



HAL
open science

Logics for Contact and Measure

Philippe Balbiani, Quentin Gougeon, Tinko Tinchev

► **To cite this version:**

Philippe Balbiani, Quentin Gougeon, Tinko Tinchev. Logics for Contact and Measure. 2024. hal-04544145v1

HAL Id: hal-04544145

<https://hal.science/hal-04544145v1>

Preprint submitted on 12 Apr 2024 (v1), last revised 24 Nov 2024 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Logics for Contact and Measure

Philippe Balbiani^a

Quentin Gougeon^a

Tinko Tinchev^b

^aInstitut de recherche en informatique de Toulouse
CNRS-INPT-UT3
Toulouse, France

`philippe.balbiani@irit.fr, quentin.gougeon@irit.fr`

^bFaculty of Mathematics and Informatics
Sofia University St. Kliment Ohridski
Sofia, Bulgaria
`tinko@fmi.uni-sofia.bg`

Abstract

We enrich contact algebras with a new binary relation that compares the size of regions, and provide axiom systems for various logics of contact and measure. Our contribution is three-fold: (1) we characterize the relations on a Boolean algebra that derive from a measure, thereby improving an old result of Kraft, Pratt and Seidenberg; (2) for all $n \geq 1$, we axiomatize the logic of regular closed sets of \mathbb{R}^n with null boundary; (3) considering a broad class of equational theories that contains all logics of contact, we prove that they all have unary or finitary unification, and that unification and admissibility are decidable.

1 Introduction

This paper attempts to combine two lively areas of research in logic that share similar motivations. On the one hand, *measurement theory* investigates the relationships between the qualitative and quantitative aspects of measurement. Qualitative statements are those of the form “the table is heavier than the chair”, while quantitative statements involve numerical measures, as in “the temperature of the room is 19°C”. Even though modern science allowed us to quantify a lot of our intuitions, it acts as a complement rather than a replacement of common sense – seeing that our daily life remains indeed full of qualitative reasoning. The mathematical foundations of measurement have been studied extensively in a series of volumes [KLTS71], and results from this field are notably relevant to *probability logic*, a bundle of formal systems for probabilistic reasoning [ORM16].

On the other hand, *contact algebra* is a theory of space that makes no mention of points, but instead deals with *regions* as a primitive concept, and aims to describe the environment in terms of which regions are in contact with each other. Formally, the collection of all regions forms a Boolean algebra equipped with a binary relation of contact. This framework is thus meant to be practical for both human minds and artificial agents, while remaining general enough to describe a wide variety of structures. Its origins can be traced back to the philosophical work of de Laguna [dL22] and Whitehead [Whi29], and this line of research is nowadays carried by a branch of Artificial Intelligence known as *Qualitative Spatial Reasoning (QSR)* [CH01]. We

refer to [Vak07] for a comprehensive survey of the field. Here we will be mostly interested in the work of Balbiani, Tinchev and Vakarelov [BTV07b], which introduced various logics of contact.

Thus measurement theory and contact algebra both seem to have been designed with a similar vision, and a common purpose of describing the world through the lens of qualitative concepts. In this regard, it is only natural to attempt to connect the two frameworks. In this paper, we undertake to design a logic of contact and measure, by building on the existing work on contact logic and enriching the language with a binary relation \preceq that allows to compare the size of regions. As we soon realized when working on axiomatization, this project brings up the challenge of finding necessary and sufficient conditions for \preceq to derive from a measure. This question is not new, and was addressed by Kraft, Pratt and Seidenberg in their 1959 paper [KPS59]. Their characterization, however, involves infinitely many conditions, which is awkward in many respects. Notably, it fails to provide a finite axiomatization for logics of measure – not only those introduced in this paper, but also various examples found across the literature [Seg71, Gär75, vdH96]. It is surprising, perhaps, that this result has never been improved in sixty years, nor proved to be optimal. In Section 2, we break this uncomfortable status quo by proposing a new characterization consisting of only two elementary conditions, called *comparability* and *linearity*. We also extend this result to measures with infinite values.

In Section 3, we essentially expand the content of [BTV07b] by introducing logics for contact and measure, along with semantics, axiom systems, and soundness and completeness results. Our main logics are designed to describe abstract and general spaces, ranging from actual topological spaces to more minimalistic structures called *adjacency frames*. However, it is also desirable to axiomatize concrete spaces, like the real line. In Section 4, we thus give special attention to the *logic of the unbounded space* \mathbf{S} , and prove that it is complete for the space \mathbb{R} with the Lebesgue measure. More precisely, regions are interpreted as regular closed sets of \mathbb{R} with boundary of measure 0. This choice is inspired from [LS19], and is meant to ensure that the Lebesgue measure behaves properly, while being lax enough to model most ‘natural’ physical bodies. Likewise, we show that the logic \mathbf{BS} (the bounded counterpart of \mathbf{S}) is complete for the unit segment $[0, 1]$, and we also generalize these results to the n -dimensional case.

In Section 5, we address the fundamental problem of *unification*. Unification problems are systems of equations involving logical formulas, and their solutions (called *unifiers*) are Boolean substitutions. A logic is *unary* or *finitary* when the solution space of any unification problem can be generated by finitely many unifiers. Unification is usually motivated by the closely related problem of *admissibility*, which is the question of deciding whether an inference rule preserves validity. In [BG20], Balbiani and Gencer investigated parameter-free unification and admissibility in contact logic, and proved that the two problems are decidable, but left open the question of the unification type. In this paper, we generalize their work to the parametric setting, and show that unification is unary or finitary in *all* logics of contact. However, upon closer examination of our proof, we also realized that these results are not tied to contact logic in particular, but stem from more fundamental properties of Boolean substitutions. By carefully identifying the minimal properties required by our arguments, we managed to extend our results to a considerably larger class of abstract equational theories, which we dub *Boolean-valued theories*. Finally, we give in Section 6 some directions for future work.

2 Measurable Boolean algebras

2.1 Preliminaries

Formally, a Boolean algebra is a tuple $\langle B, \sqcup, \sqcap, \cdot, *, 0, 1 \rangle$ defined as in [DP02, Ch. 4], and with the usual requirement that $0 \neq 1$. For convenience, we will identify $\langle B, \sqcup, \sqcap, \cdot, *, 0, 1 \rangle$ to its underlying set B . We write $a - b := a \sqcap b^*$. We write $a \sqsubseteq b$ whenever $a \sqcap b = a$. We let $a \sqsubset b$ stand for $a \sqsubseteq b$ and $b \not\sqsubseteq a$. We call $a, b \in B$ *disjoint* if we have $a \sqcap b = 0$.

Let \preceq be a binary relation on B . We let $a \prec b$ stand for $a \preceq b$ and $b \not\preceq a$, and $a \simeq b$ for $a \preceq b$ and $b \preceq a$. A *measure* on B is a function $\mu : B \rightarrow [0, \infty]$ satisfying $\mu(0) = 0$ and $\mu(a \sqcup b) = \mu(a) + \mu(b)$ whenever $a \sqcap b = 0$. We call μ *bounded* if in addition we have $\mu(a) < \infty$ for all $a \in B$, and *unbounded* if it is not bounded. We call μ *definite* if $\mu(a) = 0$ implies $a = 0$ for all $a \in B$. We define the binary relation \preceq_μ by $a \preceq_\mu b \iff \mu(a) \leq \mu(b)$ for all $a, b \in B$. We call μ *compatible* with \preceq if the relations \preceq and \preceq_μ are equal. The pair $\langle B, \preceq \rangle$ is said to be (bounded, unbounded, definite) *measurable* if there exists a (bounded, unbounded, definite) measure μ compatible with \preceq .

In the rest of this section, we assume that B is finite. It is then well known [DP02, Ch. 5] that we can represent B as a powerset algebra $\langle 2^X, \cup, \cap, *, 0, 1 \rangle$, where X is a finite set.¹ Kraft, Pratt and Seidenberg characterized bounded measurable algebras in [KPS59], and their result was later rewritten by Scott [Sco64] in a clearer manner, presented below. Given a_1, \dots, a_m and $x \in X$, we write $\mathbf{count}_x(a_1, \dots, a_m) := |\{i \in [1, m] : x \in a_i\}|$.

Theorem 2.1 ([Sco64]). *The pair $\langle B, \preceq \rangle$ is bounded measurable if and only if the following conditions are satisfied, for all $m \geq 1$ and for all $a, b, a_1, \dots, a_m, b_1, \dots, b_m \in B$:*

- **Positivity:** $0 \preceq a$;
- **Comparability:** $a \preceq b$ or $b \preceq a$;
- **Cancellation:** if $\mathbf{count}_x(a_1, \dots, a_m) = \mathbf{count}_x(b_1, \dots, b_m)$ for all $x \in X$ and $a_i \preceq b_i$ for all $i \in [1, m - 1]$, then $b_m \preceq a_m$.

Remark 2.2. Kraft, Pratt and Seidenberg were initially interested in the existence of a *probability measure* μ , which additionally satisfies $\mu(1) = 1$. This is why the original result contains the extra condition that $0 \prec 1$, which we omit here in order to remain as general as possible.

However, this result is not fully satisfying for a number of reasons, related to the cancellation conditions. First, they involve the high-level operator \mathbf{count}_x , and even though they can be rewritten in a purely Boolean manner [Seg71], they remain quite awkward to read and compute. Second, they come in infinite number, and are thus helpless when it comes to design finite axiomatizations, as mentioned in the introduction. In order to improve this situation, we propose the following new characterization.

Theorem 2.3. *The pair $\langle B, \preceq \rangle$ is bounded measurable if and only if the following conditions are satisfied, for all $a, b, c, d \in B$:*

- **Comparability:** $a \preceq b$ or $b \preceq a$;
- **Linearity:** if $a \sqcap c = 0$ and $a \cup c \preceq b \cup d$ and $d \preceq c$, then $a \preceq b$.

Let us briefly explain the proof strategy for the right-to-left implication of Theorem 2.3. The basic idea is to rely on Theorem 2.1, so we want to derive the cancellation conditions from comparability and linearity. To prove cancellation, assume that $\mathbf{count}_x(a_1, \dots, a_m) = \mathbf{count}_x(b_1, \dots, b_m)$ for all $x \in X$, and $a_i \preceq b_i$ for all $i \in [1, m - 1]$. Consider for a moment the case where $a_1, \dots, a_m, b_1, \dots, b_m$ are all pairwise disjoint. Then, the counting assumption yields $b_1 \cup \dots \cup b_m \preceq a_1 \cup \dots \cup a_m$, and by applying linearity $m - 1$ times we arrive at $b_m \preceq a_m$. This does not work in the general case, because when $\mathbf{count}_x(a_1, \dots, a_m) \geq 2$, the large union $a_1 \cup \dots \cup a_m$ fails to keep track of the different repetitions of x . However, we can bypass this issue, and fall back to the previous case, by ‘duplicating’ the elements of X . In Section 2.2, we will show that we can introduce an equivalent copy of any element of X , in a way that preserves positivity, comparability, and a weaker version of linearity. To keep things organized, we enumerate all the conditions on \preceq that we will use in this paper:

- **Positivity:** $0 \preceq a$;

¹While it is natural to replace \sqcup and \sqcap by their set-theoretic counterparts \cup and \cap , we prefer to keep the symbols 0 and 1 which we find more meaningful and visually more appealing than \emptyset and X .

- **Reflexivity:** $a \preceq a$;
- **Comparability:** $a \preceq b$ or $b \preceq a$;
- **Transitivity:** if $a \preceq b$ and $b \preceq c$, then $a \preceq c$;
- **Monotonicity:** if $a \subseteq b$ then $a \preceq b$;
- **Additivity:** if $a \cap c = b \cap c = 0$, then $a \preceq b \iff a \cup c \preceq b \cup c$;
- **Linearity:** if $a \cap c = 0$ and $a \cup c \preceq b \cup d$ and $d \preceq c$, then $a \preceq b$;
- **Weak Linearity:** if $a \cap c = b \cap d = a \cap d = c \cap b = 0$ and $a \cup c \preceq b \cup d$ and $d \preceq c$, then $a \preceq b$;
- **Bounded Linearity:** if $1 \not\preceq c$ and $a \cap c = 0$ and $a \cup c \preceq b \cup d$ and $d \preceq c$, then $a \preceq b$;
- **Definiteness:** if $a \preceq 0$ then $a = 0$;
- **Unboundedness:** $1 \preceq a$ or $1 \preceq a^*$;
- **Cancellation:** if $\text{count}_x(a_1, \dots, a_m) = \text{count}_x(b_1, \dots, b_m)$ for all $x \in X$ and $a_i \preceq b_i$ for all $i \in [1, m-1]$, then $b_m \preceq a_m$;

with $m \geq 1$ and $a, b, c, d, a_1, \dots, a_m, b_1, \dots, b_m \in B$. First, let us prove some basic properties.

Proposition 2.4.

1. If \preceq has comparability then \preceq is reflexive.
2. If \preceq is monotone then \preceq is positive and reflexive.
3. If \preceq is linear then \preceq is weakly linear.
4. If \preceq is reflexive and linear, then \preceq is positive.
5. If \preceq is reflexive and linear, then \preceq is monotone.
6. If \preceq is reflexive and weakly linear, then \preceq is additive.
7. If \preceq is reflexive and weakly linear, then \preceq is transitive.

Proof.

1. Simply apply comparability with $a = b$.
2. For all $a \in B$ we have $0 \subseteq a$ and $a \subseteq a$, and thus $0 \preceq a$ and $a \preceq a$.
3. Clear.
4. Let $a \in B$. By assumption we have $a \preceq a$, and thus $0 \cup a \preceq a \cup a$. From $0 \cap a = 0$ and $a \preceq a$, we then obtain $0 \preceq a$ by linearity.
5. Let $a, b \in B$ be such that $a \subseteq b$. Then we have $a \cup c = b \cup 0$, where $c := b - a$. By reflexivity, it follows that $a \cup c \preceq b \cup 0$. By 4, we have $0 \preceq c$, and thus $a \preceq b$ by linearity.
6. Let $a, b, c \in B$ be such that $a \cap c = b \cap c = 0$. First assume that $a \cup c \preceq b \cup c$. By reflexivity we have $c \preceq c$, and by weak linearity it follows that $a \preceq b$.

Conversely, assume that $a \preceq b$. Let $a' := a - b$, $b' := b - a$ and $c' := c \cup (a \cap b)$. From $a \preceq b$ we get $a' \cup (a \cap b) \preceq b' \cup (a \cap b)$. We also have $a' \cap (a \cap b) = b' \cap (a \cap b) = 0$ and $a \cap b \preceq a \cap b$, whence $a' \preceq b'$ by weak linearity. Next we have $(a' \cup c') \cup b' \preceq (b' \cup c') \cup a'$ by reflexivity, and we can also check that $a' \cap b' = a' \cap c' = b' \cap c' = 0$. By applying weak linearity, we then obtain $a' \cup c' \preceq b' \cup c'$, which boils down to $a \cup c \preceq b \cup c$, as desired.

7. Assume that $a \preceq b$ and $b \preceq c$. Let us introduce

$$\begin{aligned} u &:= (a \cap b) - c, \\ v &:= (a \cap c) - b, \\ w &:= (b \cap c) - a, \\ d &:= a \cap b \cap c, \\ a' &:= a - (b \cup c), \\ b' &:= b - (a \cup c), \\ c' &:= c - (a \cup b), \end{aligned}$$

depicted in Figure 1. We can check that u, v, w, d, a', b', c' are pairwise disjoint. By 6, the relation \preceq is additive. Hence, from $a \preceq b$ and $b \preceq c$ we obtain $a' \cup v \preceq b' \cup w$ and $b' \cup u \preceq c' \cup v$. By reflexivity, we also have $a' \cup b' \cup c' \cup u \cup v \cup w \preceq a' \cup b' \cup c' \cup u \cup v \cup w$. After applying weak linearity twice, we obtain $a' \cup u \preceq c' \cup w$. By additivity, it follows that $a' \cup u \cup v \cup d \preceq c' \cup w \cup v \cup d$, that is, $a \preceq c$.

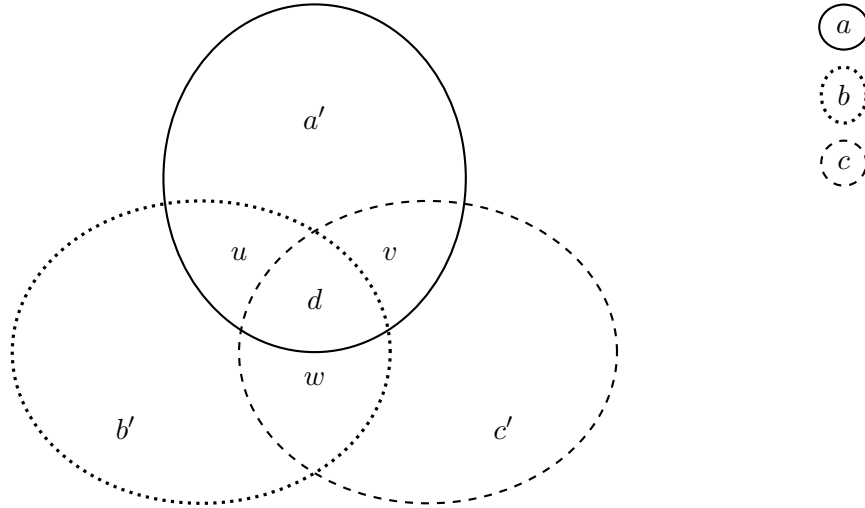


Figure 1: Venn diagram of a, b, c

□

2.2 Duplicating atoms

In this subsection, we assume that \preceq satisfies monotonicity, comparability and weak linearity. By Proposition 2.4, we know that \preceq is reflexive, positive, additive and transitive, and we will use these properties freely in the sequel. Let us select an element $x^1 \in X$. Our goal is to extend B by introducing a fresh element x^2 such that $\{x^1\}$ and $\{x^2\}$ have equal measure, while preserving the properties of monotonicity, comparability and weak linearity. Let thus write $X' := X \cup \{x^2\}$ and $B' := 2^{X'}$. We observe that $B \subseteq B'$, and that B and B' share the same operators \cup, \cap and 0 . We aim to prove the following.

Proposition 2.5. *There exists a binary relation \preceq' on B' such that:*

1. \preceq' satisfies monotonicity, comparability and weak linearity,
2. for all $a, b \in B$ we have $a \preceq b \iff a \preceq' b$,

3. $\{x^1\} \simeq' \{x^2\}$.

