Completely rethinking physics to understand the universe

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Abstract

Humans have developed mathematical theories to explain observed physical phenomena in nature. However, the links between these theories are not fully understood in 2024. Apparent incompatibilities between quantum mechanics and relativity, for instance, still exist. This paper proposes a comprehensive theory of the universe that includes gravitation, quantum mechanics, electromagnetism, special relativity, and general relativity. All of these theories are derived from the single postulate that the universe is a four-dimensional ball undergoing accelerating inflation, the surface of which is our three-dimensional world. The paper shows how each theory is derived from this postulate.

Keywords *– Universe, four-dimensional space, mechanical deformation, fluid mechanics, added mass, compressibility*

1 Introduction

In the mid-17$^{\text{th}}$ century, Isaac Newton published his law of gravitation in *Philosophiae Naturalis Principia Mathematica*. He assumed that the gravitational forces experienced by the planets were reciprocally proportional to the squares of their distances. The proportional constant was first measured by Henry Cavendish in 1797. Classical mechanics remained the mainstay of science until the end of the 19$^{\text{th}}$ century. The wave aspect of light, described by electromagnetism and Maxwell's equations, suggested that light must propagate in a medium that scientists called the luminiferous ether. In the 1880s, while most of them were convinced of its existence, A. Michelson and E. Morley [1] carried out several experiments that challenged this hypothesis. In particular, they did not measure "ether wind". Classical mechanics could not explain this result and more and more scientists abandoned the ether concept. It was the case of A. Einstein who, together with M. Grossmann, D. Hilbert, H. Lorentz and H. Poincaré (for the most famous scientists), contributed to the special and general theories of relativity. These theories did not require a tangible material such as the ether medium. They were followed by the discovery of new physical phenomena. One example is the discovery of the expansion of the universe based on the redshift. In 1913, Slipher observed the universe's expansion [2]. In 1922, Friedman formulated mathematically a dynamic universe based on general relativity [3]. Then, Lemaître [4] and Hubble [5] independently formulated a law according to which galaxies move away from each other at a rate proportional to their distance. In 1998, two independent teams discovered that this expansion was accelerating [6, 7]. Scientists explain this phenomenon with dark energy [8] but its physical nature remains unknown. Many experimenters have carried out tests to check these theories. So far, special relativity and quantum mechanics have passed these tests.

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General relativity may also be consistent with observations. The precession of Mercury's perihelion [9, 10, 11], the bending of light by the Sun [12, 13, 14], gravitational redshift [15], and gravitational waves [16, 17] are successes of general relativity. However, this theory requires dark matter [18], for example to match the rotation curves of galaxies [19]. Scientists are trying to detect this hypothetical matter where it should be, without success so far. Others are focusing on Modified Newtonian Dynamics (MOND) [20], a theory in which Newton's laws are modified to fit the properties of galaxies.

In 1901, Max Planck [21] solved the problem of black-body radiation by quantizing energy that would later lead to quantum mechanics. This theory correctly explains the behaviour of atoms and molecules, quantum tunnelling, supraconductivity, *etc.* Mathematically, the theory is described by the Schrödinger equation, governing wave functions. Max Born interpreted the square of the complex modulus of the wave function as the probability density of finding the particle at a given time and position. From this point on, the scientific community was divided into two categories, deterministic and non-deterministic physicists. With the Copenhagen interpretation of quantum mechanics, the non-deterministic physicists won the debate. This way of thinking considers that there is no hidden variable responsible for the random behaviour of particles. In the mid-20^{th} century, the success of quantum electrodynamics led to the development of quantum field theory for the other fundamental interactions, the weak and strong interactions. Despite many attempts, gravitation has not been successfully embedded in such a formalism. Theoretical physicists are developing ideas to couple gravity to the other three fundamental interactions through the so-called "theories of everything". The first attempt to unify gravitation and electromagnetism was the Kaluza-Klein theory, which adds a fifth dimension to spacetime [22, 23]. The two most famous families of theories of everything are string theory [24] and loop quantum gravity [25]. Like string theory, many other theories take extra dimensions into account. For instance, extra dimensions are added to spacetime in the Randall-Sundrum models [26, 27].

Therefore, the question of understanding the universe remains open at the beginning of the 21^{st} century. This is likely due to the intrinsic dual nature of a physical theory, which consists of two elements, a mathematical framework and an interpretation of that framework. Thus, the construction of mathematical frameworks based on experiments, such as Newtonian mechanics, special relativity or quantum mechanics, is the easier part of the process and ensures predictions, but not necessarily understanding. As a result, theories may have multiple interpretations. The current main interpretation of the mathematical framework of relativity is time relativity, while the main interpretation for quantum mechanics is the Copenhagen interpretation. However, neither has been rigorously proven. When experimenters make measurements that are consistent with the predictions of these theories, the measurements are often taken to confirm them. Instead, these measurements only indicate that the mathematical frameworks are correct, possibly up to higher order terms that are negligible in this range of measurements. It is worth noting that this does not necessarily validate our physical interpretation. For instance, measuring a frequency difference with an atomic clock does not necessarily support any particular interpretation of the mathematical framework of relativity. Regardless of the interpretation, the mathematical framework inherently results in a variation of the measured frequency depending on the relative velocity through the Lorentz transformation. Therefore, an atomic clock is only a real clock under the assumption that time is relative. In a nutshell, measuring effects and not the cause of those effects never proves the cause, but only the existence of the effects. Thus, it might be possible to interpret relativity differently from time relativity and still predict the relativistic effects that follow from the equations. Besides, this was the objective of several physicists around 1900, but they failed to find a convincing explanation, like Lorentz's theory.

Based on this observation, this article proposes alternative interpretations for existing theories, extending some works already existing in the literature but within a more global and structured framework. The aim is to propose an original and common explanation for the origin of different physical theories that are often considered independent or without discovered links. Based on one postulate and

several secondary hypotheses relative to each theory, we successively propose a fundamental reason for the existence of gravitation, quantum mechanics, electromagnetism, special and general relativity. We have developed a mathematical framework that leads to the fundamental laws of each theory, except for quantum mechanics. In the case of quantum mechanics, we rely on the well-known pilot-wave interpretation. In that sense, the idea discussed throughout this paper should be understood as a hypothetical unification of existing fundamental physical theories. The postulate and the reasons leading to it are presented first. Following this, each physical theory is detailed in a dedicated section, outlining the corresponding mathematical approach.

2 The postulate

Before giving the postulate of the theory developed in the paper, its origin is explained based on three points. First, general relativity is based on the curvature of spacetime caused by the presence of matter. To aid in understanding this concept, the literature often uses the analogy of a two-dimensional space being deformed into a three-dimensional one (Fig. 1). But this is only an illustration of how three-dimensional space can be warped into four-dimensional space.

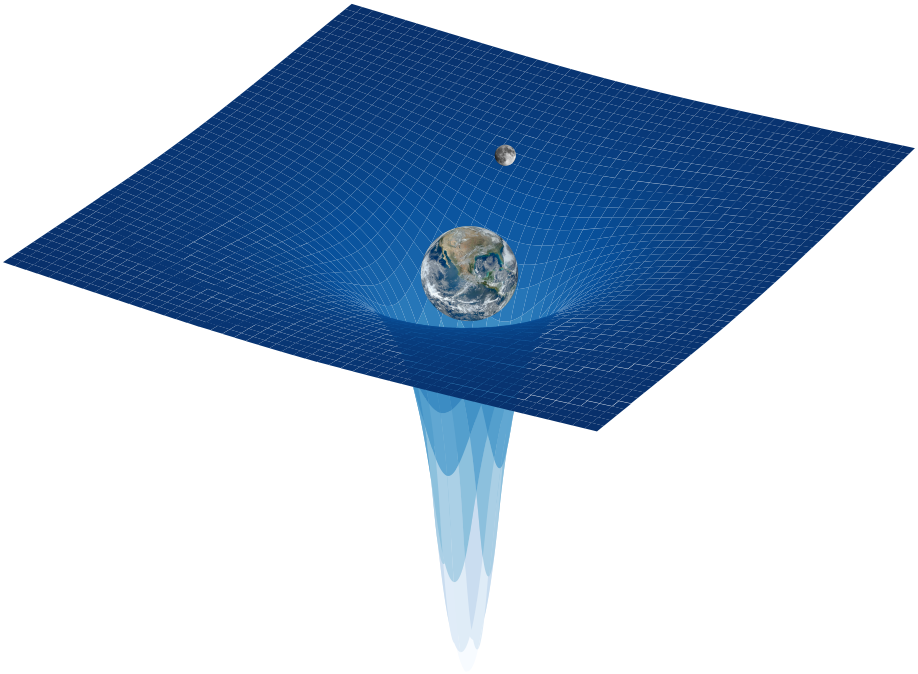

Figure 1: Deformation of spacetime due to the presence of a heavy body. The light body rolls on the curved spacetime. Spacetime is represented as a 2D surface but is actually a 3D spatial + 1D temporal surface.

