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On the friable mean-value of the Erdős-Hooley Delta function

B. Martin, G. Tenenbaum, and J. Wetzer

ABSTRACT. For integer n and real u , define $\Delta(n, u) := |\{d : d \mid n, e^u < d \leq e^{u+1}\}|$. Then, the Erdős-Hooley Delta function is defined as $\Delta(n) := \max_{u \in \mathbb{R}} \Delta(n, u)$. We provide uniform upper and lower bounds for the mean-value of $\Delta(n)$ over friable integers, i.e. integers free of large prime factors.

1. INTRODUCTION AND STATEMENT OF RESULTS

For integer $n \geq 1$ and real u , put

$$\Delta(n, u) := |\{d : d \mid n, e^u < d \leq e^{u+1}\}|, \quad \Delta(n) := \max_{u \in \mathbb{R}} \Delta(n, u).$$

The Δ -function was introduced by Erdős in 1974 and was highlighted in 1979 by Hooley [13]. It turned out to be a key-concept in many branches of analytic number theory such as Waring type problems, circle method, Diophantine approximation, distribution of prime factors in polynomial sequences, etc.

However, the behaviour of $\Delta(n)$ remains rather mysterious. For instance, the average order is still not known with desirable precision. Hall and Tenenbaum [8] obtained in 1982 the lower bound

$$(1.1) \quad D(x) := \sum_{n \leq x} \Delta(n) \gg x \log_2 x \quad (x \geq 3),$$

whereas Tenenbaum [16] showed in 1985 that for suitable $c > 0$ we have

$$(1.2) \quad D(x) \ll x e^{c \sqrt{\log_2 x \log_3 x}} \quad (x \geq 16).$$

Here and in the sequel, we let \log_k denote the k -fold iterated logarithm. Recently, La Bretèche and Tenenbaum [3, th.1.1] obtained a slight improvement to (1.2) by removing the triple logarithm in the exponent and, even more recently, Koukoulopoulos and Tao [14] obtained the remarkable bound

$$D(x) \ll x (\log_2 x)^{11/4} \quad (x \geq 3).$$

A few months later, Ford, Koukoulopoulos and Tao [7] improved (1.1) by showing

$$D(x) \gg x (\log_2 x)^{1+\eta+o(1)} \quad (x \geq 3),$$

where the exponent $\eta \approx 0.3533227$ appears in the work of Ford, Green and Koukoulopoulos [6] on the normal order of $\Delta(n)$. Both bounds have been recently improved by La Bretèche and Tenenbaum [4]: we have

$$(1.3) \quad x (\log_2 x)^{3/2} \ll D(x) \ll x (\log_2 x)^{5/2} \quad (x \geq 3),$$

which constitutes the current state of the art.

Let $P^+(n)$ denote the largest prime factor of an integer $n > 1$ and let us agree that $P^+(1) = 1$. Following usual notation, we define $S(x, y)$ as the set of y -friable integers not exceeding x , and denote by $\Psi(x, y)$ its cardinality, viz.

$$S(x, y) := \{n \leq x : P^+(n) \leq y\}, \quad \Psi(x, y) = |S(x, y)| \quad (x \geq 1, y \geq 1).$$

Structural properties of the set $S(x, y)$ motivated a vast array of the literature in the last forty years. The applications are indeed numerous and significant: circle method, Waring-type problems, cryptology, sieve theory, probabilistic models in number theory.

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Given an arithmetical function f , let us use the notation $\Psi(x, y; f) := \sum_{n \in S(x, y)} f(n)$. In this work we investigate bounds for the friable mean-value

$$(1.4) \quad \mathfrak{S}(x, y) := \frac{\Psi(x, y; \Delta)}{\Psi(x, y)} \quad (x \geq y \geq 2).$$

We now define some quantities arising in our statements. Given $\kappa > 0$, denote by ϱ_κ the continuous solution on $]0, \infty[$ of the delay differential system

$$\begin{cases} \varrho_\kappa(v) = v^{\kappa-1}/\Gamma(\kappa) & (0 < v \leq 1), \\ v\varrho'_\kappa(v) + (1 - \kappa)\varrho_\kappa(v) + \kappa\varrho_\kappa(v-1) = 0 & (v > 1), \end{cases}$$

and set $\varrho_\kappa(v) := 0$ for $v < 0$.

Thus (see, e.g., [12]) ϱ_κ is the order κ fractional convolution power of $\varrho := \varrho_1$, the Dickman function, which provides a continuous approximation to $\Psi(x, y)$ in

$$(1.5) \quad H_\varepsilon := \left\{ (x, y) : x \geq 3, e^{(\log_2 x)^{5/3+\varepsilon}} \leq y \leq x \right\} \quad (\varepsilon > 0).$$

Indeed, improving on results by Dickman and de Bruijn, Hildebrand [10] proved the asymptotic formula

$$(1.6) \quad \Psi(x, y) = x\varrho(u) \left\{ 1 + O\left(\frac{\log(2u)}{\log y}\right) \right\} \quad ((x, y) \in H_\varepsilon),$$

with the standard notation

$$u = \frac{\log x}{\log y}.$$

The asymptotic behaviour of the functions ϱ_κ (and in fact of more general delay differential equations, as displayed in [12]) may be described in terms of the function $\xi(t)$ defined as the unique positive solution to $e^\xi = 1 + t\xi$ for $t \neq 1$ and by $\xi(1) = 0$. From [17, lemma III.5.11] and the remark following [17, th. III.5.13], we quote the estimates

$$(1.7) \quad \xi(t) = \log t + \log_2 t + O\left(\frac{\log_2 t}{\log t}\right), \quad \xi'(t) = \frac{1}{t} + \frac{1}{t \log t} + O\left(\frac{\log_2 t}{t(\log t)^2}\right) \quad (t \rightarrow \infty).$$

