



HAL
open science

Invariant measures for substitutions on countable alphabets

Weberty Domingos, Sébastien Ferenczi, Ali Messaoudi, Glauco Valle

► **To cite this version:**

Weberty Domingos, Sébastien Ferenczi, Ali Messaoudi, Glauco Valle. Invariant measures for substitutions on countable alphabets. *Ergodic Theory and Dynamical Systems*, 2024, 44 (9), pp.2533 - 2564. 10.1017/etds.2023.113 . hal-04543820

HAL Id: hal-04543820

<https://hal.science/hal-04543820v1>

Submitted on 15 Apr 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

INVARIANT MEASURES FOR SUBSTITUTIONS ON COUNTABLE ALPHABETS

WEBERTY DOMINGOS, SÉBASTIEN FERENCZI, ALI MESSAOUDI, AND GLAUCO VALLE

ABSTRACT. In this work, we study ergodic and dynamical properties of symbolic dynamical system associated to substitutions on an infinite countable alphabet. Specifically we consider shift dynamical systems associated to irreducible substitutions which have well-established properties in the case of finite alphabets. Based on dynamical properties of a countable integer matrix related to the substitution, we obtain results on existence and uniqueness of shift invariant measures.

1. INTRODUCTION

Let A be a countable set (called *alphabet*), A^* be the set of finite words on A and $A^{\mathbb{Z}_+}$ be the set of infinite words on A , where $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. A *substitution* is a map $\sigma : A \rightarrow A^*$. We assume that for every letter $a \in A$, $\sigma(a)$ is not empty. We extend σ to A^* and $A^{\mathbb{Z}_+}$ by concatenation and, to simplify the notation, we also denote these extensions by σ . Hence $\sigma(u_0 \dots u_n) = \sigma(u_0) \dots \sigma(u_n)$ for all $u_0 \dots u_n \in A^*$ and $\sigma(u_0 u_1 \dots) = \sigma(u_0) \sigma(u_1) \dots$ for all $u_0 u_1 \dots \in A^{\mathbb{Z}_+}$. We assume that there exists a letter a in A such that the length of the finite word $\sigma^n(a)$ converges to infinity as n goes to infinity.

To any substitution σ , we can associate a *shift dynamical system* (Ω_σ, S) , where

$\Omega_\sigma = \{u \in A^{\mathbb{Z}_+} : \text{any finite factor of } u \text{ occurs in } \sigma^n(a) \text{ for some } n \in \mathbb{N} \text{ and } a \in A\}$,
 $\mathbb{N} = \{1, 2, \dots\}$, and S is the shift map given by

$$S(u_0 u_1 \dots) = u_1 u_2 \dots \text{ for all } u = u_0 u_1 \dots \in A^{\mathbb{Z}_+}.$$

Shift dynamical systems associated to substitutions provide many important examples in ergodic theory and they have been well studied in the literature when the alphabet is finite (see for instance [20, 21]). It is classical that if σ is a *primitive substitution* on $A = \{0, \dots, d-1\}$, $d \geq 2$, i.e there exists $k \in \mathbb{N}$ such that for all $a, b \in A$ the letter b occurs in the word $\sigma^k(a)$, then the dynamical system is minimal, uniquely ergodic with topological entropy 0 (see [18] and [21, Chapter 5]). Moreover Ω_σ is the closure of the orbit of any periodic point of σ .

The unique shift invariant probability measure μ is given on cylinders $[w]$, where $w = w_0 \dots w_n$, $w_i \in A$ for $i = 0, \dots, n$, is a finite word that occurs in u and $[w] = \{u_0 u_1 \dots \in \Omega_\sigma, u_i = w_i, i = 0, \dots, n\}$, by $\mu[w]$ which is the frequency of occurrences of w in the periodic point u . Moreover the vector $(\mu[0], \dots, \mu[d-1])$ is the normalized left Perron eigenvector associated to the dominant Perron-Frobenius eigenvalue of the matrix $M_\sigma = (M_{ij})_{0 \leq i, j \leq d-1}$ associated to σ , where $M_{ij} := |\sigma(i)|_j$ is the number of occurrences of the

2020 *Mathematics Subject Classification.* 37A05, 37A25, 37B10, 37A50.

Key words and phrases. substitutions on infinite alphabet, unique ergodicity, countable nonnegative matrices, shift dynamical systems.

Sébastien Ferenczi and Ali Messaoudi were partially supported by the Brazilian Federal Agency for Support and Evaluation of Graduate Education (CAPES), in the scope of the Program CAPES-PrInt, grant number 88887.310463/2018-00 and International Cooperation grant number 88881.310741/2018-01. Ali Messaoudi was partially supported by CNPq grant 310784/2021-2 and by Fapesp grant 2019/10269-3. Glauco Valle was supported by CNPq grant 307938/2022-0 and by FAPERJ grant 26/202.636/2019.

letter j in the word $\sigma(i)$. On the other hand, it is known (see [2]) that if σ is of Pisot type, then the dynamical system (Ω_σ, S) has good geometrical properties, in particular it is semi-conjugated to a translation on the torus \mathbb{T}^{d-1} .

When the alphabet A is a topological compact set, many results are given in [6, 15, 21].

When A is countably infinite, the situation is more complicated and there are already some work on the subject, see for instance [1, 6, 9, 15]. One of the difficulties in studying ergodic properties of the dynamical system (Ω_σ, S) in such case lies in the fact that the countably infinite matrix M_σ may present a larger number of possible behaviors. Specifically consider an irreducible countably infinite matrix $M = (M_{ij})_{i,j \in \mathbb{Z}_+}$, that means for all $i, j \in \mathbb{Z}_+$, there exists an integer $n \geq 1$ such that for all $k \geq n$, $M_{ij}^k > 0$, where for the sake of simplicity we write $(M^n)_{ij} = M_{ij}^n$. It is known that for all $i, j \in \mathbb{Z}_+$, $\lim_{n \rightarrow \infty} (M_{ij}^n)^{1/n} = \lambda$ exists. We say that M is *transient* if and only if $\sum_{n=0}^{+\infty} \frac{M_{ij}^n}{\lambda^n} < +\infty$, otherwise M is said to be *recurrent*. It is known that if M is recurrent there are left and right eigenvectors l and r associated to λ and when the scalar product $l \cdot r$ is finite, we say that M is *positive recurrent*, otherwise M is said to be *null recurrent*. Thus for instance if the countably infinite matrix M_σ is irreducible then it could be either transient or null recurrent or positive recurrent and each of these cases may be associated to a distinct behavior of (Ω_σ, S) .

For substitutions on countably infinite alphabets an important study was initiated by Ferenczi in [9]. In that paper several results were proved, in particular it is considered the squared drunken substitution defined on $A = 2\mathbb{Z}$ by $\sigma(n) = (n-2)nn(n+2)$, $n \in A$ and proved that the dynamical system (Ω_σ, S) is not minimal and has non finite invariant measure. However it is also shown that (Ω_σ, S) has an infinite invariant measure μ which is shift ergodic and has Krengel entropy equal to 0.

Let us recall that σ is called left determined or determined to order 1 if there exists a nonnegative integer N such that every w of length at least N which occurs on some element of Ω_σ , has a unique decomposition $w = w_1 \dots w_s$, where each $w_i = \sigma(a_i)$ for some $a_i \in A$, except that w_1 may be only a suffix of $\sigma(a_1)$ and w_s may be only a prefix of $\sigma(a_s)$, and the a_i , $1 \leq i \leq s-1$, are unique.

The definition of determined to order 1 was introduced in [16] (see also definition 1 in [19]). In [9], the author used the same definition and called it left determined. It is known that this condition is stronger than recognizability, see [19].

In [9] it is also proved that if σ is of constant length, left determined and has an irreducible aperiodic positive recurrent matrix M_σ , then the associated shift dynamical system admits an ergodic probability invariant measure.

In [1], the authors constructed stationary and non-stationary generalized Bratteli-Vershik models for left determined, irreducible, aperiodic and recurrent substitutions on an infinite countable alphabet. As a consequence, they proved that for a left determined substitution $\sigma : \mathbb{Z} \rightarrow \mathbb{Z}$ with M_σ irreducible, aperiodic and recurrent which is also of *bounded size* (the letters of all $\sigma(n)$ belong to the set $\{n-t, n-t+1, \dots, n+t\}$ where $t \in \mathbb{Z}$ is independent of n), there exists a shift invariant measure μ on Ω_σ .

It is also worth mentioning that an arithmetic study of substitutions on countably infinite alphabets was done in [17].

In this paper, unless explicitly indicated we consider $A = \mathbb{Z}_+$ and $\sigma : A \rightarrow A^*$ a bounded length substitution ($\sup\{|\sigma(a)|, a \in A\}$ is finite) such that σ has a periodic point u and $M = M_\sigma$ is irreducible and aperiodic. We prove that if M_σ satisfies

$$\lim_{n \rightarrow +\infty} \sup_{i \in A} \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} = 0 \text{ for all } j \in A, \quad (1.1)$$

then the dynamical system (Ω_σ, S) has no finite invariant measure. In particular, the last result holds for a subclass of substitutions σ such that M_σ is transient and σ has constant length, or M_σ is recurrent and has a left Perron eigenvector $l = (l_i)_{i \geq 0} \notin l^1$.

We also prove that if M_σ is positive recurrent, then the dynamical system (Ω_σ, S) has a shift invariant measure μ which is finite if and only if M_σ has a left Perron eigenvector $l \in l^1$. Moreover, if σ has constant length and M_σ has a power that is scrambling, then (Ω_σ, S) has a unique shift invariant probability measure μ . Let us recall that a nonnegative matrix $M = (M_{ij})_{i,j \geq 0}$ is said to be *scrambling* if there exists $a > 0$ such that

$$\sum_{j=0}^{+\infty} \min(M_{ij}, M_{kj}) \geq a \text{ for all } i \neq k \in \mathbb{Z}_+.$$

Scrambling stochastic infinite countable matrices are very important, since a stochastic matrix $P = (P_{ij})_{i,j \geq 0}$ is strongly ergodic (see definition 2.12) if and only if a power of P is scrambling.

We also consider the case where σ is not a constant length substitution. We introduce the notions of strongly ergodic and \star -strongly ergodic matrices M_σ related to the convergence of

$$\frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n}, \quad i, j \in A,$$

as $n \rightarrow \infty$. Then we show that if M_σ has a right Perron eigenvector in l^∞ and has a power that is scrambling (M_σ strongly ergodic), then (Ω_σ, S) is minimal and has a unique shift invariant probability measure μ .

A difference concerning substitutions on countable infinite alphabets that we should point out is that substitutions may not have a periodic point. In this paper we consider M_σ irreducible and suppose the existence of a periodic point u , thus Ω_σ is the closure of the orbit of any periodic point of σ . However our results will remain valid for σ that have no periodic point, since instead of using the left determined condition, we use the true fact that any finite word V occurring in some element of Ω_σ , has a decomposition (not necessarily unique) as $V = v_0 \sigma(Z) w_0$ where v_0 , w_0 and Z finite words occurring in some elements of Ω_σ and $\max(|v_0|, |w_0|) \leq \sup\{|\sigma(a)|, a \in A\}$, where for all finite word $z \in A^*$, $|z|$ denotes the length of z .

The paper is organized as follows. In section 2, we give notations, definitions and preliminary results. Section 3, is devoted to the main results of the paper.

2. PRELIMINARIES AND NOTATIONS

As in Section 1, let A be a countable set (called alphabet), A^* be the set of finite words on A and $A^{\mathbb{Z}^+}$ the set of infinite words on A . We denote a finite word on A by $u_0 \dots u_{n-1}$ for some $n \geq 1$ and we call $n = |u_0 \dots u_{n-1}|$ its length. An infinite word on A will be denoted by $u = u_0 u_1 \dots$. For $U = u_0 \dots u_{n-1}$ and $V = v_0 \dots v_{m-1}$ in A^* , where $n \geq m$ are positive integers, we denote

$$|U|_V = \{0 \leq k \leq n - m, u_k \dots u_{k+m-1} = v_0 \dots v_{m-1}\}$$

which is the number of occurrences of V in U . Let $u = u_0 u_1 \dots \in A^{\mathbb{Z}^+}$ and $V \in A^*$, we say that V occurs in u or V is a factor of u if $V = u_k \dots u_l$ for some integers $0 \leq k \leq l$. We denote by F_u the set of all factors of u .

On $A^{\mathbb{Z}^+}$ we consider the discrete product topology, which is metrizable and generated by the metric d defined on $A^{\mathbb{Z}^+}$ by:

$$d(u_0 u_1 \dots, v_0 v_1 \dots) = 0 \text{ if } u_0 u_1 \dots = v_0 v_1 \dots$$

and

$$d(u_0u_1\dots, v_0v_1\dots) = \frac{1}{2^{k_0}} \text{ where } k_0 = \min\{i \geq 0, u_i \neq v_i\} \text{ otherwise.}$$

A base for the discrete product topology is given by the cylinders

$$[w] = \{u_0u_1\dots \in A^{\mathbb{Z}_+}, u_i = w_i, \forall 0 \leq i \leq k\},$$

for $w = w_0\dots w_k \in A^*$. The cylinders are clopen sets. When the alphabet A is finite, the set $A^{\mathbb{Z}_+}$ is compact and is homeomorphic to a Cantor set. If A is infinite, $A^{\mathbb{Z}_+}$ is closed but not compact.