We write $\bar{x}^2 := x^1$, and $\bar{x} := x$ for all $x \in X$. Given $a \in B'$, we then define $\bar{a} := \{\bar{x} : x \in a\}$. We call $a \in B'$ *saturated* if we have $x^1 \in a$ and $x^2 \in a$. A pair $\langle a_0, a_1 \rangle \in B'^2$ is said to be a *partition* of a if we have $a_0 \cap a_1 = 0$ and $a_0 \cup a_1 = a$, and *proper* if in addition we have $a_0 \neq 0$ and $a_1 \neq 0$. Given $a, b \in B'$, we now define $a \preceq' b$ by induction on $|a| + |b|$ as follows.

1. If $a \cap b \neq 0$ then $a \preceq' b \iff a \setminus b \preceq' b \setminus a$.
2. Otherwise, if $a \cup b$ is non-saturated, then $a \preceq' b \iff \bar{a} \preceq \bar{b}$.
3. Otherwise, $a \preceq' b$ if and only if for all partitions $\langle a_0, a_1 \rangle$ of a and all partitions $\langle b_0, b_1 \rangle$ of b such that at least one of $\langle a_0, a_1 \rangle$ or $\langle b_0, b_1 \rangle$ is proper, we have $b_1 \preceq' a_1 \implies a_0 \preceq' b_0$.

First, let us check that \preceq' is well defined. In case 1, $a \cap b \neq 0$ entails $a \setminus b \subset a$ and $b \setminus a \subset b$, and thus $|a \setminus b| + |b \setminus a| < |a| + |b|$. Hence, we can legally use the truth value of $a \setminus b \preceq' b \setminus a$. In case 3, if the partition $\langle a_0, a_1 \rangle$ is proper, then $a_1 \subset a$ and $b_1 \subseteq b$. If instead the partition $\langle b_0, b_1 \rangle$ is proper, then $a_1 \subseteq a$ and $b_1 \subset b$. In both cases, we have $|a_1| + |b_1| < |a| + |b|$, and we can thus refer to $b_1 \preceq' a_1$. The same reasoning applies to $a_0 \preceq' b_0$.

Next, we prove that \preceq' has the required properties.

Lemma 2.6. *We have $\{x^1\} \simeq' \{x^2\}$.*

Proof. We have $\{x^1\} \cap \{x^2\} = 0$, and $\{x^1, x^2\}$ is saturated, so $\{x^1\}$ and $\{x^2\}$ fall in case 3. However, since $\{x^1\}$ and $\{x^2\}$ are singletons, they have no proper partition, and so the condition of case 3 is vacuously true. Hence $\{x^1\} \simeq' \{x^2\}$. \square

Lemma 2.7. *For all $a, b \in B$, we have $a \preceq b \iff a \preceq' b$.*

Proof. We proceed by induction on $|a| + |b|$. First assume that a and b fall in case 1. Then $a \preceq' b \iff a \setminus b \preceq' b \setminus a$. By the induction hypothesis, we have $a \setminus b \preceq' b \setminus a \iff a \setminus b \preceq b \setminus a$. By applying additivity with $c := a \cap b$, we obtain $a \setminus b \preceq b \setminus a \iff a \preceq b$. After combining all these equivalences, we end up with $a \preceq' b \iff a \preceq b$. Otherwise, since $a \cup b$ is a member of B , it is non-saturated, and thus a and b fall in case 2. Then $\bar{a} = a$ and $\bar{b} = b$, and the conclusion follows. \square

Lemma 2.8. *The relation \preceq' is positive.*

Proof. Let $b \in B'$. We proceed by induction on $|b|$, and consider all three cases on 0 and b . Since $0 \cap b = 0$, we are not in case 1. In case 2, $0 \preceq' b$ follows from $0 \preceq \bar{b}$. So assume that we are in case 3. Consider a partition $\langle 0, 0 \rangle$ of 0 and a partition $\langle b_0, b_1 \rangle$ of b such that at least one of $\langle 0, 0 \rangle$ or $\langle b_0, b_1 \rangle$ is proper (obviously, in this case $\langle b_0, b_1 \rangle$ must be proper). By the induction hypothesis we have $0 \preceq' b_0$, and in particular the implication $b_1 \preceq' 0 \implies 0 \preceq' b_0$ is true. Therefore $0 \preceq' b$. \square

Lemma 2.9. *The relation \preceq' is monotone.*

Proof. Let $a, b \in B'$ and assume that $a \subseteq b$. We proceed by induction on $|a| + |b|$. First assume that a and b fall in case 1. Then $a \setminus b = 0$, and so by Lemma 2.8 we have $a \setminus b \preceq' b \setminus a$, whence $a \preceq' b$. Otherwise, we have $a = a \cap b = 0$ and thus $a \preceq' b$ by Lemma 2.8 again. \square

Lemma 2.10. *If $b \cap c = 0$ and $a \subseteq b$ and $b \preceq' c$, then $a \preceq' c$.*

Proof. We proceed by induction on $|b| + |c|$, and distinguish cases on $b \preceq' c$. Since $b \cap c = 0$, we are not in case 1. In case 2, $b \cup c$ is not saturated, and so $a \cup c$ is not saturated either. By assumption we have $\bar{b} \preceq \bar{c}$. Since $a \subseteq b$, we have $\bar{a} \subseteq \bar{b}$, and thus $\bar{a} \preceq \bar{b}$. By transitivity, it follows that $\bar{a} \preceq \bar{c}$. Hence $a \preceq' c$.

Now assume that we are in case 3. If $a = 0$ or $a = b$ then $a \preceq' c$ is immediate, so we assume that $a \neq 0$ and $a \subset b$. Then $\langle a, b \setminus a \rangle$ is a proper partition of b , and $\langle c, 0 \rangle$ is a partition of c . By Lemma 2.8 we have $0 \preceq' b \setminus a$, and by applying the definition of \preceq' in case 3 to $\langle a, b \setminus a \rangle$ and $\langle c, 0 \rangle$, it follows that $a \preceq' c$. \square

Lemma 2.11. *If $a \cap b = 0$ and $a \preceq' b$ and $b \preceq' 0$, then $a \preceq' 0$.*

Proof. By induction on $|a| + |b|$. We distinguish cases on $a \preceq' b$. Since $a \cap b = 0$, we are not in case 1. In case 2, a and b are not saturated and $\bar{a} \preceq \bar{b}$. Since $b \cap 0 = 0$ and b and 0 are not saturated, we obtain $\bar{b} \preceq 0$ from $b \preceq' 0$. Then $\bar{a} \preceq 0$ by transitivity. Since $a \cap 0 = 0$ and a and 0 are not saturated, it follows that $a \preceq' 0$.

In case 3, $a \cup b$ is saturated. Suppose that a is saturated. Then a and 0 fall in case 3 as well. To show that $a \preceq' 0$, we consider a proper partition $\langle a_0, a_1 \rangle$ of a and the unique partition $\langle 0, 0 \rangle$ of 0 . We have $a_0 \subseteq a$ and $a \preceq' b$ and $a \cap b = 0$, so by Lemma 2.10 we obtain $a_0 \preceq' b$. We also have $a_0 \cap b = 0$, and so by the induction hypothesis we get $a_0 \preceq' 0$. In particular, this makes the implication $0 \preceq' a_1 \implies a_0 \preceq' 0$ true, and this proves that $a \preceq' 0$.

Now suppose that a is not saturated. Since $a \cup b$ is saturated, there must exist $i \in \{1, 2\}$ such that $x^i \in b$. Let $b_0 := \{x^i\}$ and $b_1 := b \setminus \{x^i\}$. Then $\langle a, 0 \rangle$ is a partition of a , and $\langle b_0, b_1 \rangle$ is a proper partition of b . Since $b_0, b_1 \subseteq b$ and $b \cap 0 = 0$ and $b \preceq' 0$, we obtain $b_0 \preceq' 0$ and $b_1 \preceq' 0$ by Lemma 2.10. By applying the definition of \preceq' in case 3 to $\langle a, 0 \rangle$ and $\langle b_0, b_1 \rangle$, it follows that $a \preceq' b_0$. By applying the induction hypothesis, we then obtain $a \preceq' 0$, as desired. \square

Lemma 2.12. *If $a \cap b = a \cap c = b \cap c = 0$, then $a \preceq' b$ iff $a \cup c \preceq' b \cup c$.*

Proof. If $a \cup c$ and $b \cup c$ fall in case 1, we have $a \cup c \preceq' b \cup c \iff (a \cup c) \setminus (b \cup c) \preceq' (b \cup c) \setminus (a \cup c)$. Since $a \cap b = a \cap c = b \cap c = 0$, we have $(a \cup c) \setminus (b \cup c) = a$ and $(b \cup c) \setminus (a \cup c) = b$, and the equivalence simplifies to $a \cup c \preceq' b \cup c \iff a \preceq' b$. Otherwise, we have $(a \cup c) \cap (b \cup c) = 0$, whence $c = 0$, and the result is immediate. \square

Lemma 2.13. *We have $a \preceq' b$ if and only if $a \setminus b \preceq' b \setminus a$.*

Proof. Apply Lemma 2.12 with $a' := a \setminus b$, $b' := b \setminus a$ and $c := a \cap b$. \square

Lemma 2.14. *The relation \preceq' is weakly linear.*

Proof. Suppose that $a \cap c = b \cap d = a \cap d = c \cap b = 0$ and $a \cup c \preceq' b \cup d$ and $d \preceq' c$. Writing $u := a \cup c$ and $v := b \cup d$, we have $u \preceq' v$, and we proceed by induction on $|u| + |v|$. Again, we distinguish cases on $u \preceq' v$. In case 1, $u \preceq' v$ yields $u \setminus v \preceq' v \setminus u$, that is, $(a \setminus v) \cup (c \setminus v) \preceq' (b \setminus u) \cup (d \setminus u)$. We aim to apply the induction hypothesis to $a \setminus v$, $b \setminus u$, $c \setminus v$ and $d \setminus u$. Since $d \preceq' c$, we obtain $d \setminus c \preceq' c \setminus d$ by Lemma 2.13. Since $a \cap d = c \cap b = 0$, it follows that $d \setminus u \preceq' c \setminus v$. From $a \cap c = b \cap d = a \cap d = c \cap b = 0$ we immediately deduce $(a \setminus v) \cap (c \setminus v) = (b \setminus u) \cap (d \setminus u) = (a \setminus v) \cap (d \setminus u) = (c \setminus v) \cap (b \setminus u) = 0$, and so by the induction hypothesis it follows that $a \setminus v \preceq' b \setminus u$. Since $a \cap d = b \cap c = 0$, this amounts to $a \setminus b \preceq' b \setminus a$, whence $a \preceq' b$ by Lemma 2.13.

In case 2, we have $u \cap v = 0$ and $u \cup v$ is non-saturated. Then a, c, b, d are pairwise disjoint, and $\bar{a} \cup \bar{c} \preceq \bar{b} \cup \bar{d}$. We want to apply the weak linearity of \preceq to $\bar{a}, \bar{b}, \bar{c}, \bar{d}$. Suppose that there exists $x \in \bar{a} \cap \bar{c}$. Then $x = \bar{y} = \bar{z}$ for some $y \in a$ and $z \in c$, and since $a \cap c = 0$ we must have $\{y, z\} = \{x^1, x^2\}$. However, this leads to $x^1, x^2 \in a \cup c \subseteq u \cup v$, contradicting the fact that $u \cup v$ is not saturated. Therefore $\bar{a} \cap \bar{c} = 0$. By the same reasoning, we obtain $\bar{b} \cap \bar{d} = \bar{a} \cap \bar{d} = \bar{c} \cap \bar{b} = 0$. Further, $d \cap c = 0$ and $d \cup c \subseteq u \cup v$ is non-saturated, so d and c fall in case 2, and we have $\bar{d} \preceq \bar{c}$. By weak linearity, it follows that $\bar{a} \preceq \bar{b}$. As in the case of d and c , we can show that a and b fall in case 2, and thus obtain $a \preceq' b$.

Finally, in case 3, $\langle a, c \rangle$ is a partition of u , and $\langle b, d \rangle$ is a partition of v . If one of these partitions is proper, then since $d \preceq' c$ we obtain $a \preceq' b$ by definition of \preceq' . So there remains

to check the cases where both partitions are non-proper. If $a = 0$, then $a \preceq' b$ follows from Lemma 2.8. If $c = d = 0$, then $u \preceq' v$ directly simplifies to $a \preceq' b$. Finally, if $c = b = 0$, then $a \preceq' d$ and $d \preceq' 0$, and by Lemma 2.11 it follows that $a \preceq' 0$, that is, $a \preceq' b$. \square

Lemma 2.15. *The relation \preceq' satisfies comparability.*

Proof. Let $a, b \in B'$. We proceed by induction on $|a| + |b|$. If a and b fall in case 1, then we have either $a \setminus b \preceq' b \setminus a$ or $b \setminus a \preceq' a \setminus b$ by the induction hypothesis, and thus either $a \preceq' b$ or $b \preceq' a$. If a and b fall in case 2, then either $\bar{a} \preceq \bar{b}$ or $\bar{b} \preceq \bar{a}$, and thus either $a \preceq' b$ or $b \preceq' a$. Now assume that a and b fall in case 3. If $a \preceq' b$ we are done, so assume the opposite. Then there exist a partition $\langle a_0, a_1 \rangle$ of a and a partition $\langle b_0, b_1 \rangle$ of b such that at least one of $\langle a_0, a_1 \rangle$ or $\langle b_0, b_1 \rangle$ is proper, and such that $b_1 \preceq' a_1$ and $a_0 \not\preceq' b_0$. Then by the induction hypothesis, we have $b_0 \preceq' a_0$. By Proposition 2.4, Lemma 2.9 and Lemma 2.14, the relation \preceq' is additive and transitive. Since we are in case 3, we know that $a \cap b = 0$, whence $a_0 \cap b_1 = 0$. We also have $b_0 \cap b_1 = 0$ by construction. From $b_0 \preceq' a_0$, we thus obtain $b_0 \cup b_1 \preceq' a_0 \cup b_1$ by additivity. By the same reasoning, we deduce $a_0 \cup b_1 \preceq' a_0 \cup a_1$ from $b_1 \preceq' a_1$. Then by transitivity, it follows that $b_0 \cup b_1 \preceq' a_0 \cup a_1$, that is, $b \preceq' a$. \square

This concludes the proof of Proposition 2.5.

2.3 The bounded case

We are now ready to make the final steps toward the proof of Theorem 2.3. First, we will need the following lemma.

Lemma 2.16. *Suppose that B has monotonicity, comparability and weak linearity. Then B satisfies cancellation.*

Let us write $X := \{x_1, \dots, x_n\}$. Let $a_1, \dots, a_m, b_1, \dots, b_m \in B$ and assume that for all $i \in [1, n]$ we have $\text{count}_{x_i}(a_1, \dots, a_m) = \text{count}_{x_i}(b_1, \dots, b_m)$, and for all $j \in [1, m-1]$ we have $a_j \preceq b_j$. By iterating Proposition 2.5 $n \times (2m-1)$ times, we obtain a Boolean algebra $B^\bullet = 2^{X^\bullet}$ and a relation \preceq^\bullet on B^\bullet such that:

1. $X^\bullet = \{x_i^j : 1 \leq i \leq n \text{ and } 1 \leq j \leq 2m\}$;
2. \preceq^\bullet satisfies monotonicity, comparability and weak linearity;
3. for all $a, b \in B$ we have $a \preceq b \iff a \preceq^\bullet b$;
4. we have $\{x_i\} = \{x_i^1\} \simeq^\bullet \dots \simeq^\bullet \{x_i^{2m}\}$ for all $i \in [1, n]$.

By Proposition 2.4, we know that \preceq and \preceq^\bullet are additive and transitive, and we will use these properties freely in the sequel. For all $i \in [1, n]$ and $j \in [1, 2m]$, we write $\bar{x}_i^j = x_i^1 = x_i$. Again, for all $a \in B^\bullet$, we write $\bar{a} := \{\bar{x} : x \in a\}$. We call $a \in B^\bullet$ *saturated* if there exist $i \in [1, n]$ and $j, k \in [1, 2m]$ such that $j \neq k$ and $x_i^j, x_i^k \in a$.

Claim 2.17. *If $a \in B^\bullet$ is non-saturated, then $a \simeq^\bullet \bar{a}$.*

Proof. Let us write $a = \{x_{i_1}^{j_1}, \dots, x_{i_p}^{j_p}\}$. We know that we have $\{x_{i_l}^{j_l}\} \simeq^\bullet \{x_{i_l}^1\} = \{x_{i_l}\}$ for all $l \in [1, p]$. Since a is non-saturated, the $x_{i_l}^1$'s are pairwise distinct, and so by iterating additivity we obtain $\{x_{i_1}^{j_1}\} \cup \dots \cup \{x_{i_p}^{j_p}\} \simeq^\bullet \{x_{i_1}\} \cup \dots \cup \{x_{i_p}\}$, that is, $a \simeq^\bullet \bar{a}$. \square

Claim 2.18. *If $a, b \in B^\bullet$ are non-saturated, then $a \preceq^\bullet b \iff \bar{a} \preceq \bar{b}$.*

Proof. By Claim 2.17 and transitivity, we have $a \preceq^\bullet b \iff \bar{a} \preceq^\bullet \bar{b}$. By construction of \preceq^\bullet , we also have $\bar{a} \preceq^\bullet \bar{b} \iff \bar{a} \preceq \bar{b}$, and we are done. \square

Given $i \in [1, n]$ and $j \in [1, m]$, we say that a_j is the k -th occurrence of x_i if we have $x_i \in a_j$ and $\text{count}_{x_i}(a_1, \dots, a_j) = k$. We then define $a_j^\bullet := \{x_i^{k_i} : 1 \leq i \leq n \text{ and } x_i \in a_j\}$ where for all $i \in [1, n]$, the integer k_i is such that a_j is the k_i -th occurrence of x_i in case $x_i \in a_j$. Likewise, we set $b_j^\bullet := \{x_i^{m+k_i} : 1 \leq i \leq n \text{ and } x_i \in b_j\}$ where for all $i \in [1, n]$, the integer k_i is such that b_j is the k_i -th occurrence of x_i in case $x_i \in b_j$.