Applying [12, cor. 2] in the case $(a, b) = (1 - \kappa, \kappa)$, we have

$$(1.8) \quad \varrho_\kappa(v) = \sqrt{\frac{\xi'(v/\kappa)}{2\pi\kappa}} \exp\left\{ \kappa\gamma - \kappa \int_1^{v/\kappa} \xi(t) dt \right\} \left\{ 1 + O\left(\frac{1}{v}\right) \right\} \quad (v \geq 1 + \kappa),$$

where γ denotes Euler's constant. We put

$$(1.9) \quad \mathfrak{r}(v) := \frac{\varrho_2(v)}{\sqrt{v}\varrho(v)} \asymp \frac{1}{\sqrt{v}} \exp\left(\int_1^v \{\xi(t) - \xi(t/2)\} dt \right) \asymp 2^{v+O(v/\log 2v)} \quad (v \geq 1),$$

while a genuine asymptotic formula follows from (1.8).

Let $\tau(n)$ denote the total number of divisors of an integer n . We trivially have

$$(1.10) \quad \tau(n)/\log 2n \ll \Delta(n) \leq \tau(n) \quad (n \geq 1),$$

where the lower bounds follows from the pigeon-hole principle. Since, by [19, cor. 2.3], we have

$$\Psi(x, y; \tau) = \left\{ 1 + O\left(\frac{\log(2u)}{\log y}\right) \right\} x\varrho_2(u) \log y \quad ((x, y) \in H_\varepsilon),$$

we may state as a benchmark that

$$\frac{\mathfrak{r}(u)}{\sqrt{u}} \ll \mathfrak{S}(x, y) \ll 2^{u+O(u/\log 2u)} \log y \quad ((x, y) \in H_\varepsilon).$$

We obtain the following results, where the following notation is used:

$$(1.11) \quad \bar{u} := \min\left(\frac{y}{\log y}, u\right) \quad (x \geq y \geq 2),$$

$$(1.12) \quad g(t) := \log\left(\frac{(1+2t)^{1+2t}}{(1+t)^{1+t}(4t)^t}\right) \quad (t > 0),$$

$$(1.13) \quad \varepsilon_y := \frac{1}{\sqrt{\log y}} \quad (y \geq 2).$$

Theorem 1.1. (i) Let $\varepsilon > 0$. For a suitable absolute constant $c > 0$ and uniformly for $(x, y) \in H_\varepsilon$, we have

$$(1.14) \quad \log_2 y + \mathfrak{r}(u) \ll \mathfrak{S}(x, y) \ll 2^{u+O(u/\log 2u)} e^{c\sqrt{\log_2 y \log_3 y}}.$$

(ii) For $2 \leq y \leq x^{1/(2\log_2 x \log_3 x)}$, and with $\lambda := y/\log x$, we have

$$(1.15) \quad \mathfrak{S}(x, y) \asymp e^{\{1+O(\varepsilon_y+1/\log 2u)\}g(\lambda)u}.$$

Note that g is positive and strictly increasing on $(0, +\infty)$. The asymptotic behaviour of this function is given by

$$(1.16) \quad g(\lambda) = \begin{cases} \log 2 - 1/(4\lambda) + O(1/\lambda^2) & \text{as } \lambda \rightarrow \infty, \\ \lambda \log(1/\lambda) - \lambda(\log 4 - 1) + O(\lambda^2) & \text{as } \lambda \rightarrow 0. \end{cases}$$

Moreover, the lower bound $g(\lambda)u \gg \bar{u}$ holds on the whole range $x \geq y \geq 2$.

The error term in (1.15) may be simplified to $1/\log 2u$ if $\log y > (\log_2 x)^2$ and to ε_y otherwise.

Note that (1.10) implies

$$\mathfrak{S}(x, y) \asymp \frac{\Psi(x, y; \tau)}{u^K \Psi(x, y)} \quad \left(\log y \leq \sqrt{\log x} \right),$$

with $K = K(x, y) \in [0, 2]$, so that, to the stated accuracy, the evaluation of $\mathfrak{S}(x, y)$ reduces in this range to that of $\Psi(x, y; \tau)/\Psi(x, y)$. This is consistent with the Gaussian tendency of the distribution of the divisors of friable integers: as the friability parameter y decreases, the divisors of friable n concentrate around the mean-value \sqrt{n} and $\Delta(n)$ resembles more and more to $\tau(n)$, the total number of divisors. Another description of this phenomenon appears in [5].

Considering available methods, Theorem 1.1 essentially agrees with standard expectations regarding methodology. We leave to a further project the task of adapting the method of [14] or [4] in the upper bound of (1.14). We note right away that, in the present context, such an improvement would only be relevant for very large values of y since the exponent $\sqrt{\log_2 y \log_3 y}$ is absorbed by the remainder $O(u/\log 2u)$ as soon as $y \leq x^{1/(\log_2 x)^c}$ with $c > 1/2$.