Let $\sigma : A \rightarrow A^*$ be a substitution. We will assume without loss of generality that $A = \mathbb{Z}_+$ (and occasionally $A = \mathbb{Z}$ in some examples). We define the infinite matrix $M_\sigma = (M_{ij})_{i,j \in \mathbb{Z}_+}$ by $M_{ij} = |\sigma(i)|_j$. Observe that M_σ is the transpose of the substitution matrix given in [21]. It is easy to prove by induction that for all $i, j \in A$ and for all integer $n \in \mathbb{N}$,

$$|\sigma^n(i)|_j = M_{ij}^n, \quad |\sigma^n(i)| = \sum_{j=1}^{\infty} M_{ij}^n.$$

For example, if $\sigma(n) = 0(n+1)$ for all $n \in \mathbb{Z}_+$, then

$$M_\sigma = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \quad (2.1)$$

We say that a substitution $\sigma : A \rightarrow A^*$ is of *constant length* (resp. *bounded length*) if there exists an integer $L \geq 1$ such that $|\sigma(a)| = L$ (resp. $|\sigma(a)| \leq L$) for all $a \in A$.

Observe that if σ has constant length L (resp. bounded length by L), then the sum of the coefficients of each line of the matrix M_σ^n , $n \in \mathbb{N}$ equals L^n (resp. $\leq L^n$).

In this paper, we will assume that σ is a bounded length substitution and there exists $a \in A$ such that $|\sigma^n(a)|$ tends to infinity as n converges to infinity.

We define *the language* of a substitution σ on a A as the set F_σ of finite factors of $\sigma^n(a)$ for some integer $n \geq 0$ and $a \in A$.

We will need some classical definitions from the theory of countable nonnegative matrices, see [11, 26].

Definition 2.1. Let $M = (M_{ij})_{i,j \in \mathbb{Z}_+}$ be an infinite nonnegative matrix (not necessarily a substitution matrix). We say that M is *irreducible* if for all $i, j \in \mathbb{Z}_+$, there exists an integer $k = k(i, j) \geq 1$ such that $M_{ij}^k > 0$. Let $i \in \mathbb{Z}_+$, the number

$$p_i = \gcd\{n \in \mathbb{N}, M_{ii}^n > 0\}$$

is called the *period* of the state i . If M is irreducible, then there exists $p \geq 1$ such that $p_i = p$ for every $i \in \mathbb{Z}_+$ and we say that M has *period* $p \geq 1$. We say that an irreducible matrix M is *aperiodic* if $p = 1$ and *periodic* otherwise.

Observe that M_σ in (2.1) is irreducible and aperiodic and σ is a constant length substitution which has a fixed point $u = \lim_{n \rightarrow \infty} \sigma^n(0)$ since $\sigma(0) = 01$ begins with 0.

Remark 2.2. (see [11]) If a matrix $M = (M_{ij})_{i,j \in \mathbb{Z}_+}$ is irreducible and aperiodic, then for all $i, j \in \mathbb{Z}_+$, there exists an integer $n = n(i, j) \geq 1$ such that for all $k \geq n$, $M_{ij}^k > 0$.

Remark 2.3. Let $\sigma : A \rightarrow A^*$ be a substitution which has a fixed point and M_σ is irreducible. Since there exists $i \in \mathbb{Z}_+$ such that $M_{ii} > 0$, we deduce that M_σ is aperiodic.

Assume that $M = (M_{ij})_{i,j \in \mathbb{Z}_+}$ is an irreducible and aperiodic nonnegative matrix until the end of this section. It is known (see [21, 22]) that there exists $\lambda_M \in [0, \infty]$, called the Perron value of M , such that for all $i, j \in \mathbb{Z}_+$

$$\lim_{n \rightarrow \infty} (M_{ij}^n)^{1/n} = \lambda_M. \quad (2.2)$$

For all $i, j \in \mathbb{Z}_+$ put as usual $M_{ij}^0 = \delta_{ij}$, then consider the series

$$\overline{M}_{ij}(z) = \sum_{n=0}^{+\infty} M_{ij}^n z^n, \quad z \in \mathbb{C}.$$

Observe that the convergence radius of the series $\overline{M}_{ij}(z)$ is equal to λ_M^{-1} . When there is no possibility of confusion we will omit the subscript in λ_M and write simply λ .

Remark 2.4. Directly from the definition, if $\hat{M} = CM$ for some $C > 0$, then $\lambda_{\hat{M}} = C\lambda_M$. If σ is a substitution with constant length L , then $P = M/L$ is a stochastic matrix and $\lambda_M = L\lambda_P$. Moreover for the stochastic matrix P , clearly $\lambda_P \leq 1$ and if $\overline{P}_{ij}(1) = +\infty$, then $\lambda_P = 1$. Thus $\lambda_M \leq L$. Indeed it is enough to have σ with bounded length L , see Lemma 2.10.

We either have $\overline{M}_{ij}(1/\lambda) < \infty$ for every $i, j \in \mathbb{Z}_+$, in this case we say that M is *transient*, or $\overline{M}_{ij}(1/\lambda) = \infty$ for every $i, j \in \mathbb{Z}_+$, and we say that M is *recurrent*. The class of irreducible, aperiodic recurrent matrices can be divided into two classes: Positive recurrent matrices and null recurrent ones. To present the definitions, we need to introduce the series

$$L_{ij}(M, z) = L_{ij}(z) = \sum_{n=0}^{+\infty} l_{ij}(M, n) z^n,$$

where $l_{ij}(M, n) = l_{i,j}(n)$ is defined by: $l_{ij}(0) = 0$, $l_{ij}(1) = M_{ij}$ and

$$l_{ij}(n+1) = \sum_{s \neq i}^{+\infty} l_{is}(n) M_{sj} \text{ for all } n \geq 1.$$

The matrix M is said to be *positive recurrent* if

$$\sum_{n=0}^{+\infty} \frac{nl_{ii}(n)}{\lambda^n} < +\infty,$$

otherwise we say that M is null recurrent.

An interesting result is that if M is an irreducible, aperiodic and recurrent matrix with finite Perron value $\lambda > 0$, then λ has strictly positive left and right eigenvectors l and r , unique up to multiples by a constant. Moreover the scalar product $l \cdot r$ is finite if and only if M is positive recurrent.

Remark 2.5. In Section 3.2 we will give examples of null recurrent nonnegative matrices with constant length L having Perron value strictly smaller than L . These cases are associated to stochastic matrices with Perron value strictly smaller than 1, so they are transient in probabilistic sense (see [8]), but they might be null recurrent according to the above definition. This is not a novelty, see [11]. What is important here is also that we provide substitution matrices in our examples.

To state the next result, we still need to introduce another important series

$$R_{ij}(M, z) = R_{ij}(z) = \sum_{n=0}^{+\infty} r_{ij}(M, n) z^n,$$

where $r_{ij}(M, n) = r_{ij}(n)$ is defined by: $r_{ij}(0) = 0$, $r_{ij}(1) = M_{i,j}$ and

$$r_{ij}(n+1) = \sum_{s \neq j}^{+\infty} M_{is} r_{sj}(n) \text{ for all } n \geq 1.$$

Lemma 2.6. (see [23] and [11, page 211]). *Let M be a nonnegative, irreducible and aperiodic matrix, with finite Perron value $\lambda > 0$. Let $i, j \in \mathbb{Z}_+$.*

(1) *If M is positive recurrent, then*

$$\lim_{n \rightarrow \infty} \frac{M_{ij}^n}{\lambda^n} = \frac{L_{ij}(1/\lambda)}{\mu(i)} = \frac{R_{ij}(1/\lambda)}{\mu(j)} > 0,$$

where $\mu(i) = \sum_{n=1}^{+\infty} n l_{ii}(n) / \lambda^n$.

(2) *If M is transient or null recurrent, then $\lim_{n \rightarrow \infty} M_{ij}^n / \lambda^n = 0$.*

For all $i, j \in \mathbb{Z}_+$, let

$$l^{(i)} = (L_{ik}(1/\lambda))_{k \geq 0} \text{ and } r^{(j)} = (R_{sj}(1/\lambda))_{s \geq 0}.$$

Lemma 2.7. (see [11, page 203]). *Let M be a nonnegative, irreducible, aperiodic matrix, with finite Perron value $\lambda > 0$.*

(1) *If M is recurrent, then for all $i, j \in \mathbb{Z}_+$,*

$$l^{(i)} M = \lambda l^{(i)} \text{ and } M r^{(j)} = \lambda r^{(j)}.$$

(2) *If M is transient then for all $i, j \in \mathbb{Z}_+$,*

$$l^{(i)} M \leq \lambda l^{(i)} \text{ and } M r^{(j)} \leq \lambda r^{(j)}.$$

Remark 2.8. Let $M = (M_{ij})_{i,j \geq 0}$ be a nonnegative, irreducible, aperiodic positive recurrent matrix, with finite Perron value $\lambda > 0$. By item (1) of Lemma 2.6 and item (1) of Lemma 2.7, the vector

$$T_i = (t_{ij})_{j \geq 0} \text{ where } t_{ij} = \lim_{n \rightarrow \infty} M_{ij}^n / \lambda^n \tag{2.3}$$

is a left eigenvector for λ associated to M . Moreover, we have

$$\lim_{n \rightarrow \infty} \frac{M_{i,j}^n}{\lambda^n} = \frac{l_j r_i}{\sum_{k=0}^{+\infty} l_k r_k} \tag{2.4}$$

where $l = (l_k)_{k \geq 0}$ and $r = (r_k)_{k \geq 0}$ are respectively a left and a right Perron eigenvector of M (see [26]).

Lemma 2.9. (see [11, Proposition 7.1.11, page 204]). *Let $M = (M_{ij})_{i,j \geq 0}$ be a nonnegative, irreducible, aperiodic and recurrent matrix with finite Perron value λ . Let $Z = (z_i)_{i \geq 0}$ be a sub invariant nonnegative and non zero eigenvector of M_σ associated to λ , that is $(ZM)_i \leq \lambda z_i$ for all $i \geq 0$ and $Z \neq 0$, then Z is a left Perron eigenvector associated to M .*

Lemma 2.10. *Let $M = (M_{ij})_{i,j \in \mathbb{Z}_+}$ be a nonnegative, irreducible and aperiodic matrix with finite Perron value λ . The following results hold:*

(1) *If M has line sums uniformly bounded by $L > 0$, then $\lambda \leq L$.*

(2) *If M is positive recurrent and has constant line sums equal to L , then $\lambda = L$. Moreover L is the unique eigenvalue of M having nonnegative probability left eigenvector.*

Proof. (1) Suppose that M has line sums bounded by L , then for all integers $j \geq 0$ and $n \geq 1$, we have

$$M_{jj}^n \leq \sum_{k=0}^{+\infty} M_{jk}^n \leq L^n.$$

We deduce by (2.2), that $\lambda \leq L$.

(2) If M is positive recurrent and has constant line sums equal to L , then there exists $l = (l_i)_{i \geq 0} \in l^1$ such that $\sum_{i=0}^{\infty} l_i = 1$ and $lM = \lambda l$, then $\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} l_i M_{ij} = \lambda$, then $L = \lambda$. Using the same idea, we obtain that L is the unique eigenvalue of M having nonnegative probability left eigenvector. \square

Definition 2.11. Let $M = (M_{ij})_{i,j \geq 0}$ be a nonnegative matrix. We say that M is scrambling if there exists $a > 0$ such that

$$\sum_{j=0}^{+\infty} \min(M_{ij}, M_{kj}) \geq a \text{ for all } i \neq k \in \mathbb{Z}_+.$$

Note that a substitution matrix M_σ has a power that is scrambling, if and only if for some $n \geq 1$

$$\forall i, k \in A, \exists j \in A \text{ which occurs in } \sigma^n(i) \text{ and } \sigma^n(k). \quad (2.5)$$

Definition 2.12. Let $P = (P_{ij})_{i,j \geq 0}$ be a nonnegative stochastic matrix. We say that P is

- *ergodic* if $\lim_{n \rightarrow \infty} P_{ij}^n = \pi_j > 0$ for all $i, j \in \mathbb{N}$, where $(\pi_j)_{j \geq 0}$ is a probability vector;
- *strongly ergodic* if P is ergodic and if there exists a probability vector $(\pi_i)_{i \geq 0}$ of nonnegative real numbers such that $\lim_{n \rightarrow \infty} \|P^n - Q\|_s = 0$ where Q is the infinite stochastic matrix with rows equal to $(\pi_j)_{j \geq 0}$ and $\|N\|_s = \sup_{i \geq 0} \sum_{j=0}^{+\infty} |N_{ij}|$ for any infinite complex matrix $N = (N_{ij})_{i,j \geq 0}$. In other words

$$\lim_{n \rightarrow \infty} \sup_{i \geq 0} \sum_{j=0}^{\infty} |P_{ij}^n - \pi_j| = 0.$$

Remark 2.13. It was proved in [10] that if P is strongly ergodic, then P is uniformly geometrically ergodic, i.e., there exist $\beta \in (0, 1)$ and a constant $C > 0$ such that

$$|P_{ij}^n - \pi_j| \leq C\beta^n, \forall i, j, n \in \mathbb{Z}_+.$$

The converse is proved in [14]. In particular it is showed that P is strongly ergodic if and only if for some $j \geq 0$ with $\pi_j > 0$, we have

$$\lim_{n \rightarrow \infty} \sup_{i \geq 0} |P_{ij}^n - \pi_j| = 0. \quad (2.6)$$

There is a nice characterisation of the strong ergodicity (see [25]). It is defined as follows. If $P = (P_{ij})_{i,j \in \mathbb{N}}$ is a stochastic nonnegative countable matrix, then P is strongly ergodic if and only if there exists an integer $n \geq 1$, such that $\delta(P^n) < 1$, where the δ coefficient of any nonnegative countable stochastic matrix $N = (N_{ij})_{i,j \in \mathbb{N}}$ is

$$\delta(N) = \frac{1}{2} \sup_{i,k \in \mathbb{N}} \sum_{j=0}^{+\infty} |N_{ij} - N_{kj}|. \quad (2.7)$$

The number $\delta(N)$ is called Dobrushin coefficient of N or coefficient of ergodicity of N (see for instance [4], [10], [13], [14]). It's not difficult to show that

$$\delta(N) = 1 - \inf_{i \neq k} \sum_{j=0}^{+\infty} \min(N_{ij}, N_{kj}). \quad (2.8)$$

Observe that $\delta(N) < 1$ if and only if N is scrambling. Hence P is strongly ergodic if and only if there exists an integer $n \geq 1$, such that P^n is scrambling.

