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REPRESENTATIONS OF KNOT GROUPS INTO $\mathrm{SL}(n, \mathbb{C})$

MICHAEL HEUSENER AND LEILA BEN ABDELGHANI

ABSTRACT. We study the local structure of the representation variety of a knot group into $\mathrm{SL}(n, \mathbb{C})$ at certain diagonal representations. In particular we determine the tangent cone of the representation variety at these diagonal representations, and show that the latter can be deformed into irreducible representations. Furthermore, we use Luna's slice theorem to analyze the local structure of the character variety.

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Keywords: knot group ; variety of representations ; variety of characters.

1. INTRODUCTION

The aim of this paper is to analyze the local structure of the varieties of representations and of characters of a knot group into the Lie group $\mathrm{SL}(n, \mathbb{C})$, $n \geq 2$, in the neighborhood of certain diagonal representations. It is a well known fact that if Γ is a finitely generated group and G is a linear algebraic group then the set $R(\Gamma, G)$ of representations $\Gamma \rightarrow G$ has the natural structure of an affine algebraic scheme (see [25]). A first approximation to the local structure of the representation variety at a given representation ρ is given by its Zariski tangent space. Two finer approximations are the *quadratic* and the *tangent cones* (see works of W. Goldman and J. Millson [14, 15]).

Given a finitely generated group Γ , we denote by $R_n(\Gamma) := R(\Gamma, \mathrm{SL}(n, \mathbb{C}))$ the $\mathrm{SL}(n, \mathbb{C})$ -representation variety of Γ . The group $\mathrm{SL}(n, \mathbb{C})$ acts on $R_n(\Gamma)$ by conjugation, and the algebraic quotient of this action is the variety of characters which will be denoted by $X_n(\Gamma) := R_n(\Gamma) // \mathrm{SL}(n, \mathbb{C})$.

In this article, we focus on fundamental groups of exteriors of knots in homology spheres, and determine explicitly the quadratic and tangent cones of the representation variety at certain diagonal representations. For the fundamental group Γ of the exterior of a knot K in a homology sphere, we obtain a canonical surjection $h: \Gamma \rightarrow \mathbb{Z} \cong \Gamma/\Gamma'$. The representation space $R_n(\mathbb{Z})$ is isomorphic to $\mathrm{SL}(n, \mathbb{C})$, and the surjection h induces a *closed immersion* $R_n(\mathbb{Z}) \hookrightarrow R_n(\Gamma)$. In fact, $R_n(\mathbb{Z}) \subset R_n(\Gamma)$ forms an $(n^2 - 1)$ -dimensional algebraic component of $R_n(\Gamma)$. A particular representation $\rho_D \in R_n(\Gamma)$ is given by a *regular* diagonal matrix $D = D(\lambda_1, \dots, \lambda_n) \in \mathrm{SL}(n, \mathbb{C})$ i.e. all the diagonal elements are pairwise different. More precisely, we let $\rho_D: \Gamma \rightarrow \mathrm{SL}(n, \mathbb{C})$ denote the diagonal representation which factors through the abelianization Γ/Γ' , and which is given by $\rho_D(\gamma) = D^{h(\gamma)}$ (see Section 2 for details). If none of the quotients λ_i/λ_j is a root of the Alexander polynomial Δ_K , then ρ_D is a smooth point of $R_n(\mathbb{Z})$, and there exists a neighborhood of ρ_D which consists only of representations with abelian images. On the other hand, if all quotients $\alpha_i = \lambda_i/\lambda_{i+1}$, $i = 1, \dots, n - 1$, are simple roots of the Alexander polynomial Δ_K and if $\Delta_K(\lambda_i/\lambda_j) \neq 0$ for all $|i - j| \geq 2$, then there exists a representation $\rho_D^{(n)}$ into the solvable group of upper triangular matrices. The representation $\rho_D^{(n)} \in R_n(\Gamma)$ is reducible and non-semisimple. Its existence was motivated by a result of G. Burde [9] and G. de Rham [10]. The orbit of the representation $\rho_D^{(n)}$

under the action of conjugation of $\mathrm{SL}(n, \mathbb{C})$ is not closed, but is of maximal dimension and the closure of its orbit contains the diagonal representation ρ_D .

We obtain our main result which generalizes [22, Cor. 1.3], and [20, Thm. 1.2].

Theorem 1.1. *Let K be a knot in a three-dimensional integer homology sphere, and let $D = D(\lambda_1, \dots, \lambda_n) \in \mathrm{SL}(n, \mathbb{C})$ be a regular diagonal matrix. If all quotients $\alpha_i = \lambda_i/\lambda_{i+1}$, $i = 1, \dots, n-1$, are simple roots of the Alexander polynomial Δ_K and if $\Delta_K(\lambda_i/\lambda_j) \neq 0$ for all $|i-j| \geq 2$, then there exists a unique algebraic component $R_D^{(n)}$ of $R_n(\Gamma)$ which passes through ρ_D and which contains irreducible representations.*

Moreover, $\dim R_D^{(n)} = (n^2 + n - 2)$, and $\rho_D \in R_D^{(n)}$ is a smooth point. The components $R_n(\mathbb{Z})$ and $R_D^{(n)}$ intersect transversally at the orbit of ρ_D .

Due to a result of A. Weil [33], the Zariski tangent space at a representation $\rho \in R_n(\Gamma)$ is contained in the space of crossed morphisms or 1-cocycles $Z^1(\Gamma; \mathrm{Ad} \rho)$. We call a cocycle $U \in Z^1(\Gamma; \mathrm{Ad} \rho)$ *integrable* if there exists an analytic path ρ_t in $R_n(\Gamma)$ such that $\rho_0 = \rho$, and which is tangent to U i.e. for all $\gamma \in \Gamma$ we have:

$$U(\gamma) = \left. \frac{d\rho_t(\gamma)}{dt} \right|_{t=0} \rho(\gamma)^{-1}.$$

By work of W. Goldman [14], the first obstruction to integrability of the cocycle U is given by the second cohomology class represented by its cup product $[U \smile U]$. We prove that for the diagonal representation ρ_D and under the hypothesis of Theorem 1.1 this obstruction is sufficient:

Theorem 1.2. *Under the hypothesis of Theorem 1.1, a tangent vector $U \in Z^1(\Gamma; \mathrm{Ad} \rho_D)$ is integrable if and only if its cup product $[U \smile U]$ represents the trivial cohomology class in $H^2(\Gamma; \mathrm{Ad} \rho_D)$.*

Moreover, under the hypothesis of Theorem 1.1, we will show (see Section 4) that the *tangent cone* $TC_{\rho_D}(R_n(\Gamma))$ at ρ_D is the union of 2^{n-1} affine subspaces. It follows that for $n > 2$ the representation ρ_D is contained in at least n and in at most 2^{n-1} components (see Remark 4.14). The component $R_n(\mathbb{Z})$ is the unique component of dimension $(n^2 - 1)$, and $R_D^{(n)}$ is the unique component of dimension $(n^2 + n - 2)$. The component $R_n(\mathbb{Z})$ contains only abelian representations and $R_D^{(n)}$ contains irreducible ones. All the other components contain non-abelian, reducible representations and their dimension ranges between n^2 and $n^2 + n - 3$.

Furthermore, we analyse in this paper (see Section 5) the local structure of the character variety $X_n(\Gamma)$ at the character χ_D of ρ_D . Let $t: R_n(\Gamma) \rightarrow X_n(\Gamma)$ be the canonical projection and denote by $X_n(\mathbb{Z}) := t(R_n(\mathbb{Z}))$ and $X_D^{(n)} := t(R_D^{(n)})$ the quotients of the components $R_n(\mathbb{Z})$ and $R_D^{(n)}$ respectively. In fact $t: R_n(\mathbb{Z}) \rightarrow X_n(\mathbb{Z}) \cong \mathbb{C}^{n-1}$ maps the matrix $A \in \mathrm{SL}(n, \mathbb{C})$ onto the coefficients of the characteristic polynomial of A (see [11, Example 1.2]).

As the set $R_n^{irr}(\Gamma)$ of irreducible representations in $R_n(\Gamma)$ is Zariski open and invariant by conjugation, its image $X_n^{irr}(\Gamma) = t(R_n^{irr}(\Gamma))$ is Zariski open in $X_n(\Gamma)$. Their Zariski closures in $R_n(\Gamma)$ and $X_n(\Gamma)$ will be denoted by $\overline{R_n^{irr}(\Gamma)}$ and $\overline{X_n^{irr}(\Gamma)}$ respectively.

Theorem 1.3. *Under the hypothesis of Theorem 1.1 we obtain:*

- (1) *The character $\chi_D = t(\rho_D)$ is a smooth point of $\overline{X_n^{irr}(\Gamma)}$.*
- (2) *The character χ_D is a smooth point of each of the two $(n-1)$ -dimensional components $X_n(\mathbb{Z})$ and $X_D^{(n)}$.*

(3) The intersection of the two above components at χ_D is trivial:

$$T_{\chi_D}^{Zar} X_n(\mathbb{Z}) \cap T_{\chi_D}^{Zar} X_D^{(n)} = \{0\}.$$

(4) Let $ad: \mathfrak{sl}(n, \mathbb{C}) \rightarrow T_{\rho_D^{(n)}}^{Zar} R_n(\Gamma)$ and $dt: T_{\rho_D^{(n)}}^{Zar} R_n(\Gamma) \rightarrow T_{\chi_D}^{Zar} X_n(\Gamma)$ be the tangent morphisms induced by the conjugation action of $\mathrm{SL}(n, \mathbb{C})$ on $\rho_D^{(n)}$ and by the canonical projection $t: R_n(\Gamma) \rightarrow X_n(\Gamma)$. Then we have the exact sequence:

$$0 \rightarrow \mathfrak{sl}(n, \mathbb{C}) \xrightarrow{ad} T_{\rho_D^{(n)}}^{Zar} R_n(\Gamma) \xrightarrow{dt} T_{\chi_D}^{Zar} \overline{X_n^{irr}(\Gamma)} \rightarrow 0,$$

$$\text{and } H^1(\Gamma, \mathrm{Ad} \rho_D^{(n)}) \cong T_{\chi_D}^{Zar} \overline{X_n^{irr}(\Gamma)}.$$

Plan of the article. In Section 2 we recall some results that we will need in our study, and we introduce the necessary notations. The experienced reader can skip this section and use it as a reference. In Section 3 we construct the non-semisimple representation $\rho_D^{(n)}$, and we show that it is a smooth point of $R_n(\Gamma)$ contained in a unique component $R_D^{(n)}$ of dimension $n^2 + n - 2$. Furthermore, we prove that the component $R_D^{(n)}$ contains irreducible representations. In Section 4 we determine the quadratic cone of the diagonal representation ρ_D and prove that it coincide with the tangent cone. Finally, in Section 5 we use our results to analyze the local structure of the character variety.

2. NOTATIONS AND BACKGROUND RESULTS

For the convenience of the reader, we state in this section the definitions and results that will be used in the sequel. More precisely, subsection 2.1 is devoted to collect results and definitions from group cohomology. In subsection 2.2 we review some general facts about affine algebraic schemes, Zariski tangent space, quadratic and tangent cones. In subsection 2.4, we focus on the representation schemes and varieties. Finally, in Section 2.5 we recall some facts about the Alexander module, as well as certain cohomological results.

2.1. Group cohomology. The general reference for group cohomology is K. Brown's book [8, Chap. III]. Let A be a Γ -module. We denote by $C^*(\Gamma; A)$ the cochain complex and by $\delta: C^n(\Gamma; A) \rightarrow C^{n+1}(\Gamma; A)$ the coboundary operator. The coboundaries (respectively cocycles, cohomology) of Γ with coefficients in A are denoted by $B^*(\Gamma; A)$ (respectively $Z^*(\Gamma; A)$, $H^*(\Gamma; A)$). In what follows 1-cocycles and 1-coboundaries will be also called *derivations* and *principal derivations* respectively.

A short exact sequence:

$$0 \rightarrow A_1 \xrightarrow{i} A_2 \xrightarrow{p} A_3 \rightarrow 0$$

of Γ -modules gives rise to a short exact sequence of cochain complexes:

$$0 \rightarrow C^*(\Gamma; A_1) \xrightarrow{i^*} C^*(\Gamma; A_2) \xrightarrow{p^*} C^*(\Gamma; A_3) \rightarrow 0.$$

We will make use of the corresponding long exact cohomology sequence (see [8, III. Proposition 6.1]):

$$0 \rightarrow H^0(\Gamma; A_1) \rightarrow H^0(\Gamma; A_2) \rightarrow H^0(\Gamma; A_3) \xrightarrow{\beta^0} H^1(\Gamma; A_1) \rightarrow \dots$$

Recall that the Bockstein homomorphism $\beta^n: H^n(\Gamma; A_3) \rightarrow H^{n+1}(\Gamma; A_1)$ is determined by the snake lemma: if $z \in Z^n(\Gamma; A_3)$ is a cocycle and if $\tilde{z} \in (p^*)^{-1}(z) \subset C^n(\Gamma; A_2)$ is any lift of z then $\delta_2(\tilde{z}) \in \mathrm{Im}(i^*)$ where δ_2 denotes the coboundary operator of $C^*(\Gamma; A_2)$. It follows that any cochain $z' \in C^{n+1}(\Gamma; A_1)$ such that $i^*(z') = \delta_2(\tilde{z})$ is a cocycle and

that its cohomology class does only depend on the cohomology class represented by z . The cocycle z' represents the image of the cohomology class represented by z under β^n .

By abuse of notation and if no confusion can arise, we will sometimes identify a cocycle with its cohomology class and will write $\beta^n(z)$ for a cocycle $z \in Z^n(\Gamma; A_3)$ even if the map β^n is only well defined on cohomology classes. This will simplify the notations.

Cup products. Let A_1 , A_2 and A_3 be Γ -modules. The cup product of two cochains $u_1 \in C^p(\Gamma; A_1)$ and $u_2 \in C^q(\Gamma; A_2)$ is the cochain $u_1 \smile u_2 \in C^{p+q}(\Gamma; A_1 \otimes A_2)$ defined by:

$$u_1 \smile u_2(\gamma_1, \dots, \gamma_{p+q}) := u_1(\gamma_1, \dots, \gamma_p) \otimes \gamma_1 \cdots \gamma_p \cdot u_2(\gamma_{p+1}, \dots, \gamma_{p+q}).$$

Here $A_1 \otimes A_2$ is a Γ -module via the diagonal action.

The cup-product is compatible with the boundary operator in the following sense:

$$\delta(u_1 \smile u_2) = \delta(u_1) \smile u_2 + (-1)^p u_1 \smile \delta(u_2).$$

When combined with any Γ -invariant bilinear map $A_1 \otimes A_2 \rightarrow A_3$, this gives a cup product:

$$(1) \quad \smile^b: C^1(\Gamma; A_1) \times C^1(\Gamma; A_2) \rightarrow C^2(\Gamma; A_3).$$

In this paper, we will mainly make use of two bilinear forms: the multiplication of complex numbers and the Lie bracket when $A = \mathfrak{g}$ is a Lie algebra. We will denote, respectively, $\cdot \smile \cdot$ and $[\cdot \smile \cdot]$ the corresponding cup products.

Let $b: A_1 \otimes A_2 \rightarrow A_3$ be bilinear and let $z_i \in Z^1(\Gamma; A_i)$, $i = 1, 2$, be cocycles. We define $f: \Gamma \rightarrow A_3$ by $f(\gamma) := b(z_1(\gamma) \otimes z_2(\gamma))$. A direct calculation gives: $df(\gamma_1, \gamma_2) + b(z_1(\gamma_1) \otimes \gamma_1 \cdot z_2(\gamma_2)) + b(\gamma_1 \cdot z_1(\gamma_2) \otimes z_2(\gamma_1)) = 0$. This shows that:

$$(2) \quad df + z_1 \smile^b z_2 + z_2 \smile^{b \circ \tau} z_1 = 0$$

where $\tau: A_1 \otimes A_2 \rightarrow A_2 \otimes A_1$ is the twist operator.

2.2. Affine algebraic schemes. This section can be skipped by readers familiar with schemes and can be used as a reference. For preliminaries to affine algebraic schemes we follow the development in Appendix A in [27]. For a summary of basic results the reader is invited to consult [21, Section 2.1]. General references for schemes include the books of D. Eisenbud and J. Harris [13], E. Kunz [24] and D. Mumford [28].

We are only interested in subschemes of a complex affine space \mathbb{A}^N . The coordinate algebra of \mathbb{A}^N will be denoted by S and it is isomorphic to the polynomial ring $\mathbb{C}[x_1, \dots, x_N]$. A subscheme X of \mathbb{A}^N is given by an ideal $J := J(X) \subset S$. Formally, an affine subscheme is a pair $X = (\text{Spm}(S/J), S/J)$ where $|X| := \text{Spm}(S/J)$ is the maximal spectrum of the \mathbb{C} -algebra S/J which carries the Zariski topology. The quotient S/J is called the *coordinate algebra* or the *algebra of regular functions* of X , and will be denoted by $\mathcal{O}(X)$. A point p of $|X|$ corresponds to a maximal ideal \mathfrak{m}_p in $\mathcal{O}(X)$, and the value of the regular function $f \in \mathcal{O}(X)$ at p is the reduction $f(p) := f \bmod \mathfrak{m}_p$. For any ideal $I \subset \mathcal{O}(X)$ we define $V(I) = \{p \in |X| \mid I \subset \mathfrak{m}_p\}$, and we can define the subscheme $Y = (V(I), \mathcal{O}(X)/I)$ (see [13, Section I.2.1]).

Let $X \subset \mathbb{A}^N$ be an affine subscheme and $p \in |X|$ a point. We denote $J := J(X)$ the corresponding ideal of S . In the definition of the tangent space $T_p(X)$ one takes all $f \in J$, expanded around p and take their linear parts. It is sufficient to take only the linear part of a finite set of generators of J . The tangent space $T_p(X)$ is the subscheme corresponding to the ideal $J_p^{(1)}$ generated by these homogeneous linear forms, i.e. $T_p(X) = V(J_p^{(1)})$. In fact it is an affine subspace of \mathbb{A}^N .

The difference with the definition of the tangent cone $TC_p(X)$ to X at p is this: we take all $f \in J$, expand around p , and look not at their linear terms but at their *leading*

terms, all these leading terms generate an ideal $J_p^{(\infty)}$. Notice that the leading terms of a set of generators of J is in general not sufficient.

Similar, for the quadratic cone $TQ_p(X)$ we take all $f \in J$, expand around p , and look at their leading term if its degree is at most two, all these leading terms generate an ideal $J_p^{(2)}$. We obtain:

$$J_p^{(1)} \subset J_p^{(2)} \subset J_p^{(\infty)} \quad \text{and} \quad T_p(X) \supset TQ_p(X) \supset TC_p(X).$$

Notice that the tangent space is the smallest affine subspace of \mathbb{A}^N which contains the tangent cone $TC_p(X)$ (see [28, III, §4] for more details).

Moreover, for each point $p \in |X|$ we have the following inequalities:

$$(3) \quad \dim_p X = \dim TC_p(X) \leq \dim TQ_p(X) \leq \dim T_p(X).$$

The point p is called *nonsingular* or *smooth* if the tangent cone equals the tangent space $TC_p(X) = T_p(X)$. Notice that $\dim TC_p(X) = \dim T_p(X)$ implies that p is nonsingular.

An algebra is called *reduced* if it has no non-zero nilpotent elements. A scheme X is called *reduced* if its coordinate algebra $\mathcal{O}(X)$ is reduced, and a point p of X is called *reduced* if its local ring $\mathcal{O}_p(X)$ is reduced. It follows from the local to global principle that X is reduced if and only if every point of $|X|$ is reduced [24, IV.1].

In what follows we will give a sufficient condition on a point $p \in X$ to be reduced. More precisely, we will make use of the fact that the coordinate algebra of $TC_p(X)$ is isomorphic to the associated graded algebra of the local ring of $p \in X$ (see [28, III.§3], [24, Chap. 6, Prop. 1.2] and [21]).