2. PRELIMINARY ESTIMATES

Here and throughout, the letter p denotes a prime number. In [11], Hildebrand and Tenenbaum provided a universal estimate for $\Psi(x, y)$ by the saddle-point method. Define

$$\zeta(s, y) := \prod_{p \leq y} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \varphi_y(s) := -\frac{\zeta'(s, y)}{\zeta(s, y)} \quad (\Re s > 0, y \geq 2),$$

and, for $2 \leq y \leq x$, let $\alpha = \alpha(x, y)$ denote the unique positive solution to the equation $\varphi_y(\alpha) = \log x$. According to [11, th. 1], we have

$$(2.1) \quad \Psi(x, y) = \frac{x^\alpha \zeta(\alpha, y)}{\alpha \sqrt{2\pi |\varphi_y'(\alpha)|}} \left\{ 1 + O\left(\frac{1}{u} + \frac{\log y}{y}\right) \right\} \quad (x \geq y \geq 2).$$

By [11, (2.4)], we have

$$(2.2) \quad \alpha = \frac{\log(1 + y/\log x)}{\log y} \left\{ 1 + O\left(\frac{\log_2 y}{\log y}\right) \right\} \quad (x \geq y \geq 2).$$

Moreover, by [11, (7.8)], we have, for any given $\varepsilon > 0$,

$$(2.3) \quad \alpha = 1 - \frac{\xi(u)}{\log y} + O\left(e^{-(\log y)^{(3/5)-\varepsilon}} + \frac{1}{u(\log y)^2}\right) \quad (x \geq x_0(\varepsilon), (\log x)^{1+\varepsilon} \leq y \leq x).$$

Finally, by [11, (2.5)], we have

$$(2.4) \quad |\varphi_y'(\alpha)| = \left(1 + \frac{\log x}{y}\right) \log x \log y \left\{ 1 + O\left(\frac{1}{\log(u+1)} + \frac{1}{\log y}\right) \right\} \quad (x \geq y \geq 2).$$

3. PROOF OF THEOREM 1.1(i): LOWER BOUND

Let $\tau(n)$ denote the total number of divisors of a natural integer n . The following inequality is established in [9, lemma 60.1]

$$\Delta(n)\tau(n) \geq \sum_{\substack{d, d' | n \\ 0 < \log(d'/d) \leq 1}} 1 = \sum_{\substack{dd' | n \\ (d, d') = 1 \\ 0 < \log(d'/d) \leq 1}} \tau\left(\frac{n}{dd'}\right) \quad (n \geq 1),$$

the equality above being obtained by representing the ratios d'/d in reduced form.

Put

$$u_t := \frac{\log t}{\log y} \quad (t \geq 1, y \geq 2), \quad \Omega(n) := \sum_{p^\nu || n} \nu \quad (n \geq 1).$$

Since $\tau(ab) \leq \tau(a)2^{\Omega(b)}$ ($a, b \geq 1$), we have, for $(x, y) \in H_\varepsilon$,

$$(3.1) \quad \mathfrak{S}(x, y) \geq \frac{1}{\Psi(x, y)} \sum_{\substack{dd' \in S(x, y) \\ (d, d') = 1 \\ 0 < \log(d'/d) \leq 1}} \frac{1}{2^{\Omega(dd')}} \Psi\left(\frac{x}{dd'}, y\right) \gg \sum_{\substack{dd' \in S(x, y) \\ (d, d') = 1 \\ 0 < \log(d'/d) \leq 1}} \frac{\varrho(u - u_{dd'})}{\varrho(u) dd' 2^{\Omega(dd')}},$$

where the last inequality follows from (1.6). To evaluate the double sum in (3.1), we establish an asymptotic formula for

$$T_d(x, y) := \sum_{\substack{m \in S(x, y) \\ (m, d) = 1}} \frac{1}{2^{\Omega(m)}}.$$

We shall make use of the following notation

$$C := \prod_p \frac{\sqrt{1 - 1/p}}{1 - 1/2p}, \quad \kappa_y := \frac{1}{(\log y)^{2/5}},$$

$$\varphi_y(d) := \prod_{p|d} \left(1 + \frac{1}{2p^{1-\kappa_y}}\right), \quad \vartheta_y(d) := \sum_{p|d} \frac{\log p}{p^{1-\kappa_y}}, \quad \mathfrak{q}(d) := \prod_{p|d} \left(1 - \frac{1}{2p}\right) \quad (d \geq 1).$$

Lemma 3.1. *Let $\varepsilon > 0$. For $x \geq 1$, $y > \exp\{(\log_2 3x)^{5/3+\varepsilon}\}$, $d \in S(x, y)$, we have*

$$(3.2) \quad T_d(x, y) = \frac{Cx\varrho_{1/2}(u)}{\sqrt{\log y}} \left\{ \mathfrak{q}(d) + O\left(\kappa_y \varphi_y(d) \{1 + \vartheta_y(d)\}\right) \right\}.$$

Proof. We have

$$T_d(x, y) = \sum_{m \in S(x, y)} \frac{1}{2^{\Omega(m)}} \sum_{t|(m, d)} \mu(t) = \sum_{t|d} \frac{\mu(t)}{2^{\Omega(t)}} T_1\left(\frac{x}{t}, y\right).$$

An estimate for the inner T_1 -term follows from [19, cor. 2.3], which, in the domain

$$x \geq 1, \quad y > \exp\{(\log_2 3x)^{5/3+\varepsilon}\},$$

we rewrite as

$$(3.3) \quad T_1(x, y) = \frac{Cx\varrho_{1/2}(u)}{\sqrt{\log y}} \left\{ 1 + O\left(\frac{\log(u+1)}{\log y} + \frac{1}{\sqrt{\log y}} + \frac{1}{\log(2x)}\right) \right\}.$$