Second, the expansion of the universe shows that the observable universe does not contain a point that could be considered as the origin of the big bang. The Hubble-Lemaître law states that all points are moving away from Earth and there is no reason for our planet to be at the centre of the universe. To understand this process, scientists again use an analogy with two-dimensional space. For example, if the universe has a positive curvature, our habitat can be likened to the surface of an inflating ball. All points on the surface of an inflating ball move away from each other, while the centre is not part of this surface. Thus, the universe would be a 3-sphere (sphere in three dimensions), locally deformed due to mass, embedded in a four-dimensional space.

Third, it is widely believed that relativity involves the propagation of light in vacuum, making it a non-material wave. This would be due to the impossibility for an absolute frame of reference to coexist with relativity. In special relativity, there is indeed no absolute frame of reference, but this does not mean that it cannot exist, this means that if it does exist, it does not take part in the mathematical framework of special relativity. Light could propagate in matter while obeying the laws of relativity, there is no contradiction, as already noted by Einstein himself, who returned to a concept of aether in 1920 [28].

Based on these three points, it is proposed that **the universe is a four-dimensional ball undergoing accelerating inflation and containing matter that is a continuous medium, at least near its surface**. Both the observable universe and ourselves would exist on this surface.

From this single hypothesis, we will derive the same mathematical framework as special relativity, understand in a new way the pilot wave theory, which is a candidate interpretation of quantum mechanics, and mathematically demonstrate the relativistic effects in gravity. In addition, in the next section, we will show that this structure of the Universe, with additional hypotheses, involves Newton's law of gravitation.

Before going any further, let us give a preliminary argument in favour of the proposed theory. The electromagnetic and gravitational waves become material waves in this theory. The speed of light should thus be given by a common relation such as $c \propto \sqrt{\frac{P}{\rho}}$ with P a parameter homogeneous to a pressure and ρ a density, both characterising the continuous material at the surface of the four-dimensional universe, so cosmic vacuum from our point of view. This contradicts the current hypothesis about the speed of light, which is supposed to be a fundamental constant of the universe and therefore not given by a formula. However, we have shown in [29] that such a relation exists:

$$c \sim \sqrt{10\frac{(\varepsilon_0 e^{-2})^3 (k_B T_{CMB})^4}{\rho_c}} \sim 3 \times 10^8 \ m.s^{-1} \tag{2.1}$$

where ε_0 is the vacuum permittivity, e is the elementary charge, k_B is the Boltzmann constant, T_{CMB} is the temperature of the CMB (Cosmic Microwave Background) and ρ_c is the critical density of the universe. The latter is close to the mean vacuum density and can be considered as such in an order-of-magnitude analysis. The parameter $(\varepsilon_0 e^{-2})^3 (k_B T_{CMB})^4$ is homogeneous to a pressure in Pa. Note that because of the 4D hypothesis, the density should be in $kg.m^{-4}$ and the pressure in $N.m^{-3}$. However, the 4D density would be given by a relation like $\frac{\rho_c}{L}$ with L the thickness of our "three-dimensional" world (in the direction of the fourth dimension), like the pressure term. As the ratio of both is considered in the formula, the thickness L disappears, making it possible to use 3D terms to express the speed of light. Finally, it is worth noting that because our world appears to be 3D, L must be very weak. This point is also necessary to make this material continuous enough to propagate waves. Too low a 4D density would indeed not characterize a continuous medium.

3 Gravitation

3.1 Preliminary considerations

3.1.1 The nature of gravitation

A direct consequence of the postulate of the theory is the nature of gravitation. Three-dimensional high-mass bodies evolve on the surface of four-dimensional continuous matter. As the universe inflation is accelerating, high-mass bodies experience this acceleration along the fourth dimension and involve mechanical deformations of the 4D material. This looks like Fig. 1 except spacetime is replaced by the 3D surface of the 4D material. In the vicinity of high-mass bodies, an object therefore

also experiences the acceleration of the universe's expansion along the fourth dimension, but due to the surface deformation, experiences an acceleration in the 3-sphere. In the framework of the theory developed in the paper, this is the origin of gravitation. Gravitation and the universe's expansion would have the same origin. This involves that gravitation is not a fundamental interaction, but the result of accelerating inflation and mechanical deformation.

In the following of this section, we focus on the surface profile equation to be associated with Newton's law of universal gravitation. The continuum mechanics theory to be chosen to describe the physical deformation of the four-dimensional material is then discussed. The second part deals with the mathematical equations of the gravitation theory and their resolution. This theory of gravitation is finally discussed.

3.1.2 The deformation profile

This section concerns the deformation profile associated with Newton's law of universal gravitation. The heavy body inducing gravitational forces experiences the acceleration of the universe's expansion (**g**) perpendicular to the undeformed 3D surface. Fig. 2 displays such a configuration, where a body of mass M experiences an acceleration **g** and bends the surface profile. A lighter body then experiences an acceleration **a** due to the curvature of the surface.

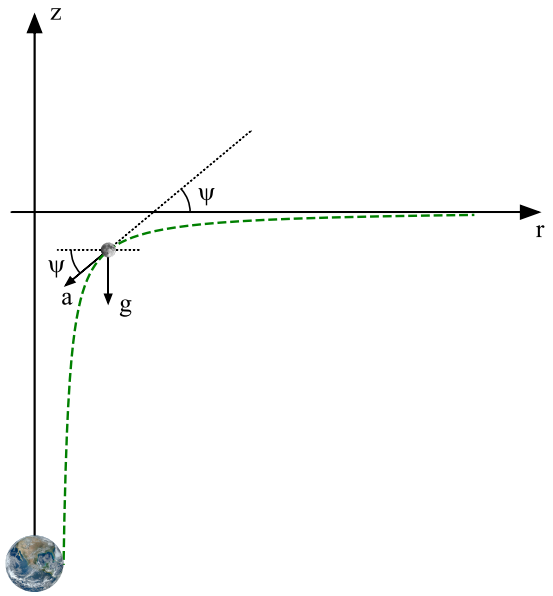

Figure 2: Section of the three-dimensional profile (in green) deformed by a high-mass body (grey disk). According to Newton's law of universal gravitation, the acceleration a of a light body (red star) is equal to $\frac{GM}{r^2}$. g is the acceleration perpendicular to the undeformed surface profile.

We note the surface equation f. The z-coordinate of the light body is $z = f(r, \theta, \varphi)$. Due to the spherical symmetry, the equation is independent of θ (polar angle) and φ (azimuthal angle), so $z = f(r)$. According to Newton, the acceleration of a body in the gravitational field of a heavy body is $a = \frac{GM}{r^2}$. Inside it, assuming the heavy body is homogeneous, $a = \frac{GMr}{R^3}$ (where R is the radius of the heavy body). Using the notations in Fig. 2, $tan\ \Psi = \frac{\mathrm{d}z}{\mathrm{d}r}$ and $sin\ \Psi = \frac{a}{g}$.

Both relations yield

$$\frac{\mathrm{d}z}{\mathrm{d}r} = \begin{cases} \tan\left(\arcsin\left(\frac{GMr}{gR^3}\right)\right) & \text{if } r \leq R \\ \tan\left(\arcsin\left(\frac{GM}{gr^2}\right)\right) & \text{if } r \geq R \end{cases} \tag{3.1}$$

Assuming that $\frac{GM}{R^2} \ll g$, equation (3.1) becomes

$$\frac{\mathrm{d}z}{\mathrm{d}r} \sim \begin{cases} \frac{GMr}{gR^3} & \text{if } r \leq R \\ \frac{GM}{gr^2} & \text{if } r \geq R \end{cases}$$

The deformation profile is thus given by

$$z \sim \begin{cases} \frac{GMr^2}{2gR^3} - \frac{3}{2}\frac{GM}{gR} & \text{if } r \leq R \\ -\frac{GM}{gr} & \text{if } r \geq R \end{cases} \tag{3.2}$$

3.1.3 Which material properties?

This section focuses on the choice of the most suitable mechanical theory to describe the hypothetical deformation process. First, Newton's law of universal gravitation states that gravitation forces are proportional to the mass of bodies. Then, the force experienced by a light body in the gravitational field of two heavy bodies is equal to the sum of both. The theory of deformation must therefore be linear. Second, no gravitational field has ever been observed without matter. So the deformation caused by bodies disappears when they move away. The deformation must therefore be elastic. Third, gravitational attraction outside a body does not depend on its radius. Therefore, the hypothetical deformation of four-dimensional matter cannot derive from the surface tension phenomenon of a fluid. In fact, the deformation of a fluid by this process depends on the load surface. For a solid, the deformation does not always depend on the load surface. For instance, the deformation of thin plates or beams depends only on the magnitude of the applied force. Thus, for the deformation to be independent of the body radius, it would seem that the hypothetical four-dimensional material must be thin in the fourth dimension. Given the previous considerations, continuum mechanics with Hooke's law will be the adopted theory. Therefore, the gravitation theory is based on a linear elastic thin four-dimensional solid material.