Lemma 2.1. *Let $X = \text{Spm}(k[X_1, \dots, X_N]/I) \subset \mathbb{A}^N$ be an affine scheme, and let $p \in X$ be a point. If the tangent cone $TC_p(X)$ is reduced then the local ring $\mathcal{O}_p(X)$ is also reduced and thus p is reduced.*

If X is a scheme there exists a unique reduced subscheme X_{red} of X with reduced coordinate algebra $\mathcal{O}(X_{red}) = \mathcal{O}(X)/N$ (here N is the nilradical of $\mathcal{O}(X)$, see [13, 21]). The topological spaces $|X|$ and $|X_{red}|$ are homeomorphic. Notice that if p is a reduced point then the tangent spaces $T_p(X_{red})$ and $T_p(X)$ coincide.

2.3. Curve tangents and quadratic cone. We recall in this section the link between the quadratic cone and the curve tangents up to second order. In what follows we consider the Artin algebras $A_n := \mathbb{C}[t]/(t^{n+1})$, $n \in \mathbb{N}$. The algebra A_0 is identified with \mathbb{C} , and A_1 is commonly identified with dual numbers $\mathbb{C}[\epsilon]$, $\epsilon^2 = 0$. We let $\pi_n: A_{n+1} \rightarrow A_n$ denote the canonical projection.

As in Section 2.2 we let $X \subset \mathbb{A}^n$ denote an affine algebraic subscheme with coordinate ring $\mathcal{O}(X)$. For very point $p \in X$ we let $\mathfrak{m}_p \subset \mathcal{O}(X)$ denote the corresponding maximal ideal and $\alpha_p: \mathcal{O}(X) \rightarrow \mathbb{C}$ the corresponding algebra morphism with $\text{Ker}(\alpha_p) = \mathfrak{m}_p$. A tangent vector $v \in T_p X$ corresponds to a morphism $\alpha_v: \mathcal{O}(X) \rightarrow A_1$ such that $\pi_0 \circ \alpha_v = \alpha_p$. We say that $v \in T_p X$ is a *curve tangent of order two at p* if there exists a morphism $\alpha_2: \mathcal{O}(X) \rightarrow A_2$ such that $\pi \circ \alpha_2 = \alpha_v$.

Lemma 2.2. *Let $p \in X$ be a point. The space of curve tangents of order two at p has the structure of an affine algebraic subscheme of $T_p X$. More precisely, it coincides with the quadratic cone of X at p .*

Proof. The coordinate ring of $X \subset \mathbb{A}^n$ has a presentation $\mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_m)$. After an affine change of coordinates, and a change of generators of the ideal $J = (f_1, \dots, f_m)$, we can assume that $\mathfrak{m}_p = (x_1, \dots, x_n) \supseteq J$,

$$f_i = x_i + f_i^{(2)} \pmod{\mathfrak{m}_p^3} \quad \text{for } 1 \leq i \leq r, \text{ and } f_i = f_i^{(2)} \pmod{\mathfrak{m}_p^3} \quad \text{for } r < i \leq m.$$

Here $f_i^{(2)}$ is the second homogenous component of $f_i = f_i^{(1)} + f_i^{(2)} \bmod \mathfrak{m}_p^3$.

The tangent space $T_p X$ is then given by the equations $x_i = 0$, $1 \leq i \leq r$. Hence a substitution $x_i = a_i t$, $a_i \in \mathbb{C}$, defines an algebra morphism

$$\alpha_v: \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_m) \rightarrow A_1$$

if and only if $a_i = 0$, $1 \leq i \leq r$. The corresponding tangent vector is $v = (a_{r+1}, \dots, a_n) \in T_p X$. Now, α_v extends to a morphism $\alpha_2: \mathcal{O}(X) \rightarrow A_2$ which is defined by the substitution $x_i = a_i t + b_i t^2$, $b_i \in \mathbb{C}$, if and only if

$$b_i + f_i^{(2)}(0, \dots, 0, a_{r+1}, \dots, a_n) = 0 \quad \text{for } 1 \leq i \leq r,$$

and

$$f_i^{(2)}(0, \dots, 0, a_{r+1}, \dots, a_n) = 0 \quad \text{for } r+1 \leq i \leq m.$$

Hence the coordinate algebra of second order curve tangents at p is isomorphic to

$$(4) \quad \mathbb{C}[t_1, \dots, t_{n-r}]/(g_1, \dots, g_{m-r})$$

where $g_j(t_1, \dots, t_{n-r}) = f_{r+j}^{(2)}(0, \dots, 0, t_1, \dots, t_{n-r})$, $1 \leq j \leq m-r$, are homogenous polynomials of degree two.

The ideal of the quadratic cone $J^{(2)}$ is the ideal generated by the leading terms of polynomials of J of degree lower or equal to two. We will show that $J^{(2)}$ coincides with the ideal K which is generated by the leading terms of the generators of J i.e. $J^{(2)} = K = (x_1, \dots, x_r, f_{r+1}^{(2)}, \dots, f_m^{(2)})$. From this it follows directly that the algebra of the quadratic cone is isomorphic to the algebra defined in equation (4) i.e.

$$\mathbb{C}[x_1, \dots, x_n]/J^{(2)} \cong \mathbb{C}[t_1, \dots, t_{n-r}]/(g_1, \dots, g_{m-r}).$$

It is clear that $K \subset J^{(2)}$. In order to prove that $K \supset J^{(2)}$ we consider a polynomial $g \in J$, $g = \sum_{i=1}^m a_i f_i$ with polynomials $a_i \in \mathbb{C}[x_1, \dots, x_n]$. Now a direct calculation shows that

$$g^{(1)} = \sum_{i=1}^r a_i^{(0)} f_i^{(1)} \quad \text{and} \quad g^{(2)} = \sum_{i=1}^m a_i^{(0)} f_i^{(2)} + \sum_{i=1}^r a_i^{(1)} x_i.$$

It follows from the first equation that the linear form $g^{(1)}$ is a linear combination of x_1, \dots, x_r . Observe that the leading term of $g^{(1)}$ is zero if and only if $a_i^{(0)} = 0$, $1 \leq i \leq r$. In this case the leading term of g is $g^{(2)} = \sum_{j=r+1}^m a_j^{(0)} f_j^{(2)} + \sum_{i=1}^r a_i^{(1)} x_i$ is in the ideal K . \square

2.4. Representation scheme and variety. The principal reference for this section is Lubotzky's and Magid's book [25]. To shorten notation we will simply write $\mathrm{SL}(n)$ and $\mathrm{GL}(n)$ instead of $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{GL}(n, \mathbb{C})$ respectively. The same notation applies for the Lie algebras $\mathfrak{sl}(n) = \mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{gl}(n) = \mathfrak{gl}(n, \mathbb{C})$.

Let Γ be a finitely generated group. The set of all homomorphisms of Γ into $\mathrm{SL}(n)$ has the structure of an *affine scheme*. More precisely, the $\mathrm{SL}(n)$ *scheme of representations* $R_n(\Gamma) := R(\Gamma, \mathrm{SL}(n))$ is defined as the *spectrum* of the universal algebra $A_n(\Gamma)$:

$$R_n(\Gamma) := \mathrm{Spm}(A_n(\Gamma))$$

(see [25] or [21] for more details). The scheme $R_n(\Gamma)$ is equipped with the *Zariski topology*. A single representation $\rho: \Gamma \rightarrow \mathrm{SL}(n)$ corresponds to a \mathbb{C} -algebra morphism $A_n(\Gamma) \rightarrow \mathbb{C}$ i.e. to a maximal ideal $\mathfrak{m}_\rho \subset A_n(\Gamma)$, and each maximal ideal $\mathfrak{m} \subset A_n(\Gamma)$ determines a representation $\rho_\mathfrak{m}: \Gamma \rightarrow \mathrm{SL}(n)$ (this follows from Zariski's Lemma). Hence a maximal ideal in $A_n(\Gamma)$ corresponds exactly to a representation $\Gamma \rightarrow \mathrm{SL}(n)$, and we will use this identification in what follows.

Notice that the universal algebra $A_n(\Gamma)$ might be not reduced. Therefore, one has to consider the underlying affine scheme with a possible non-reduced coordinate ring. For more details and a presentation of these notions see also [18, §2].

Associated to the representation scheme is the representation variety $R_n(\Gamma)_{red}$ with reduced coordinate ring. The representation variety $R_n(\Gamma)_{red}$ is a closed subscheme of $R_n(\Gamma)$. The underlying topological spaces of $R_n(\Gamma)$ and $R_n(\Gamma)_{red}$ are homeomorphic and hence $\dim R_n(\Gamma) = \dim R_n(\Gamma)_{red}$.

Let $\rho: \Gamma \rightarrow \mathrm{SL}(n)$ be a representation. The Lie algebra $\mathfrak{sl}(n)$ turns into a Γ -module via $\mathrm{Ad} \rho$. A 1-cocycle or derivation $d \in Z^1(\Gamma; \mathrm{Ad} \rho)$ is a map $d: \Gamma \rightarrow \mathfrak{sl}(n)$ satisfying

$$d(\gamma_1 \gamma_2) = d(\gamma_1) + \mathrm{Ad}_{\rho(\gamma_1)}(d(\gamma_2)) \quad \forall \gamma_1, \gamma_2 \in \Gamma.$$

It was observed by A. Weil [33] that the tangent space $T_\rho R_n(\Gamma)_{red}$ of the representation variety at ρ embeds into $Z^1(\Gamma; \mathrm{Ad} \rho)$. More precisely, there is an isomorphism between the tangent space of the representation scheme $T_\rho R_n(\Gamma)$ at ρ and the space of 1-cocycles $Z^1(\Gamma; \mathrm{Ad} \rho)$ (see [25, Proposition 2.2]):

$$T_\rho R_n(\Gamma) \cong Z^1(\Gamma; \mathrm{Ad} \rho).$$

Informally speaking, given a smooth curve ρ_t of representations through $\rho_0 = \rho$ one gets a 1-cocycle $d: \Gamma \rightarrow \mathfrak{sl}(n)$ by defining:

$$d(\gamma) := \left. \frac{d \rho_t(\gamma)}{dt} \right|_{t=0} \rho(\gamma)^{-1}, \quad \forall \gamma \in \Gamma.$$

We obtain the following inequalities:

$$\dim_\rho R_n(\Gamma) = \dim_\rho R_n(\Gamma)_{red} \leq \dim T_\rho R_n(\Gamma)_{red} \leq \dim T_\rho R_n(\Gamma) = \dim Z^1(\Gamma; \mathrm{Ad} \rho).$$

The two inequalities are independent and if $\dim_\rho R_n(\Gamma) = \dim T_\rho R_n(\Gamma)_{red}$ then ρ is a smooth point of the representation variety $R_n(\Gamma)_{red}$. We call $\rho \in R_n(\Gamma)$ *scheme smooth* if $\dim_\rho R_n(\Gamma) = \dim Z^1(\Gamma; \mathrm{Ad} \rho)$. The representation $\rho \in R_n(\Gamma)$ is called *reduced* if the local ring $\mathcal{O}_\rho(R_n(\Gamma))$ is reduced. In this case $\dim T_\rho R_n(\Gamma)_{red} = \dim T_\rho R_n(\Gamma)$ holds independent from the first inequality. Therefore, a reduced representation is scheme smooth if and only if it is a smooth point of the representation variety.

It is easy to see that the tangent space to the orbit by conjugation $O(\rho)$ of ρ corresponds to the space of principal derivations $B^1(\Gamma; \mathrm{Ad} \rho)$. Here a principal derivation is a map $b: \Gamma \rightarrow \mathfrak{sl}(n)$ for which there exists $X \in \mathfrak{sl}(n)$ such that for all $\gamma \in \Gamma$ we have $b(\gamma) = \mathrm{Ad}_{\rho(\gamma)}(X) - X$. A detailed account can be found in [25].

2.5. The Alexander module. In this subsection we recall the main results from [4] concerning the Alexander module and certain cohomological results that we will use in the sequel (see also [1] and [2]).

Given a knot K in a three-dimensional integer homology sphere M^3 , we denote by $X = \overline{M^3} \setminus V(K)$ its exterior. Moreover, we let $\Gamma = \pi_1(X)$ denote the fundamental group and let $h: \Gamma \rightarrow \mathbb{Z}$ denote the abelianization morphism, so that $h(\gamma) = \mathrm{lk}(\gamma, K)$ is the linking number in M^3 between any loop realizing $\gamma \in \Gamma$ and K . There is a short exact splitting sequence:

$$1 \rightarrow \Gamma' \rightarrow \Gamma \rightarrow Z \rightarrow 1$$

where $\Gamma' = [\Gamma, \Gamma]$ denotes the commutator subgroup of Γ , and $Z = \langle t \mid - \rangle$ is a free cyclic group generated by a distinguished generator t represented by a meridian of K . The surjection $\Gamma \rightarrow Z$ is given by $\gamma \mapsto t^{h(\gamma)}$. Hence Γ can be written as a semi-direct product $\Gamma = \Gamma' \rtimes Z$. Note that Γ' is the fundamental group of the infinite cyclic covering

X_∞ of X . So $H_1(X_\infty; \mathbb{Z}) \cong \Gamma'/\Gamma''$. Moreover, the previous short exact splitting sequence induces the sequence:

$$1 \rightarrow \Gamma'/\Gamma'' \rightarrow \Gamma/\Gamma'' \rightarrow Z \rightarrow 1.$$

Hence Γ/Γ'' can be written as $\Gamma/\Gamma'' = \Gamma'/\Gamma'' \rtimes Z$.

As $\Lambda := \mathbb{C}[t^{\pm 1}]$ is a principal ideal domain, the complex version $\Gamma'/\Gamma'' \otimes \mathbb{C} \cong H_1(\Gamma; \Lambda)$ of the Alexander module decomposes into a direct sum of cyclic modules of the form $\Lambda/(t - \alpha)^k$, $\alpha \in \mathbb{C}^* \setminus \{1\}$ i.e. there exist $\alpha_1, \dots, \alpha_s \in \mathbb{C}^* \setminus \{1\}$ such that:

$$H_1(\Gamma; \Lambda) \cong \tau_{\alpha_1} \oplus \dots \oplus \tau_{\alpha_s} \quad \text{where} \quad \tau_{\alpha_j} = \bigoplus_{i_j=1}^{n_{\alpha_j}} \Lambda/(t - \alpha_j)^{r_{i_j}}$$

denotes the $(t - \alpha_j)$ -torsion of $H_1(\Gamma; \Lambda)$.

A generator of the order ideal of $H_1(\Gamma; \Lambda)$ is called the *Alexander polynomial* $\Delta_K(t) \in \Lambda$ of K i.e. $\Delta_K(t)$ is the product:

$$\Delta_K(t) = \prod_{j=1}^s \prod_{i_j=1}^{n_{\alpha_j}} (t - \alpha_j)^{r_{i_j}}.$$

Notice that the Alexander polynomial is symmetric and is well defined up to multiplication by a unit ct^k of Λ , $c \in \mathbb{C}^*$, $k \in \mathbb{Z}$. It is well known that Δ_K can be normalized such that Δ_K has only integer coefficients, and $\Delta_K(1) = 1$. Hence the $(t - 1)$ -torsion of the Alexander module is trivial.

If α is a simple root of $\Delta_K(t)$ then the $(t - \alpha)$ -torsion of the Alexander module is of the form $\mathbb{C}_\alpha = \Lambda/(t - \alpha)$. It follows from Blanchfield-duality [16, Chapter 7] that in this case the $(t - \alpha^{-1})$ -torsion of the Alexander module is of the same form $\mathbb{C}_{\alpha^{-1}} = \Lambda/(t - \alpha^{-1})$, thus α^{-1} is also a simple root of $\Delta_K(t)$.

For completeness we will state the next result which shows that the cohomology groups $H^*(\Gamma; \Lambda/(t - \alpha)^k)$ are determined by the Alexander module $H_1(\Gamma; \Lambda)$. Recall that the action of Γ on $\Lambda/(t - \alpha)^k$ is induced by $\gamma \cdot p(t) = t^{h(\gamma)}p(t)$. Denote by π_2 the composition:

$$\pi_2: \Gamma' \rightarrow \Gamma'/\Gamma'' \rightarrow \Gamma'/\Gamma'' \otimes \mathbb{C} \cong H_1(\Gamma; \Lambda).$$

Proposition 2.3. *Let $K \subset M^3$ be a knot in an integer homology sphere and $\Gamma = \Gamma' \rtimes Z$ its group. Let $\alpha \in \mathbb{C}^*$ and let $\tau_\alpha = \bigoplus_{i=1}^{n_\alpha} \Lambda/(t - \alpha)^{r_i}$ denote the $(t - \alpha)$ -torsion of the Alexander module $H_1(\Gamma; \Lambda)$.*

(1) *If $\alpha = 1$ then $\mathbb{C} = \mathbb{C}_1$ is the trivial Γ -module and*

$$H^q(\Gamma; \mathbb{C}) \cong \begin{cases} \mathbb{C} & \text{for } q = 0, 1, \\ 0 & \text{for } q = 2. \end{cases}$$

(2) *For $\alpha \neq 1$ with $\Delta_K(\alpha) \neq 0$ we have:*

$$H^q(\Gamma; \mathbb{C}_\alpha) \cong 0 \quad \text{for } q \geq 0.$$

(3) *If $\alpha \in \mathbb{C}^*$ is a root of the Alexander polynomial Δ_K then the morphism*

$$\begin{aligned} \Phi: \text{Hom}_\Lambda(\Gamma'/\Gamma'' \otimes \mathbb{C}, \mathbb{C}_\alpha) &\rightarrow H^1(\Gamma' \rtimes Z; \mathbb{C}_\alpha) \\ U &\mapsto [\bar{U}]; \quad \bar{U}(x, t^k) = U(\pi_2(x)) \quad \forall x \in \Gamma' \end{aligned}$$

is an isomorphism and

$$\begin{aligned} \psi := \Phi^{-1}: H^1(\Gamma' \rtimes Z; \mathbb{C}_\alpha) &\rightarrow \text{Hom}_\Lambda(\Gamma'/\Gamma'' \otimes \mathbb{C}, \mathbb{C}_\alpha) \\ [V] &\mapsto \tilde{V}; \quad \tilde{V}(\pi_2(x)) = V(x, 1) \quad \forall x \in \Gamma'. \end{aligned}$$

(4) For α a simple root of $\Delta_K(t)$ we have:

$$H^q(\Gamma; \mathbb{C}_\alpha) \cong \begin{cases} 0 & \text{for } q = 0, \\ \mathbb{C} & \text{for } q = 1, 2. \end{cases}$$

In particular, as $H_1(\Gamma; \Lambda) \cong \Gamma'/\Gamma'' \otimes \mathbb{C} = \Lambda/(t - \alpha) \oplus \tau$ where τ is a torsion Λ -module with no $(t - \alpha)$ -torsion, $H^1(\Gamma; \mathbb{C}_\alpha)$ is generated by the non-principal derivation u_α uniquely defined by the condition $\tilde{u}_\alpha(e_\alpha) = 1$ and $\tilde{u}_\alpha(y) = 0 \forall y \in \tau$ where (e_α) be a basis of $\Lambda/(t - \alpha)$.

Proof. For the proof see [2, Proposition 2.1 and Proof of Theorem 3.1]. □

The next lemma forms the basis for our subsequent results and is implicitly contained in [4, Proposition 3.5] (see also [2, Theorem 3.1], [1, Theorem 3.2]).

Lemma 2.4. *Let $K \subset M^3$ be a knot in an integer homology sphere and Γ its group. Let $\alpha \in \mathbb{C}^*$ be a simple root of the Alexander polynomial Δ_K and $h: \Gamma \rightarrow \mathbb{Z}$ the canonical epimorphism. The cup product:*

$$h \smile \cdot: H^1(\Gamma; \mathbb{C}_{\alpha^\pm}) \rightarrow H^2(\Gamma; \mathbb{C}_{\alpha^\pm})$$

is an isomorphism. In other words, if u^\pm is a generator of $H^1(\Gamma; \mathbb{C}_{\alpha^\pm})$ then $H^2(\Gamma; \mathbb{C}_{\alpha^\pm})$ is generated by $h \smile u^\pm$.

Remark 2.5. Observe that by Equation 2 the cocycles $h \smile u^\pm$ and $-u^\pm \smile h$ are cohomologous.