Here the error term $1/\log(2x)$ enables to include the case $1 \leq x < y$: the corresponding estimate follows from [17, th. II.6.2]. Since $\log(u+1) \ll (\log y)^{3/5}$ in H_ε , we get

$$(3.4) \quad T_d(x, y) = \frac{Cx}{\sqrt{\log y}} \sum_{\substack{t|d \\ t \leq x/\sqrt{y}}} \frac{\mu(t)\varrho_{1/2}(u - u_t)}{t^{2\Omega(t)}} + R_1 + R_2,$$

with

$$R_1 \ll \frac{x}{(\log y)^{9/10}} \sum_{\substack{t|d \\ t \leq x/\sqrt{y}}} \frac{\mu(t)^2 \varrho_{1/2}(u - u_t)}{t^{2\Omega(t)}} \ll \frac{x\varrho_{1/2}(u)}{(\log y)^{9/10}} \sum_{t|d} \frac{\mu(t)^2}{2^{\Omega(t)} t^{1-\xi(2u)/\log y}},$$

$$R_2 \ll \sum_{\substack{t|d \\ x/\sqrt{y} < t \leq x}} \frac{x}{t\sqrt{\log 2x/t}},$$

where the bound for R_1 follows from

$$(3.5) \quad \varrho_{1/2}(u-v) \ll \varrho_{1/2}(u)e^{v\xi(2u)} \quad (u \geq 1, 0 \leq v \leq u - \tfrac{1}{2})$$

proved in [15]¹. By multiplicativity, we thus get

$$(3.6) \quad R_1 \ll \frac{x\varrho_{1/2}(u)\varphi_y(d)}{(\log y)^{9/10}}.$$

Since $d \leq x$, we have $p_{\omega(d)} \ll \log x$, where $p_{\omega(d)}$ denotes the $\omega(d)$ th prime number. Hence, using de Bruijn's estimate for $\log \Psi(x, y)$ as refined in [17, th. III.5.2], we plainly obtain, for a suitable absolute constant $c > 0$,

$$(3.7) \quad \sum_{t|d, t \leq z} 1 \leq \Psi(z, p_{\omega(d)}) \leq z^{c/\log_2 x} \quad (\sqrt{x} \leq z \leq x).$$

As a consequence

$$R_2 \ll x \int_{\sqrt{x}}^x \frac{1}{z} dO(z^{c/\log_2 x}) \ll \sqrt{x} e^{c \log x / \log_2 x},$$

and we conclude that

$$(3.8) \quad R_1 + R_2 \ll \frac{x\varrho_{1/2}(u)\varphi_y(d)}{(\log y)^{9/10}}.$$

To estimate the main term of (3.4), we approximate $\varrho_{1/2}(u-u_t)$ by $\varrho_{1/2}(u)$, using the bound

$$\varrho'_{1/2}(w) \ll \varrho_{1/2}(w) \log(1+w) \quad (w \geq \tfrac{1}{2})$$

which, with an appropriate modification of the range of validity, is also proved in [15, lemma 6.2]. In view of (3.5), this implies that

$$\varrho_{1/2}(u-u_t) - \varrho_{1/2}(u) \ll u_t \varrho_{1/2}(u) t^{\kappa_y} \log(u+1).$$

Thus,

$$\begin{aligned} \sum_{\substack{t|d \\ t \leq x/\sqrt{y}}} \frac{\mu(t)\varrho_{1/2}(u-u_t)}{t^{2\Omega(t)}\varrho_{1/2}(u)} &= \sum_{\substack{t|d \\ t \leq x/\sqrt{y}}} \frac{\mu(t)}{t^{2\Omega(t)}} + O\left(\sum_{\substack{t|d \\ t \leq x/\sqrt{y}}} \frac{\mu(t)^2(\log t) \log(u+1)}{t^{1-\kappa_y} 2^{\Omega(t)} \log y}\right) \\ &= \mathfrak{q}(d) + O\left(\sum_{\substack{t|d \\ x/\sqrt{y} < t \leq x}} \frac{1}{t} + \kappa_y \sum_{t|d} \frac{\mu(t)^2 \log t}{t^{1-\kappa_y} 2^{\Omega(t)}}\right). \end{aligned}$$

By (3.7), the first error term is $\ll \sqrt{y}x^{-1+c/\log_2 x}$, which is compatible with (3.2). To estimate the second, we write $\log t = \sum_{p|t} \log p$ since $\mu^2(t) = 1$ and invert summations. This yields the required estimate (3.2). \square

By (3.1), we have

$$\mathfrak{S}(x, y) \gg \sum_{d \in S(\sqrt{x}/e, y)} \frac{\varrho(u-2u_d)\{T_d(ed, y) - T_d(d, y)\}}{\varrho(u)d^{2\Omega(d)}}.$$

We insert (3.2) to evaluate the difference between curly brackets and sum separately the resulting main term and the remainder terms. This can be done by partial summation, using a variant of (3.3) in which the inclusion of the factors $\mathfrak{q}(d)$ or $\varphi_y(d)\{1 + \vartheta_y(d)\}$ has as sole effects to alter the value of the constant C . This yields

$$(3.9) \quad \begin{aligned} \mathfrak{S}(x, y) &\gg \sum_{d \in S(\sqrt{x}/e, y)} \frac{\mathfrak{q}(d)\varrho(u-2u_d)\varrho_{1/2}(u_d)}{\varrho(u)2^{\Omega(d)}d\sqrt{\log y}} \\ &\gg \frac{1}{\varrho(u)} \int_{1/\log y}^{u/2} \varrho(u-2v)\varrho_{1/2}(v)^2 dv = \frac{1}{2\varrho(u)} \int_{2/\log y}^u \varrho(u-v)\varrho_{1/2}(\tfrac{1}{2}v)^2 dv. \end{aligned}$$

The contribution of the interval $[2/\log y, 2]$ to the last integral is

$$(3.10) \quad \geq 2\varrho(u) \int_{1/\log y}^1 \varrho_{1/2}(v)^2 dv = \frac{2\varrho(u)}{\pi} \int_{1/\log y}^1 \frac{dv}{v} = \frac{2\varrho(u)}{\pi} \log_2 y.$$

¹In [15, lemma 6.1], this bound is claimed for $0 \leq v \leq u$, but it is necessary to exclude the case when $u-v$ is small.