Example 2.19. Below, a_1 is the first occurrence of x_3 , and a_3 is the second occurrence of x_3 . Accordingly, we have $a_3^\bullet = \{x_3^2\}$ with superscript 2.

$$\begin{array}{ll|ll} a_1 = \{x_1, x_3\} & b_1 = \{x_1, x_2\} & a_1^\bullet = \{x_1^1, x_3^1\} & b_1^\bullet = \{x_1^5, x_2^5\} \\ a_2 = \{x_1, x_2\} & b_2 = \{x_3\} & a_2^\bullet = \{x_1^2, x_2^1\} & b_2^\bullet = \{x_3^5\} \\ a_3 = \{x_3\} & b_3 = \{x_1, x_2, x_3\} & a_3^\bullet = \{x_3^2\} & b_3^\bullet = \{x_1^6, x_2^6, x_3^6\} \\ a_4 = \{x_1\} & b_4 = \{x_1\} & a_4^\bullet = \{x_1^3\} & b_4^\bullet = \{x_1^7\} \end{array}$$

We can see that if $\text{count}_{x_i}(a_1, \dots, a_m) = k$ – that is, x_i occurs k times among a_1, \dots, a_m – then each one of x_i^1, \dots, x_i^k will occur exactly once among $a_1^\bullet, \dots, a_m^\bullet$. Likewise, if $\text{count}_{x_i}(b_1, \dots, b_m) = k$, then each one of $x_i^{m+1}, \dots, x_i^{m+k}$ will occur exactly once among $b_1^\bullet, \dots, b_m^\bullet$. Note that we must have $k \leq m$, which means that there is no overlap between the sequences x_i^1, \dots, x_i^k and $x_i^{m+1}, \dots, x_i^{m+k}$. We also have $\{x_i^1\} \simeq^\bullet \dots \simeq^\bullet \{x_i^k\} \simeq^\bullet \{x_i^{m+1}\} \simeq^\bullet \dots \simeq^\bullet \{x_i^{m+k}\}$, so by iterating additivity we obtain $\bigcup_{j=1}^m a_j^\bullet \simeq^\bullet \bigcup_{j=1}^m b_j^\bullet$.

Further, the elements $a_1^\bullet, \dots, a_m^\bullet, b_1^\bullet, \dots, b_m^\bullet$ are all non-saturated and pairwise disjoint, by construction. For all $j \in [1, m]$ we also have $\overline{a_j^\bullet} = a_j$ and $\overline{b_j^\bullet} = b_j$. When $j < m$, we have $a_j \preceq b_j$, whence $\overline{a_j^\bullet} \preceq \overline{b_j^\bullet}$ and thus $a_j^\bullet \preceq^\bullet b_j^\bullet$ by Claim 2.18. Hence, by iterating weak linearity on $\bigcup_{j=1}^m b_j^\bullet \preceq^\bullet \bigcup_{j=1}^m a_j^\bullet$, we obtain $b_m^\bullet \preceq^\bullet a_m^\bullet$. By Claim 2.18 it follows that $b_m \preceq a_m$, as desired. This concludes the proof of Lemma 2.16, and we finally prove the main result of this section.

Proof of Theorem 2.3. Assume that $\langle B, \preceq \rangle$ is bounded measurable, and let μ be a bounded measure compatible with \preceq . It is then obvious that \preceq has comparability. For linearity, assume that $a \cap c = 0$ and $a \cup c \preceq b \cup d$ and $d \preceq c$. Then $\mu(a \cup c) \leq \mu(b \cup d)$. Since $a \cap c = 0$, we have $\mu(a \cup c) = \mu(a) + \mu(c)$. The identity $\mu(b \cup d) = \mu(b) + \mu(d) - \mu(b \cap d)$ is also well known, whence $\mu(a) + \mu(c) \leq \mu(b) + \mu(d)$. Since $\mu(d) \leq \mu(c)$, it follows that $\mu(a) \leq \mu(b)$. Therefore $a \preceq b$. Conversely, suppose that \preceq has comparability and linearity. Then by Proposition 2.4, \preceq is also monotone, positive and weakly linear. By Lemma 2.16, it follows that \preceq satisfies cancellation. By Theorem 2.1, we conclude that $\langle B, \preceq \rangle$ is bounded measurable. \square

Note that Theorem 2.3 provides a straightforward procedure for checking whether a relation is bounded measurable, which in addition can be performed in logarithmic space. This is thus a direct improvement on the polynomial space algorithm of Kraft, Pratt and Seidenberg [KPS59].

Corollary 2.20. *Given a finite Boolean algebra B and a binary relation \preceq on B , one can decide whether $\langle B, \preceq \rangle$ is bounded measurable in logarithmic space.*

Proof. Let $n := |X|$. The elements of $B = 2^X$ can be represented as n -tuples of bits. Since \preceq is a subset of $B \times B$, it consists of at most $2^n \times 2^n$ pairs of n -tuples, and thus has size $\leq 2^{2n} \cdot n$. Hence, the input (n, \preceq) of the problem has size $\leq \log n + 4^n \cdot n$. We can then check comparability and linearity by letting four variables a, b, c, d range over B . They have total size $\leq 4n$, which means that we use an amount of space logarithmic in the size of (n, \preceq) . \square

2.4 The general case

While Kraft, Pratt and Seidenberg only considered bounded measures in their work, we may occasionally want some elements to have infinite measure. Having improved their characterization, we are now also in a position to generalize it, and we provide necessary and sufficient

conditions for $\langle B, \preceq \rangle$ to be measurable. To achieve this, we apply Theorem 2.3 to a well-chosen algebra $B' \subseteq B$, and we extend the resulting bounded measure to a measure on B .

Lemma 2.21. *Suppose that \preceq is reflexive and bounded linear, and that $1 \not\preceq 0$. If $b \preceq 0$, then $a \cup b \preceq a$.*

Proof. Assume that $b \preceq 0$. By reflexivity, we have $(a \cup b) \cup 0 \preceq a \cup b$. Since $1 \not\preceq 0$ and $(a \cup b) \cap 0 = 0$, we can apply bounded linearity and obtain $a \cup b \preceq a$. \square

Theorem 2.22. *The pair $\langle B, \preceq \rangle$ is measurable if and only if the following conditions are satisfied, for all $a, b, c, d \in B$:*

- **Comparability:** $a \preceq b$ or $b \preceq a$;
- **Transitivity:** $a \preceq b$ and $b \preceq c$ implies $a \preceq c$;
- **Monotonicity:** $a \subseteq b$ implies $a \preceq b$;
- **Bounded Linearity:** if $1 \not\preceq c$ and $a \cap c = 0$ and $a \cup c \preceq b \cup d$ and $d \preceq c$, then $a \preceq b$.

Proof. First, suppose that $\langle B, \preceq \rangle$ is measurable, with measure μ . It is easy to check that \preceq satisfies comparability, transitivity and monotonicity. Then let $a, b, c, d \in B$ be such that $1 \not\preceq c$, $a \cap c = 0$, $d \preceq c$ and $a \cup c \preceq b \cup d$. In this case we have $\mu(d) \leq \mu(c) < \infty$, otherwise $\infty = \mu(c) \leq \mu(1)$, contradicting $1 \not\preceq c$. If $\mu(b) = \infty$ then obviously $a \preceq b$, so we assume that $\mu(b) < \infty$. In case $\mu(a) = \infty$, we have $\infty = \mu(a) + \mu(c) \leq \mu(b \cup d) \leq \mu(b) + \mu(d)$, and since $\mu(d) < \infty$, it follows that $\mu(b) = \infty$, a contradiction. We thus end up with $\mu(a), \mu(b), \mu(c), \mu(d) < \infty$. From $\mu(a) + \mu(c) \leq \mu(b) + \mu(d)$ and $\mu(d) \leq \mu(c)$, we then deduce that $\mu(a) \leq \mu(b)$. Therefore, \preceq is bounded linear.

Conversely, assume that \preceq has comparability, transitivity, monotonicity and bounded linearity. If $1 \preceq 0$, then by transitivity and monotonicity, we have $a \preceq 0$ for all $a \in B$. The desired measure is then obviously the zero function on B . Hence, in the sequel we assume that $1 \not\preceq 0$. In particular, this means that we never have $1 \preceq a$ and $a \preceq 0$ for any $a \in B$. We then introduce:

- $B_\infty := 2^{X_\infty}$ where $X_\infty := \{x \in X : 1 \preceq \{x\}\}$,
- $B_0 := 2^{X_0}$ where $X_0 := \{x \in X : \{x\} \preceq 0\}$,
- $B' := 2^{X'}$ where $X' := X \setminus (X_\infty \cup X_0)$.

From our assumption, it is easy to see that every element of X belongs to exactly one of X_∞ , X_0 or X' . Basically, the singletons of X_0 are expected to have measure 0, those of X_∞ are expected to have measure ∞ , and those of X' are expected to have bounded measure – but with which values exactly? To answer this question, we need to prove that (B', \preceq) is bounded measurable. In the sequel, we will use the properties of transitivity and monotonicity without justification.

We aim to apply Theorem 2.3, and we already know that \preceq satisfies comparability. For linearity, let $a, b, c, d \in B'$ be such that $a \cap c = 0$, $d \preceq c$ and $a \cup c \preceq b \cup d$. If $1 \not\preceq c$, we apply bounded linearity and obtain $a \preceq b$. Otherwise we have $1 \preceq c$. We write $c = \{x_1, \dots, x_m\}$. We also have $1 = c^* \cup c$, and it follows that $c^* \cup \{x_1\} \cup \dots \cup \{x_m\} \preceq \{x_1\} \cup \dots \cup \{x_m\}$. Since $c \in B'$, we have $x_m \in X'$, and thus $1 \not\preceq \{x_m\}$. By comparability we also have $\{x_m\} \preceq \{x_m\}$. By bounded linearity we then obtain $c^* \cup \{x_1\} \cup \dots \cup \{x_{m-1}\} \preceq \{x_1\} \cup \dots \cup \{x_{m-1}\}$. By repeating this step, we eventually arrive at $c^* \preceq 0$. Now suppose toward a contradiction that there exists $x \in X' \setminus c$. Then $\{x\} \subseteq c^*$, and from $c^* \preceq 0$ we then obtain $\{x\} \preceq 0$, contradicting $x \in X'$. Therefore $c = X'$, and since $a \subseteq X'$, we then have $a = a \cap c = 0$. Therefore $a \preceq b$, and this proves that \preceq is linear on B' . So let μ' be a bounded measure on B' compatible with \preceq .

Now, for all $a \in B$, we can uniquely decompose a as $a = a' \cup a_0 \cup a_\infty$ where $a' \in B'$, $a_0 \in B_0$ and $a_\infty \in B_\infty$. We then set

$$\mu(a' \cup a_0 \cup a_\infty) := \begin{cases} \infty & \text{if } a_\infty \neq 0, \\ \mu'(a') & \text{otherwise.} \end{cases}$$

We prove that \preceq_μ is equal to \preceq . Consider two elements $a = a' \cup a_0 \cup a_\infty$ and $b = b' \cup b_0 \cup b_\infty$, with $a', b' \in B'$ and $a_0, b_0 \in B_0$ and $a_\infty, b_\infty \in B_\infty$. First assume that $a \cap b = 0$. If $a_\infty \neq 0$ or $b_\infty \neq 0$, then we have $a_\infty \cup b_\infty \neq 0$, and thus $\mu(a \cup b) = \infty = \mu(a) + \mu(b)$. Otherwise, we have $a_\infty = b_\infty = 0$ and it follows that $\mu(a \cup b) = \mu'(a' \cup b') = \mu'(a') + \mu'(b') = \mu(a) + \mu(b)$. Finally, we prove that $a \preceq b \iff \mu(a) \leq \mu(b)$. We consider three cases.

1. If $b_\infty \neq 0$ then $\mu(b) = \infty$, and thus $\mu(a) \leq \mu(b)$. In addition, there exists $x \in b_\infty$. Then $1 \preceq \{x\}$, and since $a \preceq 1$ and $\{x\} \subseteq b$ it follows that $a \preceq b$.
2. Assume that $a_\infty \neq 0$. Then $\mu(a) = \infty$. If $\mu(a) \leq \mu(b)$ then we have $\mu(b) = \infty$ as well, whence $b_\infty \neq 0$. From case 1, it follows that $a \preceq b$. Conversely, assume that $a \preceq b$. Reasoning as above, there exists $x \in a_\infty$, and it follows that $1 \preceq b$. Let us write $b = \{x_1, \dots, x_m\}$. Then from $1 \preceq b$ we obtain $b^* \cup \{x_1\} \cup \dots \cup \{x_m\} \preceq \{x_1\} \cup \dots \cup \{x_m\}$. By iterating bounded linearity – like we did earlier – we then obtain $b^* \preceq 0$. Suppose toward a contradiction that $b_\infty = 0$. Then since $x \in B_\infty$ we have $x \notin b$ and thus $x \in b^*$. From $b^* \preceq 0$ it follows that $\{x\} \preceq 0$, a contradiction. Hence $b_\infty \neq 0$, and we obtain $\mu(b) = \infty \geq \mu(a)$, as desired.
3. Otherwise, we have $a_\infty = b_\infty = 0$. If $\mu(a) \leq \mu(b)$, then $\mu'(a') \leq \mu'(b')$, and thus $a' \preceq b'$. By Lemma 2.21 we have $a' \cup a_0 \preceq a'$, and by monotonicity we have $b' \preceq b' \cup b_0$. By transitivity, it follows that $a' \cup a_0 \preceq b' \cup b_0$, that is, $a \preceq b$. Conversely, assume that $a' \cup a_0 \preceq b' \cup b_0$. Let us write $b_0 = \{x_1, \dots, x_m\}$. Since $\{x_1\}, \dots, \{x_m\} \preceq 0$, we can iterate Lemma 2.21 and obtain $b_0 = \{x_1\} \cup \dots \cup \{x_m\} \preceq 0$. From $0 \preceq a_0$ it follows that $b_0 \preceq a_0$. Now suppose toward a contradiction that $1 \preceq a_0$. Likewise, we have $a_0 \preceq 0$, and from $1 \cup 0 \preceq 0 \cup a_0$ and $1 \not\preceq 0$ and $1 \cap 0 = 0$, it follows that $1 \preceq 0$ by bounded linearity, a contradiction. All in all we have $1 \not\preceq a_0$, $b_0 \preceq a_0$ and $a' \cap a_0 = 0$. We can thus apply bounded linearity to $a' \cup a_0 \preceq b' \cup b_0$ and obtain $a' \preceq b'$. Thus $\mu'(a') \leq \mu'(b')$, and it follows that $\mu(a) \leq \mu(b)$. □

Again, the characterization of Theorem 2.22 immediately gives us a decision procedure.

Corollary 2.23. *Given a finite Boolean algebra B and a binary relation \preceq on B , one can decide whether $\langle B, \preceq \rangle$ is measurable in logarithmic space.*

We also characterize unbounded measurable algebras, though only in the case of definite measures.

Proposition 2.24. *Assume that $\langle B, \preceq \rangle$ is measurable. Then:*

1. $\langle B, \preceq \rangle$ is definite iff \preceq is definite;
2. if $\langle B, \preceq \rangle$ is definite, then $\langle B, \preceq \rangle$ is unbounded iff \preceq is unbounded.

Proof. Item 1 is clear. For 2, assume that there exists an unbounded measure μ compatible with \preceq . Given $a \in B$, we also have $\mu(1) = \mu(a \sqcup a^*) = \mu(a) + \mu(a^*)$. Since $\mu(1) = \infty$, we then have either $\mu(a) = \infty$ or $\mu(a^*) = \infty$, that is, $1 \preceq a$ or $1 \preceq a^*$.

From right to left, assume that \preceq is unbounded. If $B = \{0, 1\}$, then B admits an unbounded measure μ defined by $\mu(1) := \infty$ and $\mu(0) := 0$ (note that since \preceq is definite, we have $1 \not\preceq 0$). Otherwise, we can select some $a \in B \setminus \{0, 1\}$. We have $1 \preceq a$ or $1 \preceq a^*$, and without loss of generality we suppose that $1 \preceq a$. By assumption, let μ be a definite measure compatible with \preceq . Then $\mu(1) = \mu(a) + \mu(a^*) = \mu(1) + \mu(a^*)$. Since μ is definite, we also have $\mu(a^*) > 0$, and thus $\mu(1) = \infty$. □

2.5 Bounding the size of a measure

When $\langle B, \preceq \rangle$ is measurable, it may be desirable to directly represent the relation \preceq by a compatible measure. Obviously, a measure is in general not effectively representable, since it takes arbitrary real values. Nonetheless, we can prove that \preceq always admit an integer measure, whose size is polynomial in the cardinality of B .

If $v \in \mathbb{R}^n$ is a n -vector and $1 \leq i \leq k$, we will denote by v_i the i -th component of v . We then write $\|v\| := \max \{|v_i| : 1 \leq i \leq k\}$. Given $v, u \in \mathbb{R}^k$, we write $v \leq u$ in case $v_i \leq u_i$ for all $i \in [1, k]$. Likewise, we write $v < u$ in case $v_i < u_i$ for all $i \in [1, k]$. Given $x \in \mathbb{R}$, we will abuse notations and let x refer to the n -vector with all coordinates equal to x .

Lemma 2.25. *Let A be a $m \times n$ matrix with coefficients in $\{-1, 0, 1\}$, and let b be a m -vector with coefficients in $\{-1, 0\}$. If the linear system*

$$(S) : \begin{cases} Av \leq b \\ v \geq 0 \end{cases}$$

is feasible, then it admits an integer solution v such that $\|v\| \leq 2(n+m)m^{2m+3}$.

Proof. First, if (S) is feasible, then by [PS98, Th 2.1] it admits a rational solution $u = (p_1/q_1, \dots, p_n/q_n)$. If $N := q_1 \dots q_n$, then Nu is an integer vector, and since $N \geq 1$ and $b \in \{-1, 0\}^m$, it is also a solution of (S) . By the corollary of [PS98, Th. 13.4], it follows that (S) admits an integer solution v such that $\|v\| \leq 2(n+m)m^{2m+3}$. \square

Lemma 2.26. *Let A_1 be a $m_1 \times n$ matrix, and let A_2 be a $m_2 \times n$ matrix, both with coefficients in $\{-1, 0, 1\}$. Let $m := m_1 + m_2$. If the linear system*

$$(S) : \begin{cases} A_1 v \leq 0 \\ A_2 v < 0 \\ v \geq 0 \end{cases}$$

is feasible, then it admits an integer solution v such that $\|v\| \leq 2(n+m)m^{2m+3}$.