It should be noted that the linear elasticity hypothesis assumes that the strains and deformations are small. In the previous section, it was assumed that the acceleration g is much greater than the gravitational acceleration. This assumption is actually a consequence of the linear elasticity hypothesis, since for small deformations, $\frac{a}{g} = sin\ \Psi \ll 1$.

3.2 The equations of the theory

The static equations of continuum mechanics correspond to $\boldsymbol{\nabla}.\boldsymbol{\sigma} = \mathbf{0}$ with $\boldsymbol{\sigma}$ the Cauchy stress tensor. Due to the spherical symmetry of gravitation, the spherindrical coordinates lend themselves to this problem. A spherinder is a four-dimensional cylinder generated by a ball instead of a disc in 3D (Fig. A.1). The tensor analysis in spherindrical coordinates is presented in appendix A.

The Cauchy stress tensor is calculated from Hooke's law

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda tr(\boldsymbol{\varepsilon})\mathbf{I}$$

where λ and μ are the Lamé coefficients and $\boldsymbol{\varepsilon} = \frac{1}{2}\left({}^t\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\mathbf{u}\right)$ with $\mathbf{u}$ the displacement.

Due to the symmetries, we assume that $\mathbf{u} = u_r(r,z)\mathbf{e_r} + u_z(r,z)\mathbf{e_z}$. Equation (A.2) then gives the gradient of the displacement and the strain tensor

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \partial_r u_r & 0 & 0 & \frac{1}{2}(\partial_r u_z + \partial_z u_r) \\ 0 & \frac{u_r}{r} & 0 & 0 \\ 0 & 0 & \frac{u_r}{r} & 0 \\ \frac{1}{2}(\partial_r u_z + \partial_z u_r) & 0 & 0 & \partial_z u_z \end{pmatrix}$$

Considering the symmetries, the strain tensor expression and Hooke's law, relation (A.3) leads to

$$\boldsymbol{\nabla}.\boldsymbol{\sigma} = \begin{pmatrix} \partial_r \sigma_{rr} + \partial_z \sigma_{rz} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + \frac{\sigma_{rr}-\sigma_{\varphi\varphi}}{r} \\ 0 \\ 0 \\ \partial_r \sigma_{rz} + \partial_z \sigma_{zz} + \frac{2}{r}\sigma_{rz} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.3}$$

In terms of displacements, the set of equations (3.3) yields the following two equations:

$$2\mu\partial_r^2 u_r + \lambda\partial_r\left(\partial_r u_r + 2\frac{u_r}{r} + \partial_z u_z\right) + \mu\partial_z\left(\partial_r u_z + \partial_z u_r\right) + 4\frac{\mu}{r}\left(\partial_r u_r - \frac{u_r}{r}\right) = 0$$

$$\mu\partial_r\left(\partial_r u_z + \partial_z u_r\right) + 2\mu\partial_z^2 u_z + \lambda\partial_z\left(\partial_r u_r + 2\frac{u_r}{r} + \partial_z u_z\right) + 2\frac{\mu}{r}\left(\partial_r u_z + \partial_z u_r\right) = 0$$

The hypotheses given in section 3.1.3 lead to a simplification in both equations. Assuming that the material is very thin in the fourth dimension $\mathbf{e_z}$, the displacement u_z can be considered independent of z, so $u_z(r,z) = u_z(r)$. Note that for this type of thin geometry, such as beams or plates, there is no longitudinal strain along the neutral axis. Fibres above the neutral axis must be in tension and those below in compression in the configuration studied. The radial displacement u_r is therefore expected to be equal to 0 along the neutral axis ($u_r(r,0) = 0$). Finally, the equations to solve for a rectangular domain $\{(r,z) \in]0;+\infty[\times]-\frac{h}{2};\frac{h}{2}[\}$ are the following:

$$\begin{cases} 2\mu\partial_r^2 u_r + \lambda\partial_r\left(\partial_r u_r + 2\frac{u_r}{r}\right) + \mu\partial_z^2 u_r + 4\frac{\mu}{r}\left(\partial_r u_r - \frac{u_r}{r}\right) = 0 \\ \mu\partial_r\left(\partial_r u_z + \partial_z u_r\right) + \lambda\partial_z\left(\partial_r u_r + 2\frac{u_r}{r}\right) + 2\frac{\mu}{r}\left(\partial_r u_z + \partial_z u_r\right) = 0 \end{cases} \tag{3.4}$$

with the following boundary conditions, consistent with the symmetries and boundary conditions of gravitation:

$$\begin{cases} u_r(0,z) = 0 & \forall z \in [-\frac{h}{2};\frac{h}{2}] \\ \partial_r u_z|_{r=0} = 0 & \\ \lim\limits_{r\to+\infty} u_r(r,z) = 0 & \forall z \in [-\frac{h}{2};\frac{h}{2}] \\ \lim\limits_{r\to+\infty} u_z(r) = 0 & \\ \boldsymbol{\sigma}(r,-\frac{h}{2}).\mathbf{e_z} = \mathbf{0} & \forall r \in [0;+\infty[\\ \boldsymbol{\sigma}(r,\frac{h}{2}).\mathbf{e_z} = -\frac{Mg}{\frac{4}{3}\pi R^3}\mathbf{e_z} & \forall r \in [0;R] \\ \boldsymbol{\sigma}(r,\frac{h}{2}).\mathbf{e_z} = \mathbf{0} & \forall r \in]R;+\infty[\end{cases} \tag{3.5}$$

It is worth noting that the pressure load applied to the material domain in 4D must be in $N.m^{-3}$, which explains why the 3D volume of the ball is considered in the sixth boundary condition of (3.5).

3.3 Resolution

3.3.1 General solution

The second equation of (3.4) can be transformed as follows:

$$\begin{aligned}
&\mu\left[\partial_r\left(\partial_r u_z+\partial_z u_r\right)+\frac{2}{r}\left(\partial_r u_z+\partial_z u_r\right)\right]+\lambda\partial_z\left(\partial_r u_r+2\frac{u_r}{r}\right)=0\\
\Rightarrow&(\mu+\lambda)\left(\partial_{rz}^2 u_r+\frac{2}{r}\partial_z u_r\right)+\mu\left(\partial_r^2 u_z+\frac{2}{r}\partial_r u_z\right)=0\\
\overset{\times r^2}{\Rightarrow}&(\mu+\lambda)\left(r^2\partial_{rz}^2 u_r+2r\partial_z u_r\right)+\mu\left(r^2\partial_r^2 u_z+2r\partial_r u_z\right)=0\\
\Rightarrow&(\mu+\lambda)\partial_r\left(r^2\partial_z u_r\right)+\mu\partial_r\left(r^2\partial_r u_z\right)=0\\
\Rightarrow&(\mu+\lambda)\partial_z u_r+\mu\partial_r u_z=\frac{\phi(z)}{r^2}
\end{aligned}\tag{3.6}$$

with ϕ an unknown function that depends only on z. Since $\partial_z^2 u_r=\frac{1}{\mu+\lambda}\frac{\phi'(z)}{r^2}$ because $\partial_z u_z=0$, the first equation of (3.4) can be transformed as follows:

$$\begin{aligned}
&2\mu\partial_r^2 u_r+\lambda\partial_r\left(\partial_r u_r+2\frac{u_r}{r}\right)+\frac{\mu}{\mu+\lambda}\frac{\phi'(z)}{r^2}+\frac{4\mu}{r}\left(\partial_r u_r-\frac{u_r}{r}\right)=0\\
\Rightarrow&(2\mu+\lambda)\partial_r\left(\partial_r u_r+2\frac{u_r}{r}\right)+\frac{\mu}{\mu+\lambda}\frac{\phi'(z)}{r^2}=0\\
\Rightarrow&\partial_r u_r+2\frac{u_r}{r}=\frac{\mu}{(2\mu+\lambda)(\mu+\lambda)}\frac{\phi'(z)}{r}+3\alpha(z)
\end{aligned}$$

with α an unknown function that depends only on z. Note that a multiplication factor of 3 is used here for simplicity. The solution of this equation is

$$u_r(r,z)=\alpha(z)r+\frac{\beta(z)}{r^2}+\frac{\mu}{2(2\mu+\lambda)(\mu+\lambda)}\phi'(z)$$

with α and β two unknown functions. Equation (3.6) gives the vertical displacement

$$\begin{aligned}
&\partial_r u_z=\frac{\phi(z)}{\mu r^2}-\frac{\mu+\lambda}{\mu}\partial_z u_r\\
\Rightarrow&\partial_r u_z=\frac{\phi(z)-(\mu+\lambda)\beta'(z)}{\mu r^2}-\frac{\mu+\lambda}{\mu}\alpha'(z)r-\frac{\phi''(z)}{2(2\mu+\lambda)}\\
&u_z(r)=\frac{(\mu+\lambda)\beta'(z)-\phi(z)}{\mu r}-\frac{\mu+\lambda}{2\mu}\alpha'(z)r^2-\frac{\phi''(z)}{2(2\mu+\lambda)}r+\gamma
\end{aligned}$$

with γ a constant since u_z is independent of z.