Now observe that (1.8) implies

$$\varrho_{1/2}(\tfrac{1}{2}v)^2 \asymp \frac{\varrho(v)}{\sqrt{v}} \quad (v \geq 1).$$

Since ϱ_2 is the convolution square of ϱ , it follows that

$$(3.11) \quad \frac{1}{\varrho(u)} \int_2^u \varrho(u-v) \varrho_{1/2}(\tfrac{1}{2}v)^2 dv \gg \frac{\varrho_2(u)}{\sqrt{u}\varrho(u)} = \mathfrak{r}(u).$$

Carrying back into (3.9) and taking (3.10) into account, we obtain the required estimate.

4. PROOF OF THEOREM 1.1(i): UPPER BOUND

We adapt to the friable case the iterative method developed by Tenenbaum in [16] (see also [9, §7.4]) for bounding the mean-value of the Δ -function. Throughout this proof the letters c and C , with or without index, stand for absolute positive constants.

Given an integer $n \geq 2$, let us denote by $\{p_j(n)\}_{1 \leq j \leq \omega(n)}$ the increasing sequence of its distinct prime factors. Following [16] (see also [9]), define

$$M_q(n) = \int_{\mathbb{R}} \Delta(n, u)^q du,$$

and, for squarefree n , put

$$n_k := \begin{cases} \prod_{j \leq k} p_j(n) & \text{if } k \leq \omega(n), \\ n & \text{otherwise.} \end{cases}$$

Now, let

$$L_{k,q} = L_{k,q}(x, y) := \sum_{P^+(n) \leq y} \frac{\mu(n)^2 M_q(n_k)^{1/q}}{n^\beta},$$

where $\beta := \alpha(\sqrt{x}, y)$ is the saddle-point related to the friable mean-value of $\tau(n)$, the divisor function.

We aim at bounding $L_{k,q}$ from above for large k and q . The starting point is the identity

$$\Delta(mp, u) = \Delta(m, u) + \Delta(m, u - \log p) \quad (u \in \mathbb{R}, p \nmid m).$$

Apply this to $m = n_k$, $p = p_{k+1}(n)$. Raising to the power q and expanding out, we obtain

$$M_q(n_{k+1}) = 2M_q(n_k) + E_q(n_k, p_{k+1}) \quad (\omega(n) > k),$$

with

$$E_q(m, p) := \sum_{1 \leq j < q} \binom{q}{j} \int_{\mathbb{R}} \Delta(m; v)^j \Delta(m; v - \log p)^{q-j} dv.$$

It follows that

$$L_{k+1,q} \leq 2^{1/q} L_{k,q} + \sum_{\substack{P^+(m) \leq y \\ \omega(m)=k}} \mu(m)^2 \sum_{P^+(m) < p \leq y} E_q(m, p)^{1/q} \sum_{\substack{P^+(n) \leq y \\ \omega(n) \geq k+1 \\ n_{k+1}=mp}} \frac{\mu(n)^2}{n^\beta}.$$

The latter sum is

$$\ll \frac{\zeta_1(\beta, y)}{p^\beta m^\beta} \prod_{\ell \leq p} \frac{1}{1 + \ell^{-\beta}} =: \frac{\zeta_1(\beta, y) g_\beta(p)}{p^\beta m^\beta},$$

where, here and in the remainder of this proof, ℓ denotes a prime number, and

$$\zeta_1(\sigma, y) := \prod_{\ell \leq y} (1 + \ell^{-\sigma}).$$

Hölder's inequality yields

$$\sum_{z < p \leq y} \frac{E_q(m, p)^{1/q}}{p^\beta} \leq \left\{ \sum_{p \geq 2} \frac{E_q(m, p) \log p}{p} \right\}^{1/q} \left\{ \sum_{z < p \leq y} \frac{1}{p^{(q\beta-1)/(q-1)} (\log p)^{1/(q-1)}} \right\}^{(q-1)/q}.$$

and the prime number theorem enables to bound the last sum over p by

$$\ll \frac{qy^{q(1-\beta)/(q-1)}}{(\log z)^{1/(q-1)}}.$$

Now, we have (see, e.g., [9, th. 73])

$$\sum_p \frac{E_q(m, p) \log p}{p} \leq C 4^q \tau(m)^{q/(q-1)} M_q(m)^{(q-2)/(q-1)}.$$

It follows that

$$(4.1) \quad L_{k+1, q} \leq 2^{1/q} L_{k, q} + C_1 q e^{\xi(u/2)} G_k \leq 2^{1/q} L_{k, q} + C_2 q u^2 G_k,$$

with

$$G_k := \zeta_1(\beta, y) \sum_{\substack{P^+(m) \leq y \\ \omega(m) = k}} \frac{\mu(m)^2 \tau(m)^{1/(q-1)} M_q(m)^{(q-2)/q(q-1)} g_\beta(P^+(m))}{m^\beta (\log P^+(m))^{1/q}}.$$

Since

$$\frac{\mu(m)^2 \zeta_1(\beta, y) g_\beta(P^+(m))}{m^\beta} = \sum_{\substack{P^+(n) \leq y \\ n_k = m}} \frac{\mu(n)^2}{n^\beta},$$

we infer that

$$G_k \leq \sum_{\substack{P^+(n) \leq y \\ \omega(n) \geq k}} \frac{\mu(n)^2 \tau(n_k)^{1/(q-1)} M_q(n_k)^{(q-2)/q(q-1)}}{n^\beta (\log p_k(n))^{1/q}}.$$