3. IRREDUCIBLE APERIODIC SUBSTITUTIONS

3.1. Nonexistence of finite invariant measure. In [9], the author proved that if $A = \mathbb{Z}$ and $\sigma(n) = (n-1)nn(n+1)$, $n \in A$, then the dynamical system (Ω_σ, S) has no finite invariant measure. We will extend this result in the next theorem.

Theorem 3.1. *Let $\sigma : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+^*$ be a bounded length substitution such that σ has a periodic point u and $M = M_\sigma$ is irreducible and aperiodic. If M satisfies*

$$\lim_{n \rightarrow +\infty} \sup_{i \in \mathbb{Z}_+} \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} = 0 \text{ for all } j \in \mathbb{Z}_+, \quad (3.1)$$

then the dynamical system (Ω_σ, S) has no finite invariant measure.

Remark 3.2. One natural question is if condition (3.1) can be replaced by the weaker condition

$$\lim_{n \rightarrow +\infty} \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} = 0 \text{ for all } i, j \in \mathbb{Z}_+. \quad (3.2)$$

This last condition is more natural and holds for a large class of substitutions σ such that M_σ is transient or null recurrent and σ has constant length, or M_σ is positive recurrent with left Perron eigenvector $l = (l_k)_{k \geq 0} \notin l^1$, see Lemma 3.4 at the end of this section and also Remark 3.3 just below.

Proof of Theorem 3.1. Assume without loss of generality that $u = u_0u_1\dots$ is a fixed point of σ . By (3.1), we have that for all $j \in \mathbb{Z}_+$

$$\lim_{n \rightarrow +\infty} \sup_{a \in A} \frac{|\sigma^n(a)|_j}{|\sigma^n(a)|} = 0. \quad (3.3)$$

Now, assume that (Ω_σ, S) has a finite invariant measure, then there exists a finite ergodic invariant measure μ . By Birkhoff's ergodic theorem, we deduce that for μ almost all $x \in \Omega_u$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{card}\{0 \leq k \leq N-1 : S^k(x) \in [j]\} = \mu[j] \quad \forall j \in \mathbb{Z}_+. \quad (3.4)$$

Now, let $x \in \Omega_\sigma$ satisfying (3.4) and $N \in \mathbb{N}$. Let $V = u_m \dots u_{m+N-1}$, $m \in \mathbb{N}$ be a prefix of x . The word V can be written as

$$V = v_0\sigma(v_1) \dots \sigma^{n-1}(v_{n-1})\sigma^n(v_n)\sigma^{n-1}(w_{n-1}) \dots \sigma(w_1)w_0, \quad (3.5)$$

where $n \geq 1$ is an integer and v_i , $i \in \{0, \dots, n\}$, w_j , $j \in \{0, \dots, n-1\}$, are elements of F_u possibly empty words of lengths smaller or equal to $K = \max\{|\sigma(b)|, b \in A\}$ and v_n is not empty. Equality (3.5) comes from the fact that since $u = \sigma(u)$, there exists $a \in A$ and $n \in \mathbb{N}$ such that V is a factor of $\sigma^{n+1}(a)$ and V is not a factor of $\sigma^n(a)$. Hence there exist v_0, w_0, V_1 in F_u such that

$$V = v_0\sigma(V_1)w_0$$

and $|v_0|, |w_0| \leq K$. We proceed analogously with V_1 , continuing by induction until the process stops and we obtain (3.5).

With our choice of x and its prefix V , from (3.4) and (3.5) we have that

$$\frac{1}{N} \text{card}\{0 \leq k \leq N-1, S^k(x) \in [j]\} = \frac{|V|_j}{|V|} = \frac{|\sigma^n(v_n)|_j + \sum_{k=0}^{n-1} (|\sigma^k(v_k)|_j + |\sigma^k(w_k)|_j)}{|\sigma^n(v_n)| + \sum_{k=0}^{n-1} (|\sigma^k(v_k)| + |\sigma^k(w_k)|)}.$$

By (3.3), we deduce that

$$\limsup_{k \rightarrow \infty} \left\{ \frac{|\sigma^k(v)|_j}{|\sigma^k(v)|}, v \in F_u, |v| \leq K \right\} = 0. \quad (3.6)$$

Using (3.6), and the Stolz-Cesaro Theorem, we deduce that

$$\lim_{n \rightarrow \infty} \frac{|\sigma^n(v_n)|_j + \sum_{k=0}^{n-1} (|\sigma^k(v_k)|_j + |\sigma^k(w_k)|_j)}{|\sigma^n(v_n)| + \sum_{k=0}^{n-1} (|\sigma^k(v_k)| + |\sigma^k(w_k)|)} = 0.$$

Therefore

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{card}\{0 \leq k \leq N-1, S^k(x) \in [j]\} = \mu[j] = 0.$$

Since j is arbitrary $\mu(\Omega) = 0$ which yields a contradiction. \square

Remark 3.3. It is important to notice that condition (3.1) may or may not hold on both the transient and the null recurrent cases. To see this we consider examples where M is a multiple of an irreducible stochastic matrix P . In this situation (3.1) is equivalent to

$$\lim_{n \rightarrow +\infty} \sup_{i \in A} P_{ij}^n = 0 \text{ for all } j \in A. \quad (3.7)$$

It is simple to find examples of stochastic matrices for which (3.7) does not hold. So we start with a first example that can be adapted to both transient and recurrent cases. Consider $A = \mathbb{Z}$ and set $P_{-n, -n-1} = q_n = 1 - P_{-n, -n+1}$ for $n \geq 1$, where $q_n \in (0, 1)$ and $\sum_{n=1}^{+\infty} q_n < \infty$. Also put $P_{0, -1} = P_{0, 1} = 1/2$ and $P_{m, -n} = 0$ for $m, n \geq 1$. No matter how we complete the definition of P to obtain a irreducible and aperiodic matrix which may be recurrent or transient, we have that

$$\sup_{a \in A} P_{a, 0}^n \geq P_{-n, 0}^n \geq \prod_{k=1}^{+\infty} (1 - q_k) > 0, \text{ for every } n \geq 1.$$

Thus (3.7) does not hold. However, since $\lim_{n \rightarrow \infty} q_n = 0$, there is no multiple of P which is a matrix M associated to a substitution. Thus we will provide another example.

Again we consider $A = \mathbb{Z}$ and set $P_{-2^n, 0} = 1/2 = P_{-2^n, -2^{n-1}}$ and $P_{-2^{n-j}, -2^{n-j-1}} = 1$ for $j = 1, \dots, 2^n - 1$ and $n \geq 1$. We can check that $P_{-2^{2^n-1}, 0} = 1/2$. Again, no matter how we complete the definition of P , which may be recurrent or transient, we have that

$$\limsup_{n \rightarrow \infty} \sup_{a \in A} P_{a, 0}^n \geq 1/2 > 0,$$

thus (3.7) does not hold. In this case, we could define $M_{-2^n, 0} = 1 = M_{-2^n, -2^{n-1}}$ and $M_{-2^{n-j}, -2^{n-j-1}} = 2$ and complete the definition for the other entries for M in order to have an irreducible and aperiodic matrix associated to a substitution of constant length equal to two. We have that $P = M/2$, thus (3.1) does not hold.

As a third example we consider P as the transition matrix of a simple random walk on \mathbb{Z} , i.e., we fix $p \in (0, 1)$ and set $P_{n, n+1} = p = 1 - P_{n, n-1}$ for every $n \in \mathbb{Z}$ (for basic properties of random walks, the reader can check [8]). Notice that this Markov chain is irreducible with period two which is null recurrent if $p = 1/2$ and transient otherwise. The stochastic matrix P is irreducible and we can use P^2 instead of P for an example with an aperiodic chain. A standard computation using the binomial distribution and Stirling formula shows that

$$\sup_{w \in \mathbb{Z}} P_{w, \bar{w}}^n = \sup_{w \in \mathbb{Z}} P_{0, \bar{w}-w}^n \leq \max_{0 \leq k \leq n} \binom{n}{k} p^k (1-p)^{n-k} = O(n^{-\frac{1}{2}}).$$

Thus (3.7) holds. Here, we also have $P = M/2$, where M is a substitution matrix of constant length equals to 2 defined as

$$M_{n, n+1} = M_{n, n-1} = 1 \quad \forall n \in \mathbb{Z}.$$

Thus M satisfies (3.1).

Question 3.1. Under the hypothesis of Theorem 3.1, is the dynamical system (Ω_σ, S) not minimal?

Question 3.2. Is the result of Theorem 3.1 still true if M_σ is transient, or recurrent with a left Perron eigenvector $l = (l_i)_{i \geq 0} \notin l^1$ and without the condition (3.1)? Even in a little less general setting, is the result of Theorem 3.1 still true if M_σ satisfies the weaker condition (3.2)?

We finish this section proving a result with conditions that imply condition (3.2).

Lemma 3.4. *Let $M = (M_{ij})_{i,j \in \mathbb{Z}_+}$ be a nonnegative, irreducible and aperiodic matrix with finite Perron value λ . If M is transient with constant line sums, or M is positive recurrent with a left Perron eigenvector $l = (l_k)_{k \geq 0} \notin l^1$, then*

$$\lim_{n \rightarrow +\infty} \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} = 0 \text{ for all } i, j \in \mathbb{Z}_+.$$

Proof. Assume that M is transient with constant line sums equal to L . Let $i, j \in \mathbb{Z}_+$. Since $\lambda \leq L$ and $\lim_{n \rightarrow +\infty} M_{ij}^n / \lambda^n = 0$, then

$$\frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} = \frac{M_{ij}^n}{L^n} \leq \frac{M_{ij}^n}{\lambda^n} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Now, let us suppose that M is positive recurrent and $l = (l_k)_{k \geq 0} \notin l^1$ is a left Perron eigenvector. Let $i, j \in \mathbb{Z}_+$. Since M is positive recurrent, we have by remark 2.8 that

$$\lim_{n \rightarrow \infty} \frac{M_{ik}^n}{\lambda^n} = cl_k \text{ for all } k \in \mathbb{Z},$$

where $c > 0$. Using Fatou Lemma for series and the fact $l = (l_k)_{k \geq 0} \notin l^1$, we deduce that

$$\lim_{n \rightarrow +\infty} \frac{\sum_{k=0}^{+\infty} M_{ik}^n}{M_{ij}^n} \geq \frac{\sum_{k=0}^{+\infty} l_k}{l_j} = +\infty$$

and we are done. \square

Question 3.3. 1. If M is transient with non constant line sums, Is

$$\lim_{n \rightarrow +\infty} \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} = 0 \text{ for all } i, j \in \mathbb{Z}_+?$$

Note that the answer is affirmative if

$$\liminf_{n \rightarrow +\infty} \frac{M^n \mathbf{1}}{\lambda^n} > 0,$$

which is simple to verify in the finite dimensional case from linear algebra arguments. It is also true to check in the infinite dimensional case when M is transient and has a right Perron eigenvector $r = (r_i)_{i \geq 0} \in l^\infty$ such that $\inf\{r_j, j \geq 0\} > 0$, since for all $j \geq 0$,

$$\frac{\inf_j r_j}{\sup_j r_j} \leq \frac{1}{\lambda^n} \sum_{k=0}^{+\infty} M_{ik}^n \leq \frac{\sup_j r_j}{\inf_j r_j}. \quad (3.8)$$

2. Assume that M is recurrent with a left Perron eigenvector $l = (l_k)_{k \geq 0} \in l^1$. Does there exist $i, j \in A$ such that

$$\lim_{n \rightarrow +\infty} \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} > 0?$$

Again, from (2) in Lemma 2.6, (2) in Lemma 2.10 and (3.8), it is simple to check that this holds when M is positive recurrent and has a right Perron eigenvector $r = (r_i)_{i \geq 0} \in l^\infty$ such that $\inf\{r_j, j \geq 0\} > 0$. In particular in the case of lines with constant sums.

3.2. A class of examples: Let $\sigma := \sigma_{a,b,c}$ defined by

$$\sigma(0) = 0^{a+b}1^c \text{ and } \sigma(n) = (n-1)^a n^b (n+1)^c \text{ for all } n \geq 1$$

where a, b, c are nonnegative integers such that $a > 0$, $c > 0$ and $i^k = ii \dots i$ (k times). The matrix M_σ is irreducible and aperiodic. We have

$$M_\sigma = \begin{bmatrix} a+b & c & 0 & 0 & 0 & 0 & 0 & \cdots \\ a & b & c & 0 & 0 & 0 & 0 & \cdots \\ 0 & a & b & c & 0 & 0 & 0 & \cdots \\ 0 & 0 & a & b & c & 0 & 0 & \cdots \\ 0 & 0 & 0 & a & b & c & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note that σ is a substitution of constant length $L = a + b + c$. The stochastic matrix $P = M_\sigma/L$ is the transition matrix of a homogeneous nearest-neighbor random walk in $\{0, 1, 2, \dots\}$ partially reflected at the boundary, see also the last example in Remark 3.3. It is well known, see [8], that the random walk is (in the probabilistic sense) positive recurrent if $c < a$, null recurrent if $c = a$ and transient if $c > a$. The difference for the matrix theoretical definition is that we also have null recurrence in the case $c > a$, see also [11, Example 7.1.28] for the case $b = 0$ and $a = c$.