3.3.2 Determination of the unknown functions

We separate two cases, one inside and one outside the heavy body.

1. **Outside**
   The boundary conditions as r tends to infinity lead to:

   - $\forall z\in[-\frac{h}{2};\frac{h}{2}],\ \lim\limits_{r\to+\infty}u_r(r,z)=0\Rightarrow\left\{\begin{array}{l}\alpha=0\\ \phi'=0\end{array}\right.$
   - $\forall z\in[-\frac{h}{2};\frac{h}{2}],\ \lim\limits_{r\to+\infty}u_z(r)=0\Rightarrow\left\{\begin{array}{l}\alpha'=0\\ \phi''=0\\ \gamma=0\end{array}\right.$

The displacements are therefore given by

$$\begin{cases} u_r(r,z) = \frac{\beta(z)}{r^2} \\ u_z(r) = \frac{(\mu+\lambda)\beta'(z)-\phi}{\mu r} \end{cases}$$

Note that ϕ is actually a constant, since $\forall z \in [-\frac{h}{2};\frac{h}{2}],\ \phi'(z) = 0$. We also remember that u_z is independent of z, so $\forall z \in [-\frac{h}{2};\frac{h}{2}],\ \beta''(z) = 0$ and $\beta(z) = az + b$, where a and b are two constants. But the radial displacement along the neutral fibre is equal to 0, so $u_r(r,0) = 0$ and $\beta(z) = az$.

The boundary conditions at the top and bottom of the domain result in

- $\boldsymbol{\sigma}(r,z).\mathbf{e_z} = \mathbf{0}$ when $z = -\frac{h}{2}$ and $z = \frac{h}{2}$. This leads to $\sigma_{rz}(r,z) = 0$ and $\sigma_{zz}(r,z) = 0$ when $z = -\frac{h}{2}$ and $z = \frac{h}{2}$.
  - $\forall r \in]R,+\infty[,\ \sigma_{rz}(r,\pm\frac{h}{2}) = 0 \Leftrightarrow \partial_r u_z|_{r,z=\pm\frac{h}{2}} = -\partial_z u_r|_{r,z=\pm\frac{h}{2}} \Leftrightarrow \phi - (\mu+\lambda)a = -\mu a \Leftrightarrow a = \frac{\phi}{\lambda}$.
  - $\forall r \in]R,+\infty[,\ \sigma_{zz}(r,\pm\frac{h}{2}) = 0 \Leftrightarrow \partial_r u_r + 2\frac{u_r}{r} = 0$: always true

Finally, the displacements are given by

$$\begin{cases} u_r = \frac{\phi}{\lambda}\frac{z}{r^2} \\ u_z = \frac{\phi}{\lambda r} \end{cases}$$

2. **Inside**

The boundary conditions when $r = 0$ result in:

- $\forall z \in [-\frac{h}{2};\frac{h}{2}],\ u_r(r=0,z) = 0 \Rightarrow \begin{cases} \beta = 0 \\ \phi' = 0 \end{cases}$
- $\forall z \in [-\frac{h}{2};\frac{h}{2}],\ \partial_r u_z|_{r=0,z} = 0 \Rightarrow \begin{cases} \phi = (\mu+\lambda)\beta' = 0 \\ \phi'' = 0 \end{cases}$

The displacements are therefore given by

$$\begin{cases} u_r(r,z) = \alpha(z)r \\ u_z(r) = -\frac{\mu+\lambda}{2\mu}\alpha'(z)r^2 + \gamma \end{cases}$$

Since u_z is independent of z, $\forall z \in [-\frac{h}{2};\frac{h}{2}],\ \alpha''(z) = 0$, so $\alpha(z) = cz + d$, where c and d are two constants. But the radial displacement along the neutral fibre is equal to 0, so $u_r(r,0) = 0$ and $\alpha(z) = cz$. The stress boundary condition gives the value of c:

$$\begin{aligned} \boldsymbol{\sigma}\left(r,\frac{h}{2}\right).\mathbf{e_z} = -\frac{Mg}{\frac{4}{3}\pi R^3}\mathbf{e_z} \Rightarrow & \sigma_{zz}\left(r,\frac{h}{2}\right) = -\frac{Mg}{\frac{4}{3}\pi R^3} \\ \Rightarrow & \lambda\left(\partial_r u_r|_{r,z=\frac{h}{2}} + 2\frac{u_r(r,\frac{h}{2})}{r}\right) = -\frac{Mg}{\frac{4}{3}\pi R^3} \\ \Rightarrow & 3\lambda c\frac{h}{2} = -\frac{Mg}{\frac{4}{3}\pi R^3} \\ \Rightarrow & c = -\frac{Mg}{2h\pi\lambda R^3} \end{aligned}$$

Finally, the displacements are given by

$$\begin{cases} u_r(r,z) = -\frac{Mg}{2h\pi\lambda R^3} zr \\ u_z(r) = \frac{\mu+\lambda}{2\mu} \frac{Mg}{2h\pi\lambda R^3} r^2 + \gamma \end{cases}$$

3. $\mathbf{r = R}$

The displacement must be continuous when $r = R$. This gives the following relations:

- $u_r(R,z) = -\frac{Mg}{2h\pi\lambda R^3} zR = \frac{\phi}{\lambda}\frac{z}{R^2} \Rightarrow \phi = -\frac{Mg}{2h\pi}$
- $u_z(R) = \frac{\mu+\lambda}{2\mu}\frac{Mg}{2h\pi\lambda R^3}R^2 + \gamma = \frac{\phi}{\lambda R} = -\frac{Mg}{2h\pi\lambda R} \Rightarrow \gamma = -\frac{3\mu+\lambda}{2\mu}\frac{Mg}{2h\pi\lambda R}$

Eventually, the solution is

$$u_r(r,z) = \begin{cases} -\frac{Mg}{2\pi h\lambda R^3} zr & \text{if } r \leq R \\ -\frac{Mg}{2\pi h\lambda}\frac{z}{r^2} & \text{if } r \geq R \end{cases}$$

$$u_z(r) = \begin{cases} \frac{Mg}{4\pi h\lambda\mu R}\left((\mu+\lambda)\frac{r^2}{R^2} - (3\mu+\lambda)\right) & \text{if } r \leq R \\ \frac{-Mg}{2\pi h\lambda r} & \text{if } r \geq R \end{cases} \tag{3.7}$$

3.4 Comparison with Newton's law of universal gravitation

Solution (3.7) can be compared to Newton's law of universal gravitation (3.2)

$$u_z(r) \sim \begin{cases} \frac{GMr^2}{2gR^3} - \frac{3}{2}\frac{GM}{gR} & \text{if } r \leq R \\ -\frac{GM}{gr} & \text{if } r \geq R \end{cases} \tag{3.8}$$

For equations (3.7) and (3.8) to be equivalent, it is necessary that:

$$\begin{cases} \frac{G}{g} = \frac{g}{2\pi h\lambda\mu}(\mu+\lambda) \\ 3\frac{G}{g} = \frac{g}{2\pi h\lambda\mu}(3\mu+\lambda) \\ \frac{G}{g} = \frac{g}{2\pi h\lambda} \end{cases}$$

One way to satisfy these three equations is to assume that $\mu \gg \lambda$. This is equivalent to assuming a Poisson's ratio close to 0. In this case, the three previous equations become

$$\begin{cases} \frac{G}{g} \sim \frac{g}{2\pi h\lambda} \\ 3\frac{G}{g} \sim \frac{3g}{2\pi h\lambda} \\ \frac{G}{g} \sim \frac{g}{2\pi h\lambda} \end{cases}$$

With $G = \frac{g^2}{2\pi h\lambda}$, the solution is equivalent to Newton's law of universal gravitation.

