, y) := \{n \leq x : P^+(n) \leq y\}, \quad \Psi(x, y) = |S(x, y)| \quad (x \geq 1, y \geq 1).$$

Structural properties of the set  $S(x, y)$  motivated a vast array of the literature in the last forty years. The applications are indeed numerous and significant: circle method, Waring-type problems, cryptology, sieve theory, probabilistic models in number theory.

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Given an arithmetical function  $f$ , let us use the notation  $\Psi(x, y; f) := \sum_{n \in S(x, y)} f(n)$ . In this work we investigate bounds for the friable mean-value

$$(1.4) \quad \mathfrak{S}(x, y) := \frac{\Psi(x, y; \Delta)}{\Psi(x, y)} \quad (x \geq y \geq 2).$$

We now define some quantities arising in our statements. Given  $\kappa > 0$ , denote by  $\varrho_\kappa$  the continuous solution on  $]0, \infty[$  of the delay differential system

$$\begin{cases} \varrho_\kappa(v) = v^{\kappa-1}/\Gamma(\kappa) & (0 < v \leq 1), \\ v\varrho'_\kappa(v) + (1 - \kappa)\varrho_\kappa(v) + \kappa\varrho_\kappa(v-1) = 0 & (v > 1), \end{cases}$$

and set  $\varrho_\kappa(v) := 0$  for  $v < 0$ .

Thus (see, e.g., [12])  $\varrho_\kappa$  is the order  $\kappa$  fractional convolution power of  $\varrho := \varrho_1$ , the Dickman function, which provides a continuous approximation to  $\Psi(x, y)$  in

$$(1.5) \quad H_\varepsilon := \left\{ (x, y) : x \geq 3, e^{(\log_2 x)^{5/3+\varepsilon}} \leq y \leq x \right\} \quad (\varepsilon > 0).$$

Indeed, improving on results by Dickman and de Bruijn, Hildebrand [10] proved the asymptotic formula

$$(1.6) \quad \Psi(x, y) = x\varrho(u) \left\{ 1 + O\left(\frac{\log(2u)}{\log y}\right) \right\} \quad ((x, y) \in H_\varepsilon),$$

with the standard notation

$$u = \frac{\log x}{\log y}.$$

The asymptotic behaviour of the functions  $\varrho_\kappa$  (and in fact of more general delay differential equations, as displayed in [12]) may be described in terms of the function  $\xi(t)$  defined as the unique positive solution to  $e^\xi = 1 + t\xi$  for  $t \neq 1$  and by  $\xi(1) = 0$ . From [17, lemma III.5.11] and the remark following [17, th. III.5.13], we quote the estimates

$$(1.7) \quad \xi(t) = \log t + \log_2 t + O\left(\frac{\log_2 t}{\log t}\right), \quad \xi'(t) = \frac{1}{t} + \frac{1}{t \log t} + O\left(\frac{\log_2 t}{t(\log t)^2}\right) \quad (t \rightarrow \infty).$$

Applying [12, cor. 2] in the case  $(a, b) = (1 - \kappa, \kappa)$ , we have

$$(1.8) \quad \varrho_\kappa(v) = \sqrt{\frac{\xi'(v/\kappa)}{2\pi\kappa}} \exp\left\{ \kappa\gamma - \kappa \int_1^{v/\kappa} \xi(t) dt \right\} \left\{ 1 + O\left(\frac{1}{v}\right) \right\} \quad (v \geq 1 + \kappa),$$

where  $\gamma$  denotes Euler's constant. We put

$$(1.9) \quad \mathfrak{r}(v) := \frac{\varrho_2(v)}{\sqrt{v}\varrho(v)} \asymp \frac{1}{\sqrt{v}} \exp\left( \int_1^v \{\xi(t) - \xi(t/2)\} dt \right) \asymp 2^{v+O(v/\log 2v)} \quad (v \geq 1),$$

while a genuine asymptotic formula follows from (1.8).

Let  $\tau(n)$  denote the total number of divisors of an integer  $n$ . We trivially have

$$(1.10) \quad \tau(n)/\log 2n \ll \Delta(n) \leq \tau(n) \quad (n \geq 1),$$

where the lower bounds follows from the pigeon-hole principle. Since, by [19, cor. 2.3], we have

$$\Psi(x, y; \tau) = \left\{ 1 + O\left(\frac{\log(2u)}{\log y}\right) \right\} x\varrho_2(u) \log y \quad ((x, y) \in H_\varepsilon),$$

we may state as a benchmark that

$$\frac{\mathfrak{r}(u)}{\sqrt{u}} \ll \mathfrak{S}(x, y) \ll 2^{u+O(u/\log 2u)} \log y \quad ((x, y) \in H_\varepsilon).$$

We obtain the following results, where the following notation is used:

$$(1.11) \quad \bar{u} := \min\left(\frac{y}{\log y}, u\right) \quad (x \geq y \geq 2),$$

$$(1.12) \quad g(t) := \log\left(\frac{(1+2t)^{1+2t}}{(1+t)^{1+t}(4t)^t}\right) \quad (t > 0),$$

$$(1.13) \quad \varepsilon_y := \frac{1}{\sqrt{\log y}} \quad (y \geq 2).$$

**Theorem 1.1.** (i) Let  $\varepsilon > 0$ . For a suitable absolute constant  $c > 0$  and uniformly for  $(x, y) \in H_\varepsilon$ , we have

$$(1.14) \quad \log_2 y + \mathfrak{r}(u) \ll \mathfrak{S}(x, y) \ll 2^{u+O(u/\log 2u)} e^{c\sqrt{\log_2 y \log_3 y}}.$$

(ii) For  $2 \leq y \leq x^{1/(2\log_2 x \log_3 x)}$ , and with  $\lambda := y/\log x$ , we have

$$(1.15) \quad \mathfrak{S}(x, y) \asymp e^{\{1+O(\varepsilon_y+1/\log 2u)\}g(\lambda)u}.$$

Note that  $g$  is positive and strictly increasing on  $(0, +\infty)$ . The asymptotic behaviour of this function is given by

$$(1.16) \quad g(\lambda) = \begin{cases} \log 2 - 1/(4\lambda) + O(1/\lambda^2) & \text{as } \lambda \rightarrow \infty, \\ \lambda \log(1/\lambda) - \lambda(\log 4 - 1) + O(\lambda^2) & \text{as } \lambda \rightarrow 0. \end{cases}$$

Moreover, the lower bound  $g(\lambda)u \gg \bar{u}$  holds on the whole range  $x \geq y \geq 2$ .