Proof. If (S) is feasible, we claim that the system

$$(S') : \begin{cases} A_1 v \leq 0 \\ A_2 v \leq -1 \\ v \geq 0 \end{cases}$$

is also feasible. Indeed, let v be a solution of (S) , and let $M := \max \{(A_2 v)_i : 1 \leq i \leq m_2\}$. By assumption, we have $M < 0$. Then $v' := -\frac{1}{M}v$ is a solution of (S') , since for all $i \in [1, m_2]$ we have $(A_2 v')_i = -\frac{1}{M}(A_2 v)_i \leq -\frac{1}{M}M = -1$. As a result, (S') can be seen as a system of the form considered in Lemma 2.25, and it follows that it admits an integer solution v such that $\|v\| \leq 2(n+m)m^{2m+3}$. It is then clear that v is also a solution of (S) , and we are done. \square

Proposition 2.27. *Assume that $\langle B, \preceq \rangle$ is bounded measurable, and let $n := |X|$. Then there exists a measure $\mu : B \rightarrow \mathbb{N}$ compatible with \preceq and such that $\mu(1) \leq 2(n+3^n)3^{2n-3^n+3n}$.*

Proof. For all $x \in X$, we introduce a variable v_x . We then introduce the linear system (S) containing, for all $a, b \in 2^X$, the inequation $\sum_{x \in a} v_x - \sum_{x \in b} v_x < 0$ if $a \prec b$, and the two inequations $\sum_{x \in a} v_x - \sum_{x \in b} v_x \leq 0$ and $\sum_{x \in b} v_x - \sum_{x \in a} v_x \leq 0$ if $a \simeq b$. In particular, since \preceq is positive, (S) contains for all $x \in X$ the inequation $-v_x \leq 0$, which is equivalent to $v_x \geq 0$.

We then observe that all inequations in (S) involve at most n variables and have coefficients in $\{-1, 0, 1\}$, and up to removing duplicates, we can thus assume that (S) contains at most

3^n inequations. It is also clear that the solutions of (S) correspond exactly to the bounded measures compatible with \preceq . Therefore, by Lemma 2.26, there exists a measure $\mu : B \rightarrow \mathbb{N}$ compatible with \preceq and such that $\mu(1) \leq 2(n + 3^n)(3^n)^{2 \cdot 3^n + 3} = 2(n + 3^n)3^{2n \cdot 3^n + 3n}$. \square

Theorem 2.28. *Assume that $\langle B, \preceq \rangle$ is measurable, and let $n := |X|$. Then there exists a measure $\mu : B \rightarrow \mathbb{N} \cup \{\infty\}$ compatible with \preceq and representable in size $O(n^2 \cdot 3^n)$.*

Proof. Let X' be defined as in the proof of Theorem 2.22. We know that $\langle 2^{X'}, \preceq \rangle$ is bounded measurable, and by Proposition 2.27, there exists a measure $\mu : 2^{X'} \rightarrow \mathbb{N}$ compatible with \preceq and such that $\mu(X') \leq 2(n + 3^n)3^{2n \cdot 3^n + 3n}$. Then, let $\mu' : B \rightarrow \mathbb{N} \cup \{\infty\}$ be defined from μ as in the proof of Theorem 2.22. Since μ' is a measure, it can be encoded by the n -tuple $(\mu'(\{x\}))_{x \in X}$. For all $x \in X$, either $\mu'(\{x\})$ is equal to ∞ and thus representable in constant size, or $\mu'(\{x\}) = \mu(\{x\}) \leq 2(n + 3^n)3^{2n \cdot 3^n + 3n}$, in which case $\mu'(\{x\})$ has size

$$\log(2(n + 3^n)3^{2n \cdot 3^n + 3n}) = \log 2 + \log(n + 3^n) + (2n \cdot 3^n + 3n) \log 3 = O(n \cdot 3^n).$$

Since $|X| = n$, it follows that μ' is representable in size $O(n^2 \cdot 3^n)$. \square

3 Logics for contact and measure

3.1 Generalities

We consider a countable set **Prop** of Boolean variables. We fix a finite subset $P \subseteq \mathbf{Prop}$. We define the set \mathcal{B}_P of *Boolean terms* over P by the following grammar:

$$a ::= p \mid 0 \mid (a \sqcap a) \mid a^*$$

where $p \in P$. We introduce the abbreviations $1 := 0^*$, $a \sqcup b := (a^* \sqcap b^*)^*$ and $a \leftrightarrow b := (a^* \sqcup b) \sqcap (b^* \sqcup a)$. We then define the set \mathcal{L}_P of *formulas* over P by the following grammar:

$$\varphi ::= a \sqsubseteq b \mid a \preceq b \mid a C b \mid \perp \mid (\varphi \wedge \varphi) \mid \neg \varphi$$

where $a, b \in \mathcal{B}_P$. Boolean terms are also called *regions*, and $a C b$ reads “the regions a is in contact with the region b ”, while $a \preceq b$ reads “the region b is at least as big as the region a ” – the interpretation of ‘big’ being left open to the reader. We follow the usual rules for omission of parentheses. The abbreviations \top , \vee , \rightarrow and \leftrightarrow are defined as usual. We also introduce the abbreviation $a \equiv b := a \sqsubseteq b \wedge b \sqsubseteq a$. We denote by \mathcal{L}^C the fragment of \mathcal{L} without subformulas of the form $a C b$.

In the sequel, a *propositional logic* will be a subset of \mathcal{L} containing all propositional tautologies, and closed under Modus Ponens and uniform substitution. If \mathbf{L} and \mathbf{L}' are two propositional logics such that $\mathbf{L} \subseteq \mathbf{L}'$, we call \mathbf{L}' an extension of \mathbf{L} . Given a propositional logic \mathbf{L} and $\varphi_1, \dots, \varphi_n \in \mathcal{L}$, we denote by $\mathbf{L} + \varphi_1 + \dots + \varphi_n$ the smallest extension of \mathbf{L} containing $\varphi_1, \dots, \varphi_n$. Let us denote by \mathbf{LC} the minimal logic of contact (in the language \mathcal{L}^C) introduced in [BTV07b]. This logic contains all valid Boolean equivalences $a \equiv b$, the axiom $\neg(0 \equiv 1)$, and the following axioms for contact:

- Cont0 : $a C b \rightarrow \neg(a \equiv 0) \wedge \neg(b \equiv 0)$,
- ContMon : $a C b \wedge a \sqsubseteq a' \wedge b \sqsubseteq b' \rightarrow a' C b'$,
- ContJoin : $(a C (b \sqcup c) \rightarrow a C b \vee a C c) \wedge ((b \sqcup c) C a \rightarrow b C a \vee c C a)$.

We will also consider the following axioms for contact and measure:

- ContRefl : $\neg(a \equiv 0) \rightarrow a C a$,
- ContSym : $a C b \rightarrow b C a$,

- $\text{Con} : \neg(a \equiv 0) \wedge \neg(a \equiv 1) \rightarrow a C a^*$,
- $\text{Eq} : a \preceq b \wedge a \equiv a' \wedge b \equiv b' \rightarrow a' \preceq b'$,
- $\text{Comp} : a \preceq b \vee b \preceq a$,
- $\text{Trans} : a \preceq b \wedge b \preceq c \rightarrow a \preceq c$,
- $\text{Mon} : a \sqsubseteq b \rightarrow a \preceq b$,
- $\text{Lin} : (a \sqcap c \equiv 0) \wedge (a \sqcup c \preceq b \sqcup d) \wedge (d \preceq c) \rightarrow a \preceq b$,
- $\text{BLin} : \neg(1 \preceq c) \wedge (a \sqcap c \equiv 0) \wedge (a \sqcup c \preceq b \sqcup d) \wedge (d \preceq c) \rightarrow a \preceq b$,
- $\text{Def} : a \preceq 0 \rightarrow a \equiv 0$,
- $\text{Unbd} : 1 \preceq a \vee 1 \preceq a^*$.

We then introduce the following logics:

- $\text{LCM} := \text{LC} + \text{Comp} + \text{Trans} + \text{Mon} + \text{BLin}$, the minimal logic of contact and measure;
- $\text{LCMB} := \text{LC} + \text{Comp} + \text{Lin}$, the minimal logic of contact and bounded measure;
- $\text{LCM}^+ := \text{LCM} + \text{ContRefl} + \text{ContSym}$;
- $\text{LCMB}^+ := \text{LCMB} + \text{ContRefl} + \text{ContSym}$;
- $\text{S} := \text{LCM}^+ + \text{Con} + \text{Def} + \text{Unbd}$, the logic of the unbounded space;
- $\text{BS} := \text{LCMB}^+ + \text{Con} + \text{Def}$, the logic of the bounded space.

Formulas in the language \mathcal{L}^C are evaluated in contact algebras, a general semantics that encompasses both adjacency frames and topological spaces. By endowing contact algebras with a measurable relation, we then obtain a semantics for the language \mathcal{L} .

Definition 3.1. A *contact algebra* is a pair $\langle B, C \rangle$ where B is a Boolean algebra, and C is a binary relation on B satisfying the following, for all $a, b, a', b', c \in B$:

- $a C b$ implies $a \neq 0$ and $b \neq 0$,
- $a C b$ and $a \sqsubseteq a'$ and $b \sqsubseteq b'$ implies $a' C b'$,
- $a C (b \sqcup c)$ implies $a C b$ or $a C c$,
- $(b \sqcup c) C a$ implies $b C a$ or $c C a$.

A *measurable contact algebra* is a tuple $\langle B, C, \preceq \rangle$ where $\langle B, C \rangle$ is a contact algebra, and $\langle B, \preceq \rangle$ is a measurable Boolean algebra. Further, we call $\langle B, C, \preceq \rangle$:

- *spatial* if $a \neq 0$ implies $a C a$, and $a C b$ implies $b C a$, for all $a, b \in B$;
- *connected* if we have $a C a^*$ for all $a \in B \setminus \{0, 1\}$;
- *bounded* if $\langle B, \preceq \rangle$ is bounded measurable;
- *definite* if $a \preceq 0$ implies $a = 0$, for all $a \in B$.

Definition 3.2. Let $\langle B, C, \preceq \rangle$ be a measurable contact algebra, and let $P \subseteq \mathbf{Prop}$. A *valuation* on B is a map $V : \mathcal{B}_P \rightarrow B$ satisfying $V(0) = 0$, $V(a \sqcap b) = V(a) \sqcap V(b)$ and $V(a^*) = V(a)^*$ for all $a, b \in \mathcal{B}_P$. We then define by induction on $\varphi \in \mathcal{L}_P$ the statement $\langle B, C, \preceq, V \rangle \models \varphi$ as follows:

- $\langle B, C, \preceq, V \rangle \models a \sqsubseteq b$ iff $V(a) \sqsubseteq V(b)$,
- $\langle B, C, \preceq, V \rangle \models a C b$ iff $V(a) C V(b)$,
- $\langle B, C, \preceq, V \rangle \models a \preceq b$ iff $V(a) \preceq V(b)$,
- $\langle B, C, \preceq, V \rangle \not\models \perp$,
- $\langle B, C, \preceq, V \rangle \models \varphi \wedge \psi$ iff $\langle B, C, \preceq, V \rangle \models \varphi$ and $\langle B, C, \preceq, V \rangle \models \psi$,
- $\langle B, C, \preceq, V \rangle \models \neg\varphi$ iff $\langle B, C, \preceq, V \rangle \not\models \varphi$.

We say that φ is *valid* on $\langle B, C, \preceq \rangle$, and write $\langle B, C, \preceq \rangle \models \varphi$, if for all valuations V on B we have $\langle B, C, \preceq, V \rangle \models \varphi$. We say that φ is *satisfiable* on $\langle B, C, \preceq \rangle$ if there exists a valuation V on B such that $\langle B, C, \preceq, V \rangle \models \varphi$.

Satisfiability of formulas is notably preserved by homomorphisms.

Definition 3.3. Let $\langle B, C, \preceq \rangle$ and $\langle B', C', \preceq' \rangle$ be two measurable contact algebras. A *homomorphism* from $\langle B, C, \preceq \rangle$ to $\langle B', C', \preceq' \rangle$ is a function $f : B \rightarrow B'$ such that:

- f is a Boolean algebra homomorphism;
- for all $a, b \in B$, we have $a C b \iff f(a) C' f(b)$; *(contact preservation)*
- for all $a, b \in B$, we have $a \preceq b \iff f(a) \preceq' f(b)$. *(relative size preservation)*

Remark 3.4. If μ and μ' are two measures respectively compatible with \preceq and \preceq' , then a sufficient condition for f to preserve relative size is to have $\mu'(f(a)) = \mu(a)$ for all $a \in B$.

Proposition 3.5. *If $f : \langle B, C, \preceq \rangle \rightarrow \langle B', C', \preceq' \rangle$ is a homomorphism and $\varphi \in \mathcal{L}$ is satisfiable on $\langle B, C, \preceq \rangle$, then φ is satisfiable on $\langle B', C', \preceq' \rangle$.*

Proof. Assume that φ is satisfiable on $\langle B, C, \preceq \rangle$, and let V be a valuation on B such that $\langle B, C, \preceq, V \rangle \models \varphi$. We can then easily prove by induction on φ that $\langle B', C', \preceq', f \circ V \rangle \models \varphi$. \square

3.2 Relational semantics

We now introduce a more practical semantics, based on adjacency frames. An adjacency frame is simply a set of points together with a binary relation. Regions are then arbitrary sets of points, and a region a is in contact with a region b if some point of a relates to some point of b .

Definition 3.6. An *adjacency frame* is a pair $\langle W, R \rangle$ where W is a non-empty set and R is a binary relation on W . A *measurable adjacency frame* is a tuple $\mathfrak{F} = \langle W, R, \preceq \rangle$ where $\langle W, R \rangle$ is an adjacency frame and $\langle 2^W, R \rangle$ is a measurable Boolean algebra. A sequence $\bar{w} = w_0 \dots w_n \in W$ is said to be a *path* in \mathfrak{F} from w_0 to w_n if we have $w_i R w_{i+1}$ for all $i \in [0, n-1]$. We denote by $|\bar{w}| := n$ the *length* of \bar{w} . We call \mathfrak{F} *connected* if for all $w, u \in W$, there exists a path in \mathfrak{F} from w to u .

Proposition 3.7. *Let $\mathfrak{F} = \langle W, R, \preceq \rangle$ be a measurable frame. Let C_R be the binary relation on 2^W defined by*

$$U C_R V \iff \exists u \in U, \exists v \in V, u R v$$

for all $U, V \subseteq W$. Then:

1. $\langle 2^W, C_R, \preceq \rangle$ is a measurable contact algebra,
2. $\langle 2^W, C_R, \preceq \rangle$ is spatial iff \mathfrak{F} is reflexive and symmetric,
3. $\langle 2^W, C_R, \preceq \rangle$ is connected iff \mathfrak{F} is connected.

Proof. The proof of 1 is easy and left to the reader. For 2 and 3, we refer respectively to [BTV07b, Lemma 2.1] and [BTV07b, Lemma 2.2]. \square

In the sequel, we will identify every measurable adjacency frame $\langle W, R, \preceq \rangle$ to its associated measurable contact algebras $\langle 2^W, C_R, \preceq \rangle$. We also introduce adjacency maps (called *p-morphisms* in [BTV07b]), which act as the natural morphisms between measurable adjacency frames.

Definition 3.8. Let $\mathfrak{F} = \langle W, R, \preceq \rangle$ and $\mathfrak{F}' = \langle W', R', \preceq' \rangle$ be two adjacency frames. An *adjacency map* from \mathfrak{F}' to \mathfrak{F} is a function $f : W' \rightarrow W$ such that:

- if $u' R' v'$, then $f(u') R f(v')$ *(forth condition)*;

- if $u R v$, then there exist $u', v' \in W'$ such that $f(u') = u$ and $f(v') = v$ and $u' R' v'$ (back condition);
- for all $U, V \subseteq W$ we have $U \preceq V \iff f^{-1}[U] \preceq' f^{-1}[V]$ (relative size preservation).

Proposition 3.9. *If $f : W' \rightarrow W$ is an adjacency map from $\langle W', R', \preceq' \rangle$ to $\langle W, R, \preceq \rangle$, then $f^{-1} : 2^W \rightarrow 2^{W'}$ is a homomorphism from $\langle 2^W, C_R, \preceq \rangle$ to $\langle 2^{W'}, C_{R'}, \preceq' \rangle$.*

Proof. Let $U, V \subseteq W$ and assume that $U C_R V$. Then there exist $u \in U$ and $v \in V$ such that $u R v$. By the back condition, there exist $u', v' \in W'$ such that $f(u') = u$ and $f(v') = v$ and $u' R' v'$. Then we have $u' \in f^{-1}[U]$ and $v' \in f^{-1}[V]$, and it follows that $f^{-1}[U] C_{R'} f^{-1}[V]$. Conversely, suppose that $f^{-1}[U] C_{R'} f^{-1}[V]$. Then there exist $u' \in f^{-1}[U]$ and $v' \in f^{-1}[V]$ such that $u' R' v'$, and by the forth condition we obtain $f(u') R f(v')$. By construction, we have $f(u') \in U$ and $f(v') \in V$, and thus $U C_R V$. \square

As usual, we establish soundness and completeness results by constructing the canonical frame. Let us fix an extension \mathbf{L} of \mathbf{LCM} . Let $P \subseteq \mathbf{Prop}$ be finite. A *consistent set* of formulas is a set $\Gamma \subseteq \mathcal{L}_P$ such that there are no $\varphi_1, \dots, \varphi_n \in \Gamma$ such that $\not\vdash_{\mathbf{L}} \neg(\varphi_1 \wedge \dots \wedge \varphi_n)$. We call Γ a *maximal consistent set* if in addition there is no consistent set Γ' such that $\Gamma \subset \Gamma'$. From now on, we assume that Γ is maximal consistent. A set $F \subseteq \mathcal{B}_P$ is said to be a Γ -*filter* if the following conditions hold:

- $1 \in F$,
- if $a \in F$ and $(a \sqsubseteq b) \in \Gamma$ then $b \in F$,
- if $a, b \in F$ then $a \sqcap b \in F$.

We call F *proper* if in addition we have $0 \notin F$. We call F a *maximal Γ -filter* if F is proper and there is no proper Γ -filter F' such that $F \subset F'$. Now, let W_Γ be the set of all Γ -filters. We define a binary relation R_Γ on W_Γ by

$$F R_\Gamma F' \iff \forall a \in F, \forall b \in F', (a C b) \in \Gamma.$$

Further, let $V_\Gamma : \mathcal{B}_P \rightarrow 2^{W_\Gamma}$ be the valuation uniquely defined by $V_\Gamma(p) := \{F \in W_\Gamma : p \in F\}$ for all $p \in P$. We claim that V_Γ is surjective. Indeed, given $F \in W_\Gamma$, the term $a_F := (\prod_{p \in P \cap F} p) \sqcap (\prod_{p \in P \setminus F} p^*)$ fully characterizes F , in the sense that $a_F \in F' \iff F = F'$ for all $F' \in W_\Gamma$. Given $U \subseteq W_\Gamma$, we then set $a := \bigsqcup_{F \in U} a_F$, and we have $V_\Gamma(a) = U$. In addition, it follows that W_Γ is finite and has size bounded by $2^{|P|}$. Then, let \preceq_Γ be the binary relation on 2^{W_Γ} defined by

$$V_\Gamma(a) \preceq_\Gamma V_\Gamma(b) \iff (a \preceq b) \in \Gamma.$$

Note that whether $V_\Gamma(a) \preceq_\Gamma V_\Gamma(b)$ does not depend on the chosen representatives a and b , as a consequence of axiom Eq.