Proposition 3.5. *The following properties hold:*

- If $c < a$, then M_σ is positive recurrent.
- If $c \geq a$, then M_σ is null recurrent and (Ω_σ, S) has no finite invariant measure.

Proof. For the cases $c < a$ and $c = a$, we have $\lambda_P = 1$, thus $\lambda_{M_\sigma} = L$. From the probabilistic results on transience/recurrence of random walks, we have that M_σ is positive recurrent for $c < a$ and null recurrent for $c = a$.

Before we deal with the case $c > a$, let us point out that we can prove the result in the cases $c < a$ and $c = a$ by directly computing the Perron eigenvectors.

Let λ be the Perron value of M_σ , then by Lemma 2.10, we have $\lambda \leq L = a + b + c$. Let $l = (l_i)_{i \geq 0}$ be a left eigenvector of M_σ associated to L . A simple computations implies that $l_1 = \frac{c}{a}l_0$ and

$$cl_n + al_{n+2} = (a+c)l_{n+1} \text{ for all } n \geq 0.$$

Hence

$$l_n = \left(\frac{c}{a}\right)^n l_0 \text{ for all } n \geq 1.$$

Assume that M_σ is positive recurrent, then by Lemma 2.10, we deduce that $\lambda = L$. Thus $l \in l^1$ (since a right Perron eigenvector of M_σ has constant entries) and we deduce that $c < a$.

Now assume that $c \leq a$. If $\lambda = L$, then l is a left Perron eigenvector, hence M_σ is positive recurrent if $c < a$ and null recurrent if $c = a$. Now suppose that $\lambda < L$ and let $u = (u_i)_{i \geq 0}$ be a non zero nonnegative left Perron sub-invariant eigenvector of M_σ associated to λ . Thus $uM < LM$. Hence

$$u_{n+1} \leq \frac{c}{a}u_n \text{ for all } n \geq 0$$

and there exists a real number $s > 0$ and an integer $k \geq 1$ such that $u_k = \frac{c}{a}u_{k-1} - s$. Since $cu_{k-1} + au_{k+1} \leq (a+c)u_k$, we deduce that $u_{k+1} \leq \frac{c}{a}u_k - s$. Thus

$$u_{n+1} \leq \frac{c}{a}u_n - s \text{ for all } n \geq k. \quad (3.9)$$

Therefore

$$u_n \leq \left(\frac{c}{a}\right)^{n-k}u_k - s \text{ for all } n \geq k+1.$$

If $c < a$, we deduce that there exists a positive integer N such that $u_n < 0$ for all integer $n \geq N$. This is absurd, then $u = u_0l$. Therefore $\lambda = L$ and hence M_σ is positive recurrent.

If $c = a$, we deduce by (3.9) that

$$u_n \leq u_k - (n-k)s \text{ for all } n \geq k+1.$$

Then $\lambda = L$ and M_σ is null recurrent.

Now consider the case $c > a$. We will consider a probabilistic approach to show that $\lambda_P < 1$ and that $\overline{P}_{00}(1/\lambda_P) = \infty$, this implies null recurrence. Let $(X_n)_{n \geq 0}$ be a Markov chain with transition matrix P and \mathbb{P}^x the distribution of $(X_n)_{n \geq 0}$, when $X_0 = x$ for $x \in \mathbb{Z}_+$. Set $p = c/(a+c)$, which is the conditional probability that the random walk jumps to the right, when it necessarily leaves its current position and this is not 0, i.e.,

$$p = \mathbb{P}^x(X_{n+1} = X_n + 1 | X_{n+1} \neq X_n) \quad \forall x \neq 0.$$

We want to estimate P_{00}^n , i.e., the probability that the random walk is visiting state 0 at time n given that it has also started at 0. For this last event to happen, necessarily we must have a number of jumps to the right equal to the number of jumps to the left. Here we need to important observations:

- (i) Note that $\#\{1 \leq j \leq n : X_{j+1} \neq X_j\}$ counts the total number of jumps to the right or to the left. There exist strictly positive constants $c, \delta \in (0, 1)$ such that

$$\mathbb{P}^0(\#\{1 \leq j \leq n : X_{j+1} \neq X_j\} \geq cn) > 1 - (\delta q)^n.$$

- (ii) In $2k$ transitions to the left or to the right, the number of transitions to the right is distributed as a binomial random variable with parameters $2k$ and p . Thus the probability of having an equal number of jumps to the left or to the right is

$$\mathbb{P}(\text{Bin}(2k, p) = k) = \binom{2k}{k} p^k (1-p)^k \approx \frac{C}{\sqrt{k}} (4p(1-p))^k.$$

(the approximation could be appropriately described using Stirling's formula). Note that $q = 4p(1-p) < 1$.

Using (i) and (ii) we are able to show that P_{00}^n is of order $O(q^n/\sqrt{n})$. This implies that $\lambda_P = q$ and $\overline{P}_{00}(1/\lambda_P) = \overline{P}_{00}(1/q) = \infty$. Therefore P and M_σ are null recurrent matrices.

It is worth mentioning that M_σ satisfies (3.1) for every $a \leq c$ and b . This follows as in the last example in Remark 3.3 in the case $a = c$ and from computation as in the proof of Proposition 3.5. Indeed one can prove that $\sup_{i \in \mathbb{Z}_+} P_{i,j}^n$ is of order $O(1/\sqrt{n})$, which implies that M_σ satisfies (3.1). This can also be proved using the Local Central Limit Theorem for simple random walks [12, Theorem 1.2.1]. \square

Remark 3.6. It is worth mentioning that apparently small modifications on the matrix can completely change its behavior. For instance, consider the case $b = 0$ and $a = c$ which implies that M_σ is null recurrent. Instead of $\sigma(0) = 0^a 1^c$, put $\sigma(0) = 1^c$, then, from [11, (i) in example 7.1.29], we have that M_σ is transient. For the case $b > 0$, $a \leq c$ and $\sigma(0) = 1^c$, we also have transience as a consequence of our Proposition 3.5 and [11, Lemma 7.1.23].

Remark 3.7. We consider the substitution σ of [9] defined on $A = \mathbb{Z}$ by $\sigma(n) = (n - 1)nn(n + 1)$. The associated matrix is null recurrent and satisfies condition (3.1). Hence by using Theorem 3.1, we deduce that the dynamical system (Ω_σ, S) associated to σ has non finite invariant measure.

Let $\sigma := \sigma_{a_n, b_n, c_n}$ defined by

$$\sigma(0) = 0^{a_0+b_0}1^{c_0} \text{ and } \sigma(n) = (n-1)^{a_n}n^{b_n}(n+1)^{c_n} \text{ for all } n \geq 1$$

where a_n, b_n, c_n are nonnegative integers such that $a_n > 0, c_n > 0$ for every $n \geq 1$ and $L = \sup\{a_n + b_n + c_n : n \geq 1\} < \infty$. The matrix M_σ is irreducible and aperiodic with bounded length L and can be represented as

$$M_\sigma = \begin{bmatrix} a_0 + b_0 & c_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_1 & b_1 & c_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_2 & b_2 & c_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & a_3 & b_3 & c_3 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_4 & b_4 & c_4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We will see in Proposition 3.8 below that (Ω_σ, S) is not minimal for these substitutions. We does not discuss the transience/recurrence in this general case, but we discuss an example. Consider $(a_n, b_n, c_n) = (2, 1, 1)$ for n even and $(a_n, b_n, c_n) = (1, 1, 1)$ otherwise. Our first step is to compute the Perron value λ . For this we will estimate $(M_\sigma^n)_{0,0}$. Consider a matrix \hat{M}_σ^n which has the form above but with $(a_n, b_n, c_n) = (2, 1, 1)$ for every n . The Perron eigenvalue of \hat{M}_σ^n is 4, since it has constant row sums equal to 4. Now for each path of length n leaving and returning to n , we will have the number of jumps to the left equal to the number of jumps to the right. So for a total of $2m \leq n$ jumps with m jumps to the left and m jumps to right (neglecting jumps from 0 to 1 and jumps from an state to itself), $m/2$ jumps to the left are made from an odd position and these jumps contributes with a factor of $(\sqrt[4]{2})^{2m} = 2^{\frac{m}{2}}$ to the product of weights $(\hat{M}_\sigma^n)_{0,0}$. This shows that

$$\left(\frac{\sqrt[4]{2}}{4}\right)^n (M_\sigma^n)_{0,0} \geq \frac{(\hat{M}_\sigma^n)_{0,0}}{4^n},$$

which implies $\lambda \geq 4/\sqrt[4]{2}$. On the other hand, $(M_\sigma^n)_{0,0} \leq (\hat{M}_\sigma^n)_{0,0}$ and $\lambda \leq 4$. Thus $\lambda \in [4/\sqrt[4]{2}, 4]$. With this bound on λ we can show that M_σ is positive recurrence. For this we follow the computation in [11, Example 7.1.29 (iii)] to obtain that

$$\lim_{n \rightarrow \infty} \sqrt[2n]{\ell_{00}(2n)} = 2^{\frac{5}{4}} < \lambda,$$

then apply [11, Lemma 7.1.25] to conclude.

Proposition 3.8. *Let (Ω_σ, S) be the shift dynamical system associated to σ_{a_n, b_n, c_n} , then it is not minimal.*

Proof. For all $n \geq 2$, we have

$$\sigma^{k-1}(k) = (\sigma^{k-2}(k-1))^{a_n} (\sigma^{k-2}(k))^{b_n} (\sigma^{k-2}(k+1))^{c_n}.$$

Hence the infinite word w beginning with $\sigma^{k-1}(k)$ for all $k \geq 2$ is well defined and belongs to Ω_σ . Moreover the letter 0 does not occur in w . Thus the orbit of w does not visit the cylinder $[0]$, hence (Ω_σ, S) is not minimal. \square

Remark 3.9. The last proposition gives examples of positive or null recurrent, aperiodic irreducible substitutions such that its shift dynamical systems are not minimal. The first example was given by Ferenczi in [9] by considering $\sigma(n) = (n-1)nn(n+1)$, $n \in \mathbb{Z}$.

We will see in Theorem 3.33 that given a substitution σ on $A = \mathbb{Z}_+$, not necessarily with constant length, such that σ has a periodic point u and M_σ is irreducible, aperiodic and has a scrambling positive power, then (Ω_σ, S) is minimal. Observe that the matrices associated to substitutions σ_{a_n, b_n, c_n} have not a scrambling power, since for any positive integer k , there is no letter occurring both in the words $\sigma_{a_n, b_n, c_n}^k(k)$ and $\sigma_{a_n, b_n, c_n}^k(4k)$.

To end this section we describe an interesting substitution whose matrix is transient. The construction of the matrix is based on multidimensional random walks in dimension greater or equal to 3. Thus we set $A = \mathbb{Z}^d$, let $\{e_j : 1 \leq j \leq d\}$ and define the substitution

$$\sigma(x) = (x + e_1)(x - e_1)(x + e_2)(x - e_2) \dots (x + e_d)(x - e_d).$$

We have that M_σ is a matrix of length $2d$. The stochastic matrix $P = M_\sigma/2d$ is transient with $\lambda_P = 1$. Indeed from classical results in probability theory one has that $P_{00}^n \sim O(n^{-d/2})$ and $\bar{P}_{00}(1) < \infty$. Therefore, M_σ is a transient matrix with $\lambda_{M_\sigma} = 2d$. Using again the Local Central Limit Theorem [12, Theorem 1.2.1], we have that M_σ also satisfies (3.1).

3.3. Shift invariant measures and unique ergodicity. In this subsection, we prove the following results.

Theorem 3.10. *Let σ be a bounded length substitution on $A = \mathbb{Z}_+$ such that M_σ is irreducible, aperiodic, positive recurrent, then the dynamical system (Ω_σ, S) has a shift invariant measure μ which is finite if and only if any left Perron eigenvector l belongs to l^1 .*

Remark 3.11. Theorem 3.10 improves [1, Theorem 7.6], where it is assumed the additional hypothesis that σ is bounded size left determined substitution.

Theorem 3.12. *Let σ be a constant length substitution on $A = \mathbb{Z}_+$ such that σ has a periodic point u and M_σ is irreducible and aperiodic. If there exists a positive integer n such that M_σ^n is scrambling, then there exists a unique probability shift invariant measure of (Ω_σ, S) .*

Remark 3.13. The same proof of Theorem 3.12 will show that if σ is a constant length substitution on $A = \mathbb{Z}_+$ without periodic point such that M_σ is irreducible, aperiodic and M_σ^n is scrambling positive integer n , then there exists a unique probability shift invariant measure of (Ω_σ, S) .

Before proving Theorems 3.10 and 3.12 we need to introduce some notation and state some preliminary results.