The error term in (1.15) may be simplified to  $1/\log 2u$  if  $\log y > (\log_2 x)^2$  and to  $\varepsilon_y$  otherwise.

Note that (1.10) implies

$$\mathfrak{S}(x, y) \asymp \frac{\Psi(x, y; \tau)}{u^K \Psi(x, y)} \quad \left( \log y \leq \sqrt{\log x} \right),$$

with  $K = K(x, y) \in [0, 2]$ , so that, to the stated accuracy, the evaluation of  $\mathfrak{S}(x, y)$  reduces in this range to that of  $\Psi(x, y; \tau)/\Psi(x, y)$ . This is consistent with the Gaussian tendency of the distribution of the divisors of friable integers: as the friability parameter  $y$  decreases, the divisors of friable  $n$  concentrate around the mean-value  $\sqrt{n}$  and  $\Delta(n)$  resembles more and more to  $\tau(n)$ , the total number of divisors. Another description of this phenomenon appears in [5].

Considering available methods, Theorem 1.1 essentially agrees with standard expectations regarding methodology. We leave to a further project the task of adapting the method of [14] or [4] in the upper bound of (1.14). We note right away that, in the present context, such an improvement would only be relevant for very large values of  $y$  since the exponent  $\sqrt{\log_2 y \log_3 y}$  is absorbed by the remainder  $O(u/\log 2u)$  as soon as  $y \leq x^{1/(\log_2 x)^c}$  with  $c > 1/2$ .

## 2. PRELIMINARY ESTIMATES

Here and throughout, the letter  $p$  denotes a prime number. In [11], Hildebrand and Tenenbaum provided a universal estimate for  $\Psi(x, y)$  by the saddle-point method. Define

$$\zeta(s, y) := \prod_{p \leq y} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \varphi_y(s) := -\frac{\zeta'(s, y)}{\zeta(s, y)} \quad (\Re s > 0, y \geq 2),$$

and, for  $2 \leq y \leq x$ , let  $\alpha = \alpha(x, y)$  denote the unique positive solution to the equation  $\varphi_y(\alpha) = \log x$ . According to [11, th. 1], we have

$$(2.1) \quad \Psi(x, y) = \frac{x^\alpha \zeta(\alpha, y)}{\alpha \sqrt{2\pi |\varphi'_y(\alpha)|}} \left\{ 1 + O\left(\frac{1}{u} + \frac{\log y}{y}\right) \right\} \quad (x \geq y \geq 2).$$

By [11, (2.4)], we have

$$(2.2) \quad \alpha = \frac{\log(1 + y/\log x)}{\log y} \left\{ 1 + O\left(\frac{\log_2 y}{\log y}\right) \right\} \quad (x \geq y \geq 2).$$

Moreover, by [11, (7.8)], we have, for any given  $\varepsilon > 0$ ,

$$(2.3) \quad \alpha = 1 - \frac{\xi(u)}{\log y} + O\left(e^{-(\log y)^{(3/5)-\varepsilon}} + \frac{1}{u(\log y)^2}\right) \quad (x \geq x_0(\varepsilon), (\log x)^{1+\varepsilon} \leq y \leq x).$$

Finally, by [11, (2.5)], we have

$$(2.4) \quad |\varphi'_y(\alpha)| = \left(1 + \frac{\log x}{y}\right) \log x \log y \left\{ 1 + O\left(\frac{1}{\log(u+1)} + \frac{1}{\log y}\right) \right\} \quad (x \geq y \geq 2).$$

## 3. PROOF OF THEOREM 1.1(i): LOWER BOUND

Let  $\tau(n)$  denote the total number of divisors of a natural integer  $n$ . The following inequality is established in [9, lemma 60.1]

$$\Delta(n)\tau(n) \geq \sum_{\substack{d, d' | n \\ 0 < \log(d'/d) \leq 1}} 1 = \sum_{\substack{dd' | n \\ (d, d') = 1 \\ 0 < \log(d'/d) \leq 1}} \tau\left(\frac{n}{dd'}\right) \quad (n \geq 1),$$

the equality above being obtained by representing the ratios  $d'/d$  in reduced form.