A new application of Hölder's inequality yields

$$G_k \leq L_{k, q}^{(q-2)/(q-1)} S_k^{1/(q-1)},$$

where

$$\begin{aligned} S_k &:= \sum_{\substack{P^+(n) \leq y \\ \omega(n) \geq k}} \frac{\mu(n)^2 \tau(n_k)}{n^\beta \{\log p_k(n)\}^{(q-1)/q}} \\ &\leq 2 \sum_{\substack{P^+(m) \leq y \\ \omega(m) = k-1}} \frac{\mu(m)^2 \tau(m)}{m^\beta} \sum_{P^+(m) < p \leq y} \frac{1}{p^\beta (\log p)^{1-1/q}} \prod_{p < \ell \leq y} \left(1 + \frac{1}{\ell^\beta}\right) \\ &\leq \frac{\zeta_1(\beta, y)}{(k-1)!} \sum_{p \leq y} \frac{g_\beta(p)}{p^\beta (\log p)^{1-1/q}} \left(\sum_{\ell \leq p} \frac{2}{\ell^\beta}\right)^{k-1} \ll \frac{\zeta_1(\beta, y) y^{1-\beta}}{(k-1)!} \sum_{p \leq y} \frac{e^{-T(p)} \{2T(p)\}^{k-1}}{p (\log p)^{1-1/q}}, \end{aligned}$$

where we set

$$T(p) := \sum_{\ell \leq p} \frac{1}{\ell^\beta}.$$

(Recall that the letter ℓ denotes generically a prime number.)

We evaluate $T(p)$ by [2, lemma 3.6]. Writing

$$\mathcal{L}(z) := e^{(\log z)^{3/5}/(\log_2 z)^{1/5}}, \quad w(t) := \frac{t^{1-\beta} - 1}{(1-\beta) \log t},$$

we have

$$T(p) = \log_2 p + \int_1^{w(p)} t \xi'(t) dt + b + O\left(\frac{w(p)}{\mathcal{L}(p)^c} + \frac{\log(u+1)}{\log y}\right).$$

where b is a suitable constant. Note that $w(y) = u/2 + O(u/\log y)$. Defining

$$h(v) := \int_1^{w(\exp e^v)} t \xi'(t) dt + b_1,$$

with b_1 sufficiently large so that $T(p) \leq \log_2 p + h(\log_2 p)$, and writing $z_v := v + h(v)$, we have, by the prime number theorem,

$$W_k(y) := \sum_{p \leq y} \frac{e^{-T(p)} \{T(p)\}^{k-1}}{p (\log p)^{1-1/q}} \ll \int_0^{\log_2 y} e^{-(2-1/q)z_v + (1-1/q)h(\log_2 y)} z_v^{k-1} dv.$$

Since $h(\log_2 y) \leq u/2 + O(u/\log 2u)$ and since $h'(v) \geq 0$, the change of variables $z = z_v$ yields

$$W_k(y) \ll e^{u/2 + O(u/\log 2u)} \int_0^\infty e^{-(2-1/q)z} z^{k-1} dz \ll \frac{e^{u/2 + O(u/\log 2u)} (k-1)!}{(2-1/q)^{k-1}}.$$

Thus,

$$S_k \ll \frac{\zeta_1(\beta, y)e^{u/2+O(u/\log 2u)}}{(1-1/2q)^k} \ll \frac{\zeta_1(\alpha, y)e^{O(u/\log 2u)}}{(1-1/2q)^k},$$

since $\zeta(\beta, y) = \zeta(\alpha, y)e^{-u/2+O(u/\log 2u)}$ — see [18, (4.2)].

Finally, for q sufficiently large and $\frac{1}{2} < \lambda < \log 2$, we obtain

$$(4.2) \quad G_k \leq C_3 L_{k,q}^{(q-2)/(q-1)} \zeta_1(\alpha, y)^{1/(q-1)} e^{c_0 u/(q \log 2u) + \lambda k/q(q-1)}.$$

At this stage, we introduce

$$L_{k,q}^* = L_{k,q} + 2^{k/q} u^{2q} e^{c_0 u/\log 2u} \zeta_1(\alpha, y),$$

so that (4.1) still holds for $L_{k,q}^*$ in place of $L_{k,q}$. Setting $q(k) := \lfloor c_1 \sqrt{k/\log k} \rfloor$ with sufficiently small, absolute c_1 , we thus have, for large k ,

$$L_{k+1,q}^* \leq \left\{ 2^{1/q} + \frac{1}{k} \right\} L_{k,q}^* \quad (q \leq q(k)),$$

whence

$$(4.3) \quad L_{k+1,q}^* \leq 3^{1/q} L_{k,q}^* \quad (q \leq q(k)).$$

To carry out a double induction on k and q , we also need a bound on $L_{k,q+1}^*$ in terms of $L_{k,q}^*$. This is achieved by the inequality $M_{q+1}(n)^{1/(q+1)} \leq 2M_q(n)^{1/q}$ proved in [9, th. 72], which yields

$$(4.4) \quad L_{k,q+1}^* \leq 2u^2 L_{k,q}^*.$$

With the aim of bounding $L_{k,q(k)}^*$ in terms of $L_{2,q(2)}^*$, we use (4.3) to reduce the parameter k and (4.4) to secure the condition $q \leq q(k)$. The first handling provides an overall factor

$$\leq \prod_{1 \leq q \leq q(k)} q^{c_2} \leq e^{c_3 \sqrt{k \log k}}$$

whereas the second induces a global factor $\ll u^{c_4 q(k)}$.