Lemma 3.10 ([BTV07b, Lemma 3.4]). *Let $a, b \in \mathcal{B}_P$.*

1. *We have $V_\Gamma(a) \subseteq V_\Gamma(b)$ iff $(a \sqsubseteq b) \in \Gamma$.*
2. *We have $V_\Gamma(a) C_{R_\Gamma} V_\Gamma(b)$ iff $(a C b) \in \Gamma$.*

Since \mathbf{L} contains the axioms Comp, Trans, Mon and BLin, then by Lemma 3.10 it is clear that \preceq_Γ satisfies comparability, transitivity, monotonicity and linearity. By Theorem 2.22, it follows that $\langle 2^{W_\Gamma}, \preceq_\Gamma \rangle$ is measurable. We call $\mathfrak{F}_\Gamma := \langle W_\Gamma, R_\Gamma, \preceq_\Gamma \rangle$ the *canonical measurable frame* of \mathbf{L} relative to Γ .

Lemma 3.11 (Truth Lemma). *For all $\varphi \in \mathcal{L}_P$, we have $\langle W_\Gamma, R_\Gamma, \preceq_\Gamma, V_\Gamma \rangle \models \varphi$ iff $\varphi \in \Gamma$.*

Proof. By induction on φ . The non-trivial cases are given by Lemma 3.10. \square

Proposition 3.12.

1. If $\vdash_{\mathbf{L}} \text{ContRefl}$, then \mathfrak{F}_{Γ} is reflexive.
2. If $\vdash_{\mathbf{L}} \text{ContSym}$, then \mathfrak{F}_{Γ} is symmetric.
3. If $\vdash_{\mathbf{L}} \text{Con}$, then \mathfrak{F}_{Γ} is connected.
4. If $\vdash_{\mathbf{L}} \text{Lin}$, then \mathfrak{F}_{Γ} is bounded.
5. If $\vdash_{\mathbf{L}} \text{Def}$, then \mathfrak{F}_{Γ} is definite.
6. If $\vdash_{\mathbf{L}} \text{Def}$ and $\vdash_{\mathbf{L}} \text{Unbd}$, then \mathfrak{F}_{Γ} is unbounded.

Proof. For 1 and 2, see [BTV07b, Prop 3.1]. For 3, the axiom Con implies that the measurable contact algebra $\langle 2^{W_{\Gamma}}, C_{R_{\Gamma}}, \preceq_{\Gamma} \rangle$ is connected, and by Proposition 3.7 it follows that \mathfrak{F}_{Γ} is connected. For 4, the axiom Lin and the Truth Lemma entail that \preceq_{Γ} is linear, and by Theorem 2.3 it follows that $\langle 2^{W_{\Gamma}}, \preceq_{\Gamma} \rangle$ is bounded measurable. Finally, 5 and 6 follow from Proposition 2.24. \square

Theorem 3.13.

1. The logic **LCM** is sound and complete for the class of all finite measurable frames.
2. The logic **LCMB** is sound and complete for the class of all finite bounded measurable frames.
3. The logic **LCM**⁺ is sound and complete for the class of all finite, reflexive and symmetric measurable frames.
4. The logic **LCMB**⁺ is sound and complete for the class of all finite, reflexive, symmetric and bounded measurable frames.
5. The logic **S** is sound and complete for the class of all finite, reflexive, symmetric, connected and definite unbounded measurable frames.
6. The logic **BS** is sound and complete for the class of all finite, reflexive, symmetric, connected and definite bounded measurable frames.

Proof. Let us detail the proof of 1. The soundness part is immediate. Let $\varphi \in \mathcal{L}$ and assume that $\not\vdash_{\mathbf{LCM}} \neg\varphi$. Select some finite $P \subseteq \mathbf{Prop}$ such that $\varphi \in \mathcal{L}_P$. Then by the Lindenbaum Lemma, there exists a maximal consistent set Γ (with respect to **LCM**) such that $\varphi \in \Gamma$. By the Truth Lemma, we then have $\langle X_{\Gamma}, \preceq_{\Gamma}, V_{\Gamma} \rangle \models \varphi$. Thus φ is satisfiable on a finite measurable frame. For the other logics, we follow the same proof scheme, and we apply Proposition 3.12 to obtain the desired properties on the canonical frame. \square

In particular, it follows that all the logics mentioned in Theorem 3.13 have the finite model property and are decidable.

Theorem 3.14. *Validity in **LCM**, **LCMB**, **LCM**⁺, **LCMB**⁺, **S**, **BS** is decidable in double exponential space.*

Proof. Let \mathbf{L} be one these logics. Let $\varphi \in \mathcal{L}_P$ with P finite. As observed earlier, if $\Gamma \subseteq \mathcal{L}_P$ is a maximal consistent set, then the canonical measurable frame of \mathbf{L} relative to Γ contains at most $2^{|\mathcal{L}_P|}$ points. This proves that $\not\vdash_{\mathbf{L}} \varphi$ if and only if $\neg\varphi$ is satisfiable on a measurable frame with at most $2^{|\varphi|}$ points. One can decide whether this is the case by enumerating all such frames $\langle W, R, \preceq \rangle$, and checking whether φ is satisfiable on one of them. In addition, since \preceq is a relation on 2^W and $|W| \leq 2^{|\varphi|}$, the measurable frame $\langle W, R, \preceq \rangle$ has size double exponential in $|\varphi|$. This proves the claim. \square

3.3 Topological semantics

In the topological semantics, regions are regular closed sets of a topological space, and two regions are in contact when they intersect. This makes the contact relation reflexive and symmetric, and as a result the topological semantics is only suitable for extensions of \mathbf{LCM}^+ .

Definition 3.15. A *topological space* is a pair $\langle X, \tau \rangle$ where X is a non-empty set and τ is a subset of 2^X containing \emptyset and X , closed under finite intersections, and closed under arbitrary unions. The elements of τ are called *open sets*, and their complements are called *closed sets*. In the sequel, we will identify $\langle X, \tau \rangle$ to its underlying set X . We call X *connected* if there exist no non-empty open sets U and V such that $U \cup V = X$ and $U \cap V = \emptyset$.

Given $a \subseteq X$, the *interior* $\text{Int}(a)$ of a is the largest open set contained in a , and the *closure* $\text{Cl}(a)$ of a is the smallest closed set containing a . We call a *regular closed* if we have $a = \text{Cl}(\text{Int } a)$, and we denote by $\text{RC}(X)$ the family of regular closed subsets of X . The set $\text{RC}(X)$ is then known to be a Boolean algebra [Sik69, Sect. I.1] whose operators are defined as follows:

$$\begin{aligned} 0 &:= \emptyset, \\ 1 &:= X, \\ a \sqcup b &:= a \cup b, \\ a \sqcap b &:= \text{Cl}(\text{Int}(a \cap b)), \\ a^* &:= \text{Cl}(X \setminus a). \end{aligned}$$

A *general measurable space* is a tuple $\mathfrak{X} = \langle X, \mathcal{A}, \preceq \rangle$ where X is a topological space, \mathcal{A} is a Boolean subalgebra of $\text{RC}(X)$, and $\langle \mathcal{A}, \preceq \rangle$ is a measurable Boolean algebra. If $\mathcal{A} = \text{RC}(X)$, then we call \mathfrak{X} a *measurable space* and simply write it $\mathfrak{X} = \langle X, \preceq \rangle$.

Proposition 3.16 ([DW05, Prop. 3.7]). *Let $\langle X, \mathcal{A}, \preceq \rangle$ be a general measurable space. Let C_X be the binary relation on \mathcal{A} defined by $a C_X b \iff a \cap b = \emptyset$. Then $\langle \mathcal{A}, C_X, \preceq \rangle$ is a spatial contact algebra. In addition, $\langle X, \mathcal{A}, \preceq \rangle$ is connected if and only if $\langle \mathcal{A}, C_X, \preceq \rangle$ is connected.*

Again, we will identify every general measurable space $\langle X, \mathcal{A}, \preceq \rangle$ to its associated measurable contact algebras $\langle \mathcal{A}, C_X, \preceq \rangle$. In this section, we will only consider measurable spaces, since they are already flexible enough for our completeness results. However, it is not always easy to define a measure on all regular closed sets, and sometimes it is thus desirable to work with a smaller class of regular closed sets. General measurable spaces offer this option, and will be used with profit in Section 4.

Let us now construct the canonical space. We fix an extension \mathbf{L} of \mathbf{LCM}^+ , a finite set $P \subseteq \mathbf{Prop}$, and a maximal consistent set $\Gamma \subseteq \mathcal{L}_P$. A set $F \subseteq \mathcal{B}_P$ is said to be a Γ -*clan* if the following conditions hold:

- $1 \in F$,
- if $a \in F$ and $(a \sqsubseteq b) \in \Gamma$ then $b \in F$,
- if $a \sqcup b \in F$ then $a \in F$ or $b \in F$,
- if $a, b \in F$ then $(a C b) \in \Gamma$.

We call F *proper* if in addition we have $0 \notin F$. We call F a *maximal Γ -clan* if F is proper and there is no proper Γ -clan F' such that $F \subset F'$. Now, let X_Γ be the set of all maximal Γ -clans. Again, let $V_\Gamma : \mathcal{B}_P \rightarrow 2^{X_\Gamma}$ be the valuation uniquely defined by $V_\Gamma(p) := \{F \in X_\Gamma : p \in F\}$ for all $p \in P$. We endow X_Γ with a topology defined by taking $\tau_0 = \{V_\Gamma(a) : a \in \mathcal{B}_P\}$ as a basis of closed sets. This means that the closed sets of X_Γ are arbitrary intersections of elements of \mathcal{B} , but since X_Γ is finite, these intersections are in fact finite. Let us set a useful lemma.

Lemma 3.17. *Let X be a topological space.*

1. *For all U open and $A \subseteq X$, we have $\text{Cl}(A) \cap U \subseteq \text{Cl}(A \cap U)$.*

2. For all $A, B \subseteq X$, we have $\text{Cl}(\text{Int}(\text{Cl}(\text{Int } A) \cap B)) = \text{Cl}(\text{Int}(A \cap B))$.
3. For all A_1, \dots, A_n regular closed, we have $A_1 \sqcap \dots \sqcap A_n = \text{Cl}(\text{Int}(A_1 \cap \dots \cap A_n))$.

Proof.

1. We have $A \subseteq (A \cap U) \cup (X \setminus U)$, whence

$$\text{Cl}(A) \subseteq \text{Cl}((A \cap U) \cup (X \setminus U)) = \text{Cl}(U \cap A) \cup \text{Cl}(X \setminus U) = \text{Cl}(A \cap U) \cup (X \setminus \text{Int } U),$$

and it follows that $\text{Cl}(A) \cap \text{Int}(U) \subseteq \text{Cl}(A \cap U)$. Since U is open, we also have $\text{Int}(U) = U$, and therefore $\text{Cl}(A) \cap U \subseteq \text{Cl}(A \cap U)$.

2. First, we have

$$\begin{aligned} \text{Int}(A \cap B) &= \text{Int}(\text{Int}(A \cap B)) \\ &= \text{Int}(\text{Int } A \cap \text{Int } B) \\ &\subseteq \text{Int}(\text{Cl}(\text{Int } A) \cap \text{Int } B) \\ &= \text{Int}(\text{Cl}(\text{Int } A)) \cap \text{Int}(\text{Int } B) \\ &= \text{Int}(\text{Cl}(\text{Int } A)) \cap \text{Int}(B) \\ &= \text{Int}(\text{Cl}(\text{Int } A) \cap B) \end{aligned}$$

and thus $\text{Cl}(\text{Int}(A \cap B)) \subseteq \text{Cl}(\text{Int}(\text{Cl}(\text{Int } A) \cap B))$. Conversely, since $\text{Int}(B)$ is open, Item 1 gives us $\text{Cl}(\text{Int } A) \cap \text{Int}(B) \subseteq \text{Cl}(\text{Int } A \cap \text{Int } B)$. It follows that

$$\begin{aligned} \text{Int}(\text{Cl}(\text{Int } A) \cap B) &= \text{Int}(\text{Cl}(\text{Int } A)) \cap \text{Int}(B) \\ &\subseteq \text{Cl}(\text{Int } A) \cap \text{Int}(B) \\ &\subseteq \text{Cl}(\text{Int } A \cap \text{Int } B) \\ &= \text{Cl}(\text{Int}(A \cap B)) \end{aligned}$$

and therefore $\text{Cl}(\text{Int}(\text{Cl}(\text{Int } A) \cap B)) \subseteq \text{Cl}(\text{Int}(A \cap B))$.

3. By induction on $n \geq 1$. For $n = 1$, we have $A_1 = \text{Cl}(\text{Int}(A_1))$ by definition of A_1 being regular closed. Now suppose that this holds for n . We then have

$$\begin{aligned} A_1 \sqcap \dots \sqcap A_{n+1} &= \text{Cl}(\text{Int}(A_1 \cap \dots \cap A_n)) \sqcap A_{n+1} && \text{by the induction hypothesis} \\ &= \text{Cl}(\text{Int}(\text{Cl}(\text{Int}(A_1 \cap \dots \cap A_n)) \cap A_{n+1})) \\ &= \text{Cl}(\text{Int}((A_1 \cap \dots \cap A_n) \cap A_{n+1})) && \text{by 2} \\ &= \text{Cl}(\text{Int}(A_1 \cap \dots \cap A_{n+1})). \end{aligned}$$

□

We now claim that $\text{RC}(X_\Gamma) = \tau_0$. We already know that the elements of τ_0 are regular closed, by [BTV07b, Lemma 9.6]. Conversely, assume that $A \subseteq X_\Gamma$ is regular closed. We then have $A = V_\Gamma(a_1) \cap \dots \cap V_\Gamma(a_n)$ for some $a_1, \dots, a_n \in \mathcal{B}_P$. Since A is regular closed, we have $A = \text{Cl}(\text{Int } A) = \text{Cl}(\text{Int}(V_\Gamma(a_1) \cap \dots \cap V_\Gamma(a_n)))$, and by Lemma 3.17 it follows that $A = V_\Gamma(a_1) \sqcap \dots \sqcap V_\Gamma(a_n)$. By [BTV07b, Lemma 9.6], we then obtain $A = V_\Gamma(a_1 \sqcap \dots \sqcap a_n)$, and therefore $A \in \tau_0$. As a result, we can define a binary relation \preceq_Γ on $\text{RC}(X_\Gamma)$ by

$$V_\Gamma(a) \preceq_\Gamma V_\Gamma(b) \iff (a \preceq b) \in \Gamma.$$

Again, the axiom Eq guarantees that \preceq_Γ is well defined.

Lemma 3.18 ([BTV07b, Lemma 9.4]). *Let $a, b \in \mathcal{B}_P$.*

1. We have $V(a) \subseteq V(b)$ iff $(a \subseteq b) \in \Gamma$.

2. We have $V(a) C_{R_\Gamma} V(b)$ iff $(a C b) \in \Gamma$.

From the axioms **Comp**, **Trans**, **Mon** and **BLin**, we deduce as in the relational case that $\langle \text{RC}(X_\Gamma), \preceq_\Gamma \rangle$ is measurable. We then call $\mathfrak{X}_\Gamma := \langle X_\Gamma, \preceq_\Gamma \rangle$ the *canonical measurable space* of \mathbf{L} relative to Γ .

Lemma 3.19 (Truth Lemma). *For all $\varphi \in \mathcal{L}_P$, we have $\langle X_\Gamma, \preceq_\Gamma, V_\Gamma \rangle \models \varphi$ iff $\varphi \in \Gamma$.*

Proof. By induction on φ . The non-trivial cases are given by Lemma 3.18. \square

Proposition 3.20.

1. If $\vdash_{\mathbf{L}} \text{Con}$, then \mathfrak{X}_Γ is connected.
2. If $\vdash_{\mathbf{L}} \text{Lin}$, then \mathfrak{X}_Γ is bounded.
3. If $\vdash_{\mathbf{L}} \text{Def}$, then \mathfrak{X}_Γ is definite.
4. If $\vdash_{\mathbf{L}} \text{Def}$ and $\vdash_{\mathbf{L}} \text{Unbd}$, then \mathfrak{X}_Γ is unbounded.

Proof. Item 1 and Item 2 are a consequence of the Truth Lemma together with Proposition 3.16 and Theorem 2.3, respectively. Again, 3 and 4 follow from Proposition 2.24. \square

Theorem 3.21.

1. The logic \mathbf{LCM}^+ is sound and complete for the class of all finite measurable spaces.
2. The logic \mathbf{LCMB}^+ is sound and complete for the class of all finite bounded measurable spaces.
3. The logic \mathbf{S} is sound and complete for the class of all finite, connected and definite unbounded measurable spaces.
4. The logic \mathbf{BS} is sound and complete for the class of all finite, connected and definite bounded measurable spaces.

Proof. See the proof of Theorem 3.13. \square

4 Completeness for the real line

In this section, we focus on the so-called “logics of the space” \mathbf{S} and \mathbf{BS} . We denote by \mathbb{R} the set of real numbers, and we recall that the *Lebesgue measure* λ is a measure on a subalgebra B_λ of $2^{\mathbb{R}}$, which assigns to every closed interval $[a, b]$ its length $b - a$ (when $a < b$) [RF10, Ch. 2]. The elements of B_λ are said to be *Lebesgue-measurable*, and in particular B_λ contains all closed sets, including all regular closed sets. Yet, and contrary to what one may expect, λ does *not* specialize to a measure on $\text{RC}(\mathbb{R})$. This is because the Boolean operations of $\text{RC}(\mathbb{R})$ differ from the standard set-theoretic operators, and as such $\text{RC}(\mathbb{R})$ is not a subalgebra of B_λ . In [LS19, Prop. 3], we find an explicit counter-example showing that λ fails to be a measure on $\text{RC}(\mathbb{R})$. To bypass this issue, we will instead consider the Boolean algebra of regular closed sets *with null boundary* [LS19, Sect. 4.1], defined as

$$\text{RCN}(X) := \{a \in \text{RC}(X) : \lambda(a) = \lambda(\text{Int } a)\}$$

for all closed subspaces X of \mathbb{R} .² Note that when a is closed, $\lambda(a) = \lambda(\text{Int } a)$ is equivalent to $\lambda(\text{Cl } a \setminus \text{Int } a) = 0$, where $\text{Cl } a \setminus \text{Int } a$ is known as the *boundary* of the set a . Hence, elements of $\text{RCN}(X)$ have boundary of measure 0, and are thus said to have “null boundary”. It can then be showed that λ is a measure on $\text{RCN}(X)$ [LS19, Lemma 7]. Our goal is thus to prove that \mathbf{S}

²Recall that the closed sets of a subspace X of \mathbb{R} are those of the form $X \cap A$, with A closed in \mathbb{R} . So when X is closed, its closed sets are also closed in \mathbb{R} , and thus Lebesgue-measurable.

is complete for the general measurable space $\langle \mathbb{R}, \text{RCN}(\mathbb{R}), \preceq_\lambda \rangle$, and that **BS** is complete for the general measurable space $\langle [0, 1], \text{RCN}([0, 1]), \preceq_\lambda \rangle$.