3. PROOF OF THE MAIN THEOREM

This section is devoted to the proof of the first part of Theorem 1.1. Our strategy consists in introducing a certain solvable representation $\rho_D^{(n)} \in R_n(\Gamma)$ whose orbit contains in its closure the diagonal representation ρ_D . As a consequence, the representations ρ_D and $\rho_D^{(n)}$ are on the same component (see subsection 3.1). Using a theorem of [19] based on cohomological calculations, we prove that $\rho_D^{(n)}$ is a smooth point of $R_n(\Gamma)$ which is contained in a unique irreducible component $R_D^{(n)}$ of dimension $n^2 + n - 2$ (see subsection 3.2). We finish the section by proving that $R_D^{(n)}$ contains irreducible representations by making use of Burnside's matrix lemma (see subsection 3.3). It is to mention that our strategy reproduces in a certain manner the one used by [22] for the study of the local structure of the representation space $R_2(\Gamma)$ in the neighborhood of an abelian representation which corresponds to a simple root of the Alexander polynomial of the knot. It is based on the fact that $\rho_D^{(n)}$ is less singular than ρ_D in the sense that $\dim O(\rho_D^{(n)}) = n^2 - 1$ whereas $\dim O(\rho_D) = n^2 - n$ (see Proposition 3.6).

3.1. The triangular representation. Let $D = D(\lambda_1, \dots, \lambda_n) \in \text{SL}(n)$ be a diagonal matrix with $\lambda_i \neq \lambda_j$ for $i \neq j$. The aim of this section is to prove the existence of a certain upper triangular representation $\rho_D^{(n)}: \Gamma \rightarrow \text{SL}(n)$. We will show that $\rho_D^{(n)} \in R_n(\Gamma)$ is a smooth point (see Theorem 3.3), whereas the diagonal representation $\rho_D = D^h$ is not (see Remark 3.4).

Assumption. Throughout this section we will assume that

$\alpha_i := \lambda_i/\lambda_{i+1}$, $i = 1, \dots, n-1$, are simple roots of the Alexander polynomial $\Delta_K(t)$

and that

$$\Delta_K(\lambda_i/\lambda_j) \neq 0 \text{ if } |i-j| \geq 2.$$

Lemma 3.1. *If $u_i^+ \in Z^1(\Gamma; \mathbb{C}_{\alpha_i})$, $i = 1, \dots, n-1$, are derivations then there exist upper triangular representations $\rho_D^{(n)}: \Gamma \rightarrow \text{SL}(n)$ of the form:*

$$(5) \quad \rho_D^{(n)}(\gamma) = \begin{pmatrix} 1 & u_1^+(\gamma) & * & \dots & * \\ 0 & 1 & u_2^+(\gamma) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ \vdots & & \ddots & 1 & u_{n-1}^+(\gamma) \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix} D(\lambda_1, \dots, \lambda_n)^{h(\gamma)}.$$

Moreover, the closure of the orbit of $\rho_D^{(n)}$ contains the diagonal representation ρ_D .

Proof. It is easy to verify that a map given by

$$\rho(\gamma) = \begin{pmatrix} 1 & z_{12}(\gamma) & * & \dots & z_{1n}(\gamma) \\ 0 & 1 & z_{23}(\gamma) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ \vdots & & \ddots & 1 & z_{n-1,n}(\gamma) \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix} D(\lambda_1, \dots, \lambda_n)^{h(\gamma)}$$

is a representation if and only if for all $1 \leq i < j \leq n$:

$$\begin{aligned} z_{ij}(\gamma_1\gamma_2) &= \sum_{k=i}^j z_{ik}(\gamma_1) \left(\frac{\lambda_k}{\lambda_j}\right)^{h(\gamma_1)} z_{kj}(\gamma_2) \\ &= \left(\frac{\lambda_i}{\lambda_j}\right)^{h(\gamma_1)} z_{ij}(\gamma_2) + \sum_{k=i+1}^{j-1} (z_{ik} \smile z_{kj})(\gamma_1, \gamma_2) + z_{ij}(\gamma_1) \end{aligned}$$

holds. Here we have used the convention that $z_{ii}(\gamma) = 1$ for $1 \leq i \leq n$ and $\gamma \in \Gamma$. This equation is equivalent to

$$-\delta z_{ij} = \sum_{i < k < j} z_{ik} \smile z_{kj} \quad \text{for all } 1 \leq i < j \leq n.$$

It implies that $z_{i,i+1} \in Z^1(\Gamma; \mathbb{C}_{\alpha_i})$ if $1 \leq i \leq n-1$ and

$$-\delta z_{ij} = \sum_{i < k < j} z_{ik} \smile z_{kj} \in B^2(\Gamma; \mathbb{C}_{\lambda_i/\lambda_j}) \quad \text{if } j-i \geq 2.$$

By Proposition 2.3, since λ_i/λ_j is not a root of Δ_K for $|i-j| \geq 2$, we have $H^2(\Gamma; \mathbb{C}_{\lambda_i/\lambda_j}) = 0$ and the cocycle $\sum_{i < k < j} z_{ik} \smile z_{kj}$ is a coboundary. Notice also that by Proposition 2.3, there exist non principal derivations $z \in Z^1(\Gamma; \mathbb{C}_\alpha)$ as α is a root of $\Delta_K(t)$.

For $t > 0$, let C_t be the diagonal matrix $C_t := \text{diag}(t^{n-1}, t^{n-2}, \dots, 1)$. Then it is easy to see that $\lim_{t \rightarrow 0} C_t \rho_D^{(n)}(\gamma) C_t^{-1} = \rho_D(\gamma)$ for all $\gamma \in \Gamma$. This proves that $\rho_D \in \overline{O(\rho_D^{(n)})}$. \square

Remark 3.2. (1) The representation $\rho_D^{(n)}$ is non-abelian if there exists i_0 , $1 \leq i_0 \leq n-1$, such that $u_{i_0}^+$ is a non principal derivation. To see that suppose that a derivation $z_{i_0} \in Z^1(\Gamma; \mathbb{C}_{\alpha_{i_0}})$ is non-principal. By Lemma 2.1 of [22] we can find a set of generators $\gamma_1, \dots, \gamma_k$ of Γ such that $h(\gamma_l) = 1$ for $l = 1, \dots, k$. As z_{i_0} is non-principal, there exist γ_l and $\gamma_{l'}$ such that $z_{i_0}(\gamma_l) \neq z_{i_0}(\gamma_{l'})$. It follows that $\rho_D^{(n)}$ is non-abelian since the two matrices

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_{i+1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_{i+1} \end{pmatrix}$$

commute if and only if $(b-c) = 0$.

(2) Once the derivations u_i^+ , $i = 1, \dots, n-1$, fixed, two such representations $\rho_D^{(n)}$ are conjugated. This claim is a consequence of the fact that $H^2(\Gamma; \mathbb{C}_{\lambda_i/\lambda_j}) = 0$ for $|i-j| \geq 2$, and the fact that adding a coboundary $b(\gamma) = ((\lambda_i/\lambda_j)^{h(\gamma)} - 1)a$ to $z_{i,j}$, $j-i \geq 2$, can be realized by conjugation by the matrix $I_n - aE_i^j$. Here, $(E_i^j)_{1 \leq i, j \leq n}$ is the canonical basis of $\mathfrak{gl}(n)$.

The aim of this section is to prove the next theorem.

Theorem 3.3. *Suppose that $\alpha_i = \lambda_i/\lambda_{i+1}$, $i = 1, \dots, n-1$, are simple roots of the Alexander polynomial $\Delta_K(t)$, and that $\Delta_K(\lambda_i/\lambda_j) \neq 0$ if $|i-j| \geq 2$. Then for given non-principal derivations $u_i^+ \in Z^1(\Gamma; \mathbb{C}_{\alpha_i})$ the representation $\rho_D^{(n)}: \Gamma \rightarrow \text{SL}(n)$ given by (5) is a scheme smooth representation of $R_n(\Gamma)$.*

More precisely, $\rho_D^{(n)}$ is contained in a unique irreducible component $R_D^{(n)} \subset R_n(\Gamma)$ of dimension $n^2 + n - 2$ and the component $R_D^{(n)}$ contains irreducible representations.

Remark 3.4. (1) The component $R_D^{(n)} \subset R_n(\Gamma)$ depends only on $\lambda_1, \dots, \lambda_n$ and not on the cocycles u_i^+ . This follows since the u_i^+ are non-principal and $H^1(\Gamma; \mathbb{C}_{\alpha_i}) \cong \mathbb{C}$. Adding a coboundary $b(\gamma) = (\alpha_i^{h(\gamma)} - 1)a$ to u_i^+ and multiplying u_i^+ by a non-zero complex number $c \in \mathbb{C}^*$ can be realized by conjugation:

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_i^+(\gamma) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_i^{h(\gamma)} & 0 \\ 0 & \lambda_{i+1}^{h(\gamma)} \end{pmatrix} \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_i^+(\gamma) + (\alpha_i^{h(\gamma)} - 1)a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_i^{h(\gamma)} & 0 \\ 0 & \lambda_{i+1}^{h(\gamma)} \end{pmatrix}$$

and

$$\begin{pmatrix} c^{1/2} & 0 \\ 0 & c^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & u_i^+(\gamma) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_{i+1} \end{pmatrix} \begin{pmatrix} c^{-1/2} & 0 \\ 0 & c^{1/2} \end{pmatrix} = \begin{pmatrix} 1 & c u_i^+(\gamma) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_{i+1} \end{pmatrix}.$$

(2) By Lemma 3.1, the diagonal representation ρ_D is contained in the closure of the orbit of $\rho_D^{(n)}$, thus ρ_D is contained in the irreducible component $R_D^{(n)}$. On the other hand, as an abelian representation, ρ_D is contained in the irreducible component $R_n(\mathbb{Z})$. Consequently, ρ_D is not a smooth point of $R_n(\Gamma)$.

For the proof of Theorem 3.3 we will make use of the following result (see [5, 22, 20, 19, 18]).

Proposition 3.5. *Let M be a connected and orientable 3-manifold with toroidal boundary $\partial M \cong S^1 \times S^1$ and let $\rho: \pi_1(M) \rightarrow \text{SL}(n)$ be a representation.*

If $\dim H^1(\pi_1(M); \mathfrak{sl}(n)_{\text{Ad } \rho}) = n-1$ then ρ is a smooth point of the $\text{SL}(n)$ -representation variety $R_n(\pi_1(M))$. More precisely, ρ is contained in a unique component of dimension $n^2 + n - 2 - \dim H^0(\pi_1(M); \mathfrak{sl}(n)_{\text{Ad } \rho})$.

The knot complement X has a single torus boundary component. To prove the first part of Theorem 3.3, we are going to prove that $H^0(\Gamma, \mathfrak{sl}(n)_{\text{Ad } \rho_D^{(n)}}) = 0$ and $\dim H^1(\Gamma; \mathfrak{sl}(n)_{\text{Ad } \rho_D^{(n)}}) = n - 1$. This calculation will occupy the following subsection. The proof of the existence of irreducible representations in the component $R_D^{(n)}$ is given in the last one.

3.2. Cohomological calculations. Hereafter, we will fix non-principal derivations $u_i^\pm: \Gamma \rightarrow \mathbb{C}_{\alpha_i^\pm}$, $1 \leq i \leq n - 1$, and we let $\rho_D^{(n)}$ be a representation as given by (5). Moreover, we will consider $\mathfrak{sl}(n)$ (respectively $\mathfrak{gl}(n)$) as a Γ -module via $\text{Ad } \rho_D^{(n)}$ and for simplicity we will write $\mathfrak{sl}(n)$ (respectively $\mathfrak{gl}(n)$) for $\mathfrak{sl}(n)_{\text{Ad } \rho_D^{(n)}}$ (respectively $\mathfrak{gl}(n)_{\text{Ad } \rho_D^{(n)}}$). Notice that we have the following isomorphism of Γ -modules: $\mathfrak{gl}(n) \cong \mathfrak{sl}(n) \oplus \mathbb{C} I_n$ where Γ acts trivially on the center $\mathbb{C} I_n$ of $\mathfrak{gl}(n)$. It follows that:

$$(6) \quad H^*(\Gamma; \mathfrak{gl}(n)) \cong H^*(\Gamma; \mathfrak{sl}(n)) \oplus H^*(\Gamma; \mathbb{C}).$$

To compute the cohomology groups $H^*(\Gamma; \mathfrak{gl}(n))$ we will use an adequate filtration of the Γ -module $\mathfrak{gl}(n)$.

Let (E_1, \dots, E_n) denote the canonical basis of the space of column vectors. Hence $E_i^j := E_i E_j^\top$, $1 \leq i, j \leq n$, form the canonical basis of $\mathfrak{gl}(n)$. For a given family $(X_i)_{i \in I}$, $X_i \in \mathfrak{gl}(n)$, we let $\langle X_i \mid i \in I \rangle \subset \mathfrak{gl}(n)$ denote the subspace of $\mathfrak{gl}(n)$ generated by the family. For a given $k \in \mathbb{Z}$, $1 - n \leq k \leq n$, we define:

$$(7) \quad C_k = \langle E_i^j \in \mathfrak{gl}(n, \mathbb{C}) \mid j - i \geq k \rangle.$$

For all $\gamma \in \Gamma$ we have:

$$(8) \quad \gamma \cdot E_i^j = (\rho_D^{(n)}(\gamma) E_i) (E_j^\top \rho_D^{(n)}(\gamma^{-1})).$$

More precisely, we obtain for $1 \leq i, j \leq n$,

$$\rho_D^{(n)}(\gamma) E_i = \lambda_i^{h(\gamma)} E_i + \lambda_i^{h(\gamma)} u_{i-1}^+(\gamma) E_{i-1} + \dots$$

and

$$\begin{aligned} E_j^\top \rho_D^{(n)}(\gamma^{-1}) &= \lambda_j^{-h(\gamma)} E_j^\top + \lambda_{j+1}^{-h(\gamma)} u_j^+(\gamma^{-1}) E_{j+1}^\top + \dots \\ &= \lambda_j^{-h(\gamma)} E_j^\top - \lambda_{j+1}^{-h(\gamma)} \left(\frac{\lambda_j}{\lambda_{j+1}} \right)^{-h(\gamma)} u_j^+(\gamma) E_{j+1}^\top + \dots \end{aligned}$$

Here we have used the fact that, as u_j^+ is a derivation, $u_j^+(\gamma^{-1}) = -(\lambda_j/\lambda_{j+1})^{-h(\gamma)} u_j^+(\gamma)$.

Hence for $E_i^j \in C_k$ we have:

$$(9) \quad \gamma \cdot E_i^j = \left(\frac{\lambda_i}{\lambda_j} \right)^{h(\gamma)} E_i^j + \left(\frac{\lambda_i}{\lambda_j} \right)^{h(\gamma)} u_{i-1}^+(\gamma) E_{i-1}^j - \left(\frac{\lambda_i}{\lambda_j} \right)^{h(\gamma)} u_j^+(\gamma) E_i^{j+1} + c_{k+2}$$

for a certain element $c_{k+2} \in C_{k+2}$.

It follows from equation (9) that C_k is a Γ -module. Hence we obtain the descending sequence of Γ -modules:

$$\mathfrak{gl}(n, \mathbb{C}) = C_{1-n} \supset C_{2-n} \dots \supset C_0 \supset \dots \supset C_{n-1} \supset C_n = \{0\}.$$

Notice that C_0 is the Borel subalgebra and that the coordinate function $p_1^n: C_{n-1} = \langle E_1^n \rangle \rightarrow \mathbb{C}_{\lambda_1/\lambda_n}$ induces an isomorphism.

A basis of the vector space C_k/C_{k+1} is given by:

$$\begin{cases} (E_1^{k+1} + C_{k+1}, E_2^{k+2} + C_{k+1}, \dots, E_{n-k}^n + C_{k+1}) & \text{if } 0 \leq k \leq n - 1, \\ (E_{|k|+1}^1 + C_{k+1}, E_{|k|+2}^2 + C_{k+1}, \dots, E_n^{n-|k|} + C_{k+1}) & \text{if } 1 - n \leq k \leq 0. \end{cases}$$

Hence we have the following isomorphisms which are induced by the projections:

$$\begin{aligned} p_1^{k+1} \oplus p_2^{k+2} \oplus \cdots \oplus p_{n-k}^n: C_k/C_{k+1} &\xrightarrow{\cong} \bigoplus_{i=1}^{n-k} \mathbb{C}_{\lambda_i/\lambda_{k+i}} && \text{if } 0 \leq k \leq n-1, \\ p_{|k|+1}^1 \oplus p_{|k|+2}^2 \oplus \cdots \oplus p_n^{n-|k|}: C_k/C_{k+1} &\xrightarrow{\cong} \bigoplus_{i=1}^{n-|k|} \mathbb{C}_{\lambda_{i+|k|}/\lambda_i} && \text{if } 1-n \leq k < 0. \end{aligned}$$

As λ_i/λ_j , $|i-j| \geq 2$, is not a root of the Alexander polynomial Δ_K and λ_i/λ_{i+1} , $1 \leq i \leq n-1$, is a simple root of Δ_K , we obtain from Proposition 2.3

$$(10) \quad \begin{aligned} H^0(\Gamma; C_k/C_{k+1}) &\cong \begin{cases} 0 & \text{if } k \neq 0, \\ \mathbb{C}^n & \text{if } k = 0, \end{cases} \\ H^1(\Gamma; C_k/C_{k+1}) &\cong \begin{cases} 0 & \text{if } k \neq -1, 0, 1, \\ \mathbb{C}^n & \text{if } k = 0, \\ \mathbb{C}^{n-1} & \text{if } k = \pm 1, \end{cases} \\ H^2(\Gamma; C_k/C_{k+1}) &\cong \begin{cases} 0 & \text{if } k \neq \pm 1, \\ \mathbb{C}^{n-1} & \text{if } k = \pm 1. \end{cases} \end{aligned}$$

In what follows we will consider the following short exact sequence:

$$(11) \quad 0 \rightarrow C_{k+1} \xrightarrow{i_k} C_k \xrightarrow{pr_k} C_k/C_{k+1} \rightarrow 0.$$

For $|k| \leq n-1$, the sequence (11) induces a long exact sequence in cohomology:

$$(12) \quad \begin{aligned} 0 \rightarrow H^0(\Gamma; C_{k+1}) \rightarrow H^0(\Gamma; C_k) \rightarrow H^0(\Gamma; C_k/C_{k+1}) \xrightarrow{\beta_k^0} \\ H^1(\Gamma; C_{k+1}) \rightarrow H^1(\Gamma; C_k) \rightarrow H^1(\Gamma; C_k/C_{k+1}) \xrightarrow{\beta_k^1} \\ H^2(\Gamma; C_{k+1}) \rightarrow H^2(\Gamma; C_k) \rightarrow H^2(\Gamma; C_k/C_{k+1}) \rightarrow \cdots \end{aligned}$$

and we will denote by $\beta_k^*: H^*(\Gamma, C_k/C_{k+1}) \rightarrow H^{*+1}(\Gamma, C_{k+1})$ the connecting homomorphism.

Proposition 3.6. *Suppose that $\alpha_i = \lambda_i/\lambda_{i+1}$, $i = 1, \dots, n-1$, are simple roots of the Alexander polynomial $\Delta_K(t)$, and that $\Delta_K(\lambda_i/\lambda_j) \neq 0$ if $|i-j| \geq 2$. Then for given non-principal derivations $u_i^+ \in Z^1(\Gamma; \mathbb{C}_{\alpha_i})$ the representation $\rho_D^{(n)}: \Gamma \rightarrow \text{SL}(n)$ given by (5) verifies:*

$$H^0(\Gamma, \mathfrak{sl}(n)_{\text{Ad } \rho_D^{(n)}}) = \{0\} \quad \text{and} \quad \dim H^1(\Gamma, \mathfrak{sl}(n)_{\text{Ad } \rho_D^{(n)}}) = n-1.$$

Proof. It follows from (10) that $H^2(\Gamma, C_k/C_{k+1}) = 0$ for $|k| \geq 2$; hence from the long exact sequence (12) we obtain isomorphisms $H^*(\Gamma; C_{k+1}) \xrightarrow{\cong} H^*(\Gamma; C_k)$ for $|k| \geq 2$.