Let $\sigma : A \rightarrow A^*$ be a bounded length substitution, not necessarily with constant length. Let $t \geq 2$ be an integer and A_t be the set of finite words of length t that occur in u . Now, consider a substitution σ_t on the alphabet A_t defined in the following way: If $w = w_0 \dots w_{t-1} \in A_t$ and $\sigma(w) = y_0 \dots y_{|\sigma(w_0)|-1} y_{|\sigma(w_0)|} \dots y_{|\sigma(w)|-1}$, then

$$\sigma_t(w) = (y_0 \dots y_{t-1})(y_1 \dots y_t) \dots (y_{|\sigma(w_0)|-1} \dots y_{|\sigma(w_0)|+t-2}). \quad (3.10)$$

Considering that $|\sigma_t(w)|$ counts letters in A_t (not in A), note that

$$|\sigma_t(w_0 \dots w_{t-1})| = |\sigma(w_0)|, \quad (3.11)$$

and for all $i_1 \dots i_t \in A_t$, we have

$$|\sigma(w_0)|_{i_1 \dots i_t} \leq |\sigma_t(w_0 \dots w_{t-1})|_{i_1 \dots i_t} \leq |\sigma(w_0)|_{i_1 \dots i_t} + t. \quad (3.12)$$

We extend σ_t by concatenation to A_t^* and to $A_t^{\mathbb{Z}_+}$. The substitution σ_t was defined in [21] (in the case of substitutions on finite alphabets).

For example, for $A = \{0, 1\}$ and $\sigma(0) = 01$, $\sigma(1) = 0$. We have $A_2 = \{00, 01, 10\}$ and

$$\sigma_2(00) = (01)(10), \sigma_2(01) = (01)(10), \sigma_2(10) = (00).$$

If $A = \mathbb{Z}_+$ and $\tau(n) = 0(n+1)$ for all $n \in A$, then $A_2 = \{0n, n0, n \geq 1\}$ and

$$\tau_2(0n) = (01)(10), \tau_2(n0) = (0(n+1))((n+1)0) \text{ for all } n \geq 1.$$

Lemma 3.14. *The following results hold:*

- (1) For all integers $n \geq 1$ and $t \geq 2$, we have $(\sigma^n)_t = (\sigma_t)^n$.
- (2) Let $u = u_0 u_1 \dots$ be a periodic point of σ , then for all integer $t \geq 2$, the infinite word $(u_0 \dots u_{t-1}).(u_1 \dots u_t) \dots (u_i \dots u_{t+i-1}) \dots$ is a periodic point (with the same period) of σ_t .
- (3) If M_σ is irreducible and aperiodic, then so is M_{σ_t} for all integers $t \geq 2$.

Proof. The proof is analogous to that for the case of finite alphabet, which is given in [21, pages 138-139]. \square

Lemma 3.15. *Let $A = \mathbb{Z}_+$ and $\sigma : A \rightarrow A^*$ be a bounded length substitution such that M_σ irreducible and aperiodic with Perron value λ , then for all integer $t \geq 2$ the matrix $M_t = M_{\sigma_t}$ associated to σ_t also has Perron value λ . Moreover if M_σ is positive recurrent (resp. null recurrent, transient), then M_{σ_t} is also positive recurrent (resp. null recurrent, transient).*

Proof. Let $t \geq 2$ be an integer and denote by λ_t the Perron value of M_t . First observe that by (3) in Lemma 3.14, M_t is irreducible and aperiodic. For $i_1 \dots i_t, j_1 \dots j_t \in A_t$ we have

$$|\sigma_t^n(i_1 \dots i_t)|_{j_1 \dots j_t} \leq |\sigma^n(i_1)|_{j_1 \dots j_t} + t \leq |\sigma^n(i_1)|_{j_1} + t.$$

Hence

$$(M_t^n)_{i_1 \dots i_t, j_1 \dots j_t} \leq (M^n)_{i_1, j_1} + t, \text{ for all } n \in \mathbb{N}. \quad (3.13)$$

We deduce by (2.2) that $1 \leq \lambda_t \leq \lambda$.

On the other hand, let $k, m \in \mathbb{N}$ such that $j_1 \dots j_t$ is a factor of $\sigma^m(k)$. Hence

$$|\sigma^{n+m}(i_1)|_{j_1 \dots j_t} \geq |\sigma^n(i_1)|_k.$$

Thus for all $n \in \mathbb{N}$, we have $|\sigma_t^{n+m}(i_1 \dots i_t)|_{j_1 \dots j_t} \geq |\sigma^n(i_1)|_k$. Therefore

$$(M_t^{n+m})_{i_1 \dots i_t, j_1 \dots j_t} \geq (M^n)_{i_1, k} \text{ for all } n \in \mathbb{N}. \quad (3.14)$$

Thus $\lambda_t \geq \lambda$ and hence $\lambda_t = \lambda$.

Assume that M_σ is positive recurrent. By (3.14), we have

$$\liminf_{n \rightarrow +\infty} \frac{(M_t^{n+m})_{i_1 \dots i_t, j_1 \dots j_t}}{\lambda^{n+m}} \geq \lambda^{-m} \lim_{n \rightarrow +\infty} \frac{(M^n)_{i_1, k}}{\lambda^n}. \quad (3.15)$$

Hence by (3.15) and Lemma 2.6, we deduce that $\lim_{n \rightarrow +\infty} \frac{(M_t^n)_{i_1 \dots i_t, j_1 \dots j_t}}{\lambda_t^n} > 0$. Thus M_{σ_t} is positive recurrent.

Now suppose that M_σ is null recurrent, then we have by (3.14) that

$$\sum_{n=0}^{+\infty} \frac{(M_t^n)_{i_1 \dots i_t, j_1 \dots j_t}}{\lambda^n} = +\infty. \quad (3.16)$$

Hence by (3.13), we deduce that

$$\lim_{n \rightarrow \infty} \frac{(M_t^n)_{i_1 \dots i_t, j_1 \dots j_t}}{\lambda^n} = 0. \quad (3.17)$$

By (3.16) and (3.17), we deduce that M_{σ_t} is null recurrent.

Finally if M_σ is transient, we deduce by (3.13) that

$$\sum_{n=0}^{+\infty} \frac{(M_t^n)_{i_1 \dots i_t, j_1 \dots j_t}}{\lambda^n} < +\infty.$$

Hence M_{σ_t} is transient. \square

Before proving Theorem 3.10, we need the following Lemma.

Lemma 3.16. *Let σ be a bounded length substitution on $A = \mathbb{Z}_+$ such that M_σ is irreducible, aperiodic, recurrent and has finite Perron value λ . Let $r = (r_i)_{i \geq 0}$ be a right Perron eigenvector of M_σ . For all integer $t \geq 2$, Let $r^{(t)} = (r_I)_{I \in A_t}$ be an infinite vector defined by*

$$r_I = r_{i_0} \text{ for all } I = i_0 \dots i_{t-1} \in A_t,$$

then $r^{(t)}$ is a right Perron eigenvector of $M_t = M_{\sigma_t}$ associated to λ .

Proof. Let $I = i_0 \dots i_{t-1} \in A_t$, we have

$$(M_t r^{(t)})_I = \sum_{J=j_0 \dots j_{t-1} \in A_t} |\sigma_t(I)|_J r_{j_0} = \sum_{j_0 \in A} r_{j_0} \sum_{J^*=j_1 \dots j_{t-1}, j_0 J^* \in A_t} |\sigma_t(I)|_{j_0 J^*}.$$

On the other hand, for all $j_0 \in A$, we have

$$\sum_{J^*=j_1 \dots j_{t-1}, j_0 J^* \in A_t} |\sigma_t(I)|_{j_0 J^*} \leq |\sigma(i_0)|_{j_0} = M_{i_0 j_0}.$$

Thus

$$(M_t r^{(t)})_I \leq \sum_{j_0 \in A} M_{i_0 j_0} r_{j_0} = \lambda r_{i_0} = \lambda (r^{(t)})_I.$$

Since M_t is an aperiodic, irreducible and recurrent matrix, Lemma 2.9 implies that $r^{(t)}$ is a right eigenvector of M_t associated to λ . \square

Proof of Theorem 3.10. Let $u = u_0 u_1 \dots = \sigma(u)$ be an element of Ω_σ . For $j \in A$ set

$$\mu[j] := \lim_{n \rightarrow \infty} \frac{|\sigma^n(u_0)|_j}{\lambda^n} = \lim_{n \rightarrow \infty} \frac{M_{u_0, j}^n}{\lambda^n}.$$

The last limit exists since M_σ is positive recurrent with Perron eigenvalue λ . For $t \geq 2$ integer and $I_t = i_1 \dots i_t \in A_t$ set

$$\mu[i_1 \dots i_t] = \lim_{n \rightarrow \infty} \frac{|\sigma^n(u_0)|_{i_1 \dots i_t}}{\lambda^n}. \quad (3.18)$$

Applying (3.12) for σ^n in place of σ and the fact that $\lambda > 1$, we deduce that

$$\mu[i_1 \dots i_t] = \lim_{n \rightarrow \infty} \frac{|\sigma_t^n(u_0 \dots u_{t-1})|_{i_1 \dots i_t}}{\lambda^n} = \lim_{n \rightarrow \infty} \frac{(M_t^n)_{U_t, I_t}}{\lambda^n},$$

where $U_t = u_0 \dots u_{t-1}$ and $I_t = i_1 \dots i_t$. Observe that $\lim_{n \rightarrow \infty} \frac{(M_t^n)_{U_t, I_t}}{\lambda^n}$ exists since M_t is positive recurrent with Perron value $\lambda_t = \lambda$.

By Kolmogorov consistency Theorem, there exists a unique measure μ with cylinder specification (3.18) if for every integer $t \geq 1$ and $I = i_1 \dots i_t \in A_t$ we have

$$\mu[I] = \sum_{b \in A, Ib \in A_{t+1}} \mu[Ib]. \quad (3.19)$$

and

$$\mu[I] = \sum_{a \in A, aI} \mu[aI]. \quad (3.20)$$

For the proof of (3.19), let $l = (l_i)_{i \geq 0}$ and $r = (r_i)_{i \geq 0}$ be respectively left and right Perron eigenvectors of M such that the scalar product $l \cdot r = 1$. For all $t \geq 2$, let $l^{(t)} = (l_I)_{I \in A_t}$ and $r^{(t)} = (r_I)_{I \in A_t}$ be left and right Perron eigenvectors of M_t such that

$$r_{i_1 \dots i_t} = r_{i_1} \text{ for all } i_1 \dots i_t \in A_t \text{ and } l^{(t)} \cdot r^{(t)} = 1.$$

We could choose $r_{i_1 \dots i_t} = r_{i_1}$ because of Lemma 3.16.

For all $I = i_1 \dots i_t$, $t \geq 2$, we have by (2.4) that

$$\mu[I] = r_{U_t} l_I = r_{u_0} l_I.$$

Hence (3.19) is equivalent to

$$l_I = \sum_{b \in A, Ib \in A_{t+1}} l_{Ib}. \quad (3.21)$$

On the other hand, for all $I = i_1 \dots i_t \in A_t$, we have by Fatou's Lemma that

$$\sum_{b \in A, Ib \in A_{t+1}} \mu[Ib] \leq \lim_{n \rightarrow \infty} \frac{1}{\lambda^n} \sum_{b \in A, Ib \in A_{t+1}} |\sigma_{t+1}^n(u_0 \dots u_t)|_{Ib} = \lim_{n \rightarrow \infty} \frac{1}{\lambda^n} |\sigma_t^n(u_0 \dots u_{t-1})|_I.$$

Hence

$$\sum_{b \in A, Ib \in A_{t+1}} \mu[Ib] \leq \mu[I],$$

i.e.,

$$\sum_{b \in A, Ib \in A_{t+1}} l_{Ib} \leq l_I. \quad (3.22)$$

Since $\sum_{I \in A_t} r_I l_I = \sum_{J \in A_{t+1}} r_J l_J = 1$ and $r(Ib) = r(I)$ for all $I \in A_t$ and $Ib \in A_{t+1}$, we deduce that

$$\sum_{I \in A_t} r_I l_I = \sum_{I \in A_t} r_I \sum_{b \in A, Ib \in A_{t+1}} l_{Ib} = 1.$$

Using this last equality and (3.22), we obtain (3.21) and hence we get (3.19).

Analogously, (3.20) is equivalent to

$$l_I = \sum_{a \in A, aI \in A_{t+1}} l_{aI}. \quad (3.23)$$

Using Fatou's Lemma, we have for all $I \in A_t$,

$$\sum_{a \in A, aI \in A_{t+1}} \mu[aI] \leq \lim_{n \rightarrow \infty} \frac{1}{\lambda^n} \sum_{a \in A, aI \in A_{t+1}} |\sigma_{t+1}^n(u_0 \dots u_t)|_{aI}.$$

Note that

$$\beta_I := \sum_{a \in A, aI \in A_{t+1}} |\sigma_{t+1}^n(u_0 \dots u_t)|_{aI} - |\sigma_t^n(u_0 \dots u_{t-1})|_I \in \{-1, 0, 1\},$$

indeed $\beta_I = -1$ if the first letter of $\sigma_{t+1}^n(u_0 \dots u_t)$ begins with I and the last letter of $\sigma_{t+1}^n(u_0 \dots u_t)$ does not end with I . The number $\beta_I = 1$ if the first letter of $\sigma_{t+1}^n(u_0 \dots u_t)$ does not begin with I and the last letter of $\sigma_{t+1}^n(u_0 \dots u_t)$ ends with I . In the complementary case, we have $\beta_I = 0$.