Finally, we obtain

$$L_{k,q}^* \ll L_{2,q(2)}^* u^{c_5 q(k)} e^{c_5 \sqrt{k \log k}}.$$

Let $K := \log_2 y + u$. It can be shown (see [1] and use a bound similar to [9, (7.44)]) that the contribution to $L_{k,q}$ of those integers n such that $\omega(n) > CK$ is negligible, and we omit the details. Eventually, we arrive at

$$L_{k,q} \ll e^{c_5 \sqrt{K \log K}} u^{c_6 \sqrt{K/\log K}} \zeta(\alpha, y) e^{c_0 u/\log 2u} \ll \zeta(\alpha, y) e^{c_7 \sqrt{\log_2 y \log_3 y} + O(u/\log 2u)},$$

and so

$$\sum_{n \in S(x,y)} \frac{\mu(n)^2 \Delta(n)}{n^\beta} \ll \zeta(\alpha, y) e^{c \sqrt{(\log_2 y) \log_3 y} + O(u/\log 2u)}.$$

Employing the representation $n = mr^2$, $\mu(m)^2 = 1$, we obtain that the same bound holds for

$$\sum_{n \in S(x,y)} \frac{\Delta(n)}{n^\beta}.$$

This is the key to our upper bound for $D(x, y) := \sum_{n \in S(x,y)} \Delta(n)$. We have

$$\begin{aligned} D(x, y) \log x - \int_1^x \frac{D(t, y)}{t} dt &= \sum_{n \in S(x,y)} \Delta(n) \log n \leq \sum_{\substack{mp^\nu \leq x \\ P^+(mp) \leq y}} \Delta(m) (\nu + 1) \log p^\nu \\ &\ll y D\left(\frac{x}{y}, y\right) + \sum_{\substack{x/y < n \leq x \\ P^+(n) \leq y}} \frac{x \Delta(n)}{n} + \sum_{\substack{n \leq x \\ P^+(n) \leq y}} \Delta(n) \sqrt{\frac{x}{n}}. \end{aligned}$$

The trivial bound

$$D(x, y) \leq \sum_{n \in S(x,y)} \tau(n) \ll x \varrho_2(u) \log y,$$

that holds in H_ε (see [19, Cor. 2.3]), furnishes

$$\int_1^x \frac{D(t, y)}{t} dt \ll x \varrho_2(u) \log y, \quad y D(x/y, y) \ll x \varrho_2(u-1) \log y.$$

Moreover, in the same region, for y sufficiently large, $\beta > 1/2$

$$\sum_{\substack{n \leq x \\ P^+(n) \leq y}} \Delta(n) \sqrt{\frac{x}{n}} + \sum_{\substack{x/y < n \leq x \\ P^+(n) \leq y}} \frac{x \Delta(n)}{n} \ll x^\beta e^{\xi(u/2)} \sum_{n \in S(x,y)} \frac{\Delta(n)}{n^\beta}.$$

Collecting these estimates, we obtain

$$\begin{aligned} D(x, y) &\ll x \frac{\varrho_2(u)}{u} + x \varrho_2(u) \log 2u + \frac{x^\beta \zeta(\alpha, y) e^{c\sqrt{(\log_2 y) \log_3 y} + O(u/\log 2u)}}{\log x} \\ &\ll \Psi(x, y) 2^{u+O(u/\log 2u)} e^{c\sqrt{\log_2 y \log_3 y}}, \end{aligned}$$

where we used (1.9), (1.6), the estimate

$$\frac{x^\beta \zeta(\alpha, y)}{\log x} \asymp \Psi(x, y) 2^{u+O(u/\log u)},$$

which follows from (2.1), (2.4) and

$$(\beta - \alpha) \log x = -u \int_{u/2}^u \xi'(t) dt + O(1) = u \log 2 + O\left(\frac{u}{\log u}\right).$$

This concludes the proof of the upper bound included in (1.14).

5. PROOF OF THEOREM 1.1(ii)

We retain notation $g(t)$ from (1.12), ε_y from (1.13), define $\eta_y := (\log_2 y)/\log y$. Since $\max(1, \lfloor \tau(n)/\log n \rfloor) \leq \Delta(n) \leq \tau(n)$ holds for all $n \geq 1$ (see e.g. [9, th. 60, (6.7)]), we have

$$(5.1) \quad \frac{\Psi(x, y; \tau)}{2\Psi(x, y) \log x} \leq \mathfrak{S}(x, y) \leq \frac{\Psi(x, y; \tau)}{\Psi(x, y)} \quad (x \geq y \geq 2).$$

Now, by [18, th. 1.2] and [18, (1.6)], we have, with $\lambda := y/\log x$,

$$(5.2) \quad \frac{\Psi(x, y; \tau)}{\Psi(x, y)} \asymp \zeta(\alpha, y) e^{-uh(\lambda)\{1+O(\varepsilon_y)\}} \quad (x \geq y \geq 2),$$

where we have put

$$h(t) := t \log 4 - (1 + 2t) \log \left(\frac{1 + 2t}{1 + t} \right) = t \log \left(1 + \frac{1}{t} \right) - g(t) \quad (t \geq 0),$$

and, for the purpose of further reference, note that

$$(5.3) \quad uh(\lambda) \sim (1 - \log 2)u \quad (u \rightarrow \infty, \lambda \rightarrow \infty), \quad uh(\lambda) \asymp \bar{u} \quad (x \geq y \geq 2).$$

We shall show that

$$(5.4) \quad \zeta(\alpha, y) = e^{\lambda u \log(1+1/\lambda)\{1+O(\varepsilon_y+1/\log 2u)\}} \quad \left(2 \leq y \leq x^{1/(2 \log_2 x \log_3 x)} \right).$$

Since $g(\lambda)u \gg \bar{u}$ for $x \geq y \geq 2$, we see that (1.15) follows from (5.1) and (5.4) in any subregion where $\bar{u}(\varepsilon_y + 1/\log 2u) \gg \log_2 x$: the condition above corresponds to this requirement when y is large. However, for bounded y , we have $\Psi(x, y; \tau)/\Psi(x, y) \asymp (\log x)^{\pi(y)}$, and so (1.15) holds trivially. Therefore, we may assume in the sequel that y is sufficiently large.