From this and (10) we obtain:

$$0 = H^*(\Gamma; C_{n-1}) \xrightarrow{\cong} H^*(\Gamma; C_{n-2}) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H^*(\Gamma; C_2)$$

and

$$H^*(\Gamma; C_{-1}) \xrightarrow{\cong} H^*(\Gamma; C_{-2}) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H^*(\Gamma; C_{1-n}) = H^*(\Gamma; \mathfrak{gl}(n, \mathbb{C})).$$

Now, $H^*(\Gamma; C_2) = 0$ and $C_2 \subset C_{-1}$ imply that:

$$H^*(\Gamma; C_{-1}) \xrightarrow{\cong} H^*(\Gamma; C_{-1}/C_2).$$

Therefore it will be sufficient to compute the dimension of $H^*(\Gamma; C_{-1}/C_2)$.

We have the following inclusions:

$$C_{-1} \supset C_0 \supset C_1 \supset C_2 \quad \rightarrow \quad C_{-1}/C_2 \supset C_0/C_2 \supset C_1/C_2.$$

Thus we obtain two exact sequences of Γ -modules:

$$(13) \quad 0 \rightarrow C_1/C_2 \rightarrow C_0/C_2 \rightarrow C_0/C_1 \rightarrow 0,$$

$$(14) \quad 0 \rightarrow C_0/C_2 \rightarrow C_{-1}/C_2 \rightarrow C_{-1}/C_0 \rightarrow 0.$$

The short exact sequence (13) induces the following long sequence in cohomology:

$$\begin{aligned} 0 \rightarrow H^0(\Gamma; C_0/C_2) \rightarrow H^0(\Gamma; C_0/C_1) \xrightarrow{\beta^0} H^1(\Gamma; C_1/C_2) \rightarrow H^1(\Gamma; C_0/C_2) \\ \rightarrow H^1(\Gamma; C_0/C_1) \xrightarrow{\beta^1} H^2(\Gamma; C_1/C_2) \rightarrow H^2(\Gamma; C_0/C_2) \rightarrow 0 \end{aligned}$$

since by equation (10), $H^0(\Gamma; C_1/C_2) = 0$ and $H^2(\Gamma; C_0/C_1) = 0$.

In order to determine the connection operators β^* , $*$ = 0, 1, we fix a basis $E_i^i + C_1$, $i = 1, \dots, n$, of C_0/C_1 . Then $\{E_i^i + C_1\}_{1 \leq i \leq n}$ is a basis of $H^0(\Gamma; C_0/C_1)$, and the cocycles $U_j: \Gamma \rightarrow C_1/C_2$, $j = 1, \dots, n-1$, given by $U_j(\gamma) = u_j^+(\gamma)E_j^{j+1} + C_2$ represent a basis of $H^1(\Gamma; C_1/C_2)$.

Now we let δ^* denote the coboundary operator of the complex $C^*(\Gamma; \mathfrak{gl}(n))$. Then equation (9) gives directly:

$$\delta^0(E_i^i)(\gamma) = \gamma \cdot E_i^i - E_i^i = u_{i-1}^+(\gamma) E_{i-1}^i - u_i^+(\gamma) E_i^{i+1} + c_2$$

for a certain element $c_2 \in C_2$. Therefore, $\beta^0(E_i^i + C_1) = U_{i-1} - U_i + C_2$ where U_0 and U_n are understood to be zero. Hence the matrix of β^0 with respect to the basis $\{E_i^i + C_2\}_{1 \leq i \leq n}$ and $\{U_j\}_{1 \leq j \leq n-1}$ of $H^0(\Gamma; C_0/C_1)$ and $H^1(\Gamma; C_1/C_2)$ respectively is the $(n-1) \times n$ matrix:

$$(15) \quad M = \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -1 & 1 \end{pmatrix}$$

of rank $n-1$. Hence β^0 is surjective.

Similarly, $\{h E_i^i + C_2\}_{1 \leq i \leq n}$ represents a basis of $H^1(\Gamma; C_0/C_1)$, and

$$\delta^1(h E_i^i)(\gamma_1, \gamma_2) = (u_{i-1}^+ \smile h)(\gamma_1, \gamma_2) E_{i-1}^i - (u_i^+ \smile h)(\gamma_1, \gamma_2) E_i^{i+1} + c_2,$$

where u_0^+ and u_n^0 are understood to be zero. By Lemma 2.4 and Remark 2.5, the 2-cocycles $u_i^+ \smile h \in Z^2(\Gamma; \mathbb{C}_{\alpha_i})$ represent generators of $H^2(\Gamma; \mathbb{C}_{\alpha_i})$. Hence the matrix of β^1 with respect to the basis $\{h E_i^i + C_2\}_{1 \leq i \leq n}$ and $\{(u_i^+ \smile h) E_i^{i+1} + C_2\}_{1 \leq i \leq n-1}$ of $H^1(\Gamma; C_0/C_1)$ and $H^2(\Gamma; C_1/C_2)$ respectively is also the $(n-1) \times n$ matrix M of equation (15). Hence β^1 is also surjective.

As a consequence $H^2(\Gamma; C_0/C_2) = 0$ and we obtain the two short exact sequences:

$$0 \rightarrow H^0(\Gamma; C_0/C_2) \rightarrow H^0(\Gamma; C_0/C_1) \rightarrow H^1(\Gamma; C_1/C_2) \rightarrow 0,$$

and

$$0 \rightarrow H^1(\Gamma; C_0/C_2) \rightarrow H^1(\Gamma; C_0/C_1) \rightarrow H^2(\Gamma; C_1/C_2) \rightarrow 0.$$

We deduce that, $\dim H^0(\Gamma; C_0/C_2) = \dim H^1(\Gamma; C_0/C_2) = 1$ (see (10)).

Observe that $H^1(\Gamma; C_0/C_2)$ is generated by $\langle h I_n + C_2 \rangle$, where I_n is the identity matrix of order n .

Now consider the short exact sequence (14) which induces the following sequence in cohomology:

$$\begin{aligned} 0 \rightarrow H^0(\Gamma; C_0/C_2) &\rightarrow H^0(\Gamma; C_{-1}/C_2) \rightarrow H^0(\Gamma; C_{-1}/C_0) \rightarrow H^1(\Gamma; C_0/C_2) \\ &\rightarrow H^1(\Gamma; C_{-1}/C_2) \rightarrow H^1(\Gamma; C_{-1}/C_0) \rightarrow H^2(\Gamma; C_0/C_2) \\ &\rightarrow H^2(\Gamma; C_{-1}/C_2) \rightarrow H^2(\Gamma; C_{-1}/C_0) \rightarrow \cdots \end{aligned}$$

As $H^0(\Gamma; C_{-1}/C_0) = 0$ and $H^2(\Gamma; C_0/C_2) = 0$ we obtain the isomorphism:

$$H^0(\Gamma; C_0/C_2) \xrightarrow{\cong} H^0(\Gamma; C_{-1}/C_2)$$

and the short exact sequence

$$(16) \quad 0 \rightarrow H^1(\Gamma; C_0/C_2) \rightarrow H^1(\Gamma; C_{-1}/C_2) \rightarrow H^1(\Gamma; C_{-1}/C_0) \rightarrow 0.$$

In conclusion

$$\begin{cases} \dim H^0(\Gamma; \mathfrak{gl}(n)) = \dim H^0(\Gamma; C_{-1}/C_2) = \dim H^0(\Gamma; C_0/C_2) = 1, \\ \dim H^1(\Gamma; \mathfrak{gl}(n)) = \dim H^1(\Gamma; C_{-1}/C_2) = 1 + (n - 1) = n. \end{cases}$$

Now, equation (6) gives $\dim H^0(\Gamma; \mathfrak{sl}(n)) = 0$, and $\dim H^1(\Gamma; \mathfrak{sl}(n)) = n - 1$. □

Remark 3.7. The cohomology groups $H^0(\Gamma; C_0/C_2)$ and $H^1(\Gamma; C_0/C_2)$ are generated by $I_n + C_2$ and $h I_n + C_2$ respectively where I_n denotes the identity matrix. This follows since $\text{Ker } \beta^0$, and $\text{Ker } \beta^1$ are generated by $I_n + C_1$ and $h I_n + C_1$ respectively.

We call a cochain $c: \Gamma \rightarrow \mathfrak{gl}(n)$ *traceless* if $\text{tr}(c(\gamma)) = 0$ for all $\gamma \in \Gamma$. Notice that a cochain is traceless if and only if it takes values in $\mathfrak{sl}(n) \subset \mathfrak{gl}(n)$. Every principal derivation $b: \Gamma \rightarrow \mathfrak{gl}(n)$ is traceless, and hence we can define traceless cohomology classes.

It is clear that each traceless cocycle $Z^1(\Gamma; C_{-1})$ represents a cohomology class in $H^1(\Gamma; \mathfrak{sl}(n))$. Reciprocally, we have the following:

Corollary 3.8. *Each element of $H^1(\Gamma; \mathfrak{sl}(n))$ can be represented by a traceless cocycle in $Z^1(\Gamma; C_{-1})$. Moreover, two traceless cocycles in $Z^1(\Gamma; C_{-1})$ represent the same cohomology class in $H^1(\Gamma; \mathfrak{sl}(n))$ if and only if their projections represent the same cohomology class in $H^1(\Gamma; C_{-1}/C_0)$. In other words, the composition map:*

$$H^1(\Gamma; \mathfrak{sl}(n)) \hookrightarrow H^1(\Gamma; \mathfrak{gl}(n)) \cong H^1(\Gamma; C_{-1}) \rightarrow H^1(\Gamma; C_{-1}/C_0)$$

is an isomorphism.

Proof. The first part follows easily since a principal derivation with values in $\mathfrak{gl}(n)$ is traceless, and since $H^1(\Gamma; C_{-1}) \xrightarrow{\cong} H^1(\Gamma; \mathfrak{gl}(n))$ is an isomorphism. More precisely, traceless cohomology classes in $H^1(\Gamma; C_{-1})$ correspond exactly to cohomology classes in $H^1(\Gamma; \mathfrak{sl}(n)) \subset H^1(\Gamma; \mathfrak{gl}(n))$. Also, each cochain $c: \Gamma \rightarrow C_2$ is traceless, and therefore traceless cohomology classes in $H^1(\Gamma; C_{-1}/C_2)$ are defined. Next, notice that a traceless cohomology class in $H^1(\Gamma; C_{-1})$ projects onto a traceless cohomology class in $H^*(\Gamma; C_{-1}/C_2)$. We consider the sequence (16)

$$0 \rightarrow H^1(\Gamma; C_0/C_2) \rightarrow H^1(\Gamma; C_{-1}/C_2) \rightarrow H^1(\Gamma; C_{-1}/C_0) \rightarrow 0.$$

If a cohomology class in $H^1(\Gamma; C_{-1}/C_2)$ projects onto the trivial class in $H^1(\Gamma; C_{-1}/C_0)$ then it is in the image of $H^1(\Gamma; C_0/C_2) \rightarrow H^1(\Gamma; C_{-1}/C_2)$. By Remark 3.7

$\text{Im}(H^1(\Gamma; C_0/C_2) \rightarrow H^1(\Gamma; C_{-1}/C_2))$ is generated by $hI_n + C_2$. It follows that the class is trivial or non-traceless. We obtain the following isomorphism:

$$H^1(\Gamma; \mathfrak{sl}(n)) = H_0^1(\Gamma; \mathfrak{gl}(n)) \cong H_0^1(\Gamma; C_{-1}) \xrightarrow{\cong} H_0^1(\Gamma; C_{-1}/C_2) \xrightarrow{\cong} H^1(\Gamma; C_{-1}/C_0).$$

Here $H_0^1(\cdot)$ denotes the traceless cohomology classes. \square

Propositions 3.5 and 3.6 prove the first part of Theorem 3.3. The last part of Theorem 3.3 will be proved in the next subsection.

3.3. Irreducible $\text{SL}(n)$ representations.

Proof of the last part of Theorem 3.3. In order to prove that there are irreducible deformations of $\rho_D^{(n)}$ we will make use of Burnside's matrix lemma stating that a representation $\rho: \Gamma \rightarrow \text{GL}(n)$ is irreducible if and only if the image $\rho(\Gamma)$ generates the matrix algebra $M_n(\mathbb{C})$.

Let $\Gamma = \langle S_1, \dots, S_k \mid W_1, \dots, W_{k-1} \rangle$ be a presentation of the knot group Γ such that $h(S_i) = 1$ for all $1 \leq i \leq k$. Such a presentation of deficiency one exists always (see Section 2 in [22]).

Recall that $\rho_D^{(n)}: \Gamma \rightarrow \text{SL}(n)$ is a representation given by (5) and satisfying the conditions of Theorem 3.3:

$$\rho_D^{(n)}(\gamma) = \begin{pmatrix} 1 & u_1^+(\gamma) & z_{13}^+(\gamma) & \cdots & z_{1n}^+(\gamma) \\ 0 & 1 & u_2^+(\gamma) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & z_{n-2,n}^+(\gamma) \\ \vdots & & \ddots & 1 & u_{n-1}^+(\gamma) \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} D(\lambda_1, \dots, \lambda_n)^{h(\gamma)}.$$

Modulo conjugation of the representation $\rho_D^{(n)}$, we can assume that $u_1^+(S_1) = \dots = u_{n-1}^+(S_1) = 0$ and that $z_{ij}^+(S_1) = 0$ for $j - i \geq 2$, $1 \leq i < j \leq n$. This conjugation corresponds to adding principal derivations to the derivations u_i^+ , $1 \leq i \leq n - 1$, and to the cochains z_{ij}^+ ; $j - i \geq 2$. Thus we can suppose that $\rho_D^{(n)}(S_1) = D(\lambda_1, \dots, \lambda_n)$.

Since the derivations u_i^+ are not principal there exists for each i , $1 \leq i \leq n - 1$, a generator S_{j_i} , $j_i \in \{2, \dots, k\}$ such that $u_i^+(S_{j_i}) \neq 0$. Now, for each i , $1 \leq i \leq n - 1$, we chose a non-principal derivation $u_i^- \in Z^1(\Gamma; \mathbb{C}_{\lambda_{i+1}/\lambda_i})$. By Corollary 3.8, there exists a cocycle $V_{n-1} \in Z^1(\Gamma; \mathfrak{sl}(n))$ of the form:

$$V_{n-1} = u_1^- E_2^1 + u_2^- E_3^2 + \dots + u_{n-1}^- E_n^{n-1} + X,$$

where $X \in C^1(\Gamma, C_0)$. Up to adding principal derivations to the derivations u_i^- we may assume that $u_i^-(S_1) = 0$, $\forall 1 \leq i \leq n - 1$. By arguing as in the proof of Lemma 5.5 of [5] it can be proved that $u_i^-(S_{j_i}) \neq 0$, $1 \leq i \leq n - 1$.

Let ρ_t be a deformation of $\rho_D^{(n)}$ with leading term V_{n-1} :

$$\rho_t = (I_n + tV_{n-1} + o(t))\rho_D^{(n)}, \text{ where } \lim_{t \rightarrow 0} \frac{o(t)}{t} = 0.$$

As the set of diagonal matrices with non vanishing pairwise different entries is Zariski open, we may suppose that for sufficiently small t , $\rho_t(S_1) = D(a_1(t), \dots, a_n(t))$ is such that $a_i(t) \neq 0$, and $a_i(t) \neq a_j(t)$ for $i \neq j$. Recall that the diagonal elements λ_i are pairwise different. It follows that the matrix E_i^i is a polynomial in $\rho_t(S_1)$. For if $\mathbb{C}[\rho_t(S_1)]$ denotes the algebra generated by $\rho_t(S_1)$ we have $\mathbb{C}[E_1^1, \dots, E_n^n] \supset \mathbb{C}[\rho_t(S_1)]$, $\dim \mathbb{C}[\rho_t(S_1)] = n$, and hence $\mathbb{C}[E_1^1, \dots, E_n^n] = \mathbb{C}[\rho_t(S_1)]$.

Let \mathcal{A}_t denote the algebra generated by $\rho_t(\Gamma)$. Since $\rho_t(S_1) \in \mathcal{A}_t$ we have that $E_i^i \in \mathcal{A}_t$ for all $1 \leq i \leq n$. Hence if $B = (b_{ij}) \in \mathcal{A}_t$ then:

$$E_i^i B E_j^j = b_{ij} E_i^j \in \mathcal{A}_t.$$

We consider

$$E_i^i \rho_t(S_{j_i}) E_{i+1}^{i+1} = (\lambda_{i+1} u_i^+(S_{j_i}) + o(1)) E_i^{i+1}.$$

We have $\lambda_{i+1} u_i^+(S_{j_i}) \neq 0$, thus for t sufficiently small we have $E_i^{i+1} \in \mathcal{A}_t$.

Similarly,

$$E_{i+1}^{i+1} \rho_t(S_{j_i}) E_i^i = (t \lambda_i u_i^-(S_{j_i}) + o(t)) E_{i+1}^i$$

We have $\lambda_i u_i^-(S_{j_i}) \neq 0$ and hence for sufficiently small $t \neq 0$ we have $t(\lambda_i u_i^-(S_{j_i}) + o(t))/t \neq 0$. Therefore $E_{i+1}^i \in \mathcal{A}_t$.

The following Lemma implies that $\mathcal{A}_t = M_n(\mathbb{C})$ and ρ_t is irreducible. \square

Lemma 3.9. *The $2(n-1)$ matrices $\{E_i^j \mid |i-j|=1\}$ generate the full matrix algebra $M_n(\mathbb{C})$.*

Proof. Let $\mathcal{A} := \mathbb{C}[E_i^j \mid |i-j|=1]$. We will proceed by induction on $1 \leq i \leq n$ to prove that all elementary matrices E_i^j , $1 \leq j \leq n$, are in \mathcal{A} .

For the first step of the induction, a direct calculation gives:

$$E_1^1 = E_1^2 E_2^1, \quad E_1^3 = E_1^2 E_2^3, \quad E_1^4 = E_1^3 E_3^4, \quad \dots, \quad E_1^n = E_1^{n-1} E_{n-1}^n.$$

Hence $E_1^i \in \mathcal{A}$ for $1 \leq i \leq n$.

Suppose now that the result is true for $1 \leq i < n$ and let prove that $E_{i+1}^j \in \mathcal{A}$ for $1 \leq j \leq n$.

Clearly $E_{i+1}^i, E_{i+1}^{i+2}, E_{i+1}^{i+1} = E_{i+1}^i E_i^{i+1} \in \mathcal{A}$. For $1 \leq j \leq n$ and $j \notin \{i, i+1, i+2\}$, we have $E_{i+1}^j = E_{i+1}^i E_i^j \in \mathcal{A}$ by the induction hypothesis. This ends the proof. \square

4. THE DIAGONAL REPRESENTATION

The aim of this section is to prove Theorem 1.2. Throughout this section we will assume that λ_i/λ_{i+1} , $i = 1, \dots, n-1$, is a simple root of the Alexander polynomial $\Delta_K(t)$, and that $\Delta_K(\lambda_i/\lambda_j) \neq 0$ if $|i-j| \geq 2$. Notice that, by the symmetry of $\Delta_K(t)$, the quotients λ_{i+1}/λ_i are also simple roots of the Alexander polynomial (see the paragraph preceding Proposition 2.3).

Let $D = D(\lambda_1, \dots, \lambda_n) \in \mathrm{SL}(n)$ be the diagonal matrix with $\lambda_i \neq \lambda_j$ for $i \neq j$ and let ρ_D be the abelian representation given by $\rho_D(\gamma) = D^{h(\gamma)}$.

Recall that the quadratic cone at a representation $\rho \in R_n(\Gamma)$, introduced by W. Goldman [14], is an obstruction to integrability:

$$(17) \quad TQ_\rho(R_n(\Gamma)) = \{U \in Z^1(\Gamma; \mathrm{Ad} \rho) \mid [U \smile U] \sim 0\} \subset T_\rho R_n(\Gamma) \cong Z^1(\Gamma; \mathrm{Ad} \rho).$$

We will calculate the quadratic cone at the abelian representation ρ_D . More precisely, first we will compute a basis of the Zariski tangent space $Z^1(\Gamma; \mathfrak{sl}(n)_{\mathrm{Ad} \rho})$ (see Section 4.1), then we will determine the ideal I generated by the quadratic obstruction (17) (see Section 4.2). After that we will show that the ideal I is radical (see Lemma 4.9), and that I is in fact an intersection of prime ideals generated by linear forms (see Lemma 4.10). This guarantees that the quadratic cone is reduced. Finally, we will prove in Section 4.5 that every cocycle in the quadratic cone $TQ_{\rho_D}(R_n(\Gamma))$ is integrable, and hence the quadric cone coincides with the tangent cone.