Put

$$u_t := \frac{\log t}{\log y} \quad (t \geq 1, y \geq 2), \quad \Omega(n) := \sum_{p^\nu || n} \nu \quad (n \geq 1).$$

Since  $\tau(ab) \leq \tau(a)2^{\Omega(b)}$  ( $a, b \geq 1$ ), we have, for  $(x, y) \in H_\varepsilon$ ,

$$(3.1) \quad \mathfrak{S}(x, y) \geq \frac{1}{\Psi(x, y)} \sum_{\substack{dd' \in S(x, y) \\ (d, d') = 1 \\ 0 < \log(d'/d) \leq 1}} \frac{1}{2^{\Omega(dd')}} \Psi\left(\frac{x}{dd'}, y\right) \gg \sum_{\substack{dd' \in S(x, y) \\ (d, d') = 1 \\ 0 < \log(d'/d) \leq 1}} \frac{\varrho(u - u_{dd'})}{\varrho(u) dd' 2^{\Omega(dd')}},$$

where the last inequality follows from (1.6). To evaluate the double sum in (3.1), we establish an asymptotic formula for

$$T_d(x, y) := \sum_{\substack{m \in S(x, y) \\ (m, d) = 1}} \frac{1}{2^{\Omega(m)}}.$$

We shall make use of the following notation

$$C := \prod_p \frac{\sqrt{1 - 1/p}}{1 - 1/2p}, \quad \kappa_y := \frac{1}{(\log y)^{2/5}},$$

$$\varphi_y(d) := \prod_{p|d} \left(1 + \frac{1}{2p^{1-\kappa_y}}\right), \quad \vartheta_y(d) := \sum_{p|d} \frac{\log p}{p^{1-\kappa_y}}, \quad \mathfrak{q}(d) := \prod_{p|d} \left(1 - \frac{1}{2p}\right) \quad (d \geq 1).$$

**Lemma 3.1.** *Let  $\varepsilon > 0$ . For  $x \geq 1$ ,  $y > \exp\{(\log_2 3x)^{5/3+\varepsilon}\}$ ,  $d \in S(x, y)$ , we have*

$$(3.2) \quad T_d(x, y) = \frac{Cx\varrho_{1/2}(u)}{\sqrt{\log y}} \left\{ \mathfrak{q}(d) + O\left(\kappa_y \varphi_y(d) \{1 + \vartheta_y(d)\}\right) \right\}.$$

*Proof.* We have

$$T_d(x, y) = \sum_{m \in S(x, y)} \frac{1}{2^{\Omega(m)}} \sum_{t|(m, d)} \mu(t) = \sum_{t|d} \frac{\mu(t)}{2^{\Omega(t)}} T_1\left(\frac{x}{t}, y\right).$$

An estimate for the inner  $T_1$ -term follows from [19, cor. 2.3], which, in the domain

$$x \geq 1, \quad y > \exp\{(\log_2 3x)^{5/3+\varepsilon}\},$$

we rewrite as

$$(3.3) \quad T_1(x, y) = \frac{Cx\varrho_{1/2}(u)}{\sqrt{\log y}} \left\{ 1 + O\left(\frac{\log(u+1)}{\log y} + \frac{1}{\sqrt{\log y}} + \frac{1}{\log(2x)}\right) \right\}.$$

Here the error term  $1/\log(2x)$  enables to include the case  $1 \leq x < y$ : the corresponding estimate follows from [17, th. II.6.2]. Since  $\log(u+1) \ll (\log y)^{3/5}$  in  $H_\varepsilon$ , we get

$$(3.4) \quad T_d(x, y) = \frac{Cx}{\sqrt{\log y}} \sum_{\substack{t|d \\ t \leq x/\sqrt{y}}} \frac{\mu(t)\varrho_{1/2}(u - u_t)}{t 2^{\Omega(t)}} + R_1 + R_2,$$

with

$$R_1 \ll \frac{x}{(\log y)^{9/10}} \sum_{\substack{t|d \\ t \leq x/\sqrt{y}}} \frac{\mu(t)^2 \varrho_{1/2}(u - u_t)}{t 2^{\Omega(t)}} \ll \frac{x\varrho_{1/2}(u)}{(\log y)^{9/10}} \sum_{t|d} \frac{\mu(t)^2}{2^{\Omega(t)} t^{1-\xi(2u)/\log y}},$$

$$R_2 \ll \sum_{\substack{t|d \\ x/\sqrt{y} < t \leq x}} \frac{x}{t \sqrt{\log 2x/t}},$$

where the bound for  $R_1$  follows from

$$(3.5) \quad \varrho_{1/2}(u-v) \ll \varrho_{1/2}(u)e^{v\xi(2u)} \quad (u \geq 1, 0 \leq v \leq u - \tfrac{1}{2})$$

proved in [15]<sup>1</sup>. By multiplicativity, we thus get

$$(3.6) \quad R_1 \ll \frac{x\varrho_{1/2}(u)\varphi_y(d)}{(\log y)^{9/10}}.$$

Since  $d \leq x$ , we have  $p_{\omega(d)} \ll \log x$ , where  $p_{\omega(d)}$  denotes the  $\omega(d)$ th prime number. Hence, using de Bruijn's estimate for  $\log \Psi(x, y)$  as refined in [17, th. III.5.2], we plainly obtain, for a suitable absolute constant  $c > 0$ ,

$$(3.7) \quad \sum_{t|d, t \leq z} 1 \leq \Psi(z, p_{\omega(d)}) \leq z^{c/\log_2 x} \quad (\sqrt{x} \leq z \leq x).$$

As a consequence

$$R_2 \ll x \int_{\sqrt{x}}^x \frac{1}{z} dO(z^{c/\log_2 x}) \ll \sqrt{x} e^{c \log x / \log_2 x},$$

and we conclude that

$$(3.8) \quad R_1 + R_2 \ll \frac{x\varrho_{1/2}(u)\varphi_y(d)}{(\log y)^{9/10}}.$$

To estimate the main term of (3.4), we approximate  $\varrho_{1/2}(u-u_t)$  by  $\varrho_{1/2}(u)$ , using the bound

$$\varrho'_{1/2}(w) \ll \varrho_{1/2}(w) \log(1+w) \quad (w \geq \tfrac{1}{2})$$

which, with an appropriate modification of the range of validity, is also proved in [15, lemma 6.2]. In view of (3.5), this implies that

$$\varrho_{1/2}(u-u_t) - \varrho_{1/2}(u) \ll u_t \varrho_{1/2}(u) t^{\kappa_y} \log(u+1).$$