For convenience, a measurable frame $\mathfrak{F} = \langle W, R, \preceq \rangle$ is called a *concrete frame* if it is finite, reflexive, symmetric, connected and definite. A *cycle* in \mathfrak{F} is a path $w_0 \dots w_{n-1}$ in \mathfrak{F} such that $w_0 = w_{n-1}$ and $w_i \neq w_{i+1 \bmod n}$ and $w_i \neq w_{i+2 \bmod n}$ for all $i \in [1, n]$. These conditions are meant to exclude the reflexive and symmetric edges, that would otherwise trivially count as cycles of length 1 and 2. We say that \mathfrak{F} is *acyclic* if it contains no cycle.

Lemma 4.1 (Partial unraveling). *Let $\mathfrak{F} = \langle W, R, \preceq \rangle$ be a concrete frame. Then there exists an acyclic concrete frame $\mathfrak{F}' = \langle W', R', \preceq' \rangle$ and an adjacency map $f : \mathfrak{F}' \rightarrow \mathfrak{F}$. In addition, if \mathfrak{F} is bounded (resp. unbounded), then so is \mathfrak{F}' .*

Proof. Let us fix $w_0 \in W$. We denote by W' the set of all paths in \mathfrak{F} starting from w_0 and of length $\leq |W|$. We then define $f : W' \rightarrow W$ by $f(w_0 \dots w_n) := w_n$. We let R' be the smallest reflexive and symmetric relation on W' satisfying $w_0 \dots w_n R' w_0 \dots w_{n+1}$ whenever $w_0 \dots w_{n+1} \in W'$ (see Figure 2). Next, let μ be a measure compatible with \preceq . Let $w \in W$, and let $n := |f^{-1}(w)|$. If $n \neq 0$, then for all $\bar{w} \in f^{-1}(w)$ we set $\mu'(\{\bar{w}\}) := \frac{\mu(\{w\})}{n}$, so that $\mu'(f^{-1}(w)) = \mu(\{w\})$. This uniquely defines a definite measure $\mu' : 2^{W'} \rightarrow [0, \infty]$, which in addition is bounded when μ is bounded, and unbounded when μ is unbounded.

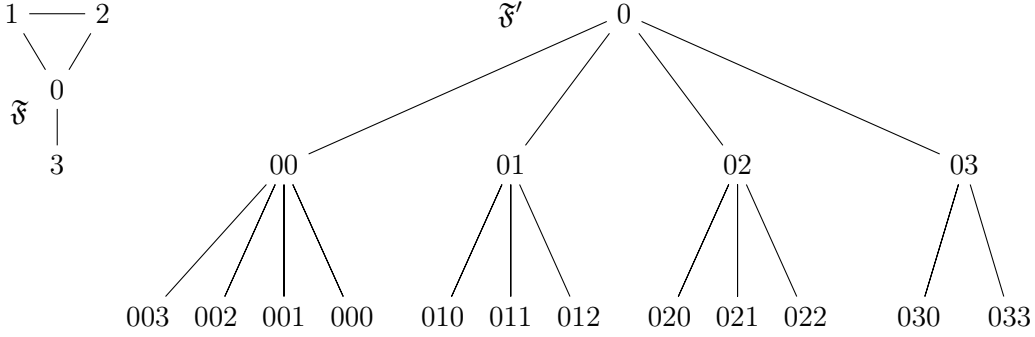


Figure 2: A frame \mathfrak{F} and its unraveling \mathfrak{F}' relative to the point 0, depicted up to the third stage (out of four). Reflexive edges are omitted.

We check that f is an adjacency map. The forth condition is immediate. For the back condition, assume that $w R u$. Since \mathfrak{F} is connected, there is a path $w_0 \dots w_n$ from w_0 to $w_n = w$, which we assume to be of minimal length. Then $n \leq |W| - 1$, for otherwise we have $w_i = w_j$ for some $i < j$, which results in a smaller path $w_0 \dots w_i w_{j+1} \dots w_n$ from w_0 to w_n , a contradiction. We then set $\bar{w} := w_0 \dots w_n$ and $\bar{u} := w_0 \dots w_n u$, and we obtain $f(\bar{w}) = w$, $f(\bar{u}) = u$ and $\bar{w} R' \bar{u}$. Finally, for all $A \subseteq W$, we have by construction that $\mu'(f^{-1}[A]) = \mu(A)$. By Remark 3.4, it follows immediately that f preserves relative size.

Finally, we check that \mathfrak{F}' is connected and acyclic. Let $\bar{w} = w_0 \dots w_n \in W'$, and let us set $\bar{w}_i = w_0 \dots w_i$ for all $i \in [0, n]$. Then $\bar{w}_0 \dots \bar{w}_n$ is a path from w_0 to \bar{w} . Since R' is symmetric, it follows that any two elements in W' are connected by a path visiting w_0 . Now, assume that $\bar{u}_0 \dots \bar{u}_{n-1}$ is a cycle in \mathfrak{F}' . Let $i \in [0, n-1]$. Since $\bar{u}_i \neq \bar{u}_{i+1 \bmod n}$, we have either $|\bar{u}_i| < |\bar{u}_{i+1 \bmod n}|$ or $|\bar{u}_i| > |\bar{u}_{i+1 \bmod n}|$. We consider two cases. First assume that $|\bar{u}_i| > |\bar{u}_{i+1 \bmod n}|$ for all $i \in [0, n-1]$. Then, since $\bar{u}_0 \dots \bar{u}_{n-1}$ is a cycle, we obtain $|\bar{u}_0| > |\bar{u}_0|$, a contradiction. Otherwise, we have $|\bar{u}_i| < |\bar{u}_{i+1 \bmod n}|$ for some i . Then by construction of R' , $\bar{u}_{i+1 \bmod n}$ is of the form $\bar{u}_i u$ with $u \in W$. As a result, we cannot have $|\bar{u}_{i+1 \bmod n}| > |\bar{u}_{i+2 \bmod n}|$, since this would implies that $\bar{u}_{i+2 \bmod n} = \bar{u}_i$, which is not permitted. Thus, we have $|\bar{u}_{i+1 \bmod n}| < |\bar{u}_{i+2 \bmod n}|$. We can iterate this reasoning and, since $\bar{u}_0 \dots \bar{u}_{n-1}$ is a

cycle, we arrive at $|\bar{u}_i| < |\bar{u}_i|$, a contradiction. Hence, we must have $|\bar{u}_i| > |\bar{u}_{i+1 \bmod n}|$ for all $i \in [0, n-1]$, and as before we obtain $|\bar{u}_0| < |\bar{u}_0|$, a contradiction. This proves that \mathfrak{F}' is acyclic. \square

Lemma 4.2. *Let $\mathfrak{F} = \langle W, R, \preceq \rangle$ be a bounded and acyclic concrete frame. Let $r \in W$, and let μ be a measure compatible with \preceq . We suppose that $\mu(W) = 1$. Then there exists a homomorphism f from \mathfrak{F} to $\langle [0, 1], \text{RCN}([0, 1]), \preceq_\lambda \rangle$ and $\varepsilon > 0$ such that:*

- (a) $[0, \varepsilon] \cup [1 - \varepsilon, 1] \subseteq f(r)$,
- (b) for all $w \in W$, we have $\lambda(f(\{w\})) = \mu(\{w\})$.

Proof. Since \mathfrak{F} is connected and acyclic, we can represent it as a tree rooted in r . Formally, given $w \in W$, we denote by $d(w)$ the length of the shortest path from r to w , the existence of which is guaranteed by the connectedness of \mathfrak{F} . Note that we always have $d(w) \leq |W| - 1$, as a consequence of the pigeonhole argument that we used in the proof of Lemma 4.1. The *children* of w are then the elements $u \in W$ such that $w R u$ and $d(u) = d(w) + 1$ – in this case we write $w \rightarrow u$. Then w will be said to be the *father* of its children. Note that since \mathfrak{F} is acyclic, the father of a point is always unique.

Next, we define $\bar{\mu} : W \rightarrow [0, 1]$ by $\bar{\mu}(w) := \mu(\{u \in W : d(w) \leq d(u)\})$ for all $w \in W$. If $[a, b]$ and $[c, d]$ are two closed intervals, we write $[a, b] \triangleleft [c, d]$ in case $c = d$. We call $[a, b]$ *trivial* if $a \geq b$. We define by induction on $d(w)$ a set $A_w \subseteq [0, 1]$, as well as $B_w^u \subseteq [0, 1]$ for every child u of w , and with the following properties:

- A_w is a finite union of closed intervals (and thus regular closed with null boundary);
- every B_w^u is a closed interval;
- if u is a child of w , then $A_w \cup \bigcup_{u \rightarrow v} B_u^v = B_w^u$ for some $i \in [1, n]$;
- $\lambda(A_w) = \mu(\{w\})$;
- $\lambda(B_w^u) = \bar{\mu}(u)$ for every child u of w .

So let $u \in W$, and assume that A_w is defined for all $w \in W$ such that $d(w) < d(u)$. If $u = r$, we set $B := [0, 1]$ and we have $\lambda(B) = 1 = \mu(W) = \bar{\mu}(u)$. Otherwise, u has a father w , and we set $B := B_w^u$. We then also have $\lambda(B) = \bar{\mu}(u)$, by the induction hypothesis. In both cases, we write $B = [a, b]$. Let us denote by v_1, \dots, v_k the children of u . We introduce $2k + 1$ closed intervals $A_u^1, B_u^{v_1}, \dots, A_u^k, B_u^{v_k}, A_u^{k+1}$ so that:

1. $\{a\} \triangleleft A_u^1 \triangleleft B_u^{v_1} \triangleleft A_u^2 \triangleleft \dots \triangleleft A_u^k \triangleleft B_u^{v_k} \triangleleft A_u^{k+1} \triangleleft \{b\}$,
2. $\lambda(A_u^1) + \dots + \lambda(A_u^{k+1}) = \mu(\{u\})$,
3. $\lambda(B_u^{v_i}) = \bar{\mu}(v_i)$ for all $i \in [1, k]$.

Conditions 2 and 3 are justified by the fact that $\lambda(B) = \bar{\mu}(u)$ and $\bar{\mu}(u) = \mu(\{u\}) + \bar{\mu}(v_1) + \dots + \bar{\mu}(v_k)$. In addition, since \mathfrak{F} is definite, we have $\mu(\{t\}) > 0$ for all $t \in W$, which ensures that we can take $A_u^1, B_u^{v_1}, \dots, A_u^k, B_u^{v_k}, A_u^{k+1}$ to be non-trivial. If $w = r$, we then set $\varepsilon := \min\{\sup A_u^1, 1 - \inf A_u^{k+1}\} > 0$. We introduce $A_w := A_u^1 \cup \dots \cup A_u^{k+1}$, and we then have $\lambda(A_w) = \mu(\{u\})$ and $A_w \cup B_u^{v_1} \cup \dots \cup B_u^{v_k} = [a, b] = B$. The construction of the A_w 's is depicted in Figure 3. Now, let us prove by induction on n the following statement:

$$\mathcal{P}(n) : \bigcup_{\substack{w \in W \\ d(w) \leq n}} A_w \cup \bigcup_{\substack{w \in W \\ d(w) = n \\ w \rightarrow u}} B_w^u = [0, 1].$$

For $n = 0$, this amounts to $A_r \cup \bigcup_{r \rightarrow u} B_r^u = [0, 1]$, which is true. Then assume that $\mathcal{P}(n)$ holds, and let us reorganize $\mathcal{P}(n+1)$ as follows:

$$\bigcup_{\substack{w \in W \\ d(w) \leq n}} A_w \cup \bigcup_{\substack{w \in W \\ d(w) = n+1}} \left(A_w \cup \bigcup_{w \rightarrow u} B_w^u \right) = [0, 1]$$

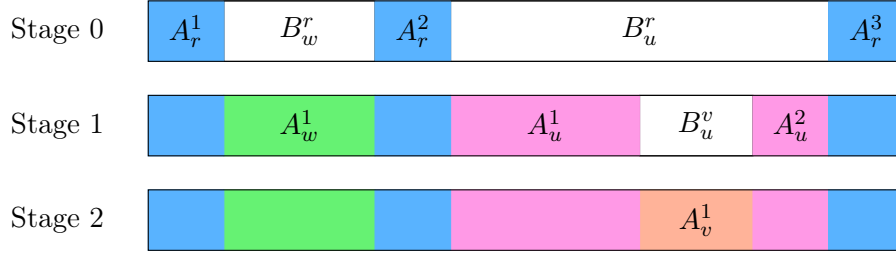


Figure 3: Step-by-step construction of a homomorphism from a frame \mathfrak{F} to the unit interval. Singletons in \mathfrak{F} are mapped to the areas of the same color.

For all $w \in W$ such that $d(w) = n + 1$, we know that $A_w \cup \bigcup_{w \rightarrow u} B_w^u = B_v^w$, for some $v \in W$ such that $d(v) = n$. Hence, the previous equality rewrites to

$$\bigcup_{\substack{w \in W \\ d(w) \leq n}} A_w \cup \bigcup_{\substack{v \in W \\ d(v) = n \\ v \rightarrow u}} B_v^u = [0, 1],$$

which is just $\mathcal{P}(n)$. Therefore $\mathcal{P}(n + 1)$ holds.

For $n = |W| - 1$, we have $d(w) \leq n$ for all $w \in W$, and when $d(w) = n$, the element w has no children. Hence $\mathcal{P}(n)$ boils down to $\bigcup_{w \in W} A_w = [0, 1]$. We then define $f : 2^W \rightarrow \text{RC}([0, 1])$ by $f(U) := \bigcup_{w \in U} A_w$ for all $U \subseteq W$. First, we check that f is a Boolean algebra homomorphism.

- Clearly, we have $f(0) = 0$.
- Let $U, V \subseteq W$. Note that when $u \neq v$, the set $A_u \cap A_v$ is finite, and thus $A_u \sqcap A_v = 0$. As a result, we have

$$f(U) \sqcap f(V) = \left(\bigcup_{u \in U} A_u \right) \sqcap \left(\bigcup_{v \in V} A_v \right) = \bigcup_{u \in U, v \in V} A_u \sqcap A_v = \bigcup_{w \in U \cap V} A_w = f(U \cap V).$$

- Let $U \subseteq W$. Since we have proved that $\bigcup_{w \in W} A_w = [0, 1]$, it follows that

$$f(U)^* = \text{Cl}([0, 1] \setminus f(U)) = \text{Cl} \left(\bigcup_{w \in W} A_w \setminus f(U) \right) = \bigcup_{w \in W} \text{Cl}(A_w \setminus f(U)).$$

Let $w \in W$. If $w \in U$, then $A_w \subseteq f(U)$ and so $A_w \setminus f(U) = \emptyset$. If instead $w \notin U$, then for all $u \in U$, the set $A_w \cap A_u$ contains only points that lie on the border of A_w , and as such we have $\text{Int}(A_w) \subseteq A_w \setminus A_u$. It follows that $\text{Int}(A_w) \subseteq A_w \setminus f(U)$, whence $\text{Cl}(\text{Int}(A_w)) \subseteq \text{Cl}(A_w \setminus f(U))$. Since $\text{Cl}(\text{Int}(A_w)) = A_w$, it follows that $A_w = \text{Cl}(A_w \setminus f(U))$. Therefore $f(U)^* = \bigcup_{w \in W \setminus U} A_w = f(U^*)$, as desired.

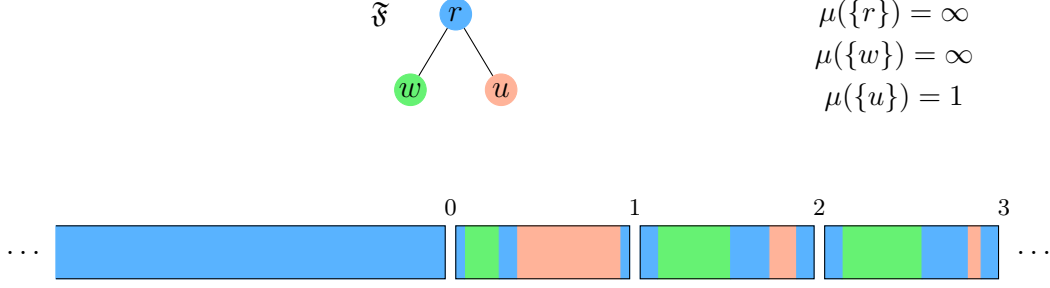


Figure 4: Homomorphism from a frame \mathfrak{F} to the real line.

Next, since $\lambda(f(\{w\})) = \lambda(A_w) = \mu(\{w\})$ for all $w \in W$, it follows once again that f preserves relative size. Finally, let $U, V \subseteq W$ and assume that $U C_R V$. Then there $u \in U$ and $v \in V$ such that $u R v$. If $u = v$ then $U \cap V \neq \emptyset$, whence $f(U) \cap f(V) \neq \emptyset$, that is, $f(U) C_{[0,1]} f(V)$. Otherwise, we have $u \rightarrow v$ or $v \rightarrow u$, and without loss of generality we assume that $u \rightarrow v$. Then by construction of A_u , there is some closed interval $A_u^i \subseteq A_u$ such that $A_u^i \triangleleft B_u^v$, that is, $\sup A_u^i = \inf B_u^v$. By construction of A_v , we also have $\inf A_v = \inf B_u^v$, and thus $\sup A_u^i = \inf A_v$. It follows that $A_u^i C_{[0,1]} A_v$, and therefore $f(U) C_{[0,1]} f(V)$. Conversely, suppose that $f(U) C_{[0,1]} f(V)$. Then there exist $u \in U$ and $v \in V$ such that $A_u \cap A_v \neq \emptyset$, which can only happens if $u R v$. Therefore $U C_R V$. \square

Lemma 4.3. *Let $\mathfrak{F} = \langle W, R, \preceq \rangle$ be a bounded and acyclic concrete frame. Then there exists a homomorphism from \mathfrak{F} to $\langle \mathbb{R}, \text{RCN}(\mathbb{R}), \preceq_\lambda \rangle$.*

Proof. Our strategy is to tile the real line with the half-line of negative integers, together with the intervals $[0, 1]$, $[1, 2]$, $[2, 3]$, \dots . By applying Lemma 4.2 to a well-chosen bounded measure μ_n on W , we can then map $\langle W, R, \preceq_{\mu_n} \rangle$ to $[n-1, n]$, for all $n \geq 1$. The idea is that when $\mu(\{w\})$ is finite, we ensure that the value of $\mu_n(\{w\})$ decreases with the inverse square of n . Since the series $\sum_{n \geq 1} \frac{1}{n^2}$ is known to converge, this will ensure that the total measure of the image of $\{w\}$ remains finite. We also map an arbitrary singleton of infinite measure to $(-\infty, 0]$. We refer to Figure 4 for a visual example.