4.1. The Zariski tangent space at the diagonal representation. Observe that we have the following decomposition:

$$(18) \quad \mathfrak{sl}(n)_{\text{Ad } \rho_D} \cong \bigoplus_{i \neq j} \mathbb{C}_{\lambda_i/\lambda_j} \oplus \mathbb{C}^{n-1}.$$

In what follows we fix the following non principal derivations:

$$h \in Z^1(\Gamma; \mathbb{C}), \quad u_i^+ \in Z^1(\Gamma; \mathbb{C}_{\lambda_i/\lambda_{i+1}}), \quad u_i^- \in Z^1(\Gamma; \mathbb{C}_{\lambda_{i+1}/\lambda_i}),$$

$i = 1, \dots, n-1$. Notice that $Z^1(\Gamma; \mathbb{C}) = H^1(\Gamma; \mathbb{C})$ since \mathbb{C} is a trivial Γ -module.

Recall that $E_i^j = E_i E_j^T$, $1 \leq i, j \leq n$, form the canonical basis of $M_n(\mathbb{C})$ and that $E_i^j E_k^l = \delta_k^j E_i^l$. Hence

$$(19) \quad [E_i^j, E_k^l] = \delta_k^j E_i^l - \delta_i^l E_k^j.$$

In what follows we will consider the cohomology classes represented by the following cocycles:

$$(20) \quad H_i = h(E_i^i - E_{i+1}^{i+1}), \quad U_i^+ = u_i^+ E_i^{i+1}, \quad \text{and } U_i^- = u_i^- E_{i+1}^i; \quad i \in \{1, \dots, n-1\}.$$

Lemma 4.1. *The $3n - 3$ cohomology classes represented by H_i , U_i^+ and U_i^- , $i \in \{1, \dots, n-1\}$, form a basis of $H^1(\Gamma; \mathfrak{sl}(n)_{\text{Ad } \rho_D})$.*

Roughly speaking, this means that the cohomology $H^1(\Gamma; \mathfrak{sl}(n)_{\text{Ad } \rho_D})$ is concentrated around the diagonal, and

$$(21) \quad \begin{aligned} \dim Z^1(\Gamma; \mathfrak{sl}(n)_{\text{Ad } \rho_D}) &= n^2 + 2n - 3, \\ \dim B^1(\Gamma; \mathfrak{sl}(n)_{\text{Ad } \rho_D}) &= n^2 - n, \\ \dim H^1(\Gamma; \mathfrak{sl}(n)_{\text{Ad } \rho_D}) &= 3(n-1). \end{aligned}$$

Moreover, we have $\dim H^0(\Gamma; \mathfrak{sl}(n)_{\text{Ad } \rho_D}) = (n-1)$. As a consequence:

$$(22) \quad \dim H^2(\Gamma; \mathfrak{sl}(n)_{\text{Ad } \rho_D}) = 2(n-1).$$

Lemma 4.2. *For the representation $\rho_D \in R_n(\Gamma)$ we have an isomorphism*

$$T_{\rho_D} R_n(\Gamma)_{\text{red}} \xrightarrow{\cong} Z^1(\Gamma; \mathfrak{sl}(n)_{\text{Ad } \rho_D}).$$

Proof. Every principal derivation is integrable, and the derivations (20) are seen to be integrable:

$$(23) \quad \rho_t^{(H_i)}(\gamma) = \exp(tH_i(\gamma))\rho_D(\gamma), \quad \text{and } \rho_t^{(U_i^\pm)}(\gamma) = \exp(tU_i^\pm(\gamma))\rho_D(\gamma).$$

This follows since diagonal matrices commute, and $(E_i^{i+1})^2 = (E_{i+1}^i)^2 = 0$. \square

Remark 4.3. The fact that the space of 1-cocycles $Z^1(\Gamma; \mathfrak{sl}(n)_{\text{Ad } \rho_D})$ is isomorphic to the Zariski tangent space $T_{\rho_D} R_n(\Gamma)$ does not imply that the representation ρ_D is scheme reduced. In order to prove this property for ρ_D , we will show in Theorem 4.13 that the tangent cone at ρ_D is reduced and then apply Lemma 2.1.

4.2. The cup product of cocycles at the diagonal representation. In general, each cocycle $U \in Z^1(\Gamma; \mathfrak{sl}(n)_{\text{Ad } \rho_D})$ has the form:

$$(24) \quad U = \sum_{i=1}^{n-1} x_i U_i^+ + y_i U_i^- + z_i H_i + C$$

where $C \in B^1(\Gamma; \mathfrak{sl}(n)_{\text{Ad } \rho_D})$ is a principal derivation, and $x_i, y_i, z_i \in \mathbb{C}$.

Lemma 4.4. *Let H_i , and U_i^\pm be the cocycles defined in Equation (20). Then*

$$[H_i \smile H_j] = 0, \quad [U_i^\pm \smile U_j^\pm] \sim 0, \quad \text{and} \quad [U_i^\mp \smile U_j^\pm] \sim 0.$$

Proof. The first relation $[H_i \smile H_j] = 0$ is obvious since diagonal matrices commute.

The next relations $[U_i^\pm \smile U_j^\pm] \sim 0$ (same sign), follow directly if $j \notin \{i+1, i-1\}$ as by (19):

$$[E_i^{i+1}, E_j^{j+1}] = \delta_j^{i+1} E_i^{j+1} - \delta_i^{j+1} E_j^{i+1} = \begin{cases} 0 & \text{if } j \notin \{i+1, i-1\}, \\ E_i^{i+2} & \text{if } j = i+1, \\ -E_{i-1}^{i+1} & \text{if } j = i-1, \end{cases}$$

and $H^2(\Gamma; \mathbb{C}_{\lambda_i/\lambda_{i+2}}) = 0$ for $|i-j| \geq 2$.

If $j = i+1$ then $u_i^+ \smile u_{i+1}^+ \in Z^2(\Gamma; \mathbb{C}_{\lambda_i/\lambda_{i+2}}) = B^2(\Gamma; \mathbb{C}_{\lambda_i/\lambda_{i+2}})$ since $\Delta_K(\lambda_i/\lambda_j) \neq 0$ if $|i-j| \geq 2$. The same argument applies for $j = i-1$. This proves $[U_i^+ \smile U_j^+] \sim 0$. The proof of $[U_i^- \smile U_j^-] \sim 0$ is similar.

Now, consider $[U_i^\pm, U_j^\mp]$ (opposite signs):

$$[E_i^{i+1}, E_{j+1}^j] = \delta_{j+1}^{i+1} E_i^j - \delta_i^j E_{j+1}^{i+1} = \begin{cases} 0 & \text{if } j \neq i, \\ E_i^i - E_{i+1}^{i+1} & \text{if } j = i. \end{cases}$$

Hence $[U_i^\pm, U_j^\mp] = 0$ if $i \neq j$, and $[U_i^\pm, U_i^\mp] = (u_i^+ \smile u_i^-)(E_i^i - E_{i+1}^{i+1})$. Now, $u_i^+ \smile u_i^- \sim 0$ since $H^2(\Gamma; \mathbb{C}) = 0$. This implies $[U_i^\pm, U_i^\mp] \sim 0$. \square

Lemma 4.5. *The $2(n-1)$ cocycles $[H_i \smile U_i^\pm]$, $i = 1, \dots, n-1$, represent a basis of $H^2(\Gamma; \mathfrak{sl}(n)_{\text{Ad } \rho_D})$.*

Proof. By (19) we obtain:

$$\begin{aligned} [H_i \smile U_i^+](\gamma_1, \gamma_2) &= h(\gamma_1) \left(\frac{\lambda_i}{\lambda_{i+1}} \right)^{h(\gamma_1)} u_i^+(\gamma_2) [E_i^i - E_{i+1}^{i+1}, E_i^{i+1}] \\ &= 2(h \smile u_i^+)(\gamma_1, \gamma_2) E_i^{i+1}. \end{aligned}$$

By a similar calculation we obtain $[H_i \smile U_i^-] = -2(h \smile u_i^-) E_{i+1}^i$. The lemma follows from the decomposition (18), Lemma 2.4, and equation (22). \square

Lemma 4.6. *For the generators (20) of $H^1(\Gamma; \mathfrak{sl}(n)_{\text{Ad } \rho_D})$, and $\epsilon \in \{\pm\}$ we have:*

$$[H_i \smile U_j^\epsilon] = \begin{cases} 0 & \text{if } |i-j| > 1, \\ -\epsilon(h \smile u_{i-1}^\epsilon) E_{i-1}^i & \text{if } j = i-1, \\ 2\epsilon(h \smile u_i^\epsilon) E_i^{i+1} & \text{if } j = i, \\ -\epsilon(h \smile u_{i+1}^\epsilon) E_{i+1}^{i+2} & \text{if } j = i+1. \end{cases}$$

Proof. Direct calculation using Equation 2. \square

We are now ready to calculate $[U \smile U]$.

Proposition 4.7. *Let $U \in Z^1(\Gamma; \mathfrak{sl}(n)_{\text{Ad } \rho_D})$ be a cocycle of the form:*

$$U = \sum_{i=1}^{n-1} x_i U_i^+ + y_i U_i^- + z_i H_i + C$$

where $C \in B^1(\Gamma; \mathfrak{sl}(n)_{\text{Ad } \rho_D})$ is a principal derivation, and $x_i, y_i, z_i \in \mathbb{C}$.

Then $[U \smile U] \sim 0$ if and only if for all $i = 1, \dots, n-1$:

$$(25) \quad (2z_i - z_{i-1} - z_{i+1})x_i = 0 \quad \text{and} \quad (2z_i - z_{i-1} - z_{i+1})y_i = 0.$$

Proof. It follows from Lemma 4.6 that $[H_i \smile U_j^\pm] \sim 0$ if $|i - j| > 1$. This and Lemma 4.4 give:

$$\begin{aligned} [U \smile U] \sim & \sum_{i=1}^{n-1} x_i z_{i-1} [H_{i-1} \smile U_i^+] + x_i z_i [H_i \smile U_i^+] + x_i z_{i+1} [H_{i+1} \smile U_i^+] \\ & + x_{i-1} z_i [H_i \smile U_{i-1}^+] + x_i z_i [H_i \smile U_i^+] + x_{i+1} z_i [H_{i+1} \smile U_{i+1}^+] \\ & + y_i z_{i-1} [H_{i-1} \smile U_i^-] + y_i z_i [H_i \smile U_i^-] + y_i z_{i+1} [H_{i+1} \smile U_i^-] \\ & + y_{i-1} z_i [H_i \smile U_{i-1}^-] + y_i z_i [H_i \smile U_i^-] + y_{i+1} z_i [H_{i+1} \smile U_{i+1}^-]. \end{aligned}$$

Here we have put $x_0 = y_0 = z_0 = x_n = y_n = z_n = 0$. With this convention we obtain:

$$(26) \quad \frac{1}{2} [U \smile U] \sim \sum_{i=1}^{n-1} (2z_i - z_{i-1} - z_{i+1}) (x_i [H_i \smile U_i^+] + y_i [H_i \smile U_i^-]).$$

Now, by Lemma 4.5, the $2(n-1)$ cocycles $[H_i \smile U_i^\pm]$, $i = 1, \dots, n-1$, represent a basis of $H^2(\Gamma; \mathfrak{sl}(n)_{\text{Ad } \rho_D})$. This proves the proposition. \square

4.3. The ideal defining the quadratic cone. In what follows we will use the following proposition:

Proposition 4.8. *Let Γ be a finitely generated group and $\rho \in R_n(\Gamma)$ a representation. We fix a basis (v_1, \dots, v_k) of $T_\rho R_n(\Gamma) \cong Z^1(\Gamma, \text{Ad } \rho)$, and we let (x_1, \dots, x_k) denote the coordinates with respect to this basis. Then the quadratic cone of $R_n(\Gamma)$ at ρ is given by the equations*

$$\sum_{i,j=1}^k x_i x_j [v_i \smile v_j] \sim 0.$$

Proof. Following W. Goldman a cocycle $u \in Z^1(\Gamma, \text{Ad } \rho)$ is a curve tangent of second order if and only if the 2-cocycle (the cup-bracketed) $[u \smile u] \sim 0$ is a coboundary. By Lemma 2.2 the curve tangents of second order coincide with the quadratic tangent cone. \square

Let R denote the polynomial ring $R := \mathbb{C}[x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}, z_1, \dots, z_{n-1}]$. In what follows we are interested in the ideal:

$$(27) \quad I^{(2)} = ((2z_i - z_{i-1} - z_{i+1})x_i, (2z_i - z_{i-1} - z_{i+1})y_i \mid i = 1, \dots, n-1)$$

which is generated by the equations (25).

Lemma 4.9. *The ideal $I^{(2)} \subset R$ is a radical ideal i.e. $I^{(2)}$ is its own radical.*

Proof. Consider the linear transformation f of $\mathbb{C}^{3(n-1)}$ given by:

$$f(x_i) = x_i, \quad f(y_i) = y_i, \quad \text{and} \quad f(z_i) = -z_{i-1} + 2z_i - z_{i+1}$$

where we have put $z_0 = z_n = 0$. The linear transformation f extends to an algebra isomorphism of R which maps the ideal J :

$$J = (x_i z_i, y_i z_i \mid i = 1, \dots, n-1) \subset R$$

to $I^{(2)}$. Notice that the ideal J is a *monomial ideal* i.e. J is an ideal generated by monomials. Hence it follows from [17, Corollary 1.2.5] that J is a radical ideal, and therefore $I^{(2)}$ also. \square

The fact that $I^{(2)}$ is a radical ideal implies that $I^{(2)}$ is the intersection of finitely many prime ideals. In order to describe these prime ideals we let 2^{n-1} denote the powerset of $\{1, \dots, n-1\}$, and for $\iota \in 2^{n-1}$ we denote by $\iota^c := \{1, \dots, n-1\} \setminus \iota$ the complement of ι . The size of $\iota \in 2^{n-1}$ will be denoted by $|\iota|$. We are interested in the ideals I_ι , $\iota \in 2^{n-1}$, given by:

$$I_\iota = (2z_i - z_{i-1} - z_{i+1}, x_j, y_j \mid i \in \iota \text{ and } j \in \iota^c).$$

Notice that $I_\iota \subset R$ is a prime ideal. In fact I_ι is generated by linear equations, and R/I_ι is isomorphic to a polynomial algebra. The scheme

$$V_\iota := (R/I_\iota, \text{Spm}(R/I_\iota)) \subset \mathbb{A}^{3(n-1)}$$

is an affine subspace of $\mathbb{A}^{3(n-1)}$ given by the equations

$$(28) \quad \begin{cases} 2z_i - z_{i-1} - z_{i+1} = 0 & \text{for } i \in \iota. \\ x_j = y_j = 0 & \text{for } j \in \iota^c. \end{cases}$$

Hence V_ι is a reduced scheme and

$$(29) \quad \dim V_\iota = 3(n-1) - |\iota| - 2|\iota^c| = 3(n-1) - |\iota| - 2((n-1) - |\iota|) = n-1 + |\iota|.$$

Given a set $E \subset R$ we will denote its *zero locus* by $V(E) \subset \mathbb{A}^{3(n-1)}$.

Lemma 4.10. *The radical ideal $I^{(2)}$ is the intersection of the prime ideals I_ι :*

$$I^{(2)} = \bigcap_{\iota \in 2^{n-1}} I_\iota.$$

Proof. Since $I^{(2)}$ and $\bigcap_{\iota \in 2^{n-1}} I_\iota$ are both radical ideals it will be sufficient to prove that

$$V(I^{(2)}) = V\left(\bigcap_{\iota \in 2^{n-1}} I_\iota\right) = \bigcup_{\iota \in 2^{n-1}} V(I_\iota).$$

If S and T are two finite sets of polynomials then $V(S) \cup V(T) = V(ST)$, where ST denotes the set of all products of an element of S by an element of T . Therefore, we obtain:

$$V((2z_i - z_{i-1} - z_{i+1})x_i, (2z_i - z_{i-1} - z_{i+1})y_i) = V(x_i, y_i) \cup V(2z_i - z_{i-1} - z_{i+1}).$$

Hence

$$\begin{aligned} V(I^{(2)}) &= \bigcap_{i=1}^{n-1} (V(x_i, y_i) \cup V(2z_i - z_{i-1} - z_{i+1})) \\ &= \bigcup_{\iota \in 2^{n-1}} \left(\left(\bigcap_{i \in \iota} V(2z_i - z_{i-1} - z_{i+1}) \right) \cap \left(\bigcap_{j \in \iota^c} V(x_j, y_j) \right) \right) \\ &= \bigcup_{\iota \in 2^{n-1}} V(2z_i - z_{i-1} - z_{i+1}, x_j, y_j \mid i \in \iota \text{ and } j \in \iota^c) \\ &= \bigcup_{\iota \in 2^{n-1}} V(I_\iota) \end{aligned}$$

which proves the lemma. □

4.4. The quadratic cone at the diagonal representation. In order to describe the quadratic cone at ρ_D we choose the basis of $Z^1(\Gamma; \text{Ad } \rho_D)$ given by the non principal derivations $H_i, U_i^\pm, i = 1, \dots, n-1$ from equation (20), and the $(n^2 - n)$ principal derivations $B_k^l, 1 \leq k, l \leq n, k \neq l$, given by

$$B_k^l(\gamma) = ((\lambda_k/\lambda_l)^{h(\gamma)} - 1)E_k^l.$$

By means of this basis we obtain an identification $\mathbb{A}^{n^2+2n-3} \cong Z^1(\Gamma; \text{Ad } \rho_D)$, and we let $(x_i, y_i, z_i, t_k^l), i = 1, \dots, n-1, 1 \leq k, l \leq n, k \neq l$, denote the coordinates of \mathbb{A}^{n^2+2n-3} .

In what follows we will identify $\mathbb{C}[x_i, y_i, z_i] \otimes \mathbb{C}[t_k^l]$ with $\mathbb{C}[x_i, y_i, z_i, t_k^l]$. This corresponds to the identification $\mathbb{A}^{3n-3} \times \mathbb{A}^{n^2-n} = \mathbb{A}^{n^2+2n-3}$. Moreover, we let $\tilde{I}^{(2)}$ and \tilde{I}_l denote the ideals

$$\tilde{I}^{(2)} := I^{(2)} \otimes \mathbb{C}[t_k^l] \subset \mathbb{C}[x_i, y_i, z_i, t_k^l] \quad \text{and} \quad \tilde{I}_l := I_l \otimes \mathbb{C}[t_k^l] \subset \mathbb{C}[x_i, y_i, z_i, t_k^l]$$

respectively. We will write $\tilde{V}_l := V(\tilde{I}_l) \subset \mathbb{A}^{n^2+2n-3}$ for short.

Proposition 4.11. *The quadratic cone $TQ_{\rho_D}R_n(\Gamma) \subset T_{\rho_D}R_n(\Gamma)$ is a union of affine subspaces, and under the identification $T_{\rho_D}R_n(\Gamma) \cong Z^1(\Gamma; \text{Ad } \rho_D) \cong \mathbb{A}^{n^2+2n-3}$, we obtain:*

$$TQ_{\rho_D}R_n(\Gamma) = V(\tilde{I}^{(2)}) = \bigcup_{\iota \in 2^{n-1}} \tilde{V}_\iota.$$

In particular, as the ideal $\tilde{I}^{(2)}$ is radical, the quadratic cone $TQ_{\rho_D}R_n(\Gamma)$ is reduced.

Proof. First we will show that if an ideal $I \subset \mathbb{C}[x_i, y_i, z_i]$ is radical, then $I \otimes \mathbb{C}[t_k^l] \subset \mathbb{C}[x_i, y_i, z_i] \otimes \mathbb{C}[t_k^l]$ is also radical. To this end, notice that for any ideal $I \subset \mathbb{C}[x_i, y_i, z_i]$ we have

$$(\mathbb{C}[x_i, y_i, z_i] \otimes \mathbb{C}[t_k^l]) / (I \otimes \mathbb{C}[t_k^l]) \cong (\mathbb{C}[x_i, y_i, z_i] / I) \otimes \mathbb{C}[t_k^l].$$

Moreover, the tensor product of two reduced \mathbb{C} -algebras is reduced [6, V, §15, Thm 3]. This implies that if $I \subset \mathbb{C}[x_i, y_i, z_i]$ is radical, then $I \otimes \mathbb{C}[t_k^l] \subset \mathbb{C}[x_i, y_i, z_i] \otimes \mathbb{C}[t_k^l]$ is also radical. Consequently the ideals \tilde{I}_l are radical, and

$$\tilde{V}_l = V(\tilde{I}_l) = V(I_l) \times \mathbb{A}^{n^2-n} \subset \mathbb{A}^{3n-3} \times \mathbb{A}^{n^2-n} = \mathbb{A}^{n^2+2n-3}$$

is an affine subspace.