Thus,

$$\begin{aligned} \sum_{\substack{t|d \\ t \leq x/\sqrt{y}}} \frac{\mu(t)\varrho_{1/2}(u-u_t)}{t^{2\Omega(t)}\varrho_{1/2}(u)} &= \sum_{\substack{t|d \\ t \leq x/\sqrt{y}}} \frac{\mu(t)}{t^{2\Omega(t)}} + O\left(\sum_{\substack{t|d \\ t \leq x/\sqrt{y}}} \frac{\mu(t)^2(\log t) \log(u+1)}{t^{1-\kappa_y} 2^{\Omega(t)} \log y}\right) \\ &= \mathfrak{q}(d) + O\left(\sum_{\substack{t|d \\ x/\sqrt{y} < t \leq x}} \frac{1}{t} + \kappa_y \sum_{t|d} \frac{\mu(t)^2 \log t}{t^{1-\kappa_y} 2^{\Omega(t)}}\right). \end{aligned}$$

By (3.7), the first error term is  $\ll \sqrt{y}x^{-1+c/\log_2 x}$ , which is compatible with (3.2). To estimate the second, we write  $\log t = \sum_{p|t} \log p$  since  $\mu^2(t) = 1$  and invert summations. This yields the required estimate (3.2).  $\square$

By (3.1), we have

$$\mathfrak{S}(x, y) \gg \sum_{d \in S(\sqrt{x}/e, y)} \frac{\varrho(u-2u_d)\{T_d(ed, y) - T_d(d, y)\}}{\varrho(u)d^2 2^{\Omega(d)}}.$$

We insert (3.2) to evaluate the difference between curly brackets and sum separately the resulting main term and the remainder terms. This can be done by partial summation, using a variant of (3.3) in which the inclusion of the factors  $\mathfrak{q}(d)$  or  $\varphi_y(d)\{1 + \vartheta_y(d)\}$  has as sole effects to alter the value of the constant  $C$ . This yields

$$(3.9) \quad \begin{aligned} \mathfrak{S}(x, y) &\gg \sum_{d \in S(\sqrt{x}/e, y)} \frac{\mathfrak{q}(d)\varrho(u-2u_d)\varrho_{1/2}(u_d)}{\varrho(u)2^{\Omega(d)}d\sqrt{\log y}} \\ &\gg \frac{1}{\varrho(u)} \int_{1/\log y}^{u/2} \varrho(u-2v)\varrho_{1/2}(v)^2 dv = \frac{1}{2\varrho(u)} \int_{2/\log y}^u \varrho(u-v)\varrho_{1/2}(\tfrac{1}{2}v)^2 dv. \end{aligned}$$

The contribution of the interval  $[2/\log y, 2]$  to the last integral is

$$(3.10) \quad \geq 2\varrho(u) \int_{1/\log y}^1 \varrho_{1/2}(v)^2 dv = \frac{2\varrho(u)}{\pi} \int_{1/\log y}^1 \frac{dv}{v} = \frac{2\varrho(u)}{\pi} \log_2 y.$$

<sup>1</sup>In [15, lemma 6.1], this bound is claimed for  $0 \leq v \leq u$ , but it is necessary to exclude the case when  $u-v$  is small.

Now observe that (1.8) implies

$$\varrho_{1/2}(\tfrac{1}{2}v)^2 \asymp \frac{\varrho(v)}{\sqrt{v}} \quad (v \geq 1).$$

Since  $\varrho_2$  is the convolution square of  $\varrho$ , it follows that

$$(3.11) \quad \frac{1}{\varrho(u)} \int_2^u \varrho(u-v) \varrho_{1/2}(\tfrac{1}{2}v)^2 dv \gg \frac{\varrho_2(u)}{\sqrt{u}\varrho(u)} = \mathfrak{r}(u).$$

Carrying back into (3.9) and taking (3.10) into account, we obtain the required estimate.

#### 4. PROOF OF THEOREM 1.1(i): UPPER BOUND

We adapt to the friable case the iterative method developed by Tenenbaum in [16] (see also [9, §7.4]) for bounding the mean-value of the  $\Delta$ -function. Throughout this proof the letters  $c$  and  $C$ , with or without index, stand for absolute positive constants.

Given an integer  $n \geq 2$ , let us denote by  $\{p_j(n)\}_{1 \leq j \leq \omega(n)}$  the increasing sequence of its distinct prime factors. Following [16] (see also [9]), define

$$M_q(n) = \int_{\mathbb{R}} \Delta(n, u)^q du,$$

and, for squarefree  $n$ , put

$$n_k := \begin{cases} \prod_{j \leq k} p_j(n) & \text{if } k \leq \omega(n), \\ n & \text{otherwise.} \end{cases}$$

Now, let

$$L_{k,q} = L_{k,q}(x, y) := \sum_{P^+(n) \leq y} \frac{\mu(n)^2 M_q(n_k)^{1/q}}{n^\beta},$$

where  $\beta := \alpha(\sqrt{x}, y)$  is the saddle-point related to the friable mean-value of  $\tau(n)$ , the divisor function.