3.5 Discussion

In summary, a light body rolling on the surface of a four-dimensional material, bent by a heavy body, is subject to the same force as gravitation, provided two assumptions are true. First, the material must be elastic and linear, with a Poisson's ratio close to 0. Second, the material must be thin so that the vertical displacement is independent of z.

If there are bodies that induce forces beyond the framework of small deformations, then the law of gravitation can no longer be considered universal. The linearity of the law of gravitation would be lost. Newton's law of gravitation could also be rendered invalid if the plasticity limit is exceeded. This

would induce permanent deformations even without the presence of bodies. The theoretical models that describe galaxies, and in particular their rotation curves, are based on the assumption that the law of gravitation is universal. They could therefore be incorrect if one of the cases mentioned above occurs for a mass of the order of magnitude of a galaxy.

4 Quantum mechanics

Mathematically, quantum mechanics is based on the Schrödinger equation

$$i\hbar\frac{\partial\Psi(\mathbf{x},t)}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi(\mathbf{x},t) + V(\mathbf{x},t)\Psi(\mathbf{x},t)$$

where Ψ is a wave function, $\hbar = \frac{h}{2\pi}$, m is the mass of the considered particle and V is a potential describing the particle environment. This equation was discovered before understanding the wave function's meaning. Max Born interpreted the square of the complex modulus of the wave function as the probability density to find the particle at time t and position $\mathbf{x}$. This interpretation caused a division within the scientific community between deterministic and non-deterministic physicists. With the Copenhagen interpretation of quantum mechanics, non-deterministic physicists won the debate, despite the physical oddities inherent to this interpretation. Among others, this interpretation posits that there are no hidden variables that can explain the non-deterministic behaviour of particles.

This still remains the most widely accepted interpretation of the Schrödinger equation. However, some scientists have proposed alternative explanations of quantum mechanics. Pilot wave theory, initially developed by De Broglie and later by Bohm [30, 31], offers a more physical explanation of quantum mechanics through a different interpretation of wave-particle duality [32]. In the conventional interpretation of quantum mechanics, the wave and the particle are two different facets of the same object. The pilot-wave theory posits the existence of both a particle and an associated matter wave, with the latter guiding the former, causing it to exhibit a wave-like behaviour.

On the fringes of theoretical physics works for decades, Couder and Fort refreshed this field in 2005 with a scientific breakthrough [33, 34]. They demonstrated experimentally that pilot wave theory also reproduces the known behaviour of quantum particles. Under specific conditions, a vibrating liquid bath allows small droplets to be propelled by the Faraday waves generated by the impacts. The particles's position on the surface of the bath followed the probability density solution of the Schrödinger equation. Pilot wave theory now tends to become a credible rival to the Copenhagen interpretation of quantum mechanics. Among others, scientists have successfully studied the interaction between two or more droplets [35, 36] or the quantum tunneling effect [37, 38].

This interpretation is consistent with the theory developed in this paper. The experimental configuration achieved by Couder and Fort, with an additional dimension, supports the general postulate in section 2. Particles would bounce off the surface of the 4D ball, generating waves and interacting with them, while being visible for us only when they are on the surface, in the 3-sphere. This means that the apparent random behaviour of particles would arise from their interaction with the particles composing the hypothetical 4D material. Therefore, similar to the speed of light, which is given by the vacuum properties, so the properties of this hypothetical material, the Planck constant should also be given by the same parameters. This is the case:

$$h \sim 21.7\frac{\sqrt{\rho_c}}{(\varepsilon_0 e^{-2})^{\frac{5}{2}}(k_B T)^2} \sim 6.6\times10^{-34}\ J.s \tag{4.1}$$

As previously mentioned in [29], the probability of finding relations for two fundamental constants based on the same five parameters and two dimensionless numbers of a reasonable order of magnitude

is mathematically close to 0. Therefore, regardless of the developed theory, both relations (2.1) and (4.1) provide insight into the true nature of the speed of light and the Planck constant.

5 Electromagnetism

5.1 Well-known fluid analogy

Maxwell's equations read:

$$\begin{cases} \boldsymbol{\nabla}.\mathbf{E} = \frac{\rho_e}{\varepsilon_0} \\ \boldsymbol{\nabla}.\mathbf{B} = 0 \\ \boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

with $\mathbf{E}$ the electric field, $\mathbf{B}$ the magnetic field, ρ_e the volume charge density, $\mu_0 = \frac{1}{\varepsilon_0 c^2}$ the vacuum permeability and $\mathbf{j}$ the current density. Given these equations, a magnetic vector potential $\mathbf{A}$ and an electric potential ϕ can be defined such that

$$\begin{aligned} \mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A} \\ \mathbf{E} &= -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} \end{aligned} \tag{5.1}$$

Let us consider the compressible equations of a perfect fluid:

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \rho \boldsymbol{\nabla} \cdot \mathbf{u} + \mathbf{u} \cdot \boldsymbol{\nabla}\rho = 0 \\ &\frac{\partial \mathbf{u}}{\partial t} + (\boldsymbol{\nabla} \times \mathbf{u}) \times \mathbf{u} + \frac{1}{2}\boldsymbol{\nabla}(\mathbf{u} \cdot \mathbf{u}) = -\frac{\boldsymbol{\nabla} P}{\rho} \end{aligned} \tag{5.2}$$

By introducing the vorticity vector $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}$ and Lamb vector $\mathbf{L} = \boldsymbol{\omega} \times \mathbf{u}$, the momentum equation of (5.2) becomes equivalent to

$$\begin{aligned} \boldsymbol{\omega} &= \boldsymbol{\nabla} \times \mathbf{u} \\ \mathbf{L} &= -\boldsymbol{\nabla}\left(\int_{P_0}^{P} \frac{dP}{\rho} + \frac{1}{2}\mathbf{u} \cdot \mathbf{u}\right) - \frac{\partial \mathbf{u}}{\partial t} \end{aligned} \tag{5.3}$$

Equations (5.3) match with equations (5.1), with

$$\begin{aligned} &\mathbf{A} \longleftrightarrow \mathbf{u} \\ &\phi \longleftrightarrow \int_{P_0}^{P} \frac{dP}{\rho} + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} \\ &\mathbf{B} \longleftrightarrow \boldsymbol{\omega} \\ &\mathbf{E} \longleftrightarrow \mathbf{L} \end{aligned} \tag{5.4}$$

Therefore, $\mathbf{L}$ and $\boldsymbol{\omega}$ satisfy Maxwell's equations, with the charge and current densities given by the potentials following the same relations as in electromagnetism:

$$\begin{aligned} \frac{\rho_e}{\varepsilon_0} &= \Box\phi - \frac{\partial}{\partial t}\left(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right) \\ \mu_0 \mathbf{j} &= \Box\mathbf{A} + \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right) \end{aligned}$$

where $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ is the d'Alembertian. Note that all the terms in analogy (5.4) can be multiplied by a constant α without affecting the equations.

5.2 The nature of electromagnetism in the developed theory

The Lorentz force experienced by a particle moving at velocity $\mathbf{v}$ corresponds to $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \longleftrightarrow q\alpha(\mathbf{v} - \mathbf{u}) \times \boldsymbol{\omega}$. Noting $\mathbf{v}_{P/f} = \mathbf{v} - \mathbf{u}$ as the velocity of the particle relatively to the fluid, the Lorentz force $\alpha q \mathbf{v}_{P/f} \times \boldsymbol{\omega}$ looks like a lift force $\rho V_p C_L \mathbf{v}_{P/f} \times \boldsymbol{\omega}$, with V_p representing the volume of the particle immersed in the fluid and C_L representing the lift coefficient. Considering this, let us itemize the different forces experienced by a solid particle in an unbounded fluid:

- Its weight $\mathbf{P} = m\mathbf{g}$
- The buoyancy force, which compensates for the weight of a partially immersed particle $\mathbf{F_b} = -\rho V_p \mathbf{g}$
- A drag force due to the viscosity of the fluid $\mathbf{F_d} = \frac{1}{2}\rho C_D S \|\mathbf{v}_{P/f}\| \mathbf{v}_{P/f}$ with C_D the drag coefficient and S a characteristic surface area of the particle
- Fluid resistance, modelled by $\mathbf{F_a} = -\rho V_p C_a \frac{d\mathbf{v}_{P/f}}{dt} + \rho V_p \frac{d\mathbf{u}}{dt}$. The theoretical explanation of this formula is given below.
- A lift force [39, 40] $\mathbf{F_l} = \rho V_p C_L \mathbf{v}_{P/f} \times \boldsymbol{\omega}$

Fluid resistance cannot really be calculated analytically, except in simple configurations such as incompressible potential flow or weakly rotational flow. It can be assessed using the added mass concept. Fluid resistance stems from the mass of fluid that a body must move away from its path. Assuming no fluid velocity, an accelerating body will cause the surrounding fluid to gain kinetic energy $T = \frac{1}{2}\rho \int_V \mathbf{u} \cdot \mathbf{u} dV = \frac{1}{2}\rho \left(\int_V \frac{\mathbf{u}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}} dV \right) \mathbf{v} \cdot \mathbf{v} = \frac{1}{2}\rho I \mathbf{v} \cdot \mathbf{v}$. The change in kinetic energy is then equal to the power of the force representing fluid resistance: $\frac{dT}{dt} = \rho I \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = \mathbf{F} \cdot \mathbf{v}$. Then, $\mathbf{F_a} = \rho I \frac{d\mathbf{v}}{dt} = \rho V_p C_a \frac{d\mathbf{v}}{dt} = m_a \frac{d\mathbf{v}}{dt}$ with m_a the virtual added mass. If the fluid has a velocity, then the force is expressed by the relation given above. Note that this force disappears at constant velocity.