Since $\lambda > 1$, we deduce that

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda^n} \sum_{a \in A, aI \in A_{t+1}} |\sigma_{t+1}^n(u_0 \dots u_t)|_{aI} = \lim_{n \rightarrow \infty} \frac{1}{\lambda^n} |\sigma_t^n(u_0 \dots u_{t-1})|_I.$$

Hence

$$\sum_{a \in A, Ia \in A_{t+1}} \mu[aI] \leq \mu(I),$$

i.e.,

$$\sum_{a \in A, Ia \in A_{t+1}} l_{aI} \leq l_I. \quad (3.24)$$

On the other hand, by (3.21), we have

$$\sum_{J \in A_{t+1}} l_J = \sum_{I \in A_t} \left(\sum_{b \in A, Ib \in A_{t+1}} l_{Ib} \right) = \sum_{I \in A_t} l_I.$$

Thus

$$\sum_{I \in A_t} \left(\sum_{a \in A, aI \in A_{t+1}} l_{aI} \right) = \sum_{I \in A_t} l_I.$$

By using (3.24), we obtain (3.20). Hence μ is an invariant measure for (Ω_u, S) . \square

3.3.1. Constant length substitution and unique ergodicity. Let σ be a substitution on $A = \mathbb{Z}_+$ with constant length $L > 0$. By (2.8), the stochastic matrix M_σ/L is strongly ergodic if and only if there exists a positive power of M_σ which is scrambling.

As an example, the dyadic substitution σ defined by $\sigma(n) = 0(n+1)$ has a strongly ergodic matrix $M_\sigma/2$ since the matrix M_σ is scrambling.

Another way to see that $M_\sigma/2$ is strongly ergodic comes from the fact that for all $i, j \in \mathbb{Z}_+$

$$\lim_{n \rightarrow \infty} \sup_{i \in \mathbb{Z}_+} \sum_{j=0}^{+\infty} \left| \frac{|\sigma^n(i)|_j}{|\sigma^n(i)|} - \frac{1}{2^{j+1}} \right| = 0. \quad (3.25)$$

Indeed, for all integers $n \in \mathbb{N}, i, j \in \mathbb{Z}_+$, we have

$$|\sigma^n(i)|_j = |\sigma^{n-1}(i)|_{j-1} = 2^{n-j-1} \text{ for all } 0 \leq j < n$$

and

$$|\sigma^n(i)|_j = |\sigma(i)|_{j+1-n} \text{ for all } j \geq n,$$

Thus for all $j \geq n$,

$$|\sigma^n(i)|_j = 1 \text{ if } j = i + n \text{ and } 0 \text{ otherwise.}$$

Hence

$$\sum_{j=0}^{+\infty} \left| \frac{|\sigma^n(i)|_j}{|\sigma^n(i)|} - \frac{1}{2^{j+1}} \right| = \sum_{j=n}^{+\infty} \frac{1}{2^{j+1}} + \left(\frac{1}{2^n} - \frac{1}{2^{i+n}} \right) \text{ for all } i \geq 0,$$

which implies that

$$\sup_{i \in \mathbb{Z}_+} \sum_{j=0}^{+\infty} \left| \frac{|\sigma^n(i)|_j}{|\sigma^n(i)|} - \frac{1}{2^{j+1}} \right| = \frac{1}{2^{n-1}}$$

and we obtain (3.25).

Another example are the substitutions $\sigma_{a,b,c}$, $a, b, c \in \mathbb{N}$ and $a > c$. We have seen in Proposition that for all positive integers a, b, c with $a > c$, the matrix $M_{\sigma_{a,b,c}}$ is positive recurrent. Furthermore the stochastic matrix $\frac{M_{\sigma_{a,b,c}}}{a+b+c}$ is not strongly ergodic since $M_{\sigma_{a,b,c}}$ does not have a scrambling power (see Remark 3.9).

Remark 3.17. Let σ be a substitution on $A = \mathbb{Z}_+$ with constant length $L > 0$ and a periodic point u such that the stochastic matrix M_σ/L is strongly ergodic, then M_σ is positive recurrent and, by Theorems 3.10, Ω_σ has a finite invariant measure.

Lemma 3.18. *Assume that σ is a constant length substitution on $A = \mathbb{Z}_+$ and M_σ is irreducible and aperiodic. If M_σ is strongly ergodic, then for all integer $t \geq 2$, M_{σ_t} is also strongly ergodic.*

Proof. Let $L > 0$ be the length of σ . Fix and interger $t \geq 2$ and $i_1 \dots i_t, k_1 \dots k_t \in A_t$. Since M_σ is strongly ergodic, then (2.5) implies that there exists an integer $n > 0$ and $j_1 \in \mathbb{N}$ such that

$$j_1 \text{ occurs in } \sigma^n(i_1) \text{ and } \sigma^n(k_1).$$

Let $m > 0$ be an integer such that

$$m = \left\lceil \frac{\ln(2t)}{\ln L} \right\rceil + 1,$$

then

$$\sigma^m(j_1) = a_1 \dots a_s \text{ where } s \geq 2t.$$

Hence $a_1 \dots a_{2t}$ occurs in $\sigma^{n+m}(i_1)$ and $\sigma^{n+m}(k_1)$. Thus

$$a_1 \dots a_t \text{ occurs in } \sigma_t^{n+m}(i_1 \dots i_t) \text{ and } \sigma_t^{n+m}(k_1 \dots k_t)$$

and we are done again by (2.5). \square

Proof of Theorem 3.12. Assume without loss of generality that σ has a fixed point $u = \sigma(u) = u_0 u_1 \dots = \lim_{n \rightarrow \infty} \sigma^n(u_0)$ and let $L > 0$ be the length of σ . Recall that for all $i, j \in \mathbb{Z}_+$ and $n \geq 0$

$$\frac{|\sigma^n(i)|_j}{|\sigma^n(i)|} = \frac{M_{ij}^n}{L^n}.$$

Since M_σ is irreducible, aperiodic and strongly ergodic, we have that $\lambda = L$ and

$$\lim_{n \rightarrow +\infty} \frac{|\sigma^n(i)|_j}{|\sigma^n(i)|} = v_j > 0$$

independently of i . Moreover strong ergodicity implies that there exist $c > 0$ and $0 < \beta < 1$ such that

$$\sup_{i \geq 0} \left| \frac{|\sigma^n(i)|_j}{|\sigma^n(i)|} - v_j \right| \leq c\beta^n \text{ for all } n \geq 0.$$

To compute, $\lim_{n \rightarrow +\infty} \frac{|\sigma^n(i)|_w}{|\sigma^n(i)|}$ where w is a word of length $t \geq 2$, we will consider a substitution σ_t on the alphabet A_t . From Lemma 3.14 and Lemma 3.18 we deduce that M_{σ_t} is irreducible, aperiodic and strongly ergodic. Thus if $w = w_0 \dots w_{t-1}$ and $B = b_0 \dots b_{t-1} \in A_t$, then there exists $d_B > 0$ such that $\lim_{n \rightarrow +\infty} \frac{|\sigma_t^n(w)|_B}{|\sigma_t^n(w)|} = d_B$ independently of w . Now, since

$$|\sigma_t^n(w)| = |\sigma^n(w_0)| = L^n \text{ and } |\sigma^n(w_0)|_B \leq |\sigma_t^n(w)|_B \leq |\sigma^n(w_0)|_B + t,$$

we obtain

$$\lim_{n \rightarrow +\infty} \frac{|\sigma^n(w_0)|_z}{|\sigma^n(w_0)|} = d_B. \quad (3.26)$$

Moreover there exists $c_t > 0$ and $0 < \beta_t < 1$ such that

$$\sup_{w_0 \geq 0} \left| \frac{|\sigma^n(w_0)|_B}{|\sigma^n(w_0)|} - d_B \right| \leq c_t \beta_t^n \text{ for all } n \geq 0. \quad (3.27)$$

To finish the proof, we have to show the following claim:

Claim: Let $t \geq 2$ and $B = b_1 \dots b_t \in F_u$, then $\lim_{N \rightarrow \infty} \frac{1}{N} |u_k \dots u_{k+N-1}|_B = d_B$ uniformly on k .

The proof is the same as that for finite alphabet and σ primitive given in [21, Theorem 4.6, page 141-142]. Indeed, let $V_k = u_k \dots u_{k+N-1} \in F_u$, $k \in \mathbb{Z}_+$, $N \in \mathbb{N}$. As cited in the proof of Theorem 3.1, the word V_k can be written as

$$V_k = v_0 \sigma(v_1) \dots \sigma^{n-1}(v_{n-1}) \sigma^n(v_n) \sigma^{n-1}(w_{n-1}) \dots \sigma(w_1) w_0 \quad (3.28)$$

where $n \geq 0$ is an integer and v_i , $i \in \{0, \dots, n\}$, w_j , $j \in \{0, \dots, n-1\}$, are elements of F_u possibly empty words of Lengths $\leq L$ and v_n is not empty.

Since $L \geq 2$, there exist $C > 0$ and $1 < \tau < L$ such that $C\tau^n \geq c_t(2n-1)((\max(\beta_t L, 1))^n)$ for every $n \geq 1$. Now for $V_k = u_k \dots u_{k+N-1}$, $k \in \mathbb{Z}_+$, for $N \in \mathbb{N}$, and $B \in F_u$ such that $|B| < N$, we use (3.28) and (3.27) to obtain that

$$\begin{aligned} ||V_k|_B - d_B N| &\leq \sum_{j=0}^n ||\sigma^j(v_j)|_B - d_B |\sigma^j(v_j)|| + \sum_{j=0}^{n-1} ||\sigma^j(w_j)|_B - d_B |\sigma^j(w_j)|| \\ &\leq \sum_{j=0}^n c_t(\beta_t L)^j + \sum_{j=0}^{n-1} c_t(\beta_t L)^j \leq c_t(2n-1)((\max(\beta_t L, 1))^n) \leq C\tau^n. \end{aligned} \quad (3.29)$$

Since $N \geq |\sigma^n(v_n)| = L^n$, we obtain that

$$\sup_{k \geq 0} \left| \frac{|V_k|_B}{N} - d_B \right| \leq C\gamma^n, \quad (3.30)$$

for some $\gamma < 1$ and we obtain the claim. \square

3.3.2. Strong ergodicity for non constant bounded length substitution.

Definition 3.19. Let $M = (M_{ij})_{i,j \geq 0}$ be a nonnegative matrix such that M is irreducible, aperiodic and positive recurrent with finite Perron value $\lambda > 0$. Let $P = (P_{ij})_{i,j \geq 0}$ be the stochastic matrix defined by

$$P_{ij} = \frac{M_{ij} r_j}{\lambda r_i} \text{ for all } i, j \geq 0,$$

where $r = (r_k)_{k \geq 0}$ is a right Perron eigenvector of M . We say that M is strongly ergodic if $P = (P_{ij})_{i,j \geq 0}$ is too.

Remark 3.20. (1) It's easy to see that the stochastic matrix P defined in the last definition satisfies $P_{ij} = \frac{M_{ij} l_j}{\lambda l_i}$ for all $i, j \geq 0$, where $l = (l_k)_{k \geq 0}$ is a left Perron eigenvector of M .

Furthermore we have that $P_{i,j}^n = \frac{M_{ij}^n r_j}{\lambda^n r_i}$ for all integer $n \geq 1$.

(2) The Definition 3.19 appeared in [26] in the case where M is a finite irreducible, aperiodic matrix, it is also an extension of the definition in the case where M has constant row sums L . This comes from the fact that $r_i = 1$ for all $i \geq 0$ and $\lambda = L$.

Remark 3.21. Let $M = (M_{ij})_{i,j \geq 0}$ be a nonnegative, irreducible, aperiodic and positive recurrent matrix with finite Perron value $\lambda > 0$. Then M is strongly ergodic if and only if there exists positive integer n and a vector of probability $(\pi_j)_{j \geq 0}$ such that

$$\lim_{n \rightarrow \infty} \sup_{i \geq 0} \sum_{j=0}^{+\infty} \left| \frac{M_{ij}^n r_j}{\lambda^n r_i} - \pi_j \right| = 0, \quad (3.31)$$

furthermore $\pi_j = l_j r_j$ where l and r are respectively Perron left and right eigenvectors such that $l \cdot r = 1$. By using Remark 2.13, we deduce that M is strongly ergodic if and only if

there exists a positive integer n and a positive constant a such that for all integer $i \neq k$, we have

$$\sum_{j=0}^{+\infty} \min\left(\frac{M_{ij}^n r_j}{\lambda^n r_i}, \frac{M_{kj}^n r_j}{\lambda^n r_k}\right) \geq a. \quad (3.32)$$

Remark 3.22. Let $M = (M_{ij})_{i,j \geq 0}$ be a nonnegative, irreducible, aperiodic and positive recurrent matrix with finite Perron value $\lambda > 0$. Assume that M has a right Perron eigenvector $r = (r_i)_{i \geq 0} \in l^\infty$ which satisfies $\inf\{r_i : i \geq 0\} > 0$, then by Remark 3.21, we deduce that M is strongly ergodic if and only if there exists a positive integer n such that M^n is scrambling.

Theorem 3.23. *Let σ be a non-constant bounded length substitution on $A = \mathbb{Z}_+$ with a periodic point u and such that $M = M_\sigma$ is irreducible, aperiodic, positive recurrent and has a finite Perron value. Assume that M_σ has a right Perron eigenvector $r = (r_i)_{i \geq 0} \in l^\infty$ and there exists a positive integer such that M_σ^n is scrambling, then the dynamical system (Ω_σ, S) has a unique invariant probability measure.*

For the proof, we need the following results.