For two closed subschemes $X_k = (R/I_k, \text{Spm}(R/I_k)), k = 1, 2$, of \mathbb{A}^N their union is defined by:

$$X_1 \cup X_2 = (R/(I_1 \cap I_2), \text{Spm}(R/(I_1 \cap I_2)))$$

(see [24, Section I. §4] and [13, Section I.2.1]). As the intersection of radical ideals is radical, the assertion follows from Lemma 4.10. \square

In what follows, given an element $\iota \in 2^{n-1}$, we will identify the linear subspace of $Z^1(\Gamma; \text{Ad } \rho_D)$ defined by the radical ideal \tilde{I}_l with the affine subspace \tilde{V}_l . Notice that by equation (29) we have:

$$\dim \tilde{V}_l = n^2 - n + (n - 1 + |\iota|) = n^2 - 1 + |\iota|.$$

Therefore, there is exactly one component \tilde{V}_\emptyset of dimension $n^2 - 1$ and exactly one component $\tilde{V}_{\{1, \dots, n-1\}}$ of dimension $n^2 + n - 2$. These components are of minimal, respectively, maximal dimension. If $n = 2$ then there are no other components, but for $n > 2$ there are.

We can easily identify the component of minimal dimension $n^2 - 1$: the surjection $h: \Gamma \rightarrow \mathbb{Z}$ induces a closed immersion (see Proposition 1.7 of [25]) $h^*: R_n(\mathbb{Z}) \hookrightarrow R_n(\Gamma)$. Its image $h^*(R_n(\mathbb{Z}))$ is the subvariety of abelian representations of Γ and will be denoted

by R_{ab} . Notice that $R_{ab} \cong \mathrm{SL}(n)$ is an irreducible $(n^2 - 1)$ -dimensional component of $R_n(\Gamma)$.

Proposition 4.12. *Let $\tilde{V}_{\{1, \dots, n-1\}}$ and \tilde{V}_\emptyset be the components of extreme dimension in the quadratic cone $TQ_{\rho_D}R_n(\Gamma)$. Then*

$$T_{\rho_D}R_D^{(n)} = \tilde{V}_{\{1, \dots, n-1\}} \quad \text{and} \quad T_{\rho_D}R_{ab} = \tilde{V}_\emptyset.$$

Therefore, ρ_D is a smooth point of each of the two components $R_D^{(n)}$ and R_{ab} which intersect transversely at the orbit of ρ_D .

Proof. The irreducible algebraic component R_{ab} has dimension $n^2 - 1$, and is isomorphic to $\mathrm{SL}(n)$. Hence $\rho_D \in R_{ab}$ is a smooth point. A tangent vector in $T_{\rho_D}R_{ab}$ is a linear combination of $\{H_1, \dots, H_{n-1}\}$ and $\{B_k^l\}$ therefore $T_{\rho_D}R_{ab}$ corresponds to a subspace of \tilde{V}_\emptyset . Now, $T_{\rho_D}R_{ab} = \tilde{V}_\emptyset$ follows since

$$\dim \tilde{V}_\emptyset = n^2 - 1 = \dim T_{\rho_D}R_{ab}.$$

By Lemma 3.1, the representation ρ_D is contained in the closure of the orbit of $\rho_D^{(n)}$ and hence is contained in the component $R_D^{(n)}$ of maximal dimension $n^2 + n - 2$.

The tangent cone $TC_{\rho_D}R_D^{(n)}$ has only components of dimension $(n^2 + n - 2)$, and is contained in the quadratic cone $TQ_{\rho_D}R_D^{(n)}$. On the other hand, $TQ_{\rho_D}R_D^{(n)}$ is contained in $TQ_{\rho_D}R_n(\Gamma)$, and therefore $TC_{\rho_D}R_D^{(n)}$ is contained in $\tilde{V}_{\{1, \dots, n-1\}}$ since this is the unique $(n^2 + n - 2)$ -dimensional component of maximal dimension of $TQ_{\rho_D}R_n(\Gamma)$ (see Lemma 4.11). Hence $TC_{\rho_D}R_D^{(n)} = \tilde{V}_{\{1, \dots, n-1\}}$ is a linear subspace. This implies that the tangent space $T_{\rho_D}R_D^{(n)}$ coincides with the tangent cone $TC_{\rho_D}R_D^{(n)}$, and $\rho_D \in R_D^{(n)}$ is a smooth point. Finally, for \tilde{V}_\emptyset and $\tilde{V}_{\{1, \dots, n-1\}}$ we have

$$\begin{aligned} \dim(\tilde{V}_{\{1, \dots, n-1\}} + \tilde{V}_\emptyset) &= \dim(\tilde{V}_{\{1, \dots, n-1\}}) + \dim(\tilde{V}_\emptyset) - \dim B^1(\Gamma, \mathrm{Ad} \rho) \\ &= n^2 + n - 2 + n^2 - 1 - (n^2 - n) \\ &= n^2 + 2n - 3 = \dim T_{\rho_D}R_n(\Gamma). \end{aligned} \quad \square$$

4.5. The tangent cone at the diagonal representation. In this section, we will prove that every cocycle in the quadratic cone at ρ_D is an integrable cocycle. More precisely, we construct an arc of representations with endpoint ρ_D and tangent to a given cocycle in the quadratic cone.

Theorem 4.13. *Every component \tilde{V}_l is contained in the tangent cone $TC_{\rho_D}R_n(\Gamma)$. Therefore, the tangent cone $TC_{\rho_D}R_n(\Gamma)$ and quadratic cone $TQ_{\rho_D}R_n(\Gamma)$ coincide. Hence the tangent cone $TC_{\rho_D}R_n(\Gamma)$ is reduced, and consequently the diagonal representation ρ_D is a reduced point of the representation variety.*

In what follows we let $A \oplus B$ denote the direct sum of the matrices A and B :

$$A \oplus B = \left(\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{0} & B \end{array} \right).$$

The case $n = 3$. Before treating the general case, we shall consider the case $n = 3$ i.e. $D = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$.

As in Theorem 3.3 we suppose that λ_1/λ_2 and λ_2/λ_3 are simple roots of the Alexander polynomial $\Delta_K(t)$, and that λ_1/λ_3 is not a root of $\Delta_K(t)$. By Proposition 4.11 the quadratic cone $TQ_D R_3(\Gamma)$ has four components \tilde{V}_\emptyset , $\tilde{V}_{\{1\}}$, $\tilde{V}_{\{2\}}$ and $\tilde{V}_{\{1,2\}}$. Let us consider these components :

\tilde{V}_\emptyset :: The component \tilde{V}_\emptyset is 8-dimensional. Up to principal derivations the vector space \tilde{V}_\emptyset is generated by the two cocycles $H_1 = h(E_1^1 - E_2^2)$ and $H_2 = h(E_2^2 - E_3^3)$. Each linear combination of these cocycles is integrable and

$$\rho_t^{(c_1 H_1 + c_2 H_2)}(\gamma) = \exp(t(c_1 H_1 + c_2 H_2)(\gamma)) \rho_D(\gamma)$$

is clearly an arc of representations as diagonal matrices commute.

$\tilde{V}_{\{1,2\}}$:: This component is a 10-dimensional vector space. Up to principal derivations it is generated by the four cocycles U_k^\pm for $k = 1, 2$. By Proposition 4.12 the representation ρ_D is a smooth point of the component $R_D^{(3)}$, and $\tilde{V}_{\{1,2\}}$ is the tangent space $T_{\rho_D} R_D^{(3)}$. Hence every vector in $\tilde{V}_{\{1,2\}}$ is integrable.

$\tilde{V}_{\{1\}}$:: This component is a 9-dimensional vector space which is generated, up to coboundaries, by the three cocycles $H := H_1 + 2H_2 = \begin{pmatrix} h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & -2h \end{pmatrix}$ and by U_1^\pm . In order to show that every cocycle in $\tilde{V}_{\{1\}}$ is integrable we have to show that the linear combination

$$cH + d^+ U_1^+ + d^- U_1^- = \begin{pmatrix} ch & d^+ u_1^+ & 0 \\ d^- u_1^- & ch & 0 \\ 0 & 0 & -c2h \end{pmatrix}, \quad c, d^+, d^- \in \mathbb{C},$$

is integrable. We will see that the cocycle

$$U = \begin{pmatrix} 0 & d^+ u_1^+ & 0 \\ d^- u_1^- & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is integrable in a specific way. To this end, we choose a complex number z_1 such that $z_1^2 = \lambda_1 \lambda_2$, and we consider the decomposition of the matrix $D = z_1 D_1 \oplus z_2 D_2$ where $D_1 = \text{diga}(\lambda_1/z_1, \lambda_2/z_1) \in \text{SL}(2)$, $z_2 = \lambda_3$, and $D_2 = (1) \in \text{SL}(1)$. Now, $\rho_{D_1}: \Gamma \rightarrow \text{SL}(2)$ is a representation which verifies the hypothesis of Theorem 1.1, and the cocycle $U_1 := \begin{pmatrix} 0 & d^+ u_1^+ \\ d^- u_1^- & 0 \end{pmatrix}$ is integrable i.e. there exist an analytic path of representations $\rho_t^{(U_1)}: \Gamma \rightarrow \text{SL}(2)$ such that $\rho_0^{(U_1)} = \rho_{D_1}$, and

$$\frac{d}{dt} \rho_t^{(U_1)}(\gamma) \Big|_{t=0} \rho_{D_1}(\gamma)^{-1} = U_1(\gamma) = \begin{pmatrix} 0 & d^+ u_1^+(\gamma) \\ d^- u_1^-(\gamma) & 0 \end{pmatrix}$$

for all $\gamma \in \Gamma$. Finally we define a path of representations $\rho_t^{(cH+U)}: \Gamma \rightarrow \text{SL}(3)$ by

$$\rho_t^{(cH+U)}(\gamma) = \exp(tcH(\gamma)) (z_1^{h(\gamma)} \rho_t^{(2)}(\gamma) \oplus z_2^{h(\gamma)}).$$

Notice that $\rho_t^{(cH+U)}$ is a representation since the block matrices commute. Moreover,

$$\rho_0^{(cH+U)}(\gamma) = I_3 (z_1^{h(\gamma)} \rho_{D_1}(\gamma) \oplus z_2^{h(\gamma)}) = \begin{pmatrix} z_1 \begin{pmatrix} \lambda_1/z_1 & 0 \\ 0 & \lambda_2/z_1 \end{pmatrix} & 0 \\ 0 & 0 & z_2 \end{pmatrix}^{h(\gamma)} = \rho_D(\gamma),$$

and

$$\frac{d}{dt} \rho_t^{(cH+U)}(\gamma) \Big|_{t=0} \rho_D(\gamma)^{-1} = (cH(\gamma) + (U_1(\gamma) \oplus 0)) = (cH + U)(\gamma).$$

$\tilde{V}_{\{2\}}$:: This case is similar to the previous case.

The general case. In order to describe the general case we need some notations. First we describe the cocycles in \tilde{V}_ι for a proper subset $\iota \in 2^{n-1}$, $\iota \neq \emptyset$. For such ι we let denote $\iota^c = \{j_1, \dots, j_l\} \subset \{1, \dots, n-1\}$ where $0 = j_0 < j_1 < j_2 < \dots < j_l < j_{l+1} = n$, and we define $n_s := j_s - j_{s-1}$, $s = 1, \dots, l+1$. Notice that $\sum_{s=1}^{l+1} n_s = n$.

Now, equations (28) give that for each $i \in \iota$ we have $U_i^\pm \in \tilde{V}_\iota$, and for $s = 1, \dots, l$ we have $F_{n_s} \in \tilde{V}_\iota$ where F_{n_s} denotes the diagonal matrix

$$F_{n_s} = \text{diag}(\underbrace{0, \dots, 0}_{n_1 + \dots + n_{s-1}}, \underbrace{n_{l+1}h}_{n_s}, \underbrace{0, \dots, 0}_{n_{s+1} + \dots + n_l}, \underbrace{-n_s h}_{n_{l+1}}).$$

Therefore,

$$\tilde{V}_\iota = \langle U_i^\pm, F_{n_s} \mid i \in \iota, s = 1, \dots, l \rangle \oplus B^1(\Gamma; \text{Ad } \rho_D).$$

Notice that we have for all U_i^\pm with $i \in \iota$ and all F_{n_s} that $F_{n_s}(\gamma)$ and $U_i^\pm(\gamma)$ commute for all $\gamma \in \Gamma$. Then a cocycle $U \in \tilde{V}_\iota$ can be written as a linear combination of the diagonal matrices F_{n_s} , and block matrices:

$$(30) \quad \begin{pmatrix} U_{n_1} & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & U_{n_{l+1}} \end{pmatrix} = \bigoplus_{s=1}^{l+1} U_{n_s}$$

where for all $1 \leq s \leq l+1$:

$$U_{n_s} = \begin{pmatrix} 0 & u_{j_{s-1}+1}^+ & 0 & \dots & 0 \\ u_{j_{s-1}+1}^- & 0 & u_{j_{s-1}+2}^+ & \ddots & \vdots \\ 0 & u_{j_{s-1}+2}^- & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & u_{j_s-1}^+ \\ 0 & \dots & 0 & u_{j_s-1}^- & 0 \end{pmatrix}.$$

Here, $u_k^+ \in Z^1(\Gamma, \mathbb{C}_{\lambda_k/\lambda_{k+1}})$ and $u_k^- \in Z^1(\Gamma, \mathbb{C}_{\lambda_{k+1}/\lambda_k})$ where $j_{s-1} + 1 \leq k \leq j_s - 1$.

Proof of Theorem 4.13. First we notice that every diagonal cocycle is integrable, and for $F = \sum_{s=1}^l c_s F_s \in \tilde{V}_\iota$ we define

$$\rho_t^{(F)}(\gamma) = \exp(tF(\gamma)).$$

For the non diagonal cocycles we proceed as in Example 4.5. Let $U \in \tilde{V}_\iota$ be a given cocycle. We will use the decomposition (30)

$$U = \bigoplus_{s=1}^{l+1} U_{n_s},$$

we split up a diagonal matrix $D = D(\lambda_1, \dots, \lambda_n) \in \text{SL}(n)$ accordingly as $D = \tilde{D}_1 \oplus \dots \oplus \tilde{D}_{l+1}$, where $\tilde{D}_s = D(\lambda_{j_{s-1}+1}, \dots, \lambda_{j_s})$ is build from n_s consecutive diagonal entries of D . We choose $z_s \in \mathbb{C}^*$ such that $z_s^{n_s} = \det(\tilde{D}_s)$, and we define $D_s := z_s^{-1} \tilde{D}_s \in \text{SL}(n_s)$. We obtain

$$D = z_1 D_1 \oplus \dots \oplus z_{l+1} D_{l+1}.$$

Now, ρ_{D_s} verifies the hypotheses of Lemma 3.1 and Theorem 3.3, and therefore there is a $(n_s^2 + n_s - 2)$ -dimensional irreducible component $R_{D_s}^{(n_s)} \subset R_{n_s}(\Gamma)$. By Lemma 4.2 we have

$$T_{\rho_{D_s}} R_{D_s}^{(n_s)} = \langle u_{j_{s-1}+1}^\pm, \dots, u_{j_s-1}^\pm \rangle \oplus B^1(\Gamma; \text{Ad } \rho_{D_s}).$$

Notice that in the case $n_s = 1$ we have $D_s = (1) \in \mathrm{SL}(1) = \{(1)\}$ is the trivial group and $R_{D_s}^{(1)}$ is zero-dimensional and contains only the trivial representation.

We define representations $\rho_t^{(U)}: \Gamma \rightarrow \mathrm{SL}(n)$ by

$$\rho_t^{(U)}(\gamma) = \bigoplus_{s=1}^{l+1} z_s^{h(\gamma)} \rho_t^{(U_s)}(\gamma)$$

where $\rho_t^{(U_s)}: \Gamma \rightarrow \mathrm{SL}(n_s)$ is a family such that

$$\frac{d}{dt} \rho_t^{(U_s)}(\gamma) \Big|_{t=0} \cdot \rho_{D_{n_s}}(\gamma)^{-1} = U_s(\gamma).$$

Notice that $\rho_0^{(U)} = \rho_D$, and it follows directly that

$$\frac{d}{dt} \rho_t^{(U)}(\gamma) \Big|_{t=0} \cdot \rho_D(\gamma)^{-1} = U(\gamma).$$

Finally notice that the matrices $\rho_t^{(U)}(\gamma)$ and $\rho_t^{(F)}(\gamma)$ commute for all F and U in \tilde{V}_l . Therefore, $\rho_t^{(F+U)}: \Gamma \rightarrow \mathrm{SL}(n)$ given by

$$\rho_t^{(F+U)}(\gamma) = \rho_t^{(F)}(\gamma) \cdot \rho_t^{(U)}(\gamma)$$

is a path of representation, and its derivative in $t = 0$ gives the cocycle $F + U \in \tilde{V}_l$.

The last assertion is an immediate consequence of Proposition 4.11 and Lemma 2.1. \square

Remark 4.14. For each k , $0 \leq k \leq n - 1$, there is at least one irreducible algebraic component of $R_n(\Gamma)$ of dimension $n^2 - 1 + k$ which contain ρ_D . So there are at least n irreducible components of $R_n(\Gamma)$ which contain ρ_D . An example is the torus knot $T(3, 2)$, and $n = 3$. The representation variety $R_3(\Gamma_{3,2})$ has exactly three algebraic components (see [29, Theorem 1.1]). All the components contain the diagonal representation ρ_D where $D = \mathrm{diag}(\exp(\pi/3), 1, \exp(-\pi/3))$ (see Section 6.3).

On the other hand, there are at most $\binom{n-1}{k}$ components of dimension $n^2 - 1 + k$ which contain ρ_D . Hence there are at most 2^{n-1} components through ρ_D . An example is given by the torus knot $T(3, 4)$, and $n = 3$. The representation variety $R_3(\Gamma_{3,4})$ has exactly two algebraic components of dimension 9 [29, Theorem 1.1]. With help of [29, Proposition 9.1] it is possible to prove that the diagonal representation of $\Gamma_{3,4}$ which maps the meridian to $D = \mathrm{diag}(\xi^4, \xi, \xi^{-5})$ for $\xi = \exp(i\pi/18)$ is contained in both components. Moreover, the representation ρ_D verifies the hypothesis of Theorem 3.3, and therefore ρ_D is contained in 4 components. (See the discussion in Section 6.2.)

5. LOCAL STRUCTURE OF THE CHARACTER VARIETY

Our aim in this section is to study the local structure of the character variety in a neighborhood of the abelian character χ_D . Most results in the literature concern the characters of irreducible representations. Very little is known in the case where the character corresponds to that of an abelian representation.

We will make use of the Luna's Étale Slice Theorem which is a tool for the local study of the character variety $X_n(\Gamma)$.

The general references for this section are [26], [12], [7], [32], [25].

5.1. **A slice étale.** Let G be a reductive algebraic group acting on an affine variety X and let $X // G$ be the quotient variety.

The following result is of independent importance and it appears implicitly in [32].

Theorem 5.1. *Let G be a reductive algebraic group acting on an affine variety X . Let $x \in X$ be such that the orbit $G(x)$ is closed in X .*

Then there exist a locally closed subvariety \mathcal{S} of X such that:

$$T_x^{Zar} X \cong T_x^{Zar}(G(x)) \oplus T_x^{Zar} \mathcal{S}.$$

Moreover, if x is a smooth point of X then

$$T_{t(x)}^{Zar}(X // G) \cong T_0(T_x^{Zar} \mathcal{S} // G_x)$$

where $t: X \rightarrow X // G$ is the quotient morphism.