Formally, let μ be a measure compatible with \preceq , and let $W_\infty := \{w \in W : \mu(\{w\}) = \infty\}$. By assumption, μ is not bounded, so there exists $r \in W_\infty$. Let $M := |W_\infty| > 0$ and $\alpha := \mu(W \setminus W_\infty)$. Up to renormalization, we can assume that $\frac{6}{\pi^2} \alpha < 1$. Let us fix $n \in \mathbb{N}$, and let $\alpha_n := \frac{6}{\pi^2 n^2} \alpha < 1$. For all $n \in \mathbb{N}$, we define a definite bounded measure μ_n on W by

$$\mu_n(\{w\}) := \begin{cases} \frac{1-\alpha_n}{M} & \text{if } w \in W_\infty \\ \frac{6}{\pi^2 n^2} \mu(\{w\}) & \text{otherwise} \end{cases}$$

for all $w \in W$, so that $\mu_n(W) = \mu_n(W_\infty) + \mu_n(W \setminus W_\infty) = M \frac{1-\alpha_n}{M} + \frac{6}{\pi n^2} \alpha = 1 - \alpha_n + \alpha_n = 1$. Let f_n be the homomorphism from $\langle W, R, \preceq_{\mu_n} \rangle$ to $\langle [n-1, n], \text{RCN}([n-1, n]), \preceq_\lambda \rangle$ obtained by applying Lemma 4.2 to μ_n and $\langle W, R, \preceq_{\mu_n} \rangle$. We then define a map $f : 2^W \rightarrow \text{RC}(\mathbb{R})$ by

$$f(U) := \begin{cases} (-\infty, 0] \cup \bigcup_{n \geq 1} f_n(U) & \text{if } r \in U \\ \bigcup_{n \geq 1} f_n(U) & \text{otherwise} \end{cases}.$$

In particular, we have $f(W) = (-\infty, 0] \cup \bigcup_{n \geq 1} f_n(W) = (-\infty, 0] \cup \bigcup_{n \geq 1} [n-1, n] = \mathbb{R}$. We then prove that f preserves Boolean operators as in Lemma 4.2.

Now let $w \in W$. We know that $\lambda(f_n(\{w\})) = \mu_n(w)$ for all $n \geq 1$. If $\mu(\{w\}) = \infty$, then $\lambda(f(\{w\})) \geq \sum_{n \geq 1} \mu_n(w) = \sum_{n \geq 1} \frac{1-\alpha_n}{M} = \infty$, since $\lim_{n \rightarrow \infty} \frac{1-\alpha_n}{M} = \frac{1}{M} > 0$. If instead

$\mu(\{w\}) < \infty$, then $\lambda(f(\{w\})) = \sum_{n \geq 1} \mu_n(w) = \frac{6}{\pi^2} \mu(\{w\}) \cdot \sum_{n \geq 1} \frac{1}{n^2} = \mu(\{w\})$. So in all cases we have $\lambda(f(\{w\})) = \mu(\{w\})$, and thus f preserves relative size.

Finally, let $U, V \subseteq W$ and assume that $U C_R V$. Since f_1 is a homomorphism, we have $f_1(U) C_{[n-1, n]} f_1(V)$, and thus $f(U) C_{\mathbb{R}} f(V)$. Conversely, suppose that $f(U) C_{\mathbb{R}} f(V)$. Then there exist $u \in U$ and $v \in V$ such that $f(\{u\}) C_{\mathbb{R}} f(\{v\})$. By examining the large unions in $f(\{u\})$ and $f(\{v\})$, we identify the following possibilities.

1. There exist $n, m \geq 1$ such that $f_n(\{u\}) C_{\mathbb{R}} f_m(\{v\})$. If $n = m$, then $f_n(\{u\}) C_{[n-1, n]} f_n(\{v\})$, and since f_n is a homomorphism it follows that $u R v$. If $n \neq m$, then we must have either $m = n+1$ and $f_n(\{u\}) \cap f_m(\{v\}) = \{n\}$, or $n = m+1$ and $f_n(\{u\}) \cap f_m(\{v\}) = \{m\}$. However, due to the condition (a), this can only happen if $u = v = r$.
2. Either $u = r$ and $(-\infty, 0] C_{\mathbb{R}} f_1(\{v\})$, or $v = r$ and $f_1(\{u\}) C_{\mathbb{R}} (-\infty, 0]$. Again, the condition (a) implies that $u = v = r$.
3. We have $u = v = r$ and $(-\infty, 0] C_{\mathbb{R}} (-\infty, 0]$.

In all cases it follows that $u R v$, and therefore $U C_R V$. □

Theorem 4.4. *The logic **BS** is sound and complete for the general measurable space $\langle [0, 1], \text{RCN}([0, 1]), \preceq_\lambda \rangle$. The logic **S** is sound and complete for the general measurable space $\langle \mathbb{R}, \text{RCN}(\mathbb{R}), \preceq_\lambda \rangle$.*

Proof. Let $\varphi \in \mathcal{L}$ and assume that $\not\vdash_{\mathbf{BS}} \neg\varphi$. By Theorem 3.13, there exists a bounded concrete frame \mathfrak{F} on which φ is satisfiable. Then by Lemma 4.1, there exists a bounded and acyclic concrete frame \mathfrak{F}' and an adjacency map f from \mathfrak{F}' to \mathfrak{F} . By Proposition 3.9 and Proposition 3.5, it follows that φ is satisfiable on \mathfrak{F}' as well. By Lemma 4.2, we obtain a homomorphism from \mathfrak{F}' to $\langle [0, 1], \text{RCN}([0, 1]), \preceq_\lambda \rangle$. By Proposition 3.5, it follows that φ is satisfiable on $\langle [0, 1], \text{RCN}([0, 1]), \preceq_\lambda \rangle$, as desired. For **S**, we follow the same proof scheme, and rely on Lemma 4.3. □

We now show that we can adapt the proof of Theorem 4.4 to higher dimensions, that is, we can replace $[0, 1]$ by $[0, 1]^n$ and \mathbb{R} by \mathbb{R}^n , where $n \geq 1$. For the sake of concision, we will not repeat all the steps in details, but we will only sketch the changes that allow the reasoning to scale to arbitrary dimensions. We call a *cuboid* a set of the form $[a_1, b_1] \times \cdots \times [a_n, b_n]$, and we recall that its Lebesgue measure is its volume $(b_1 - a_1) \times \cdots \times (b_n - a_n)$. First, let us examine Lemma 4.2. Here the goal remains the same: we construct a homomorphism $f : \langle W, R, \preceq \rangle \rightarrow \langle [0, 1]^n, \text{RCN}([0, 1]^n), \preceq_\lambda \rangle$ with the condition that $f(r)$ acts as a ‘buffer’ separating its complement from the border of $[0, 1]^n$. Formerly expressed by the condition (a), this statement can more generally be phrased as:

$$\exists \varepsilon > 0, \{x_i \in [0, 1]^n : \exists i \in [1, n], x_i \leq \varepsilon \text{ or } 1 - x_i \leq \varepsilon\} \subseteq f(r).$$

In the proof of Lemma 4.2, the A_w ’s are described as finite unions of closed interval, and the B_w^u ’s as closed intervals. In general, the B_w^u ’s will be cuboids, and A_w will just be defined as $A_w := (\bigsqcup_{w \rightarrow u} B_w^u)^* = \text{Cl}([0, 1]^n \setminus \bigcup_{w \rightarrow u} B_w^u)$. We refer to Figure 5 for an example in dimension 2. As for Lemma 4.3, we apply the same strategy, but we now tile the space with the complete n -dimensional grid. Given $\bar{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$, we introduce the tile $T_{\bar{k}} := [k_1, k_1 + 1] \times \cdots \times [k_n, k_n + 1]$. Let us select a bijection $\Delta : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{Z}^n$. For all $k \geq 1$, we define a bounded measure μ_k as before, and obtain a homomorphism $f_k : \langle W, R, \preceq_{\mu_k} \rangle \rightarrow T_{\Delta(k)}$. We then define a homomorphism $f : \langle W, R, \preceq \rangle \rightarrow \mathbb{R}^n$ by $f(U) := \bigcup_{k \geq 1} f_k(U)$. Finally, the reasoning of Theorem 4.4 remains unchanged, and we obtain the following result.

Theorem 4.5. *Let $n \geq 1$. The logic **BS** is sound and complete for the general measurable space $\langle [0, 1]^n, \text{RCN}([0, 1]^n), \preceq_\lambda \rangle$, and the logic **S** is sound and complete for the general measurable space $\langle \mathbb{R}^n, \text{RCN}(\mathbb{R}^n), \preceq_\lambda \rangle$.*

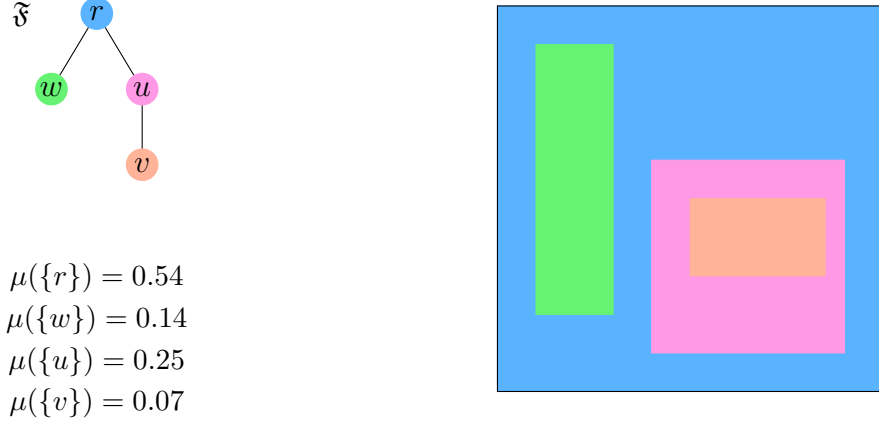


Figure 5: Homomorphism from a frame \mathfrak{F} to the unit square.

5 Unification and admissibility

In this section, we investigate unification and admissibility in so-called *Boolean-valued theories*. Those are ‘abstract’ equational theories in the same sense as abstract logics, in that they are not defined by a precise syntax, but as a set of valid equations with general and minimalistic principles. The fundamental quality that these theories have in common is that Boolean substitutions can be applied to their terms, and preserve valid equations. In the sequel, we fix a set $\mathbf{Param} \subseteq \mathbf{Prop}$ of *parameters*, so that both \mathbf{Param} and $\mathbf{Prop} \setminus \mathbf{Param}$ are infinite. As usual, a *substitution* is a function $\sigma : \mathcal{B}_P \rightarrow \mathcal{B}_Q$ with $P, Q \subseteq \mathbf{Prop}$ finite such that:

- $\sigma(0) = 0$ and $\sigma(a \sqcap b) = \sigma(a) \sqcap \sigma(b)$ and $\sigma(a^*) = \sigma(a)^*$ for all $a, b \in \mathcal{B}_P$;
- $P \cap \mathbf{Param} \subseteq Q$, and $\sigma(p) = p$ for all $p \in P \cap \mathbf{Param}$.

Given $a, b \in \mathcal{B}$, we write $a \simeq b$ if a and b are two equivalent Boolean terms.³ Given two substitutions $\sigma : \mathcal{B}_P \rightarrow \mathcal{B}_Q$ and $\tau : \mathcal{B}_P \rightarrow \mathcal{B}_{Q'}$, we write $\sigma \simeq \tau$ if we have $\sigma(p) \simeq \tau(p)$ for all $p \in P$. We write $\sigma \preceq \tau$ if there exists a substitution $\mu : \mathcal{B}_Q \rightarrow \mathcal{B}_{Q''}$ such that $\tau \simeq \mu\sigma$.

Definition 5.1. A *Boolean-valued theory* is a tuple $\langle \mathcal{T}, \mathcal{E}, \text{Var}, \cdot \rangle$ where:

- \mathcal{T} is a set of *terms*;
- $\mathcal{E} \subseteq \mathcal{T} \times \mathcal{T}$ is an *equational theory*;
- $\text{Var} : \mathcal{T} \rightarrow 2^{\mathbf{Prop}}$ assigns to every term a *finite* subset of \mathbf{Prop} ;
- the operator \cdot assigns to every term $t \in \mathcal{T}$ and every substitution $\sigma : \mathcal{B}_P \rightarrow \mathcal{B}_Q$ such that $\text{Var}(t) \subseteq P$, a term $\sigma \cdot t \in \mathcal{T}$;
- if $\sigma \preceq \tau$ and $(\sigma \cdot t, \sigma \cdot t') \in \mathcal{E}$ then $(\tau \cdot t, \tau \cdot t') \in \mathcal{E}$.

We call $\langle \mathcal{T}, \mathcal{E}, \text{Var}, \cdot \rangle$ *effective* if in addition \mathcal{T} is countable, \mathcal{E} is decidable, and the functions Var and $(\sigma, t) \mapsto \sigma \cdot t$ are computable. Given $(t, t') \in \mathcal{T} \times \mathcal{T}$, we let $t \approx t'$ denote the pair (t, t') , also called an *equation*. For convenience, we identify the tuple $\langle \mathcal{T}, \mathcal{E}, \text{Var}, \cdot \rangle$ to \mathcal{E} . If $\Gamma \subseteq \mathcal{T} \times \mathcal{T}$, we write $\text{Var}(\Gamma) := \bigcup_{(t, t') \in \Gamma} \text{Var}(t) \cup \text{Var}(t')$.

Example 5.2. A very simple example is Boolean equational theory: here $\mathcal{T} = \mathcal{B}$, $\mathcal{E} := \{(a, b) \in \mathcal{B}^2 : a \simeq b\}$, $\text{Var}(t)$ is the set of all variables occurring in t , and $\sigma \cdot t := \sigma(t)$.

Every propositional logic \mathbf{L} (as defined in Section 3) is also Boolean-valued. In this case we set $\mathcal{T} = \mathcal{L}$ and $\mathcal{E} := \{(\varphi, \psi) \in \mathcal{L}^2 : \vdash_{\mathbf{L}} \varphi \leftrightarrow \psi\}$. Again, $\text{Var}(\varphi)$ is the set of all variables occurring

³Note the difference between \simeq and \equiv : $a \simeq b$ is as an assertion, while $a \equiv b$ is a formula in the object language \mathcal{L} .

in φ , and $\sigma \cdot \varphi$ is the formula φ in which $\sigma(p)$ is substituted for every variable p . This is more generally the case of any propositional or first-order logic with predicates on Boolean terms. Likewise, one can see modal logics as Boolean-valued theories, since Boolean substitutions are special cases of modal substitutions – but as we will see below, this example is actually not very interesting, as far as unification is concerned.

Definition 5.3 (Boolean unification). Let \mathcal{E} be a Boolean-valued theory, and let $\Gamma \subseteq \mathcal{T} \times \mathcal{T}$ be a finite set of equations, also called a *unification problem*. Then a *unifier* of Γ is a substitution $\sigma : \mathcal{B}_{\text{Var}(\Gamma)} \rightarrow \mathcal{B}_Q$ such that $\sigma \cdot t \approx \sigma \cdot t' \in \mathcal{E}$ for all $t \approx t' \in \Gamma$. We call Γ *unifiable* if it admits a unifier. A set Σ of unifiers of Γ is said to be *complete* if for all unifiers τ of Γ , there exists $\sigma \in \Sigma$ such that $\sigma \preceq \tau$. When Σ is a singleton $\{\sigma\}$, we call σ a *most general unifier* of Γ . We say that \mathcal{E} is *unary* if every unifiable problem admits a most general unifier. We say that \mathcal{E} is *finitary* if it is not unary, and every unification problem admits a finite complete set of unifiers.⁴

Example 5.4. In the logic **LC**, consider the unification problem $\Gamma := \{p C 1 \vee q C 1 \approx 1 C 1\}$, where p and q are not parameters. Then Γ admits two unifiers σ_1 and σ_2 defined by $\sigma_1(p) := 1$, $\sigma_1(q) := q$, $\sigma_2(p) := p$, $\sigma_2(q) := 1$. We claim that $\{\sigma_1, \sigma_2\}$ is a complete set of unifiers of Γ . Indeed, let σ be a unifier of Γ , and let us write $a := \sigma(p)$ and $b := \sigma(q)$. Then we have $\vdash_{\text{LC}} a C 1 \vee b C 1 \leftrightarrow 1 C 1$. Toward a contradiction, suppose that $a \not\approx 1$ and $b \not\approx 1$. Then there exists a valuation from $\text{Var}(a)$ to $\{0, 1\}$ that falsifies a , and this valuation can be seen a substitution τ_1 with $\tau_1(a) = 0$. Likewise, there exists a substitution τ_2 such that $\tau_2(b) = 0$. By applying the substitution $\tau_1 \tau_2$ to $\vdash_{\text{LC}} a C 1 \vee b C 1 \leftrightarrow 1 C 1$, we then obtain $\vdash_{\text{LC}} 0 C 1 \vee 0 C 1 \leftrightarrow 1 C 1$, which simplifies to $\vdash_{\text{LC}} \neg(1 C 1)$. Yet $1 C 1$ is satisfiable on any adjacency frame with a non-empty relation, a contradiction. Therefore, we have either $a \simeq 1$ or $b \simeq 1$, and thus either $\sigma_1 \preceq \sigma$ or $\sigma_2 \preceq \sigma$.