Let us now embark on the proof of (5.4).

Observe that

$$(5.5) \quad \zeta(\alpha, y) = \zeta(1, y) \exp \left\{ \int_\alpha^1 \varphi_y(\sigma) d\sigma \right\}.$$

Using the estimate for $\varphi_y(\sigma)$ given in [11, lemma 13], we may write

$$(5.6) \quad \int_\alpha^1 \varphi_y(\sigma) d\sigma = \left\{ 1 + O\left(\frac{1}{\log y}\right) \right\} \int_\alpha^1 \frac{y^{1-\sigma} - 1}{(1-\sigma)(1-y^{-\sigma})} d\sigma.$$

By inspection of the proof of (2.2) in [11, pp. 285-7], we see that, for a suitable constant C , we have

$$(5.7) \quad \alpha(x, y) = 1 - \frac{\xi(u)}{\log y} + O\left(\frac{1}{(\log y)^2}\right) \quad \left(C(\log x)(\log_2 x)^3 < y \leq x \right).$$

This implies $y^\alpha \gg ye^{-\xi(u)} \gg \log y$ in the same domain, so the contribution of the term $1 - y^{-\sigma}$ in (5.6) is absorbed by the error term. The change of variables defined by $(1 - \sigma) \log y = \xi(t)$ then provides, in view of (5.7) and (1.7),

$$\begin{aligned} \int_\alpha^1 \varphi_y(\sigma) d\sigma &= \left\{1 + O\left(\frac{1}{\log y}\right)\right\} \int_1^u t \xi'(t) dt \\ &= u + \frac{u}{\log u} + O\left(\frac{u}{\log y}\right) = u + O\left(\frac{u}{\log 2u} + \varepsilon_y u\right). \end{aligned}$$

Since, in the domain of (5.7),

$$u\lambda \log\left(1 + \frac{1}{\lambda}\right) = u + O\left(\frac{u}{\log 2u}\right),$$

we obtain (5.4) in the range $C(\log x)(\log_2 x)^3 < y \leq x^{1/(2\log_2 x \log_3 x)}$. Indeed the factor $\zeta(1, y) \asymp \log y$ appearing in (5.5) is absorbed by the error term.

When $2 \leq y \leq C(\log x)(\log_2 x)^3$, we put $t = y^\sigma$ in (5.6) to get

$$\int_\alpha^1 \varphi_y(\sigma) d\sigma = \left\{1 + O\left(\frac{1}{\log y}\right)\right\} \int_{y^\alpha}^y \frac{y/t - 1}{(t-1) \log(y/t)} dt.$$

Note that (2.2) now implies $\alpha \log y \ll \log_2 2y$. Put $T := (\log y)^K$, where K is so large so that $T > y^\alpha$. The contribution of the interval $[T, y]$ to the above integral is

$$\ll \int_T^\infty \frac{y}{t^2} dt \ll \frac{y}{(\log y)^K},$$

where we used the bound $e^v - 1 \ll ve^v$ ($v \geq 0$). Then,

$$\begin{aligned} \int_{y^\alpha}^T \frac{y/t - 1}{(t-1) \log(y/t)} dt &= \left\{1 + O\left(\frac{\log_2 y}{\log y}\right)\right\} \frac{y}{\log y} \int_{y^\alpha}^T \frac{1}{t(t-1)} dt \\ &= \left\{1 + O(\eta_y)\right\} \frac{y}{\log y} \log\left(\frac{1 - 1/T}{1 - 1/y^\alpha}\right). \end{aligned}$$

>From [11, (7.18)], it follows that

$$\log\left(\frac{1}{1 - y^{-\alpha}}\right) = \log\left(1 + \frac{1}{\lambda}\right) \left\{1 + O\left(\frac{\log_2 y}{\log y}\right)\right\} \quad (2 \leq y \leq C(\log x)(\log_2 x)^3).$$

Therefore

$$\begin{aligned} \int_\alpha^1 \varphi_y(\sigma) d\sigma &= \left\{1 + O(\eta_y)\right\} \frac{y}{\log y} \log\left(1 + \frac{1}{\lambda}\right) + O\left(\frac{y}{(\log y)^K}\right) \\ &= \left\{1 + O(\eta_y)\right\} u\lambda \log\left(1 + \frac{1}{\lambda}\right). \end{aligned}$$

This establishes (5.4) in the complementary range $2 \leq y \leq C(\log x)(\log_2 x)^3$.

This completes the proof of theorem 1.1(ii).

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LABORATOIRE DE MATHÉMATIQUES PURES ET APPLIQUÉES, CNRS, UNIVERSITÉ DU LITTORAL CÔTE D’OPALE,
50 RUE F. BUISSON, BP 599, CALAIS, 62228, FRANCE
Email address: bruno.martin@univ-littoral.fr

INSTITUT ÉLIE CARTAN, UNIVERSITÉ DE LORRAINE, B.P. 70239,
F-54506 VANDEUVRE-LÈS-NANCY CEDEX, FRANCE
Email address: gerald.tenenbaum@univ-lorraine.fr

LABORATOIRE DE MATHÉMATIQUES PURES ET APPLIQUÉES, CNRS, UNIVERSITÉ DU LITTORAL CÔTE D’OPALE,
50 RUE F. BUISSON, BP 599, CALAIS, 62228, FRANCE
Email address: julie.wetzer@univ-littoral.fr