We aim at bounding  $L_{k,q}$  from above for large  $k$  and  $q$ . The starting point is the identity

$$\Delta(mp, u) = \Delta(m, u) + \Delta(m, u - \log p) \quad (u \in \mathbb{R}, p \nmid m).$$

Apply this to  $m = n_k$ ,  $p = p_{k+1}(n)$ . Raising to the power  $q$  and expanding out, we obtain

$$M_q(n_{k+1}) = 2M_q(n_k) + E_q(n_k, p_{k+1}) \quad (\omega(n) > k),$$

with

$$E_q(m, p) := \sum_{1 \leq j < q} \binom{q}{j} \int_{\mathbb{R}} \Delta(m; v)^j \Delta(m; v - \log p)^{q-j} dv.$$

It follows that

$$L_{k+1,q} \leq 2^{1/q} L_{k,q} + \sum_{\substack{P^+(m) \leq y \\ \omega(m)=k}} \mu(m)^2 \sum_{P^+(m) < p \leq y} E_q(m, p)^{1/q} \sum_{\substack{P^+(n) \leq y \\ \omega(n) \geq k+1 \\ n_{k+1}=mp}} \frac{\mu(n)^2}{n^\beta}.$$

The latter sum is

$$\ll \frac{\zeta_1(\beta, y)}{p^\beta m^\beta} \prod_{\ell \leq p} \frac{1}{1 + \ell^{-\beta}} =: \frac{\zeta_1(\beta, y) g_\beta(p)}{p^\beta m^\beta},$$

where, here and in the remainder of this proof,  $\ell$  denotes a prime number, and

$$\zeta_1(\sigma, y) := \prod_{\ell \leq y} (1 + \ell^{-\sigma}).$$

Hölder's inequality yields

$$\sum_{z < p \leq y} \frac{E_q(m, p)^{1/q}}{p^\beta} \leq \left\{ \sum_{p \geq 2} \frac{E_q(m, p) \log p}{p} \right\}^{1/q} \left\{ \sum_{z < p \leq y} \frac{1}{p^{(q\beta-1)/(q-1)} (\log p)^{1/(q-1)}} \right\}^{(q-1)/q}.$$

and the prime number theorem enables to bound the last sum over  $p$  by

$$\ll \frac{qy^{q(1-\beta)/(q-1)}}{(\log z)^{1/(q-1)}}.$$

Now, we have (see, e.g., [9, th. 73])

$$\sum_p \frac{E_q(m, p) \log p}{p} \leq C 4^q \tau(m)^{q/(q-1)} M_q(m)^{(q-2)/(q-1)}.$$

It follows that

$$(4.1) \quad L_{k+1, q} \leq 2^{1/q} L_{k, q} + C_1 q e^{\xi(u/2)} G_k \leq 2^{1/q} L_{k, q} + C_2 q u^2 G_k,$$

with

$$G_k := \zeta_1(\beta, y) \sum_{\substack{P^+(m) \leq y \\ \omega(m) = k}} \frac{\mu(m)^2 \tau(m)^{1/(q-1)} M_q(m)^{(q-2)/q(q-1)} g_\beta(P^+(m))}{m^\beta (\log P^+(m))^{1/q}}.$$

Since

$$\frac{\mu(m)^2 \zeta_1(\beta, y) g_\beta(P^+(m))}{m^\beta} = \sum_{\substack{P^+(n) \leq y \\ n_k = m}} \frac{\mu(n)^2}{n^\beta},$$

we infer that

$$G_k \leq \sum_{\substack{P^+(n) \leq y \\ \omega(n) \geq k}} \frac{\mu(n)^2 \tau(n_k)^{1/(q-1)} M_q(n_k)^{(q-2)/q(q-1)}}{n^\beta (\log p_k(n))^{1/q}}.$$

A new application of Hölder's inequality yields

$$G_k \leq L_{k, q}^{(q-2)/(q-1)} S_k^{1/(q-1)},$$

where

$$\begin{aligned} S_k &:= \sum_{\substack{P^+(n) \leq y \\ \omega(n) \geq k}} \frac{\mu(n)^2 \tau(n_k)}{n^\beta \{\log p_k(n)\}^{(q-1)/q}} \\ &\leq 2 \sum_{\substack{P^+(m) \leq y \\ \omega(m) = k-1}} \frac{\mu(m)^2 \tau(m)}{m^\beta} \sum_{P^+(m) < p \leq y} \frac{1}{p^\beta (\log p)^{1-1/q}} \prod_{p < \ell \leq y} \left(1 + \frac{1}{\ell^\beta}\right) \\ &\leq \frac{\zeta_1(\beta, y)}{(k-1)!} \sum_{p \leq y} \frac{g_\beta(p)}{p^\beta (\log p)^{1-1/q}} \left(\sum_{\ell \leq p} \frac{2}{\ell^\beta}\right)^{k-1} \ll \frac{\zeta_1(\beta, y) y^{1-\beta}}{(k-1)!} \sum_{p \leq y} \frac{e^{-T(p)} \{2T(p)\}^{k-1}}{p (\log p)^{1-1/q}}, \end{aligned}$$

where we set

$$T(p) := \sum_{\ell \leq p} \frac{1}{\ell^\beta}.$$

(Recall that the letter  $\ell$  denotes generically a prime number.)

We evaluate  $T(p)$  by [2, lemma 3.6]. Writing

$$\mathcal{L}(z) := e^{(\log z)^{3/5}/(\log_2 z)^{1/5}}, \quad w(t) := \frac{t^{1-\beta} - 1}{(1-\beta) \log t},$$

we have

$$T(p) = \log_2 p + \int_1^{w(p)} t \xi'(t) dt + b + O\left(\frac{w(p)}{\mathcal{L}(p)^c} + \frac{\log(u+1)}{\log y}\right).$$

where  $b$  is a suitable constant. Note that  $w(y) = u/2 + O(u/\log y)$ . Defining

$$h(v) := \int_1^{w(\exp e^v)} t \xi'(t) dt + b_1,$$

with  $b_1$  sufficiently large so that  $T(p) \leq \log_2 p + h(\log_2 p)$ , and writing  $z_v := v + h(v)$ , we have, by the prime number theorem,

$$W_k(y) := \sum_{p \leq y} \frac{e^{-T(p)} \{T(p)\}^{k-1}}{p (\log p)^{1-1/q}} \ll \int_0^{\log_2 y} e^{-(2-1/q)z_v + (1-1/q)h(\log_2 y)} z_v^{k-1} dv.$$