Lemma 3.24. *Let $M = (M_{ij})_{i,j \in \mathbb{Z}_+}$ be an irreducible, aperiodic, positive recurrent non-negative matrix such that $\|M\| = \sup\{\sum_{j=0}^{+\infty} M_{ij}, i \in \mathbb{Z}_+\} < \infty$ and $\inf\{M_{ij} : i, j \in \mathbb{Z}_+, M_{ij} > 0\} > 0$. Assume that there exists a positive integer such that M^n is scrambling, then M has a right Perron eigenvector $r = (r_i)_{i \geq 0}$ which satisfies $\inf\{r_i : i \geq 0\} > 0$.*

Proof. Assume without loss of generality that M is scrambling. Let $r = (r_i)_{i \geq 0}$ be a nonnegative right Perron eigenvector of M . Since M is irreducible, $r_i > 0$ for every $i \geq 0$. Moreover $\|M\| < \infty$ and $\inf\{M_{ij} : i, j \in \mathbb{Z}_+, M_{ij} > 0\} > 0$ imply that there exists $L > 0$ such that $M_{0k} = 0$ for all $k > L$. Since M is scrambling, then for all $i \in \mathbb{Z}_+$, there exists $k_i \in \{0, \dots, L\}$ such that $M_{i,k_i} > 0$. Since $\sum_{k=0}^{+\infty} M_{ik} r_k = \lambda r_i$, we deduce that

$$r_i \geq \frac{M_{i,k_i} r_{k_i}}{\lambda} \geq \frac{C \inf\{r_k : 0 \leq k \leq L\}}{\lambda} > 0 \text{ for all } i \in \mathbb{Z}_+,$$

where $C = \inf\{M_{ij} : i, j \in \mathbb{Z}_+, M_{ij} > 0\}$. \square

Lemma 3.25. *Let σ be a nonconstant bounded length substitution on $A = \mathbb{Z}_+$ with a periodic point u such that $M = M_\sigma$ is irreducible, aperiodic and has a finite Perron value. Assume that M_σ is strongly ergodic and has a right Perron eigenvector $r = (r_i)_{i \geq 0} \in l^\infty$ which satisfies $\inf\{r_i : i \geq 0\} > 0$, then for all integer $t \geq 2$, M_{σ_t} is strongly ergodic and has a right Perron eigenvector $r^{(t)} = (r_I)_{I \in A_t} \in l^\infty$ such that $\inf\{r_I : I \in A_t\} > 0$.*

Proof. Using the same proof given in Lemma 3.18, we can show that M_σ is strongly ergodic, then M_{σ_t} is also strongly ergodic. Moreover since $r = (r_i)_{i \geq 0} \in l^\infty$ and $\inf\{r_i : i \geq 0\} > 0$, Lemma 3.16 implies that $r^{(t)} = (r_I)_{I \in A_t} \in l^\infty$ and $\inf\{r_I : I \in A_t\} > 0$. \square

Lemma 3.26. *Let $M = (M_{ij})_{i,j \geq 0}$ be a nonnegative strongly ergodic matrix with finite Perron value $\lambda > 0$. Assume that M has a right Perron eigenvector $r = (r_i)_{i \geq 0} \in l^\infty$ which satisfies $\inf\{r_i : i \geq 0\} > 0$, then*

$$\lim_{n \rightarrow +\infty} \sup_{i \geq 0} \sum_{j=0}^{\infty} \left| \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} - z_j \right| = 0,$$

where $z_j = \frac{l_j}{\sum_{k=0}^{+\infty} l_k}$ and $l = (l_i)_{i \geq 0} \in l^1$ is a left Perron eigenvector of M .

Proof. Since M is strongly ergodic, we deduce by (3.31) that for all $i, j \in \mathbb{Z}_+$,

$$\lim_{n \rightarrow \infty} \frac{M_{ij}^n}{\lambda^n} = \pi_j \frac{r_i}{r_j}$$

and

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{+\infty} \frac{M_{ik}^n}{\lambda^n} = \sum_{k=0}^{+\infty} \pi_k \frac{r_i}{r_k},$$

where the last two limits are finite and uniform on i . Hence

$$\lim_{n \rightarrow +\infty} \sup_{i \geq 0} \sum_{j=0}^{\infty} \left| \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} - z_j \right| = 0,$$

where

$$z_j = \frac{\pi_j / r_j}{\sum_{k=0}^{+\infty} \pi_k / r_k} = \frac{l_j}{\sum_{k=0}^{+\infty} l_k}, \quad \text{for all } j \geq 0.$$

□

Corollary 3.27. *Let $M = (M_{ij})_{i,j \geq 0}$ be a nonnegative strongly ergodic matrix with finite Perron value $\lambda > 0$. Assume that M has a right Perron eigenvector $r = (r_i)_{i \geq 0} \in l^\infty$ which satisfies $\inf\{r_i : i \geq 0\} > 0$, then*

$$\lim_{n \rightarrow \infty} \frac{\sum_{j=1}^{+\infty} M_{ij}^n}{\lambda^n} = cr_i,$$

for some $c > 0$.

Proof of Theorem 3.23. Without loss of generality, assume that σ has a fixed point $u = \sigma(u) = u_0 u_1 \dots$. For all $i, j \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, we have

$$\frac{|\sigma^n(i)|_j}{|\sigma^n(i)|} = \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n}.$$

Hence by Lemma 3.26, we have

$$\lim_{n \rightarrow +\infty} \sup_{i \geq 0} \sum_{j=0}^{\infty} \left| \frac{|\sigma^n(i)|_j}{|\sigma^n(i)|} - \frac{l_j}{\sum_{k=0}^{+\infty} l_k} \right| = 0.$$

Let $j \in \mathbb{Z}_+$ and put

$$\mu[j] = \lim_{n \rightarrow \infty} \frac{|\sigma^n(u_0)|_j}{|\sigma^n(u_0)|} = \frac{l_j}{\sum_{k=0}^{+\infty} l_k}.$$

Let $t \geq 2$ be an integer and $I_t = i_1 \dots i_t \in A_t$ and put

$$\mu[i_1 \dots i_t] = \lim_{n \rightarrow \infty} \frac{|\sigma^n(u_0)|_{i_0 \dots i_{t-1}}}{|\sigma^n(u_0)|}.$$

By (3.12) and the fact that $\lambda > 1$, we deduce that

$$\mu[i_1 \dots i_t] = \lim_{n \rightarrow \infty} \frac{|\sigma_t^n(u_0 \dots u_{t-1})|_{i_1 \dots i_t}}{|\sigma_t^n(u_0 \dots u_{t-1})|} = \lim_{n \rightarrow \infty} \frac{(M_t^n)_{U_t, I_t}}{\sum_{J \in A_t} M_{U_t, J}^n},$$

where $U_t = u_0 \dots u_{t-1}$, $I_t = i_1 \dots i_t$. By the Proposition 3.25 and Lemma 3.26, we have

$$\mu[i_1 \dots i_t] = \frac{l_{I_t}^{(t)}}{\sum_{J \in A_t} l_J^{(t)}},$$

where $(l_I^{(t)})_{I \in A_t}$ is a left Perron eigenvector of M_{σ_t} associated to its Perron value λ .

The measure μ is the same given in the proof of Theorem 3.10. Hence μ is a shift invariant measure. The uniqueness is a direct consequence of the following claim:

Claim: Let E be a measurable subset of Ω_σ such that $\mu(E) > 0$. For all $x \in E$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{card}\{0 \leq k \leq N-1, S^k(x) \in E\} = \mu(E). \quad (3.33)$$

It remains to prove the claim. First assume that $E = [i_0]$. Suppose $x = u = \sigma(u) = u_0 u_1 \dots$ and $N = |\sigma^n(u_0)|$, then

$$\lim_{n \rightarrow \infty} \frac{1}{N} \text{card}\{0 \leq k \leq N-1, S^k(x) \in E\} = \lim_{n \rightarrow \infty} \frac{|\sigma^n(u_0)|_{i_0}}{|\sigma^n(u_0)|} = \mu(E).$$

Now, let $x \in \Omega_\sigma$ and $N \in \mathbb{N}$. Let $V = u_k \dots u_{k+N-1} \in F_u$, $k \in \mathbb{Z}_+$, be a prefix of x . As seen before, the word V can be written as a concatenation of at most $2n+1$ words $v_0, \sigma(v_1), \dots, \sigma^n(v_n), \sigma^{n-1}(w_{n-1}) \dots w_0$ that is

$$V = v_0 \sigma(v_1) \dots \sigma^{n-1}(v_{n-1}) \sigma^n(v_n) \sigma^{n-1}(w_{n-1}) \dots \sigma(w_1) w_0$$

where $n \geq 1$ is an integer and v_i , $i \in \{0, \dots, n\}$, w_j , $j \in \{0, \dots, n-1\}$, are elements of F_u possibly empty words of lengths $\leq K = \max\{|\sigma(b)|, b \in A\}$ and v_n is not empty. Thus

$$\begin{aligned} \frac{1}{N} \text{card}\{0 \leq k \leq N-1, S^k(x) \in E\} &= \frac{|V|_{i_0}}{|V|} \\ &= \frac{|\sigma^n(v_n)|_{i_0} + \sum_{i=0}^{n-1} (|\sigma^i(v_i)|_{i_0} + |\sigma^i(w_i)|_{i_0})}{|\sigma^n(v_n)| + \sum_{i=0}^{n-1} (|\sigma^i(v_i)| + |\sigma^i(w_i)|)}. \end{aligned} \quad (3.34)$$

Since M_σ is strongly ergodic, we have

$$\limsup_{k \rightarrow \infty} \left\{ \frac{|\sigma^k(j)|_{i_0}}{|\sigma^k(j)|}, j \in A \right\} = \limsup_{k \rightarrow \infty} \left\{ \frac{M_{j, i_0}^k}{\sum_{i=0}^{+\infty} M_{j, i}^k}, j \in A \right\} = \mu[i_0].$$

We deduce that

$$\limsup_{k \rightarrow \infty} \left\{ \frac{|\sigma^k(v)|_{i_0}}{|\sigma^k(v)|}, v \in F_u, |v| \leq K \right\} = \mu[i_0]. \quad (3.35)$$

Using (3.34), (3.35), and the Stolz-Cesaro Theorem, we obtain that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{card}\{0 \leq k \leq N-1, S^k(x) \in E\} = \mu[i_0] = \mu(E).$$

Hence we obtain the claim for $E = [i_0]$.

Now, suppose that $I = i_0 \dots i_{t-1}$ and $E = [I]$. Proceeding as in the case $E = [i_0]$, we have

$$\frac{\text{card}\{0 \leq k \leq N-1, S^k(x) \in E\}}{N} = \frac{|\sigma^n(v_n)|_I + \sum_{i=0}^{n-1} (|\sigma^i(v_i)|_I + |\sigma^i(w_i)|_I) + C_n}{|\sigma^n(v_n)| + \sum_{i=0}^{n-1} (|\sigma^i(v_i)| + |\sigma^i(w_i)|)},$$

where C_n is the cardinality of times such that $i_0 \dots i_{t-1}$ occurs in the concatenation of at least two consecutive words among the $2n+1$ words forming V . Observe that $0 \leq C_n \leq 2n$.

Now for all $j \in A$, we have

$$\lim_{k \rightarrow \infty} \frac{|\sigma^k(j)|_{i_0 \dots i_{t-1}}}{|\sigma^k(j)|} = \lim_{k \rightarrow \infty} \frac{|\sigma_t^k(j z_1 \dots z_{t-1})|_{i_0 \dots i_{t-1}}}{|\sigma_t^k(j z_1 \dots z_{t-1})|} = \mu[i_0 \dots i_{t-1}],$$

where $j z_1 \dots z_{t-1} \in F_u$, we deduce by using the fact that σ_t is strongly ergodic that

$$\limsup_{k \rightarrow \infty} \left\{ \frac{|\sigma^k(j)|_{i_0 \dots i_{t-1}}}{|\sigma^k(j)|}, j \in \mathbb{Z}_+ \right\} = \mu[i_0 \dots i_{t-1}].$$

Thus

$$\limsup_{k \rightarrow \infty} \left\{ \frac{|\sigma^k(v)|_{i_0 \dots i_{t-1}}}{|\sigma^k(v)|}, v \in F_u, |v| \leq K \right\} = \mu[i_0 \dots i_{t-1}]. \quad (3.36)$$

Using (3.36), and the Stolz-Cesaro Theorem, we deduce that

$$\lim_{n \rightarrow \infty} \frac{|\sigma^n(v_n)|_I + \sum_{i=0}^{n-1} (|\sigma^i(v_i)|_I + |\sigma^i(w_i)|_I)}{|\sigma^n(v_n)| + \sum_{i=0}^{n-1} (|\sigma^i(v_i)| + |\sigma^i(w_i)|)} = \mu[i_0 \dots i_{t-1}].$$

Since $0 \leq C_n \leq 2n$, $\lim_{n \rightarrow \infty} \frac{|\sigma^n(v_n)|_0}{\lambda^n} > 0$ and $\lambda > 1$, we deduce that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{card}\{0 \leq k \leq N-1, S^k(x) \in E\} = \mu[i_0 \dots i_{t-1}]$$

and this finishes the proof. \square

Proposition 3.28. *Let σ be a bounded length substitution on $A = \mathbb{Z}_+$ such that σ has a periodic point u and M_σ is irreducible and aperiodic. Assume that there exists an integer n such that M_σ^n is scrambling, then (Ω_σ, S) is minimal.*