However, Γ does not have a most general unifier. Indeed, assume toward a contradiction that σ is a most general unifier of Γ . Then we have either $\sigma_1 \preceq \sigma$ or $\sigma_2 \preceq \sigma$. Assume that $\sigma_1 \preceq \sigma$. Since σ is most general, we also have $\sigma \preceq \sigma_2$, and thus $\sigma_1 \preceq \sigma_2$. Then there exists a substitution μ such that $\mu \sigma_1 \simeq \sigma_2$. It follows that $p = \sigma_2(p) \simeq \mu \sigma_1(p) = \mu(1) = 1$, a contradiction. If $\sigma_2 \preceq \sigma$, the reasoning is the same. Since Γ is unifiable and lacks a most general unifier, this proves that **LC** is not unary – and this argument also applies to all the logics considered in Section 3.

To avoid confusion, one should keep in mind that the problem of Boolean unification is concerned with Boolean unifiers only, and ignores other types of substitutions. For this reason, Boolean unification in modal logics (for instance) does *not* coincide with standard modal unification. Boolean equational theory is known to be unary, seeing that it enjoys the even stronger property of *projective unification* [BG11, Sect. 3.3].

Definition 5.5 (Boolean admissibility). Let \mathcal{E} be a Boolean-valued theory, and consider an *inference rule* $\Gamma/t \approx t'$, where $\Gamma \subseteq \mathcal{T} \times \mathcal{T}$ and $t, t' \in \mathcal{T}$. Then $\Gamma/t \approx t'$ is said to be *admissible* if every unifier of Γ is also a unifier of $\{t \approx t'\}$.

In a propositional logic **L**, a special case of admissibility happens when a rule $\Gamma/\varphi \approx \varphi'$ is *derivable*, that is, when $(\bigwedge_{\psi \approx \psi' \in \Gamma} \psi \leftrightarrow \psi') \rightarrow (\varphi \leftrightarrow \varphi')$ is a theorem of **L**. When *all* admissible rules are derivable, the logic **L** is said to be *structurally complete*, and admissibility basically dissolves into the problem of validity. For logics that do not fall in this special case, the problem of admissibility needs to be taken seriously. For instance, classical propositional logic is structurally complete [CZ97, Sect 1.4], but many modal logics are not [Ryb97]. The following example shows that contact logic is not structurally complete either.

Example 5.6. In **LC**, the inference rule $\frac{p C 1 \approx \top}{p \approx 1 \approx \top}$ is admissible (when p is not a parameter). Indeed, let $a \in \mathcal{B}$ and assume that $\vdash_{\text{LC}} a C 1$. If $a \not\approx 1$, then there exists a substitution σ such that $\sigma(a) \simeq 0$. By applying σ to $\vdash_{\text{LC}} a C 1$, we then obtain $\vdash_{\text{LC}} 0 C 1$, a contradiction.

⁴Note that \emptyset is a complete set of unifiers of Γ if and only if Γ is not unifiable.

Therefore $a \simeq 1$, as desired. However, it is easy to see that $p \text{ C } 1 \rightarrow p \equiv 1$ is not valid in **LC**, and as such $\frac{p \text{ C } 1 \simeq \top}{p \equiv 1 \simeq \top}$ is not derivable. This example remains valid for all the logics considered in Section 3.

We are going to prove that Boolean unification is always unary or finitary, regardless of the selected Boolean-valued theory. Our strategy is to show that every unifier of a problem Γ is equivalent to some unifier that does not use more variables than those already present in Γ . To achieve this, we are going to algebraize our problem. Let $P \subseteq \mathbf{Prop}$ be finite. Let A_P be the free Boolean algebra generated by P (also called the *Lindenbaum algebra* over P), which is known to be finite [HG09, Ch. 28]. Since A_P contains the elements of P , we can abuse notations and identify every term $a \in \mathcal{B}_P$ to an element of A_P . As a result, it is clear that the substitutions from \mathcal{B}_P to \mathcal{B}_Q can be identified to the homomorphisms from A_P to A_Q . Given a substitution $\sigma : A_P \rightarrow A_Q$, we then introduce the *kernel* $\text{Ker } \sigma := \{(a, b) \in A_P \times A_P : \sigma(a) = \sigma(b)\}$ of σ .

Lemma 5.7. *Let A, B, C be three finite Boolean algebras. Let $\alpha : A \rightarrow B$ be an injective homomorphism, and $\gamma : A \rightarrow C$ be a homomorphism. Then there exists a homomorphism $\beta : B \rightarrow C$ such that $\beta\alpha = \gamma$.*

Proof. Using the duality between the category of finite Boolean algebras and the category of finite sets [HG09, Ch. 34 & 36], this amounts to prove that for all finite sets X, Y, Z and functions $f : Y \rightarrow X$ and $h : Z \rightarrow X$ with f surjective, there exists $g : Z \rightarrow Y$ such that $fg = h$ (see Figure 6). This is easily achieved as follows: given $z \in Z$, select $y \in Y$ such that $f(y) = h(z)$, and set $g(z) := y$.



Figure 6: Duality in action

□

Lemma 5.8. *Let $P \subseteq \mathbf{Prop}$ be finite and let $\sigma : A_P \rightarrow A_Q$ and $\tau : A_P \rightarrow A_{Q'}$ be two substitutions. Then $\sigma \simeq \tau$ if and only if $\text{Ker } \sigma = \text{Ker } \tau$.*

Proof. From left to right, this is clear. Conversely, suppose that $\text{Ker } \sigma = \text{Ker } \tau$. The set $\text{Ker } \sigma$, seen as a binary relation, is a congruence on A_P , and gives rise to a quotient algebra $A_P/\text{Ker } \sigma$. Given $a \in A_P$, we denote by $[a]$ the equivalence class of a modulo $\text{Ker } \sigma$. We then have a surjective homomorphism $\pi_\sigma : A_P \rightarrow A_P/\text{Ker } \sigma$ defined by $\pi_\sigma(a) := [a]$, and an injective homomorphism $\hat{\sigma} : A_P/\text{Ker } \sigma \rightarrow A_Q$ defined by $\hat{\sigma}([a]) := \sigma(a)$, so that $\sigma = \hat{\sigma}\pi_\sigma$ (see Figure 7). For details, we refer to [HG09, Ch. 21]. By assumption, we have $A_P/\text{Ker } \sigma = A_P/\text{Ker } \tau$. Since $\hat{\sigma}$ is injective, we can apply Lemma 5.7 to $\hat{\sigma}$ and $\hat{\tau}$ to obtain the existence of a substitution $\mu : A_Q \rightarrow A_{Q'}$ such that $\hat{\tau} = \mu\hat{\sigma}$. As a result, we obtain $\tau = \mu\sigma$, and therefore $\sigma \preceq \tau$. Likewise we can prove that $\tau \preceq \sigma$, and we are done.

□

Proposition 5.9. *Let P be finite and let $\sigma : A_P \rightarrow A_Q$ be a substitution. Then there exists a substitution $\tau : A_P \rightarrow A_P$ such that $\sigma \simeq \tau$.*

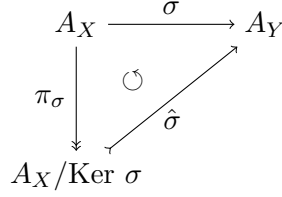


Figure 7: Canonical factorization of σ

Proof. Obviously, σ is a unifier of $\text{Ker } \sigma$ (seen as a unification problem). Since Boolean equational theory has unary unification, we obtain the existence of a most general unifier $\tau : A_P \rightarrow A_P$ of $\text{Ker } \sigma$. That τ is a unifier of $\text{Ker } \sigma$ implies $\text{Ker } \sigma \subseteq \text{Ker } \tau$. Moreover, the most general character of τ yields $\tau \preceq \sigma$, and thus $\text{Ker } \tau \subseteq \text{Ker } \sigma$. Therefore $\text{Ker } \sigma = \text{Ker } \tau$, and by Lemma 5.8 it follows that $\sigma \simeq \tau$, as desired. \square

Theorem 5.10. *Every Boolean-valued theory \mathcal{E} has unary or finitary Boolean unification. More precisely, if Γ is a unification problem with n variables, then Γ admits a complete set of at most $2^{(n \cdot 2^n)}$ unifiers. Further, if \mathcal{E} is effective, then Boolean unification and admissibility in \mathcal{E} are decidable. In addition, given a unification problem Γ , one can effectively compute a complete set of unifiers for Γ .*

Proof. Let Γ be a unification problem. Let $P := \text{Var}(\Gamma)$ and $n := |P|$. Let $\sigma_1, \dots, \sigma_m$ be an enumeration of all substitutions from \mathcal{B}_P to \mathcal{B}_P , up to equivalence. By Proposition 5.9, the set $\Sigma := \{\sigma_i \mid 1 \leq i \leq m \text{ and } \sigma_i \text{ is a unifier of } \Gamma\}$ is a complete sets of unifiers for Γ . Since \mathcal{B}_P contains 2^{2^n} terms (up to equivalence), we have $m = (2^{2^n})^n$, and therefore $|\Sigma| \leq 2^{(n \cdot 2^n)}$, as desired. Now assume that \mathcal{E} is effective. The following algorithms address the three problems.

```

Input:  $\Gamma = \{t_1 \approx u_1, \dots, t_m \approx u_m\}$  unification problem
 $P \leftarrow \text{Var}(\Gamma)$ ;
Non-deterministically select  $\sigma : \mathcal{B}_P \rightarrow \mathcal{B}_P$ ;
if  $\sigma \cdot t_1 \approx \sigma \cdot u_1, \dots, \sigma \cdot t_m \approx \sigma \cdot u_m \in \mathcal{E}$  then
  | Accept;
else
  | Reject;
end

```

Algorithm 1: Algorithm for unification

```

Input:  $\Gamma = \{t_1 \approx u_1, \dots, t_m \approx u_m\}$  finite set of equations
 $P \leftarrow \text{Var}(\Gamma)$ ;
 $\Sigma \leftarrow \emptyset$ ;
foreach  $\sigma : \mathcal{B}_P \rightarrow \mathcal{B}_P$  do
  | if  $\sigma \cdot t_1 \approx \sigma \cdot u_1, \dots, \sigma \cdot t_m \approx \sigma \cdot u_m \in \mathcal{E}$  then
    | |  $\Sigma \leftarrow \Sigma \cup \{\sigma\}$ ;
  | end
end
Output:  $\Sigma$ 

```

Algorithm 2: Algorithm for computing a complete set of unifiers

Input: $t_1 \approx u_1, \dots, t_m \approx u_m / t \approx u$ inference rule
 $P \leftarrow \text{Var}(\{t_1 \approx u_1, \dots, t_m \approx u_m, t \approx u\})$;
 Non-deterministically select $\sigma : \mathcal{B}_P \rightarrow \mathcal{B}_P$;
if $\sigma \cdot t_1 \approx \sigma \cdot u_1, \dots, \sigma \cdot t_m \approx \sigma \cdot u_m \in \mathcal{E}$ and $\sigma \cdot t \approx \sigma \cdot u \notin \mathcal{E}$ **then**
 | Accept;
else
 | Reject;
end

Algorithm 3: Algorithm for deciding whether a rule is *not* admissible

In Algorithm 2 we assume that the substitutions enumerated in the main loop are pairwise non-equivalent (so that there are finitely many of them). Note that Algorithm 3 decides non-admissibility instead of admissibility – but after the usual procedure of determinizing the algorithm and inverting the output, one obtains an algorithm for admissibility. \square

By Theorem 3.14 (and [BTV07b, Th 4.3] for **LC**), the logics **LC**, **LCM**, **LCMB**, **LCM**⁺, **LCMB**⁺, **S**, **BS** are effective Boolean-valued theories, and thus fall under the scope of Theorem 5.10. From Example 5.4, we also know that none of them are unary, which means that they are all finitary. As a case study, we examine the complexity of unification and admissibility in the logic **LC**.

Corollary 5.11. *Unification and admissibility in **LC** are decidable in exponential space. In addition, given a unification problem Γ , one can compute a complete set of unifiers for Γ in double exponential space.*

Proof. For admissibility, we analyze the complexity of Algorithm 3. Here the size of the input is $n := |t| + |u| + \sum_{i=1}^m |t_i| + |u_i|$, and we have $|P| \leq n$. Thus, a term in \mathcal{B}_P has size $\leq 2^n$, and the substitution σ selected by the algorithm has size $\leq n \cdot 2^n$. Since the satisfiability problem of **LC** is in NP [BTV07a], its validity problem is in PSPACE. Then, since the size of the formulas $\sigma \cdot t_1 \leftrightarrow \sigma \cdot u_1, \dots, \sigma \cdot t_m \leftrightarrow \sigma \cdot u_m$ and $\sigma \cdot t \leftrightarrow \sigma \cdot u$ is bounded by $n^2 \cdot 2^n$, their validity in **LC** can be checked in space polynomial in $n^2 \cdot 2^n$, and thus exponential⁵ in n . Therefore, admissibility in **LC** is in co-NEXPSpace, and thus in EXPSPACE by Savitch's theorem. The case of unification is similar. Next, we analyze the complexity of Algorithm 2. Reasoning as above, we see that whether $\sigma : \mathcal{B}_P \rightarrow \mathcal{B}_P$ is a unifier of Γ can be checked in space exponential in n . However, since Σ contains up to $n \cdot 2^{2^n}$ substitutions, its size is bounded by $(n \cdot 2^{2^n}) \cdot (n \cdot 2^n)$, hence the double exponential space complexity. \square

6 Conclusion

In this paper, we have laid the first foundations of a theory of contact and measure, our central result being the axiomatization of the space \mathbb{R}^n for all $n \geq 1$. While our characterizations of measurable Boolean algebras are only auxiliary here, their usefulness may extend beyond the scope of this paper, and be of interest to e.g. probability logic. Likewise, our results on unification and admissibility are framed in a very general setting, so that to be applied to as many equational theories as possible. As far as logics and contact and measure are concerned, the most pressing issue is the complexity of the validity problem, the current 2-EXPSPACE upper bound (Theorem 3.14) being excessively high. When working on this problem, we were able to polynomially reduce the satisfiability of a formula φ to that of a formula φ' in which the contact relation does not occur. The satisfiability of φ' can, in turn, be reduced to the feasibility of a linear system. However, we found ourselves unable to control the size of this system, and we thus defer this problem to future work. Another line of inquiry is the duality theory of contact

⁵Here there are $m + 1$ validities to check, but since $m + 1 \leq n$ the cost in space remains exponential in n .

algebras, initiated by Dimov and Vakarelov [DV05], and which could be profitably extended to measurable contact algebras.

Acknowledgments

We would like to thank Arnaud Lequen and Victor Careil for stimulating discussions about the characterization of measurable relations. This work was supported by the project 48055QG “Advanced Languages of Terms for Ontologies”, as part of the Hubert Curien RILA 2022-2023 partnership.

References

- [BG11] Franz Baader and Silvio Ghilardi. Unification in modal and description logics. *Logic Journal of IGPL*, 19(6):705–730, 2011.
- [BG20] Philippe Balbiani and Çiğdem Gencer. Contact logic is finitary for unification with constants. In *Nonclassical Logics and Their Applications: Post-proceedings of the 8th International Workshop on Logic and Cognition*, pages 85–104. Springer, 2020.
- [BTV07a] Philippe Balbiani, Tinko Tinchev, and Dimiter Vakarelov. Dynamic logics of the region-based theory of discrete spaces. *Journal of Applied Non-Classical Logics*, 17(1):39–61, 2007.
- [BTV07b] Philippe Balbiani, Tinko Tinchev, and Dimiter Vakarelov. Modal logics for region-based theories of space. *Fundamenta Informaticae*, 81(1-3):29–82, 2007.
- [CH01] Anthony G. Cohn and Shyamanta M. Hazarika. Qualitative spatial representation and reasoning: An overview. *Fundamenta informaticae*, 46(1-2):1–29, 2001.
- [CZ97] Alexander Chagrov and Michael Zakharyashev. *Modal Logic*. Oxford Logic Guides. Clarendon Press. Consecutive, Oxford, 1997.
- [dL22] Theodore de Laguna. Point, line, and surface, as sets of solids. *The journal of Philosophy*, pages 449–461, 1922.
- [DP02] Brian Davey and Hilary Priestley. *Introduction to lattices and order*. Cambridge university press, 2002.
- [DV05] Georgi Dimov and Dimiter Vakarelov. Topological representation of precontact algebras. In *International Conference on Relational Methods in Computer Science*, pages 1–16. Springer, 2005.
- [DW05] Ivo Düntsch and Michael Winter. A representation theorem for Boolean contact algebras. *Theoretical Computer Science*, 347(3):498–512, 2005.
- [Gär75] Peter Gärdenfors. Qualitative probability as an intensional logic. *Journal of Philosophical Logic*, 4(2):171–185, 1975.
- [HG09] Paul Halmos and Steven Givant. *Introduction to Boolean algebras*. Springer, 2009.
- [KLTS71] David H. Krantz, R. Duncan Luce, Amos Tversky, and Patrick Suppes. *Foundations of Measurement, Volume I*. New York Academic Press. 1971.
- [KPS59] Charles H. Kraft, John W. Pratt, and Abraham Seidenberg. Intuitive probability on finite sets. *The Annals of Mathematical Statistics*, 30(2):408–419, 1959.
- [LS19] Tamar Lando and Dana Scott. A calculus of regions respecting both measure and topology. *Journal of Philosophical Logic*, 48(5):825–850, 2019.
- [ORM16] Zoran Ognjanović, Miodrag Rašković, and Zoran Marković. *Probability logics: probability-based formalization of uncertain reasoning*. Springer, 2016.

- [PS98] Christos H. Papadimitriou and Kenneth Steiglitz. *Combinatorial optimization: algorithms and complexity*. Courier Corporation, 1998.
- [RF10] Halsey Royden and Patrick Michael Fitzpatrick. *Real analysis*. China Machine Press, 2010.
- [Ryb97] Vladimir V Rybakov. *Admissibility of logical inference rules*. Elsevier, 1997.
- [Sco64] Dana Scott. Measurement structures and linear inequalities. *Journal of mathematical psychology*, 1(2):233–247, 1964.
- [Seg71] Krister Segerberg. Qualitative probability in a modal setting. In *Studies in Logic and the Foundations of Mathematics*, volume 63, pages 341–352. Elsevier, 1971.
- [Sik69] Roman Sikorski. *Boolean algebras*, volume 2. Springer, 1969.
- [Vak07] Dimiter Vakarelov. Region-based theory of space: algebras of regions, representation theory, and logics. In *Mathematical problems from applied logic II: Logics for the XXIst century*, pages 267–348. Springer, 2007.
- [vdH96] Wiebe van der Hoek. Qualitative modalities. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 4(01):45–59, 1996.
- [Whi29] Alfred N. Whitehead. *Process and reality*. New York, Macmillan, 1929.