Since  $h(\log_2 y) \leq u/2 + O(u/\log 2u)$  and since  $h'(v) \geq 0$ , the change of variables  $z = z_v$  yields

$$W_k(y) \ll e^{u/2 + O(u/\log 2u)} \int_0^\infty e^{-(2-1/q)z} z^{k-1} dz \ll \frac{e^{u/2 + O(u/\log 2u)} (k-1)!}{(2-1/q)^{k-1}}.$$



Thus,

$$S_k \ll \frac{\zeta_1(\beta, y)e^{u/2+O(u/\log 2u)}}{(1-1/2q)^k} \ll \frac{\zeta_1(\alpha, y)e^{O(u/\log 2u)}}{(1-1/2q)^k},$$

since  $\zeta(\beta, y) = \zeta(\alpha, y)e^{-u/2+O(u/\log 2u)}$  — see [18, (4.2)].

Finally, for  $q$  sufficiently large and  $\frac{1}{2} < \lambda < \log 2$ , we obtain

$$(4.2) \quad G_k \leq C_3 L_{k,q}^{(q-2)/(q-1)} \zeta_1(\alpha, y)^{1/(q-1)} e^{c_0 u/(q \log 2u) + \lambda k/q(q-1)}.$$

At this stage, we introduce

$$L_{k,q}^* = L_{k,q} + 2^{k/q} u^{2q} e^{c_0 u/\log 2u} \zeta_1(\alpha, y),$$

so that (4.1) still holds for  $L_{k,q}^*$  in place of  $L_{k,q}$ . Setting  $q(k) := \lfloor c_1 \sqrt{k/\log k} \rfloor$  with sufficiently small, absolute  $c_1$ , we thus have, for large  $k$ ,

$$L_{k+1,q}^* \leq \left\{ 2^{1/q} + \frac{1}{k} \right\} L_{k,q}^* \quad (q \leq q(k)),$$

whence

$$(4.3) \quad L_{k+1,q}^* \leq 3^{1/q} L_{k,q}^* \quad (q \leq q(k)).$$

To carry out a double induction on  $k$  and  $q$ , we also need a bound on  $L_{k,q+1}^*$  in terms of  $L_{k,q}^*$ . This is achieved by the inequality  $M_{q+1}(n)^{1/(q+1)} \leq 2M_q(n)^{1/q}$  proved in [9, th. 72], which yields

$$(4.4) \quad L_{k,q+1}^* \leq 2u^2 L_{k,q}^*.$$

With the aim of bounding  $L_{k,q(k)}^*$  in terms of  $L_{2,q(2)}^*$ , we use (4.3) to reduce the parameter  $k$  and (4.4) to secure the condition  $q \leq q(k)$ . The first handling provides an overall factor

$$\leq \prod_{1 \leq q \leq q(k)} q^{c_2} \leq e^{c_3 \sqrt{k \log k}}$$

whereas the second induces a global factor  $\ll u^{c_4 q(k)}$ .

Finally, we obtain

$$L_{k,q}^* \ll L_{2,q(2)}^* u^{c_5 q(k)} e^{c_5 \sqrt{k \log k}}.$$

Let  $K := \log_2 y + u$ . It can be shown (see [1] and use a bound similar to [9, (7.44)]) that the contribution to  $L_{k,q}$  of those integers  $n$  such that  $\omega(n) > CK$  is negligible, and we omit the details. Eventually, we arrive at

$$L_{k,q} \ll e^{c_5 \sqrt{K \log K}} u^{c_6 \sqrt{K/\log K}} \zeta(\alpha, y) e^{c_0 u/\log 2u} \ll \zeta(\alpha, y) e^{c_7 \sqrt{\log_2 y \log_3 y} + O(u/\log 2u)},$$

and so

$$\sum_{n \in S(x,y)} \frac{\mu(n)^2 \Delta(n)}{n^\beta} \ll \zeta(\alpha, y) e^{c \sqrt{(\log_2 y) \log_3 y} + O(u/\log 2u)}.$$

Employing the representation  $n = mr^2$ ,  $\mu(m)^2 = 1$ , we obtain that the same bound holds for

$$\sum_{n \in S(x,y)} \frac{\Delta(n)}{n^\beta}.$$

This is the key to our upper bound for  $D(x, y) := \sum_{n \in S(x,y)} \Delta(n)$ . We have

$$\begin{aligned} D(x, y) \log x - \int_1^x \frac{D(t, y)}{t} dt &= \sum_{n \in S(x,y)} \Delta(n) \log n \leq \sum_{\substack{mp^\nu \leq x \\ P^+(mp) \leq y}} \Delta(m) (\nu + 1) \log p^\nu \\ &\ll y D\left(\frac{x}{y}, y\right) + \sum_{\substack{x/y < n \leq x \\ P^+(n) \leq y}} \frac{x \Delta(n)}{n} + \sum_{\substack{n \leq x \\ P^+(n) \leq y}} \Delta(n) \sqrt{\frac{x}{n}}. \end{aligned}$$

The trivial bound

$$D(x, y) \leq \sum_{n \in S(x,y)} \tau(n) \ll x \varrho_2(u) \log y,$$

that holds in  $H_\varepsilon$  (see [19, Cor. 2.3]), furnishes

$$\int_1^x \frac{D(t, y)}{t} dt \ll x \varrho_2(u) \log y, \quad y D(x/y, y) \ll x \varrho_2(u-1) \log y.$$

Moreover, in the same region, for  $y$  sufficiently large,  $\beta > 1/2$

$$\sum_{\substack{n \leq x \\ P^+(n) \leq y}} \Delta(n) \sqrt{\frac{x}{n}} + \sum_{\substack{x/y < n \leq x \\ P^+(n) \leq y}} \frac{x \Delta(n)}{n} \ll x^\beta e^{\xi(u/2)} \sum_{n \in S(x,y)} \frac{\Delta(n)}{n^\beta}.$$