Proof. Assume without loss of generality that $u = u_0 u_1 \dots$ is a fixed point and M_σ is scrambling. Let $V = u_k \dots u_{k+N}$, $k, N \in \mathbb{Z}_+$ be a factor of u . Let us prove that V occurs infinitely on u with bounded gaps. Indeed, let $n_0 \in \mathbb{N}$ such that V occurs in $\sigma^k(u_0)$ for all $k \geq n_0$ and put

$$\sigma(u_0) = t_0 \dots t_s, \quad s \in \mathbb{N}.$$

Let $m_0 \in \mathbb{N}$ such that u_0 occurs on $\sigma^k(t_i)$ for all $k \geq m_0$ and $i = 0, \dots, s$. Hence V occurs in $\sigma^k(t_i)$ for all $k \geq n_0 + m_0$ and $i = 0, \dots, s$. On the other hand, since M_σ is scrambling, then for all $i \in \mathbb{N}$, there exists $j_i \in \{0, \dots, s\}$ such that t_{j_i} occurs in $\sigma(u_i)$. Hence V occurs in $\sigma^k(u_i)$ for all $k \geq n_0 + m_0$ and $i \in \mathbb{Z}_+$. Since $u = \sigma^k(u) = \sigma^k(u_0)\sigma^k(u_1)\dots$, then we are done. \square

Examples:

(1) Let σ (infinite Fibonacci) given by

$$\sigma(2n) = 0(2n+1), \quad \sigma(2n+1) = 2n+2 \text{ for all } n \geq 0.$$

We can prove by induction that

$$|\sigma^n(0)| = F_n \text{ and } |\sigma^n(0)|_0 = F_{n-1} \text{ for all } n \geq 1,$$

where $(F_n)_{n \geq 0}$ is the Fibonacci sequence defined by

$$F_0 = 1, F_1 = 2, F_n = F_{n-1} + F_{n-2} \text{ for all } n \geq 2.$$

The substitution matrix is given by

$$M_\sigma = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is irreducible, aperiodic and its Perron eigenvector is the Golden number $\beta = \frac{1+\sqrt{5}}{2} = \lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n}$. A Perron right and a left eigenvectors are respectively

$$l = (1, 1/\beta, \dots, 1/\beta^n, \dots) \text{ and } r = (1, 1/\beta, 1, 1/\beta, 1, 1/\beta, \dots).$$

Hence M_σ is positive recurrent. Furthermore M_σ^2 is scrambling, $r \in l^\infty$ and σ has a fixed point $u = \lim_{n \rightarrow \infty} \sigma^n(0)$, thus Theorem 3.23 implies that the dynamical system (Ω_σ, S) has a unique probability invariant measure.

(2) Let τ be given by

$$\tau(n) = 0^{a_n(n+1)}, \text{ for all } n \geq 0.$$

where $0 \leq a_i \leq C$ for all $i \geq 0$ for some fixed $C > 0$ and $a_0 > 0$ and $\limsup a_n \geq 1$.

The substitution matrix is given by

$$M_\tau = \begin{pmatrix} a_0 & 1 & 0 & 0 & 0 & 0 & \dots \\ a_1 & 0 & 1 & 0 & 0 & 0 & \dots \\ a_2 & 0 & 0 & 1 & 0 & 0 & \dots \\ a_3 & 0 & 0 & 0 & 1 & 0 & \dots \\ a_4 & 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The Perron eigenvalue of M_τ is the unique real number $\lambda > 1$ satisfying

$$1 = \sum_{i=0}^{\infty} a_i \lambda^{-i-1}.$$

A right Perron and a left Perron eigenvector are respectively

$$l = (1, 1/\lambda, \dots, 1/\lambda^n, \dots) \text{ and } r = (1, \alpha_1, \dots, \alpha_n, \dots),$$

where

$$\alpha_n = \lambda^n - \sum_{i=0}^{n-1} a_i \lambda^{n-i-1} \text{ for all } n \geq 1.$$

Observe that

$$\alpha_n = \sum_{i=1}^{+\infty} a_{n+i-1} \lambda^{-i} \text{ for all } n \geq 1.$$

Since $l \cdot r$ is finite, M_σ is positive recurrent.

If there exists $k \geq 1$ such that $a_{kn} \geq 1$ for all $n \in \mathbb{Z}_+$, then $\inf\{\alpha_n, n \in \mathbb{Z}_+\} > 0$. Moreover M_τ^k is scrambling. Furthermore τ has a fixed point $u = \lim_{n \rightarrow \infty} \tau^n(0)$, thus Theorem 3.23 implies that the dynamical system (Ω_u, S) has a unique probability invariant measure.

Question 3.4. It will be interesting to study dynamical properties of (Ω_u, S) associated to τ in the case where $\inf\{\alpha_n, n \in \mathbb{Z}_+\} = 0$.

3.3.3. \star strong ergodicity for nonconstant bounded length substitution.

Definition 3.29. Let $M = (M_{ij})_{i,j \geq 0}$ be a nonnegative matrix such that M is irreducible, aperiodic, positive recurrent and $\|M\| < +\infty$. We say that M is \star ergodic if for all $i, j \in \mathbb{Z}_+$,

$$\lim_{n \rightarrow +\infty} \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} = z_j > 0, \quad (3.37)$$

where the vector $(z_j)_{j \geq 0}$ has 1 as coordinates sum and that M is \star strongly ergodic if there exists a vector $(z_j)_{j \geq 0}$ of positive real numbers such that $\sum_{j=0}^{+\infty} z_j = 1$ and

$$\lim_{n \rightarrow \infty} \sup_{i \geq 0} \sum_{j=0}^{\infty} \left| \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} - z_j \right| = 0.$$

Remark 3.30. If M is \star strongly ergodic, then it is clear that M is \star ergodic.

Question 3.5. Is $M = (M_{ij})_{i,j \geq 0}$ \star ergodic equivalent to M positive recurrent with right Perron eigenvector in l^1 ? The last question has a positive answer when M is a multiple of a stochastic matrix.

An important result is the following:

Proposition 3.31. *Let $M = (M_{ij})_{i,j \geq 0}$ be an irreducible, aperiodic matrix with finite Perron value λ . Assume that M has a right Perron eigenvector $r = (r_i)_{i \geq 0} \in l^\infty$ satisfying $\inf\{r_i, i \in \mathbb{Z}_+\} > 0$. If M is strongly ergodic, then M is \star strongly ergodic.*

Proof. It is just the Lemma 3.26. □

Question 3.6. Does there exist a nonnegative matrix $M = (M_{ij})_{i,j \geq 0}$ which is strongly ergodic (resp. \star strongly ergodic), but not \star strongly ergodic (resp. strongly ergodic)?

Lemma 3.32. *Let $M = (M_{ij})_{i,j \geq 0}$ be a \star ergodic matrix with finite Perron value λ and with right Perron eigenvector $r = (r_i)_{i \geq 0}$, then any left Perron eigenvector of M belongs to l^1 , moreover*

$$\lim_{n \rightarrow +\infty} \sum_{k=0}^{+\infty} \frac{M_{ik}^n}{\lambda^n} = \sum_{k=0}^{+\infty} \lim_{n \rightarrow +\infty} \frac{M_{ik}^n}{\lambda^n} = c r_i, \quad \forall i \in \mathbb{Z}_+, \quad (3.38)$$

for some $c > 0$, and

$$\lim_{n \rightarrow +\infty} \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} = \frac{l_j}{\sum_{k=0}^{+\infty} l_k} > 0, \quad \forall i, j \in \mathbb{Z}_+. \quad (3.39)$$

Proof. By \star ergodicity

$$\lim_{n \rightarrow +\infty} \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} = z_j.$$

Moreover, since M is positive recurrence, we have that

$$\lim_{n \rightarrow +\infty} \frac{M_{ij}^n}{\lambda^n} = l_j r_i. \quad (3.40)$$

where $l = (l_j)_{j \geq 1}$ is a left Perron eigenvector such that $l \cdot r = 1$. Thus

$$\lim_{n \rightarrow +\infty} \sum_{k=0}^{+\infty} \frac{M_{ik}^n}{\lambda^n} = \lim_{n \rightarrow +\infty} \frac{\sum_{k=0}^{+\infty} M_{ik}^n M_{ij}^n}{M_{ij}^n \lambda^n} = \frac{l_j}{z_j} r_i.$$

the left hand side above does not depend on j , thus l is a multiple of $(z_j)_{j \geq 1} \in l^1$. Thus $l \in l^1$ and we also have (3.39), and (3.38) follows from the last equality and (3.40). □

Theorem 3.33. *Let σ be a bounded length substitution on $A = \mathbb{Z}_+$ with nonconstant length such σ has a periodic point u and M_σ is irreducible, aperiodic. If M_σ and M_{σ_t} , $t \geq 2$ are \star strongly ergodic, then the dynamical system (Ω_σ, S) has a unique probability shift invariant measure.*

Proof of Theorem 3.33. Similar to the proof of Theorem 3.23. □

REFERENCES

- [1] S. Bezugly, P. E. T. Jorgensen, S. Sanadhya, *Measures and generalized Bratteli diagrams for dynamics of infinite alphabet-substitutions*, ArXiv:2203.14127, 2022, 34 pages.
- [2] V. Canterini, A. Siegel, *Geometric representation of substitutions of Pisot type*, Trans. Amer. Math. Soc. 353 (2001), no. 12, 5121–5144.
- [3] J. Cassaigne, *Complexity and special factor*, in Bull. Belg. Math. Soc. Simon Stevin, 4, vol. 4, (Mons, 1994), 1997 (french), p. 67-88.
- [4] R. Dobrushin, *Central limit theorem for non-stationary Markov chains*, I. Teor. Veroyatnost. i Primenen., 1:72–89, 1956.
- [5] D. Dombek, *Substitutions over infinite alphabet generating $(-\beta)$ -integers*. Internat. J. Found. Comput. Sci. 23(8):1627–1639, 2012.
- [6] F. Durand, N. Ormes, S. Petite, *Self-induced systems*. J. Anal. Math. 135(2):725–756, 2018.
- [7] F. Durand, *Combinatorics on Bratteli diagrams and dynamical systems*. In Combinatorics, automata and number theory, volume 135 of Encyclopedia Math. Appl., pages 324–372. Cambridge Univ. Press, Cambridge, 2010.
- [8] R. Durrett, *Probability: Theory and Examples*, Cambridge University Press, 4^o ed., 2010.
- [9] S. Ferenczi, *Substitution Dynamical Systems on infinite alphabets*, Ann. Inst. Fourier (Grenoble) 56 (2006), no. 7, 2315–2343.
- [10] C.C. Huang, D. Isaacson, and B. Vinograd, *The rate of convergence of certain non-homogeneous Markov chains*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 35 (1976) 14 l-1 46.
- [11] B. P. Kitchens, *Symbolic Dynamics, one-sided, two-sided and countable state Markov shifts*, Universitext, Springer-Verlag, 1998.
- [12] G. Lawler, *Intersections of Random Walks*, Probability and its applications series, Springer Science+Business Media New York, 1991.
- [13] D. A. Levin, Y. Peres, and E. L. Wilmer, *Markov chains and mixing times*, American Mathematical Society, Providence, RI, 2009. With a chapter by James G. Propp and David B. Wilson.
- [14] G. R. Luecke, D. I. Isaacson, *Strongly ergodic Markov chains and rates of convergence using spectral conditions*, Stoch. Proc. Appl. 7 (1978), 113-121.
- [15] N. Manibo, D. Rust, J. J. Walton, *Substitutions on compact alphabets*, 2022, 41 pages, arXiv:2204.07516.
- [16] J.C. Martin, *Minimal Flows Arising from Substitutions of Non-Constant Length*, Math. Systems Theory 7 (1973), 72–82.
- [17] C. Mauduit, *Propriétés arithmétiques des substitutions et automates infinis*. Ann. Inst. Fourier (Grenoble) 56 (2006), no. 7, 2525–2549.
- [18] P. Michel, *Stricte ergodicité d'ensembles minimaux de substitutions*, C.R. Acad. Sc. Paris 278 (1974), 811–813.
- [19] B. Mossé, *Reconnaissabilité des substitutions et complexité des suites automatiques*, Bulletin de la Société Mathématique de France, Volume 124 (1996) no. 2, pp. 329-346.
- [20] N. Pytheas Fogg, *Substitutions in dynamics, arithmetics and combinatorics*, Lecture Notes in Math., Vol 1794, Springer-Verlag, 2002.
- [21] M. Queffélec, *Substitution Dynamical systems- Spectral analysis*, Lecture Notes in Math., Vol 1294, Springer-Verlag, 1987.
- [22] Vere-Jones, *Geometric ergodicity in denumerable Markov chains*, Quarterly Journal of Mathematics 13, 1962, 7-28.
- [23] D. Vere-Jones, *Ergodic properties of nonnegative matrices. I*, Pacific J. Math. 22, 1967, 361-386.
- [24] Vere-Jones, *Ergodic properties of nonnegative matrices. II*, Pacific J. Math. 26 (1968), 601–620.
- [25] D. Isaacson, R. Madsen, *Markov Chains*, (Wiley Interscience, New York, 1976).
- [26] E. Seneta, *Non negative matrices an Markov chains*, Springer, 1973.

WEBERTY DOMINGOS. DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE ESTADUAL PAULISTA, SÃO JOSÉ DO RIO PRETO, SP, BRASIL.

Email address: `domingos.silva@unesp.br`

SÉBASTIEN FERENCZI. AIX MARSEILLE UNIVERSITÉ, CNRS, CENTRALE MARSEILLE, INSTITUT DE MATHÉMATIQUES DE MARSEILLE, I2M - UMR 7373, 13453 MARSEILLE, FRANCE

Email address: `ferenczi@iml.univ-mrs.fr`

ALI MESSAOUDI. DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE ESTADUAL PAULISTA, SÃO JOSÉ DO RIO PRETO, SP, BRASIL.

Email address: `ali.messaoudi@unesp.br`

GLAUCO VALLE. INSTITUTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO RIO DE JANEIRO, BRASIL.

Email address: `glauco.valle@im.ufrj.br`