Given Maxwell's equations and the fluid analogy, particles must only experience a lift force. The drag force can only disappear due to a negligible viscosity, which is consistent with the observation that particles do not seem to slow down without experiencing external forces. The weight and the buoyancy force are oriented along the fourth hypothetical dimension, so they do not appear in the 3D equation of motion. This leaves two forces, $\mathbf{F_a} + \mathbf{F_l}$.

Given the postulate of the theory and the interpretation of quantum mechanics, particles must remain at the surface of the fluid. The pilot-wave theory suggests that particles must bounce to reproduce quantum effects. Therefore, the situation is more complex than a particle on the surface of a perturbed fluid. In the context of pilot-wave theory, for the theory of electromagnetism to be reliable, the fluid surface must not be modified by waves (otherwise quantum mechanics should apply), so particles must be far enough away from other bouncing particles. Given the fluid analogy, we might imagine that electromagnetism is the influence of currents in the fluid (decaying as $\frac{1}{r^2}$) induced by other particles. The compatibility between the interpretation of quantum mechanics and electromagnetism requires that particles must experience lift only upon contact with the fluid. Finally, in order to match the Lorentz force, it is necessary to assume that $\|\frac{d\mathbf{u}}{dt}\| \ll \|\frac{d\mathbf{v}}{dt}\|$. This might be attributed to the fluid's higher density, which would result in greater inertia. Thus, a simple model describing the motion of a particle bouncing on a perfect heavy fluid would be given by the following equation of motion:

$$(m + m_a)\frac{d\mathbf{v}}{dt} = \beta \mathbf{v}_{P/f} \times \boldsymbol{\omega} \tag{5.5}$$

The coefficient β should take into account the fact that the particle experiences lift only upon contact with the fluid surface, the lift coefficient, the volume of the particle immersed upon diving into the fluid, and possibly the surface tension. In this simple model, equation (5.5) is similar to the equation of motion of a charged particle in an electromagnetic field, where $\beta = q\alpha$ and the mass is replaced by $m + m_a$. This prompts the question: what mass have we assigned to the particles based on measurements? The mass is only revealed in unsteady configurations when the particle accelerates, so when there is an added mass in this theory. As such, within the framework of the developed theory, the mass assigned to particles should not be their true mass m, but rather $m + m_a$. Note that this means that the actual mass of particles would be unknown. Eventually, the assumption that $\rho V_p \gg m$ not only leads to the approximation $\|\frac{d\mathbf{u}}{dt}\| \ll \|\frac{d\mathbf{v}}{dt}\|$ as seen before, but also $m_a \gg m$ and

thus $m_a + m \sim m_a$. The following section will demonstrate that the same assumption is required for special relativity.

To know unequivocally whether equation (5.5) is sufficient to describe the physical situation assumed above, it seems that only numerical simulations or experiments can help, but this is beyond the scope of this paper.

6 Special relativity

Special relativity is based on the speed of light. Given the status of the speed of light in the presented theory, i.e. the speed of a material wave, its appearance in the mathematical framework should betray the compressibility of the 4D material. Therefore, this section deals with the effects of compressibility and their comparison with the relativistic effects.

6.1 Equations

The interpretation of quantum mechanics and electromagnetism leads us to think that the 4D material appears as a fluid from a particle's perspective. This section therefore addresses the displacement of a particle on the surface of a compressible, perfect fluid. The flow is assumed to be irrotational (no electromagnetic effect). The governing equations for the fluid are

$$\begin{array}{l} \frac{\partial \rho}{\partial t} + \rho \boldsymbol{\nabla} \cdot \mathbf{u} + \mathbf{u} \cdot \boldsymbol{\nabla} \rho = 0 \\ \frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \boldsymbol{\nabla} (\mathbf{u} \cdot \mathbf{u}) = -\frac{\boldsymbol{\nabla} P}{\rho} \end{array} \tag{6.1}$$

As the fluid is assumed to be irrotational, there is a potential ϕ such that $\mathbf{u} = \boldsymbol{\nabla}\phi$. For the sake of simplicity, we assume that the velocity of the particle is $V_\infty \mathbf{e_x}$. In the following analysis, we consider the frame of reference of the particle, where the fluid moves at a velocity of $-V_\infty \mathbf{e_x}$. Assuming small perturbations, ϕ satisfies the following equation (refer to Appendix B for the mathematical developments):

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \left(1 - Ma^2\right)^2 \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} - \frac{2}{c} Ma \frac{\partial^2 \phi}{\partial x \partial t} = 0 \tag{6.2}$$

where $Ma = \frac{V_\infty}{c}$ is the Mach number. The minus sign in front of the last term in equation (6.2) differs from equation (B.5) because the fluid velocity along x-direction is negative here. It is worth noting that the equivalent equation without the compressibility effects ($c \longrightarrow \infty$) is simply

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{6.3}$$

The compressible equation (6.2) can be written as the incompressible equation (6.3) with a coordinate change. We first apply a Galilean transformation to be in the frame of reference in which the fluid is at rest:

$$\begin{aligned} \tilde{x} &= x + V_\infty t \\ \tilde{y} &= y \\ \tilde{z} &= z \\ \tilde{t} &= t \end{aligned}$$

Then, a Lorentz transformation is applied to the coordinates associated with the fluid at rest:

$$\begin{aligned} x' &= \gamma(\tilde{x} - V_\infty \tilde{t}) \\ y' &= \tilde{y} \\ z' &= \tilde{z} \\ t' &= \gamma(\tilde{t} - \frac{V_\infty}{c^2}\tilde{x} \end{aligned}$$

where $\gamma = \left(1 - \frac{V_\infty^2}{c^2}\right)^{-\frac{1}{2}}$. This yields

$$\begin{aligned} x' &= \gamma x \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(\frac{t}{\gamma^2} - \frac{V_\infty}{c^2}x\right) \end{aligned}$$

The derivatives in this coordinate system are:

$$\begin{aligned} \frac{\partial\phi}{\partial x} &= \gamma\frac{\partial\phi}{\partial x'} - \gamma\frac{V_\infty}{c^2}\frac{\partial\phi}{\partial t'} \\ \frac{\partial^2\phi}{\partial x^2} &= \gamma^2\frac{\partial^2\phi}{\partial x'^2} - 2\frac{\gamma^2 V_\infty}{c^2}\frac{\partial^2\phi}{\partial x'\partial t'} + \left(\frac{\gamma V_\infty}{c^2}\right)^2\frac{\partial^2\phi}{\partial t'^2} \\ \frac{\partial\phi}{\partial t} &= \frac{1}{\gamma}\frac{\partial\phi}{\partial t'} \\ \frac{\partial^2\phi}{\partial t^2} &= \frac{1}{\gamma^2}\frac{\partial^2\phi}{\partial t'^2} \\ \frac{\partial^2\phi}{\partial x\partial t} &= \frac{\partial^2\phi}{\partial x'\partial t'} - \frac{V_\infty}{c^2}\frac{\partial^2\phi}{\partial t'^2} \end{aligned}$$

By introducing these relations into equation (6.2) and noting that $(1 - Ma^2)^2 = \frac{1}{\gamma^2}$, we get

$$\frac{1}{c^2\gamma^2}\frac{\partial^2\phi}{\partial t'^2} - \frac{\partial^2\phi}{\partial x'^2} + 2\frac{V_\infty}{c^2}\frac{\partial^2\phi}{\partial x'\partial t'} - \left(\frac{V_\infty}{c^2}\right)^2\frac{\partial^2\phi}{\partial t'^2} - \frac{\partial^2\phi}{\partial y'^2} - \frac{\partial^2\phi}{\partial z'^2} - 2\frac{V_\infty}{c^2}\frac{\partial^2\phi}{\partial x'\partial t'} + 2\left(\frac{V_\infty}{c^2}\right)^2\frac{\partial^2\phi}{\partial t'^2} = 0$$

As $\frac{1}{c^2\gamma^2} + \left(\frac{V_\infty}{c^2}\right)^2 = \frac{1}{c^2}$, equation (6.2) is equivalent to the following incompressible equation

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t'^2} - \frac{\partial^2\phi}{\partial x'^2} - \frac{\partial^2\phi}{\partial y'^2} - \frac{\partial^2\phi}{\partial z'^2} = 0$$

This theoretical development is drawn from aeronautical works carried out in the first half of the twentieth century. The coordinate transformation is named after Glauert (in steady state) [41] but developed by Küssner [42, 43]. Prandtl also contributed to this problem and developed the theory in [44]. These authors aimed to estimate the effects of compressibility when the velocity approaches the speed of sound without computers by using a compressible solution based on the incompressible one. The coordinate transformation has no meaning, but someone who ignores compressibility might misunderstand this transformation and attribute the presence of the additional terms in the compressible equation to time dilation and space contraction. The great similarity between special relativity and this theory has already been noted [45].