Collecting these estimates, we obtain

$$\begin{aligned} D(x, y) &\ll x \frac{\varrho_2(u)}{u} + x \varrho_2(u) \log 2u + \frac{x^\beta \zeta(\alpha, y) e^{c\sqrt{(\log_2 y) \log_3 y} + O(u/\log 2u)}}{\log x} \\ &\ll \Psi(x, y) 2^{u+O(u/\log 2u)} e^{c\sqrt{\log_2 y \log_3 y}}, \end{aligned}$$

where we used (1.9), (1.6), the estimate

$$\frac{x^\beta \zeta(\alpha, y)}{\log x} \asymp \Psi(x, y) 2^{u+O(u/\log u)},$$

which follows from (2.1), (2.4) and

$$(\beta - \alpha) \log x = -u \int_{u/2}^u \xi'(t) dt + O(1) = u \log 2 + O\left(\frac{u}{\log u}\right).$$

This concludes the proof of the upper bound included in (1.14).

## 5. PROOF OF THEOREM 1.1(ii)

We retain notation  $g(t)$  from (1.12),  $\varepsilon_y$  from (1.13), define  $\eta_y := (\log_2 y)/\log y$ . Since  $\max(1, \lfloor \tau(n)/\log n \rfloor) \leq \Delta(n) \leq \tau(n)$  holds for all  $n \geq 1$  (see e.g. [9, th. 60, (6.7)]), we have

$$(5.1) \quad \frac{\Psi(x, y; \tau)}{2\Psi(x, y) \log x} \leq \mathfrak{S}(x, y) \leq \frac{\Psi(x, y; \tau)}{\Psi(x, y)} \quad (x \geq y \geq 2).$$

Now, by [18, th. 1.2] and [18, (1.6)], we have, with  $\lambda := y/\log x$ ,

$$(5.2) \quad \frac{\Psi(x, y; \tau)}{\Psi(x, y)} \asymp \zeta(\alpha, y) e^{-uh(\lambda)\{1+O(\varepsilon_y)\}} \quad (x \geq y \geq 2),$$

where we have put

$$h(t) := t \log 4 - (1 + 2t) \log \left( \frac{1 + 2t}{1 + t} \right) = t \log \left( 1 + \frac{1}{t} \right) - g(t) \quad (t \geq 0),$$

and, for the purpose of further reference, note that

$$(5.3) \quad uh(\lambda) \sim (1 - \log 2)u \quad (u \rightarrow \infty, \lambda \rightarrow \infty), \quad uh(\lambda) \asymp \bar{u} \quad (x \geq y \geq 2).$$

We shall show that

$$(5.4) \quad \zeta(\alpha, y) = e^{\lambda u \log(1+1/\lambda)\{1+O(\varepsilon_y+1/\log 2u)\}} \quad \left( 2 \leq y \leq x^{1/(2 \log_2 x \log_3 x)} \right).$$

Since  $g(\lambda)u \gg \bar{u}$  for  $x \geq y \geq 2$ , we see that (1.15) follows from (5.1) and (5.4) in any subregion where  $\bar{u}(\varepsilon_y + 1/\log 2u) \gg \log_2 x$ : the condition above corresponds to this requirement when  $y$  is large. However, for bounded  $y$ , we have  $\Psi(x, y; \tau)/\Psi(x, y) \asymp (\log x)^{\pi(y)}$ , and so (1.15) holds trivially. Therefore, we may assume in the sequel that  $y$  is sufficiently large.

Let us now embark on the proof of (5.4).

Observe that

$$(5.5) \quad \zeta(\alpha, y) = \zeta(1, y) \exp \left\{ \int_{\alpha}^1 \varphi_y(\sigma) d\sigma \right\}.$$

Using the estimate for  $\varphi_y(\sigma)$  given in [11, lemma 13], we may write

$$(5.6) \quad \int_{\alpha}^1 \varphi_y(\sigma) d\sigma = \left\{ 1 + O\left(\frac{1}{\log y}\right) \right\} \int_{\alpha}^1 \frac{y^{1-\sigma} - 1}{(1-\sigma)(1-y^{-\sigma})} d\sigma.$$

By inspection of the proof of (2.2) in [11, pp. 285-7], we see that, for a suitable constant  $C$ , we have

$$(5.7) \quad \alpha(x, y) = 1 - \frac{\xi(u)}{\log y} + O\left(\frac{1}{(\log y)^2}\right) \quad \left( C(\log x)(\log_2 x)^3 < y \leq x \right).$$

This implies  $y^\alpha \gg ye^{-\xi(u)} \gg \log y$  in the same domain, so the contribution of the term  $1 - y^{-\sigma}$  in (5.6) is absorbed by the error term. The change of variables defined by  $(1 - \sigma) \log y = \xi(t)$  then provides, in view of (5.7) and (1.7),

$$\begin{aligned} \int_\alpha^1 \varphi_y(\sigma) d\sigma &= \left\{1 + O\left(\frac{1}{\log y}\right)\right\} \int_1^u t \xi'(t) dt \\ &= u + \frac{u}{\log u} + O\left(\frac{u}{\log y}\right) = u + O\left(\frac{u}{\log 2u} + \varepsilon_y u\right). \end{aligned}$$

Since, in the domain of (5.7),

$$u\lambda \log\left(1 + \frac{1}{\lambda}\right) = u + O\left(\frac{u}{\log 2u}\right),$$

we obtain (5.4) in the range  $C(\log x)(\log_2 x)^3 < y \leq x^{1/(2\log_2 x \log_3 x)}$ . Indeed the factor  $\zeta(1, y) \asymp \log y$  appearing in (5.5) is absorbed by the error term.