Proof. Since the orbit $G(x)$ is closed in X , by Luna's Étale Slice theorem, there exists a locally closed subvariety \mathcal{S} of X such that \mathcal{S} is affine and contains x , \mathcal{S} is G_x invariant, the image of the G -morphism $\psi: G \times_{G_x} \mathcal{S} \rightarrow X$ induced by the action of G on X is a saturated open subset U of X and the restriction of $\psi: G \times_{G_x} \mathcal{S} \rightarrow U$ is strongly étale.

Thus, we obtain the following isomorphisms:

$$(31) \quad T_{(e,x)}^{Zar} \psi: T_{(e,x)}^{Zar} G \times_{G_x} \mathcal{S} \xrightarrow{\cong} T_x^{Zar} U \cong T_x^{Zar} X$$

and

$$(32) \quad T_{[(e,x)]}^{Zar}(\psi // G): T_{[(e,x)]}^{Zar}(G \times_{G_x} \mathcal{S} // G) \cong T_{[x]}^{Zar}(S // G_x) \xrightarrow{\cong} T_{[x]}^{Zar}(U // G) = T_{[x]}^{Zar}(X // G).$$

Moreover, as x is a smooth point of X , and using Luna's Slice Theorem at smooth points, we obtain that $T_x^{Zar} X = T_x^{Zar}(G(x)) \oplus T_x^{Zar} \mathcal{S}$ and there exists an étale G_x -invariant morphism $\phi: \mathcal{S} \rightarrow T_x^{Zar} \mathcal{S}$ such that $\phi(x) = 0$, $T_x \phi = Id$, the image of ϕ is a saturated open subset W of $T_x^{Zar} \mathcal{S}$ and the restriction of $\phi: \mathcal{S} \rightarrow W$ is a strongly étale G_x -morphism. As a consequence, we have the following isomorphism:

$$(33) \quad T_{[x]}^{Zar}(\phi // G_x): T_{[x]}^{Zar}(S // G_x) \xrightarrow{\cong} T_0(W // G_x) = T_0(T_x^{Zar} \mathcal{S} // G_x).$$

By combining the preceding isomorphisms (32) and (33), we obtain the desired isomorphism:

$$T_{[x]}^{Zar}(X // G) \cong T_0(T_x^{Zar} \mathcal{S} // G_x).$$

□

5.2. **Tangent spaces to slices and cohomology.** The reductive group $G = \mathrm{SL}(n)$ acts by conjugation on $R_n(\Gamma)$. More precisely, for $A \in \mathrm{SL}(n)$ and $\rho \in R_n(\Gamma)$ we define $(A \cdot \rho)(\gamma) = A\rho(\gamma)A^{-1}$ for all $\gamma \in \Gamma$. We will write $\rho \sim \rho'$ if there exists $A \in \mathrm{SL}(n)$ such that $\rho' = A \cdot \rho$, and we will call ρ and ρ' equivalent. For $\rho \in R_n(\Gamma)$ we define its character $\chi_\rho: \Gamma \rightarrow \mathbb{C}$ by $\chi_\rho(\gamma) = \mathrm{tr}(\rho(\gamma))$. We have $\rho \sim \rho' \Rightarrow \chi_\rho = \chi_{\rho'}$. By [25, Theorem 1.28]), if ρ and ρ' are semisimple, then $\rho \sim \rho'$ if and only if $\chi_\rho = \chi_{\rho'}$.

The algebraic quotient or GIT quotient for the action of $\mathrm{SL}(n)$ on $R_n(\Gamma)$ is called the character variety. This quotient will be denoted by $X_n(\Gamma) = R_n(\Gamma) // \mathrm{SL}(n)$. The character variety is not necessary an irreducible affine algebraic set. Work of C. Procesi [31] implies that there exists a finite number of group elements $\{\gamma_i \mid 1 \leq i \leq M\} \subset \Gamma$ such

that the image of $t: \mathbb{R}_n(\Gamma) \rightarrow \mathbb{C}^M$ given by $t(\rho) = (\chi_\rho(\gamma_1), \dots, \chi_\rho(\gamma_M))$ can be identified with the affine algebraic set $X_n(\Gamma) \cong t(R_n(\Gamma))$. This justifies the name character variety.

Let $t: R_n(\Gamma) \rightarrow X_n(\Gamma)$; $\rho \mapsto \chi_\rho$ be the canonical projection. Then t is a regular morphism and for all $\gamma \in \Gamma$, $I_\gamma: X_n(\Gamma) \rightarrow \mathbb{C}$; $\chi_\rho \mapsto \chi_\rho(\gamma) = \text{tr}(\rho(\gamma))$ is a regular morphism.

For $\rho \in R_n(\Gamma)$, denote the stabilizer of ρ under the G -action by conjugation on $R_n(\Gamma)$ by G_ρ . It is the centralizer of $\rho(\Gamma)$ in G .

Proposition 5.2. *Let ρ be a representation in $R_n(\Gamma)$ such that the orbit $G(\rho)$ of ρ is closed in $R_n(\Gamma)$.*

Let R_ρ be an affine subvariety of $R_n(\Gamma)$ which contains ρ and which is invariant under the conjugation action of G . Let \mathcal{S} be a slice of R_ρ at ρ .

Then

$$T_\rho^{Zar} \mathcal{S} \cong T_\rho^{Zar} R_\rho / T_\rho^{Zar} G(\rho) \quad \text{and} \quad T_{[\rho]}^{Zar}(R_\rho // G) \cong T_{[\rho]}^{Zar}(\mathcal{S} // G_\rho).$$

Moreover, if ρ is a smooth point of R_ρ then

$$T_{[\rho]}^{Zar}(R_\rho // G) \cong T_0(T_\rho^{Zar} \mathcal{S} // G_\rho).$$

In particular, if ρ is a reduced smooth point of $R_n(\Gamma)$ such that the orbit $G(\rho)$ of ρ is closed in $R_n(\Gamma)$ then there exists a slice \mathcal{S} of $R_n(\Gamma)$ at ρ such that

$$T_\rho^{Zar} \mathcal{S} \cong H^1(\Gamma, \text{Ad } \rho) \quad \text{and} \quad T_{[\rho]}^{Zar} X_n(\Gamma) \cong T_0(H^1(\Gamma, \text{Ad } \rho) // G_\rho).$$

The proof of this result is implicitly contained in [3] and [32]. But for the sake of completeness, we will reproduce it here.

Proof. We have just to prove the first isomorphism. All the other assertions are an immediate consequence of Theorem 5.1.

Let $\psi: G \times_{G_\rho} \mathcal{S} \rightarrow R_\rho$ be the G -morphism étale given by Luna's Theorem and $U := \text{Im } \psi$. We denote by θ the isomorphism $G(\rho) \xrightarrow{\theta} G/G_\rho$; $\theta(g\rho g^{-1}) = [g]$ for $g \in G$.

As the fibration $G \rightarrow G/G_\rho$ is locally trivial, the morphism $p: G \times_\rho \mathcal{S} \rightarrow G/G_\rho$; $[g, v] \mapsto [g]$ is also a locally trivial fibration with fiber $g\mathcal{S}$ in the étale topology ([30]). Let Z be an open set trivializing G/G_ρ ; then there exists a morphism étale $\phi: Z \times \mathcal{S} \rightarrow p^{-1}(Z)$; $([g], \varphi) \mapsto [g, \varphi]$ such that $p \circ \phi = pr_1$ where $pr_1: Z \times \mathcal{S} \rightarrow \mathcal{S}$ is the projection on the first component.

Consider the following chain of morphisms:

$$\begin{aligned} T_\rho^{Zar} \mathcal{S} &\xrightarrow{j_2} T_\rho^{Zar} G(\rho) \oplus T_\rho^{Zar} \mathcal{S} \xrightarrow{d\theta \oplus id} T_{[e]}^{Zar} G/G_\rho \oplus T_\rho^{Zar} \mathcal{S} \xrightarrow{d\phi} \\ &T_{[e, \rho]}^{Zar}(G \times_{G_\rho} \mathcal{S}) \xrightarrow{d\psi} T_\rho^{Zar} R_\rho \xrightarrow{q} T_\rho^{Zar} R_\rho / T_\rho^{Zar} G(\rho) \end{aligned}$$

where j_2 is the natural inclusion and q is the canonical projection. As $p^{-1}(Z)$ is open in $G \times_{G_\rho} \mathcal{S}$ and ϕ, ψ étale, we obtain that $d\phi$ and $d\psi$ are isomorphisms. Moreover, it is easy to see that $d\psi \circ d\phi \circ (d\theta \oplus id) \circ j_2 = di$ where $i: \mathcal{S} \rightarrow R_\rho$ is the natural inclusion.

On the other hand, let $j_1: T_\rho^{Zar} G(\rho) \rightarrow T_\rho^{Zar} G(\rho) \oplus T_\rho^{Zar} \mathcal{S}$ be the inclusion map; then $d\psi \circ d\phi \circ (d\theta \oplus id) \circ j_1$ is the natural inclusion $T_\rho^{Zar} G(\rho) \rightarrow T_\rho^{Zar} R_\rho$. As a consequence, $\alpha = q \circ di: T_\rho^{Zar} \mathcal{S} \rightarrow T_\rho^{Zar} R_\rho / T_\rho^{Zar} G(\rho)$ is an isomorphism and $s = di \circ \alpha^{-1}: T_\rho^{Zar} R_\rho / T_\rho^{Zar} G(\rho) \rightarrow T_\rho^{Zar} R_\rho$ is a section for q . Thus $T_\rho^{Zar} \mathcal{S} \cong T_\rho^{Zar} R_\rho / T_\rho^{Zar} G(\rho)$. \square

5.3. Local structure of the character variety. As the set $R_n^{irr}(\Gamma)$ of irreducible representations in $R_n(\Gamma)$ is Zariski open, its image $X_n^{irr}(\Gamma) = t(R_n^{irr}(\Gamma))$ is Zariski open in $X_n(\Gamma)$. Their Zariski closures in $R_n(\Gamma)$ and $X_n(\Gamma)$ will be denoted by $\overline{R_n^{irr}(\Gamma)}$ and $\overline{X_n^{irr}(\Gamma)}$.

For the proof of the next Proposition, we will make use of the following result which was proven in [Boyer-Zhang, Lemma 4.1] for the group $G = \text{SL}(2)$ and which generalizes easily to the group $G = \text{SL}(n)$.

Lemma 5.3. *For any algebraic component $X_0 \subset X_n(\Gamma)$ such that $\dim X_0 > 0$, there is an algebraic component R_0 of $t^{-1}(X_0)$ which satisfies $t(R_0) = X_0$.*

If X_0 contains the character of an irreducible representation, then R_0 is invariant under conjugation, contains irreducible representations and is uniquely determined by X_0 .

Proposition 5.4. *Let χ_D be the character $\chi_D = \chi(\rho_D^{(n)}) = \chi(\rho_D)$. Denote by $X_n(\mathbb{Z}) := t(R_n(\mathbb{Z}))$ and $X_D^{(n)} := t(R_D^{(n)})$ the quotients of the components $R_n(\mathbb{Z})$ and $R_D^{(n)}$ respectively. Then*

- (1) *The character χ_D is a smooth point of $\overline{X_n^{irr}(\Gamma)}$.*
- (2) *The character χ_D is a smooth point of each of the two $(n-1)$ -dimensional components $X^{(n)}$ and $X_n(\mathbb{Z})$.*
- (3) *The intersection of the two above components at χ_D is transverse, i.e.*

$$T_{\chi_D}^{Zar} X_n(\mathbb{Z}) \cap T_{\chi_D}^{Zar} X^{(n)} = \{0\}.$$

Proof. (1) As the irreducible component $R_D^{(n)}$ contains irreducible representations (Theorem 1.1), $X^{(n)} := t(R_D^{(n)})$ is an irreducible component of $\overline{X_n^{irr}(\Gamma)}$ and $\dim X^{(n)} = \dim R_D^{(n)} - (n^2 - 1) = n - 1$ ([25]).

To see that $X^{(n)}$ is the only component of $\overline{X_n^{irr}(\Gamma)}$ to contain χ_D , suppose that another does as well. Then there exists an irreducible component $X_0 \subset \overline{X_n^{irr}(\Gamma)}$ distinct from $X^{(n)}$ such that $\chi_D \in X_0$.

Let $R_0 \subset R_n(\Gamma)$ be the unique algebraic component invariant under conjugation, which contains irreducible representations and such that $t(R_0) = X_0$ (Lemma 5.3). Evidently $R_0 \neq R_D^{(n)}$. Let $\phi \in R_0$ such that $t(\phi) = \chi_D = t(\rho_D)$. As R_0 is invariant under conjugation and closed, we obtain $\overline{G(\phi)} \subset R_0$. By Lemma 1.26 of [25], $\overline{G(\phi)}$ contains a semisimple representation ϕ_0 . Thus $t(\phi_0) = t(\phi) = t(\rho_D)$. As ϕ_0 and ρ_D are semisimple, we conclude that $\phi_0 \sim \rho_D$ and by the same $\rho_D \in R_0$. By our analysis of the tangent cone of ρ_D , $R_D^{(n)}$ is the only algebraic component of $R_n(\Gamma)$ containing irreducible representations and the representation ρ_D . We conclude that $R_0 = R_D^{(n)}$ which contradicts our hypothesis.

- (2) The orbit $G(\rho_D)$ is closed in $R_n(\Gamma)$ with $\dim G(\rho_D) = n^2 - n$ and, as $D = \rho_D(\mu)$ is a regular element of $\text{SL}(n)$, the stabilizer G_{ρ_D} of ρ_D is the set of diagonal matrices $\mathbb{T} := (\mathbb{C}^*)^{(n-1)}$.

Now, recall that for the abelian representation ρ_D , $H^1(\Gamma; \text{Ad } \rho_D)$ is identified with $\mathbb{A}^{3(n-1)}$ by choosing the basis represented by the cocycles (U_i^+, U_i^-, H_i) , $1 \leq i \leq n-1$.

The G_{ρ_D} -action on $Z^1(\Gamma; \text{Ad } \rho_D)$ descends to an action on $H^1(\Gamma; \text{Ad } \rho_D)$. More precisely, the action of a diagonal matrix $\text{diag}(t_1, \dots, t_n) \in \mathbb{T}$ on a cocycle in

$H^1(\Gamma; \text{Ad } \rho_D)$ is given by

$$(t_1, \dots, t_n) \cdot (U_1^+, U_1^-, \dots, U_{n-1}^+, U_{n-1}^-, H_1, \dots, H_{n-1}) = \left(\frac{t_1}{t_2} U_1^+, \frac{t_2}{t_1} U_1^-, \dots, \frac{t_n}{t_{n-1}} U_{n-1}^+, \frac{t_{n-1}}{t_n} U_{n-1}^-, H_1, \dots, H_{n-1} \right).$$

The action is then trivial on the abelian cocycles H_i , $1 \leq i \leq n-1$. Furthermore, \mathbb{C}^* acts on \mathbb{C}^2 by $\lambda \cdot (z_1, z_2) = (\lambda z_1, 1/\lambda z_2)$ then $\mathbb{C}^2 // \mathbb{C}^* \cong \mathbb{C}$.

In particular, $V_\emptyset // (\mathbb{C}^*)^{n-1} \cong V_\emptyset$ and $V_{\{1, \dots, n-1\}} // (\mathbb{C}^*)^{n-1} \cong (\mathbb{C})^{n-1}$.

It is a well known fact that $X_n(\mathbb{Z}) := t(R_n(\mathbb{Z}))$ is an irreducible component of $X_n(\Gamma)$ of dimension $n-1$ and that χ_D is a smooth point of $X_n(\mathbb{Z})$ (see [11, Example 1.2]). Thus $\dim_{\chi_D} X_n(\mathbb{Z}) = n-1$.

Observe that we can prove this known result by applying the Proposition 5.2 to the representation ρ_D as a smooth point of the affine variety $R_n(\mathbb{Z})$ to obtain a slice \mathcal{S}_{Ab} of $R_n(\mathbb{Z})$ such that

$$T_{\rho_D}^{Zar} \mathcal{S}_{Ab} \cong T_{\rho_D}^{Zar} R_n(\mathbb{Z}) / T_{\rho_D}^{Zar} G(\rho) \cong V_\emptyset$$

and

$$T_{\chi_D}^{Zar} X_n(\mathbb{Z}) \cong T_{\chi_D}^{Zar} (R_n(\mathbb{Z}) // \text{SL}(n)) \cong T_0(T_{\rho}^{Zar} \mathcal{S}_{Ab} // \mathbb{T}) \cong T_0(V_\emptyset // (\mathbb{C}^*)^{n-1}) \cong V_\emptyset.$$

So we deduce that $\dim T_{\chi_D}^{Zar} X_n(\mathbb{Z}) = n-1 = \dim X_n(\mathbb{Z})$ and χ_D is a smooth point of $X_n(\mathbb{Z})$.

Again, by applying the Proposition 5.2 to the representation ρ_D as a smooth point of the affine variety $R_D^{(n)}$ to obtain a slice \mathcal{S} of $R_D^{(n)}$ such that

$$T_{\rho_D}^{Zar} \mathcal{S} \cong T_{\rho_D}^{Zar} R_D^{(n)} / T_{\rho_D}^{Zar} G(\rho) \cong V_{\{1, \dots, n-1\}}$$

and

$$T_{\chi_D}^{Zar} X^{(n)} \cong T_{\chi_D}^{Zar} (R_D^{(n)} // \text{SL}(n)) \cong T_0(T_{\rho}^{Zar} \mathcal{S} // (\mathbb{C}^*)^{n-1}) \cong T_0(V_{\{1, \dots, n-1\}} // (\mathbb{C}^*)^{n-1}) \cong (\mathbb{C})^{n-1}.$$

So we deduce that $\dim T_{\chi_D}^{Zar} X^{(n)} = n-1 = \dim X^{(n)}$ and thus χ_D is a smooth point of $X^{(n)}$.

- (3) Note that, as $T_{\chi_D}^{Zar} X_n(\mathbb{Z}) \cong V_\emptyset$, $T_{\chi_D}^{Zar} X^{(n)} \cong V_{\{1, \dots, n-1\}}$ and $V_\emptyset \cap V_{\{1, \dots, n-1\}} = \{0\}$, we obtain that $T_{\chi_D}^{Zar} X_n(\mathbb{Z}) \cap T_{\chi_D}^{Zar} X^{(n)} = \{0\}$. This implies that the intersection of the two components $X_n(\mathbb{Z})$ and $X^{(n)}$ at χ_D is transverse. □

There is an analogue of the following theorem when the considered representation is an irreducible one ([25, Theorem 2.13]).

Theorem 5.5. *Let $ad: T_{\rho_D^{(n)}}^{Zar} G(\rho_D^{(n)}) \rightarrow T_{\rho_D^{(n)}}^{Zar} R_n(\Gamma)$ and $dt: T_{\rho_D^{(n)}}^{Zar} R_n(\Gamma) \rightarrow T_{\chi_D}^{Zar} X_n(\Gamma)$ be the tangent morphisms induced by the conjugation action of $\text{SL}(n)$ on $\rho_D^{(n)}$ and by the canonical projection $t: R_n(\Gamma) \rightarrow X_n(\Gamma)$.*

Then we have the exact sequence

$$0 \rightarrow T_{\rho_D^{(n)}}^{Zar} G(\rho_D^{(n)}) \xrightarrow{ad} T_{\rho_D^{(n)}}^{Zar} R_n(\Gamma) \xrightarrow{dt} T_{\chi_D}^{Zar} \overline{X_n^{irr}(\Gamma)} \rightarrow 0$$

and $H^1(\Gamma; \text{Ad } \rho_D^{(n)}) \cong T_{\chi_D}^{Zar} \overline{X_n^{irr}(\Gamma)}$.

Proof. Consider the commutative diagram:

$$\begin{array}{ccc} T_{\rho_D^{(n)}}^{Zar} G(\rho_D^{(n)}) & \xrightarrow{\iota} & \text{Inn}(\Gamma, \text{Ad } \rho_D^{(n)}) = B^1(\Gamma; \text{Ad } \rho_D^{(n)}) \\ \downarrow ad & & \downarrow j \\ T_{\rho_D^{(n)}}^{Zar} R(\Gamma) & \longrightarrow & \text{Der}(\Gamma, \text{Ad } \rho_D^{(n)}) = Z^1(\Gamma; \text{Ad } \rho_D^{(n)}) \end{array}$$

where $\iota: \mathfrak{sl}(n) \ni a \mapsto (\iota_a: \gamma \mapsto \text{Ad}_{\rho_D^{(n)}(\gamma)} a - a; \forall \gamma \in \Gamma)$.