6.2 Compressibility effects

6.2.1 The physical phenomenon

For an incompressible potential flow, the first-order pressure variation is given by $p = -\rho\frac{\partial\phi}{\partial t}$. Based on the coordinate change, the pressure variation for a compressible flow can be expressed as $p = -\rho\frac{\partial\phi}{\partial t'} = -\gamma\rho\frac{\partial\phi}{\partial t}$. Since $-\rho\frac{\partial\phi}{\partial t}$ is the pressure variation calculated when the compressibility effects are neglected, the compressibility of the fluid adds the γ term to the pressure variation expression: $p_{comp} = \gamma p_{incomp}$. Near the wave speed, the pressure tends to infinity. This phénomenon is known as the sound barrier in aviation. As an aircraft approaches the speed of sound, the pressure in front of it increases, making it harder and harder to go faster. However, it is possible to exceed the speed of sound despite the singularity in the pressure relation when $Ma = 1$. This is due to the limitations of the presented theory, which is based on several assumptions that do not hold close to the speed of sound. In particular, the theory assumes that the flow is irrotational and that the particle causes negligible disturbances. In air, this theory provides consistent results up to $Ma \sim 0.6$. However, for a heavier fluid, the range of validity increases depending on its properties. At the scale of particles, we can therefore imagine that they barely affect the surface of the hypothetical 4D material, making the disturbances negligible and the equations developed suitable up to very close to the speed of light. Measurements on particles could therefore show no deviation from special relativity. However, according to the theory developed in the paper, special relativity is wrong at speeds close to the speed of light. It is worth noting that the theory of compressible flow becomes applicable again for speeds above the speed of sound. $\frac{1}{\sqrt{1-\frac{V_\infty^2}{c^2}}}$ must just be replaced with $\frac{1}{\sqrt{\frac{V_\infty^2}{c^2}-1}}$ for supersonic flows.

6.2.2 Law of motion

As introduced in section 5.2, a particle in a fluid experiences fluid resistance, which can be modelled by the concept of added mass. A particle moving in a fluid at rest is therefore governed by the following equation of motion:

$$\frac{d(m+m_a)\mathbf{v}}{dt} = \mathbf{F_{ext}}$$

Fluid resistance basically corresponds to the force exerted by the fluid on the surface of a particle in the direction of motion (assumed to be $\mathbf{e_x}$): $\int p\mathbf{dS}\cdot\mathbf{e_x} = m_a\frac{d\mathbf{v}\cdot\mathbf{e_x}}{dt}$. For a compressible fluid, the force is multiplied by γ, resulting in an equation of motion for a particle that appears as follows:

$$\frac{d(m+\gamma m_a)\mathbf{v}}{dt} = \mathbf{F_{ext}}$$

As discussed in section 5.2, the only way to have the same framework as special relativity is to assume that the real mass of the particle m is small compared to the added mass m_a. This could be caused by the density of the hypothetical 4D material being much greater than that of the particles. Under this assumption, the mass attributed to particles by scientists is approximately m_a, and they are governed by equation (6.4)

$$\frac{d\gamma m_a\mathbf{v}}{dt} = \mathbf{F_{ext}} \tag{6.4}$$

which is the equation of motion in special relativity.

6.2.3 Change in frequency

Let us consider a quantity oscillating with pulsation ω for an incompressible fluid, leading to a term like $e^{i\omega t}$. The compressible effects modify this term such that $e^{i\omega t'} = e^{i\omega\left(\frac{t}{\gamma}-\gamma\frac{V_\infty}{c^2}x\right)}$, giving another pulsation $\omega' = \frac{\omega}{\gamma}$ [46]. Therefore, when an experimenter measures the frequency of a phenomenon

associated with a system at rest, there are no compressible effects because there is no velocity, so a pulsation ω is measured. However, the same phenomenon associated with a system moving at a certain speed relative to the fluid exhibits a pulsation of ω'. Therefore, the compressibility effects modify the effective frequency of the phenomenon depending on the velocity. This conclusion is qualitatively and quantitatively the same as in special relativity.

6.2.4 Discussion

Atomic clocks are based on the hypothesis that the resonant frequency of atoms is universal and the same in any frame of reference. The measurement of a variation in frequency must then be understood as a dilation of time, as it only remains this explanation. In the developed theory, time is absolute, and frequency becomes relative, while adhering to the same mathematical theory. Therefore, it is worth noting that experiments based on atomic clocks do not prove time dilation, but only the mathematical framework of the theory. The same applies to the mean lifetime of muons, which is assumed to be universal and independent of the frame of reference. However, their mean lifetime could depend on their velocity relative to the hypothetical 4D material due to the compressibility effects.

In this theory, a consequence is that the speed of light is not constant and varies in different frames of reference. In short, the invariance of the speed of light associated with time dilation in special relativity is replaced by compressibility in the developed theory, while retaining the same mathematical framework.

I would like to conclude this section with the assumption of a speed limit in relativity, which would necessarily be close to, or even equal to, the speed of light. The present theory does not include such a limit, which is actually observed in the universe. Due to the universe's expansion, some bodies are moving away from each other at speeds far greater than the speed of light. Physicists save the theory of relativity by claiming that since it is spacetime that is expanding, which is immaterial, all is well. However, the bodies in spacetime are composed of matter, and they are still moving away at faster-than-light velocities, regardless of the cause of those velocities. In addition, special relativity is used to estimate these velocities through red shift. So relativity can be applied to bodies moving at sublight speeds, where the cause of the motion is the expansion of spacetime. However, for supraluminal bodies, which violate relativity, it does not matter as the cause of the motion is the expansion of spacetime. Hopefully, ad hoc explanations will fail to remain consensual.

7 General relativity

Similar to special relativity, we will focus on the compressibility effects caused by gravitation, as general relativity is also based on the speed of light (and the gravitational constant).

7.1 Bending of light

As per the previous section, relativity effects are explained without assuming the invariance of the speed of light. The bending of light can be attributed to the variation of the speed of light, which in turn modifies the refraction index, leading to the refraction of light rays. We can imagine a modification of the refraction index due to a local variation in the density of the 4D material caused by the presence of high-mass bodies.

Gravitation has been supposed to arise from the deformation of the material surface, as discussed in section 3. The deformed profile outside bodies is $z = -\frac{GM}{gr}$, as shown in equation (3.2). The variation of z results in an overpressure $dp = \rho g dz$. The classical equation of state of a perfect fluid $p = w\rho c^2$ is assumed, where w is a dimensionless number used to determine it. Then, $dp = \frac{p}{wc^2} GMd\frac{1}{r}$

and the integration gives $ln\left(\frac{p}{p_0}\right) = \frac{GM}{wc^2r}$ with p_0 the pressure at infinity. Assuming that the pressure variation $p' = p - p_0$ is tiny and replacing it with density yields

$$\rho' = \rho - \rho_0 = \frac{GM}{wc^2r}\rho_0$$

As the variation in density is very weak, we can neglect second-order terms in the relation that provides the refraction index. Instead, we will only consider a linear variation with the density. The variation of the refraction index is therefore given by $\frac{\rho'}{\rho_0}$ so

$$n = 1 + \frac{GM}{wc^2r}$$

It is possible to find the value of w to be consistent with general relativity. The Schwarzschild metric is

$$c^2d\tau^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1}(dx^2 + dy^2 + dz^2)$$

Necessarily, we need $\left(1 - \frac{2GM}{rc^2}\right) > \left(1 - \frac{2GM}{rc^2}\right)^{-1}\frac{v^2}{c^2}$, so $v < c\left(1 - 2\frac{GM}{rc^2}\right)$, making the new speed limit, which is assumed to be the speed of light, equal to $c' = c\left(1 - 2\frac{GM}{rc^2}\right)$. In the framework of the presented theory

$$n = \frac{c}{c'} = \left(1 - 2\frac{GM}{rc^2}\right)^{-1} \sim 1 + 2\frac{GM}{rc^2}$$

To match general relativity, we need $w = \frac{1}{2}$.