When  $2 \leq y \leq C(\log x)(\log_2 x)^3$ , we put  $t = y^\sigma$  in (5.6) to get

$$\int_\alpha^1 \varphi_y(\sigma) d\sigma = \left\{1 + O\left(\frac{1}{\log y}\right)\right\} \int_{y^\alpha}^y \frac{y/t - 1}{(t-1) \log(y/t)} dt.$$

Note that (2.2) now implies  $\alpha \log y \ll \log_2 2y$ . Put  $T := (\log y)^K$ , where  $K$  is so large so that  $T > y^\alpha$ . The contribution of the interval  $[T, y]$  to the above integral is

$$\ll \int_T^\infty \frac{y}{t^2} dt \ll \frac{y}{(\log y)^K},$$

where we used the bound  $e^v - 1 \ll ve^v$  ( $v \geq 0$ ). Then,

$$\begin{aligned} \int_{y^\alpha}^T \frac{y/t - 1}{(t-1) \log(y/t)} dt &= \left\{1 + O\left(\frac{\log_2 y}{\log y}\right)\right\} \frac{y}{\log y} \int_{y^\alpha}^T \frac{1}{t(t-1)} dt \\ &= \left\{1 + O(\eta_y)\right\} \frac{y}{\log y} \log\left(\frac{1 - 1/T}{1 - 1/y^\alpha}\right). \end{aligned}$$

>From [11, (7.18)], it follows that

$$\log\left(\frac{1}{1 - y^{-\alpha}}\right) = \log\left(1 + \frac{1}{\lambda}\right) \left\{1 + O\left(\frac{\log_2 y}{\log y}\right)\right\} \quad (2 \leq y \leq C(\log x)(\log_2 x)^3).$$

Therefore

$$\begin{aligned} \int_\alpha^1 \varphi_y(\sigma) d\sigma &= \left\{1 + O(\eta_y)\right\} \frac{y}{\log y} \log\left(1 + \frac{1}{\lambda}\right) + O\left(\frac{y}{(\log y)^K}\right) \\ &= \left\{1 + O(\eta_y)\right\} u\lambda \log\left(1 + \frac{1}{\lambda}\right). \end{aligned}$$

This establishes (5.4) in the complementary range  $2 \leq y \leq C(\log x)(\log_2 x)^3$ .

This completes the proof of theorem 1.1(ii).

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#### REFERENCES

- [1] R. de la Bretèche and G. Tenenbaum. Entiers friables: inégalité de Turán-Kubilius et applications. *Invent. Math.*, 159(3):531–588, 2005.
- [2] R. de la Bretèche and G. Tenenbaum. Propriétés statistiques des entiers friables. *Ramanujan J.*, 9(1-2):139–202, 2005.
- [3] R. de la Bretèche and G. Tenenbaum. Two upper bounds for the Erdős-Hooley Delta-function. *Sci. China. Math.*, 66:2683–2692, 2023.
- [4] R. de la Bretèche and G. Tenenbaum. Note on the mean value of the Erdős-Hooley Delta-function. Preprint, *arXiv:2309.03958*, 2024.
- [5] S. Drappeau and G. Tenenbaum. Lois de répartition des diviseurs des entiers friables. *Math. Z.*, 288(3-4):1299–1326, 2018.

- [6] K. Ford, B. Green, and D. Koukoulopoulos. Equal sums in random sets and the concentration of divisors. *Invent. math.*, 232:1027–1160, 2023.
- [7] K. Ford, D. Koukoulopoulos, and T. Tao. A lower bound on the mean value of the Erdős-Hooley Delta function. Preprint, *arXiv:2308.11987*, 2023.
- [8] R. R. Hall and G. Tenenbaum. On the average and normal orders of Hooley’s  $\Delta$ -function. *J. London Math. Soc. (2)*, 25(3):392–406, 1982.
- [9] R. R. Hall and G. Tenenbaum. *Divisors*, volume 90 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1988.
- [10] A. Hildebrand. On the number of positive integers  $\leq x$  and free of prime factors  $> y$ . *J. Number Theory*, 22(3):289–307, 1986.
- [11] A. Hildebrand and G. Tenenbaum. On integers free of large prime factors. *Trans. Amer. Math. Soc.*, 296(1):265–290, 1986.
- [12] A. Hildebrand and G. Tenenbaum. On a class of differential-difference equations arising in number theory. *J. Anal. Math.*, 61:145–179, 1993.
- [13] C. Hooley. On a new technique and its applications to the theory of numbers. *Proc. London Math. Soc. (3)*, 38(1):115–151, 1979.
- [14] D. Koukoulopoulos and T. Tao. An upper bound on the mean value of the Erdős-Hooley Delta function. *Proc. London Math. Soc. (3)*, 127(6):1865–1885, 2023.
- [15] H. Smida. Valeur moyenne des fonctions de Piltz sur les entiers sans grand facteur premier. *Acta Arith.*, 63(1):21–50, 1993.
- [16] G. Tenenbaum. Sur la concentration moyenne des diviseurs. *Comment. Math. Helv.*, 60(3):411–428, 1985.
- [17] G. Tenenbaum. *Introduction to analytic and probabilistic number theory*, volume 163 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, third edition, 2015. Translated from the 2008 French edition by Patrick D. F. Ion.
- [18] G. Tenenbaum. Sur le biais d’une loi de probabilité relative aux entiers friables. *J. Théor. Nombres Bordeaux*, 35(2):481–493, 2023.
- [19] G. Tenenbaum and J. Wu. Moyennes de certaines fonctions multiplicatives sur les entiers friables. *J. Reine Angew. Math.*, 564:119–166, 2003.

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