Now j is injective and ι is injective as $H^0(\Gamma; \text{Ad } \rho_D^{(n)}) = 0$ (Proposition 3.6), so ad is injective too. By the commutativity of the above diagram and for all $a \in \mathfrak{sl}(n)$, we have $ad(a) = \frac{d\rho_s}{ds}|_{s=0}$ where $\rho_s(\gamma) = (I + s(\text{Ad } \rho_D^{(n)}(\gamma)a - a)) \rho_D^{(n)}(\gamma); \forall \gamma \in \Gamma$.

Thus, for all $\gamma \in \Gamma$, $dI_\gamma \circ dt \circ ad(a) = \frac{d}{ds}|_{s=0} (\text{tr } \rho_s(\gamma)) = \text{tr} (a \rho_D^{(n)}(\gamma) - \rho_D^{(n)}(\gamma)a) = 0$ and $dt \circ ad(a) = 0$.

To prove that $\text{Ker } dt = \text{Im } ad$, let $\frac{d\rho_s}{ds}|_{s=0} \in \text{Ker } dt$ where $\rho_s(\gamma) = (I + sU(\gamma)) \rho_D^{(n)}(\gamma)$ and $U \in Z^1(\Gamma, \text{Ad } \rho_D^{(n)})$. The relation $dt \left(\frac{d\rho_s}{ds}|_{s=0} \right) = 0$ implies that $\frac{d}{ds}|_{s=0} (\text{tr } \rho_s(\gamma)) = 0 \forall \gamma \in \Gamma$.

By Corollary 3.8 and modulo conjugation by a coboundary, we may assume that there exist complexes $\epsilon_i \in \mathbb{C}$, and cochains $a_i \in C^1(\Gamma, \mathbb{C})$, $1 \leq i \leq n-1$, such that the cocycle $U \in Z^1(\Gamma; \text{Ad } \rho_D^{(n)})$ is of the form:

$$U = \begin{pmatrix} a_1 & * & * & \dots & * \\ \epsilon_1 u_1^- & a_2 & * & \ddots & \vdots \\ 0 & \epsilon_2 u_2^- & a_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ 0 & \dots & 0 & \epsilon_{n-1} u_{n-1}^- & a_n \end{pmatrix} \in Z^1(\Gamma; \text{Ad } \rho_D^{(n)}).$$

Thus $0 = \frac{d}{ds}|_{s=0} (\text{tr } \rho_s(\gamma))$ implies that:

$$\sum_{i=1}^n \lambda_i^{h(\gamma)} a_i(\gamma) + \sum_{i=1}^{n-1} \lambda_{i+1}^{h(\gamma)} \epsilon_i u_i^-(\gamma) u_i^+(\gamma) = 0 \quad \forall \gamma \in \Gamma.$$

As a consequence, we obtain:

$$\sum_{i=1}^n a_i(\gamma) + \sum_{i=1}^{n-1} \epsilon_i u_i^-(\gamma) u_i^+(\gamma) = 0 \quad \forall \gamma \in \Gamma'.$$

Now $\sum_{i=1}^n a_i(\gamma) = 0$ as $U \in Z^1(\Gamma; \text{Ad } \rho_D^{(n)})$, so we obtain:

$$\sum_{i=1}^{n-1} \epsilon_i u_i^-(\gamma) u_i^+(\gamma) = 0 \quad \forall \gamma \in \Gamma'.$$

The fact that the derivations u_i^\pm , $1 \leq i \leq n-1$, factor through Γ'' give the equation:

$$\sum_{i=1}^{n-1} \epsilon_i u_i^-(x) u_i^+(x) = 0 \quad \forall x \in \Gamma' / \Gamma'' \otimes \mathbb{C}.$$

Recall that as $\alpha_i = \lambda_i/\lambda_{i+1}$, $1 \leq i \leq n-1$ are simple roots of the Alexander polynomial Δ_K and $\Delta_K(\lambda_i/\lambda_j) \neq 0 \ \forall |i-j| \geq 2$, the Alexander module decomposes as $\bigoplus_{i=1}^{n-1} (\Lambda/t - \alpha_i \oplus \Lambda/t - \alpha_i^{-1}) \oplus \tau$ where τ is a torsion Λ -module with no $(t - \alpha_i^\pm)$ -torsion. Choose a basis $(e_1^+, e_1^-, \dots, e_{n-1}^+, e_{n-1}^-)$ of $\bigoplus_{i=1}^{n-1} (\Lambda/t - \alpha_i \oplus \Lambda/t - \alpha_i^{-1})$.

Modulo coboundaries, we may suppose that the non principal derivations u_i^\pm ; $1 \leq i \leq n-1$ are generators of $H^1(\Gamma; \mathbb{C}_{\alpha^\pm})$ as described in Proposition 2.3. In particular,

$$u_i^\pm(e_i^\pm) = 1; \quad u_i^\pm(e_i^\mp) = 0; \quad u_i^\pm(e_j^\pm) = u_i^\pm(e_j^\mp) = 0 \ \forall 1 \leq i \neq j \leq n-1.$$

Evaluating the preceding equation at $x = e_i^+ + e_i^-$, $1 \leq i \leq n-1$, gives:

$$\begin{aligned} 0 &= \sum_{j=1}^{n-1} \epsilon_j u_j^-(e_i^+ + e_i^-) u_j^+(e_i^+ + e_i^-) \\ &= \epsilon_i u_i^-(e_i^+ + e_i^-) u_i^+(e_i^+ + e_i^-) \\ &= \epsilon_i u_i^-(e_i^-) u_i^+(e_i^+) \\ &= \epsilon_i. \end{aligned}$$

We obtain that $\epsilon_i = 0 \ \forall 1 \leq i \leq n-1$ and the cocycle $U \in Z^1(\Gamma; \text{Ad } \rho_D^{(n)})$ is thus of the form:

$$U = \begin{pmatrix} \tilde{a}_1 & * & \dots & * \\ 0 & \tilde{a}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & \tilde{a}_n \end{pmatrix} \in Z^1(\Gamma, C_0).$$

By Remark 3.7 the traceless cohomology group $H_0^1(\Gamma; C_0/C_2) \cong 0$, thus the cocycle $U \in \text{Ker } dt$ is in fact a coboundary and $\text{Ker } dt = \text{Im } ad$.

Finally, the surjectivity of the morphism dt is a consequence of the fact that

$$\text{rg}(dt) = \dim T_{\rho_D^{(n)}}^{\text{Zar}} R_n(\Gamma) - \text{rg}(ad) = n^2 - 1 + n - 1 - (n^2 - 1) = n - 1 = \dim T_{\chi_D}^{\text{Zar}} \overline{X_n^{\text{irr}}}(\Gamma).$$

□

6. EXAMPLES

In this paragraph we discuss and illustrate by some examples the significance of our results in light of what is already known in the literature.

6.1. Knots with quadratic Alexander polynomial. For $\text{SL}(3)$ and knots with a quadratic Alexander polynomial Theorem 3.3 follows from the results in [19]. More precisely, up to an action of the center $\{I_3, \omega I_3, \omega^2 I_3\}$ of $\text{SL}(3)$, every diagonal representation which verifies the conditions of Theorem 3.3 is the composition of an diagonal representation in $\text{SL}(2)$ with the canonical representation $r_3: \text{SL}(2) \rightarrow \text{SL}(3)$. To see this, let K be such a knot with a root α , and let ρ_D be an abelian representation of K which maps the meridian to

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \in \text{SL}(3)$$

with λ_i , $1 \leq i \leq 3$ pairwise distinct. Then

$$\begin{cases} \lambda_1 \lambda_2 \lambda_3 = 1 \\ \lambda_1 / \lambda_2 = \alpha^{\epsilon_1}; \epsilon_1 = \pm 1 \\ \lambda_2 / \lambda_3 = \alpha^{\epsilon_2}; \epsilon_2 = \pm 1 \end{cases}$$

We obtain that $\lambda_1 = \lambda_3 \alpha^{\epsilon_1 + \epsilon_2}$, $\lambda_2 = \lambda_3 \alpha^{\epsilon_2}$, $\epsilon_1, \epsilon_2 = \pm 1$ and $\lambda_3^3 \alpha^{\epsilon_1 + 2\epsilon_2} = 1$. The condition $\lambda_1 \neq \lambda_3$ implies that $\epsilon_1 = \epsilon_2$ and $\lambda_3 = \alpha^{-\epsilon_1} \omega^k$ with ω a primitive 3rd root of unity. As a consequence

$$\rho_D(\mu) = \omega^k \begin{pmatrix} \alpha^{\epsilon_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha^{-\epsilon_1} \end{pmatrix} = \omega^k r_2 \begin{pmatrix} \alpha^{\epsilon_1/2} & 0 \\ 0 & \alpha^{-\epsilon_1/2} \end{pmatrix}.$$

This proves the claim.

6.2. The torus knot $T(3, 4)$. On the other hand, if the Alexander polynomial is not quadratic, there might be deformations which can not be detected by the results in [19]. To see this let us consider the torus knot $T(3, 4)$ which is the knot 8_{19} in the knot tables. The knot group is $\Gamma_{3,4} = \langle a, b \mid a^3 = b^4 \rangle$, a meridian is given by $m = ab^{-1}$, and the Alexander polynomial is $\Delta_{3,4}(t) = (1 - t + t^2)(1 - t^2 + t^4)$ is the product of two cyclotomic polynomials $\phi_6(t)$ and $\phi_{12}(t)$. The abelianization $h: \Gamma_{3,4} \rightarrow \langle t \mid - \rangle$ is given by $h(a) = t^4$ and $h(b) = t^3$. The abelian representation of $\Gamma_{3,4}$ which maps the meridian to

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} \xi^4 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi^{-5} \end{pmatrix}$$

for $\xi = \exp(i\pi/18)$ verifies the hypothesis of Theorem 3.3: $\lambda_1/\lambda_2 = \xi^3$ is a primitive 12-th root of unity, $\lambda_2/\lambda_3 = \xi^6$ is a primitive 6-th root of unity, and $\lambda_1/\lambda_3 = \xi^9$ is not a root of $\Delta_{3,4}(t)$. Therefore by Theorem 3.3 the abelian representation admits irreducible deformations, and is contained in a 10-dimensional component of $R_3(\Gamma_{3,4})$. These deformations can not be detected by the results in [19].

Finally there are deformations of diagonal representations which are not detected by Theorem 3.3. The $\text{SL}(2)$ -character variety of $\Gamma_{3,4}$ has 4 one-dimensional components, three of these components V_1 , V_2 and V_3 contain characters of irreducible representations and one component contains only characters of abelian (reducible) representations. Each component V_i is parametrized by the trace of the meridian, and contains exactly two characters of abelian representations which are limit of irreducible representations (see for example [23]). In order to obtain irreducible $\text{SL}(3)$ representations of $\Gamma_{3,4}$, we can compose irreducible $\text{SL}(2)$ -representation with the canonical representation $r_3: \text{SL}(2) \rightarrow \text{SL}(3)$ (see Proposition 3.1 in [19]). In this way we obtain 6 diagonal representations which are limit of irreducible representations. For example, let $\rho_{ab}: \Gamma_{3,4} \rightarrow \text{SL}(2)$ be given by

$$a \mapsto A = \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix} \text{ and } b \mapsto B = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

where $\eta = \exp(i\pi/3)$ is a primitive 6-th root of unity, and $\zeta = \exp(i\pi/4)$ is a primitive 8-th root of unity. Then $\eta\zeta^{-1}$ is a primitive 24-th root of unity, and $r_3 \circ \rho_{ab}$ maps the meridian $m = ab^{-1}$ to the diagonal matrix

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} (\eta\zeta^{-1})^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (\eta\zeta^{-1})^{-2} \end{pmatrix}.$$

Now, $\lambda_1/\lambda_2 = \exp(i\pi/6)$ and $\lambda_2/\lambda_3 = \lambda_1$ are primitive 12-th roots of unity, and $\lambda_1/\lambda_3 = \lambda_1^2$ is a primitive 6-th root of unity. Therefore, all three quotients are roots of $\Delta_{3,4}(t)$ the deformations of this abelian representation are not detected by Theorem 3.3. On the other hand, Proposition 3.1 in [19] shows that the representation $r_3 \circ \rho_{ab}$ admits irreducible deformations. Moreover, it is possible to prove that the representation ρ_{ab} is contained in a 12-dimensional component of the $\mathrm{SL}(3)$ -representation space $R_3(\Gamma_{3,4})$ (see [29, Proposition 8.2]).

6.3. The trefoil knot. By Proposition 4.12 we know that the representation ρ_D a smooth point of the two components $R_D^{(n)}$ and R_{ab} . For other components this might be not the case.

This situation occurs already for the trefoil knot and representations of its knot group Γ to $\mathrm{SL}(3)$. It follows from work of J. Porti and V. Muñoz that $R_3(\Gamma)$ has three algebraic components. One component of irreducible representations, one component of totally reducible representations, and one component of partially reducible representations [29].

The primitive 6th root of unity $\eta = \exp(i\pi/3)$ is a simple root of the Alexander polynomial $\Delta(t) = t^2 - t + 1$. Moreover we choose λ a primitive 12th root of unity such that $\lambda^2 = \eta$. It follows that $\rho_{D(\lambda, \lambda^{-1})} \in R_2(\Gamma)$ is contained in a 4-dimensional component $R_D^{(2)} \subset R_2(\Gamma)$.

If $n = 3$ and $D = D(\lambda^2, 1, \lambda^{-2})$, we obtain two families of partially reducible representations $\phi, \psi: \mathbb{C}^* \times R_D^{(2)} \hookrightarrow R_3(\Gamma)$ given by

$$\phi(s, \varrho)(\gamma) = (s^{h(\gamma)} \varrho(\gamma)) \oplus s^{-2h(\gamma)} \quad \text{and} \quad \psi(s, \varrho)(\gamma) = s^{2h(\gamma)} \oplus (s^{-h(\gamma)} \varrho(\gamma)).$$

The two families are contained in the same component of partially reducible representation of $R_3(\Gamma)$. This component is 9-dimensional. Notice also that $\phi(s, \varrho)$ and $\psi(s^{-1}, \varrho)$ are conjugate in $R_3(\Gamma)$.

Now observe that the image of $D\phi(\lambda, \rho_{D_1})$ is contained in \tilde{V}_1 and that the image of $D\psi(\lambda, \rho_{D_1})$ is contained in \tilde{V}_2 . Hence the tangent space of ρ_D at the variety of partially irreducible representations contains $\tilde{V}_1 + \tilde{V}_2$ which is 12-dimensional. This implies that ρ_D is a singular point of the component of partially reducible representations.

REFERENCES

- [1] Leila Ben Abdelghani and Daniel Lines. Involutions on knot groups and varieties of representations in a Lie group. *J. Knot Theory Ramifications*, 11(1):81–104, 2002.
- [2] Leila Ben Abdelghani. Espace des représentations du groupe d’un nœud classique dans un groupe de Lie. *Ann. Inst. Fourier (Grenoble)*, 50(4):1297–1321, 2000.
- [3] Leila Ben Abdelghani. Variété des caractères et slice étale de l’espace des représentations d’un groupe. *Ann. Fac. Sci. Toulouse Math. (6)*, 11(1):19–32, 2002.
- [4] Leila Ben Abdelghani and Michael Heusener. Irreducible representations of knot groups into $\mathrm{SL}(n, \mathbb{C})$. *Publ. Mat.*, 61(2):751–764, 2017.
- [5] Leila Ben Abdelghani, Michael Heusener, and Hajer Jebali. Deformations of metabelian representations of knot groups into $\mathrm{SL}(3, \mathbb{C})$. *J. Knot Theory Ramifications*, 19(3):385–404, 2010.
- [6] Nicolas Bourbaki. *Algebra II. Chapters 4–7*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2003.
- [7] Michel Brion. Introduction to actions of algebraic groups. *Les cours du CIRM*, 1(1):1–22, 2010.
- [8] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *GTM*. Springer-Verlag, New York, 1982.
- [9] Gerhard Burde. Darstellungen von Knotengruppen. *Math. Ann.*, 173:24–33, 1967.
- [10] Georges de Rham. Introduction aux polynômes d’un nœud. *Enseign. Math. (2)*, 13:187–194 (1968), 1967.

- [11] Igor Dolgachev. *Lectures on invariant theory*, volume 296 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003.
- [12] Jean-Marc Drézet. Luna’s slice theorem and applications. In *Algebraic group actions and quotients*, pages 39–89. Hindawi Publ. Corp., Cairo, 2004.
- [13] David Eisenbud and Joe Harris. *The geometry of schemes*, volume 197 of *GTM*. Springer-Verlag, New York, 2000.
- [14] William M. Goldman. The symplectic nature of fundamental groups of surfaces. *Adv. in Math.*, 54(2):200–225, 1984.
- [15] William M Goldman and John J Millson. The deformation theory of representations of fundamental groups of compact kähler manifolds. *Publ. Math. Inst. Hautes Etudes Sci.*, (67):43–96, 1988.
- [16] C. McA. Gordon. Some aspects of classical knot theory. In *Knot theory (Proc. Sem., Plans-sur-Bex, 1977)*, volume 685 of *Lecture Notes in Math.*, pages 1–60. Springer, Berlin, 1978.
- [17] Jürgen Herzog and Takayuki Hibi. *Monomial ideals*, volume 260 of *GTM*. Springer-Verlag London, Ltd., London, 2011.
- [18] Michael Heusener. $SL(n, \mathbb{C})$ -representation spaces of knot groups. In *Topology, Geometry and Algebra of low-dimensional manifolds*, pages 1–26. RIMS Kôkyûroku, 2016. <http://www.kurims.kyoto-u.ac.jp/~kyodo/kokyuroku/contents/1991.html>.
- [19] Michael Heusener and Ouardia Medjerab. Deformations of reducible representations of knot groups into $SL(n, \mathbb{C})$. *Math. Slovaca*, 66(5):1091–1104, 2016.
- [20] Michael Heusener and Joan Porti. Deformations of reducible representations of 3-manifold groups into $PSL_2(\mathbb{C})$. *Algebr. Geom. Topol.*, 5:965–997, 2005.
- [21] Michael Heusener and Joan Porti. The scheme of characters in SL_2 . *Trans. Amer. Math. Soc.*, 376:6283–6313, 2023.
- [22] Michael Heusener, Joan Porti, and Eva Suárez Peiró. Deformations of reducible representations of 3-manifold groups into $SL_2(\mathbb{C})$. *J. Reine Angew. Math.*, 530:191–227, 2001.
- [23] Eric Paul Klassen. Representations in $SU(2)$ and the fundamental group of the Whitehead link and of double knots. *Forum Math.*, 5:93–109, 1993.
- [24] Ernst Kunz. *Introduction to commutative algebra and algebraic geometry*. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2013.
- [25] Alexander Lubotzky and Andy R. Magid. Varieties of representations of finitely generated groups. *Mem. Amer. Math. Soc.*, 58(336):xi+117, 1985.
- [26] Domingo Luna. Slices étales. In *Sur les groupes algébriques*, Bull. Soc. Math. France, Mém. 33, pages 81–105. Soc. Math. France, Paris, 1973.
- [27] J. S. Milne. *Algebraic groups*, volume 170 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017.
- [28] David Mumford. *The red book of varieties and schemes*, volume 1358 of *Lecture Notes in Math*. Springer-Verlag, Berlin, 1988.
- [29] Vicente Muñoz and Joan Porti. Geometry of the $SL(3, \mathbb{C})$ -character variety of torus knots. *AGT*, 16(1):397–426, Feb 2016.
- [30] A. N. Parshin, Igor Rostislavovich Shafarevich, Tonny Albert Springer, V. L. Popov, and E. B. Vinberg. *Algebraic geometry*. Springer, 1994.
- [31] C. Procesi. The invariant theory of $n \times n$ matrices. *Advances in Math.*, 19(3):306–381, 1976.
- [32] Adam Sikora. Character varieties. *Trans. Amer. Math. Soc.*, 364(10):5173–5208, 2012.
- [33] André Weil. Remarks on the cohomology of groups. *Ann. of Math. (2)*, 80:149–157, 1964.

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