The equation of light rays is

$$\frac{d}{ds}\left(n\frac{d\mathbf{r}}{ds}\right) = \boldsymbol{\nabla}n$$

with s the curvilinear abscissa and $\mathbf{r}$ the position vector. Substituting n with its corresponding expression results in

$$\frac{dn}{ds}\frac{d\mathbf{r}}{ds} + n\frac{d^2\mathbf{r}}{ds^2} = -\frac{2GM}{c^2r^2}\frac{dr}{ds}\begin{pmatrix}\frac{dr}{ds}\\ r\frac{d\theta}{ds}\end{pmatrix} + \left(1 + \frac{2GM}{rc^2}\right)\begin{pmatrix}\frac{d^2r}{ds^2} - r\left(\frac{d\theta}{ds}\right)^2\\ 2\frac{dr}{ds}\frac{d\theta}{ds} + r\frac{d^2\theta}{ds^2}\end{pmatrix} = \begin{pmatrix}-\frac{2GM}{r^2c^2}\\ 0\end{pmatrix} \tag{7.1}$$

The equation in the second direction gives

$$\begin{array}{rc} & \frac{2GM}{rc^2}\left(\frac{dr}{ds}\frac{d\theta}{ds} + r\frac{d^2\theta}{ds^2}\right) + 2\frac{dr}{ds}\frac{d\theta}{ds} + r\frac{d^2\theta}{ds^2} = 0\\ \Rightarrow & \frac{d}{ds}\left(\left(\frac{2GM}{c^2r} + 1\right)r^2\frac{d\theta}{ds}\right) = 0\\ \Rightarrow & \left(\frac{2GM}{c^2r} + 1\right)r^2\frac{d\theta}{ds} = \tilde{J}\end{array}$$

As $ds = c'dt = c\left(1 + \frac{2GM}{rc^2}\right)^{-1}dt$,

$$r^2\left(1 + \frac{2GM}{rc^2}\right)^2\frac{d\theta}{dt} = c\tilde{J} = J$$

The first equation of (7.1) gives

$$\frac{d^2r}{ds^2} - \frac{\tilde{J}^2}{r^3\left(1 + \frac{2GM}{rc^2}\right)^2} = -\frac{2GM}{c^2r^2}\left(1 + \frac{2GM}{rc^2}\right)^{-1}\left(1 - \left(\frac{dr}{ds}\right)^2\right) \tag{7.2}$$

By replacing the derivatives with respect to s with derivatives with respect to θ using the following relations

$$\frac{dr}{ds} = \frac{d\theta}{ds}\frac{dr}{d\theta} = -\frac{\tilde{J}}{1+\frac{2GM}{rc^2}}\frac{d\frac{1}{r}}{d\theta}$$

$$\frac{d^2r}{ds^2} = \frac{d\theta}{ds}\frac{d}{d\theta}\left(\frac{dr}{ds}\right) = -\frac{\tilde{J}^2}{r^2\left(1+\frac{2GM}{rc^2}\right)}\frac{d}{d\theta}\left(\left(1+\frac{2GM}{rc^2}\right)^{-1}\frac{d\frac{1}{r}}{d\theta}\right)$$

$$\Rightarrow \qquad \frac{d^2r}{ds^2} = -\frac{\tilde{J}^2}{r^2\left(1+\frac{2GM}{rc^2}\right)^2}\frac{d^2\frac{1}{r}}{d\theta^2} + \frac{2GM\tilde{J}^2}{c^2r^2\left(1+\frac{2GM}{rc^2}\right)^3}\left(\frac{d\frac{1}{r}}{d\theta}\right)^2$$

and subsequently using the notation $u = \frac{1}{r}$ in equation (7.2), the following equation is obtained

$$-\frac{\tilde{J}^2u^2}{\left(1+\frac{2GMu}{c^2}\right)^2}\frac{d^2u}{d\theta^2}+\frac{2GM\tilde{J}^2u^2}{c^2\left(1+\frac{2GMu}{c^2}\right)^3}\left(\frac{du}{d\theta}\right)^2 = \frac{\tilde{J}^2u^3}{\left(1+\frac{2GMu}{c^2}\right)^2}-\frac{\frac{2GMu^2}{c^2}}{1+\frac{2GMu}{c^2}}\left(1-\frac{\tilde{J}^2}{\left(1+\frac{2GMu}{c^2}\right)^2}\left(\frac{du}{d\theta}\right)^2\right)$$

Simplifying this expression finally gives

$$\frac{d^2u}{d\theta^2}+u = \frac{2GM}{\tilde{J}^2c^2}\left(1+\frac{2GMu}{c^2}\right) = \frac{2GM}{J^2}\left(1+\frac{2GMu}{c^2}\right) \sim \frac{2GM}{J^2}$$

since $\left(\frac{GM}{Jc}\right)^2$ is very small for light because $J \sim R_\odot c$ with $R_\odot$ the body radius. The bending of light is thus twice that predicted by Newton's law of gravitation, which is consistent with the value observed for the Sun [12, 13, 14]. Note that the equation obtained differs from that of general relativity, indicating a departure from this theory, in contrast to special relativity.

7.2 Shapiro time delay

In the framework of general relativity, time dilation due to high-mass bodies retards photons [47]. Here, the increase in density leads to a decrease of light speed, which retards photon. For instance, let us consider a sginal travelling from Earth to a source (Fig. 3). As in general relativity, the curvature of the trajectory has a negligible effect on time delay. A "straight path" is therefore assumed in the calculations.

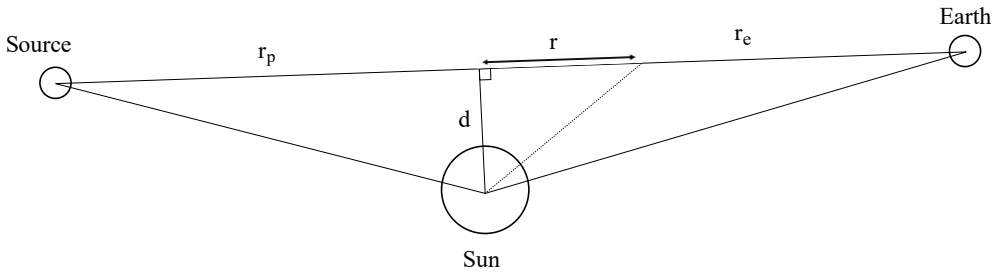

Figure 3: Geometrical configuration leading to Shapiro time delay.

It follows

$$dt = \frac{1}{c'}dr \Rightarrow cdt = \left(1+\frac{2GM}{c^2\sqrt{r^2+d^2}}\right)dr$$

Integrating this relation between Earth and the Sun, and then between the Sun and the source yields

$$\begin{aligned} c\Delta t &= \int_{r_e}^{0}\left(1+\tfrac{2GM}{c^2\sqrt{r^2+d^2}}\right)dr + \int_{0}^{r_p}\left(1+\tfrac{2GM}{c^2\sqrt{r^2+d^2}}\right)dr \\ &= r_p + r_e + \tfrac{2GM}{c^2}\left(ln(\sqrt{d^2+r_e^2}+r_e) + ln(\sqrt{d^2+r_p^2}+r_p) - 2ln(d)\right) \\ &= r_p + r_e + \tfrac{2GM}{c^2} ln\left(\tfrac{(\sqrt{d^2+r_e^2}+r_e)(\sqrt{d^2+r_p^2}+r_p)}{d^2}\right) \end{aligned}$$

The first term corresponds to the Newtonian term. Therefore, the additional time delay for a round-trip travel is

$$\frac{4GM}{c^3} ln\left(\frac{(\sqrt{d^2+r_e^2}+r_e)(\sqrt{d^2+r_p^2}+r_p)}{d^2}\right) \sim \frac{4GM}{c^3} ln\left(\frac{4r_e r_p}{d^2}\right) \tag{7.3}$$

when d is small compared to r_e and r_p. Equation (7.3) is the same relation as that obtained with general relativity [11, 48], which is consistent with measurements [49].

7.3 Gravitational redshift

For a system at rest in a weakly compressible fluid, its natural frequency differs from that in vacuum due to the surrounding fluid that must be displaced. This can be calculated using the added mass concept with the relation [50]:

$$\nu = \frac{\nu_{vacuum}}{\sqrt{1+\frac{\rho}{\rho_p}C_a}}$$

The frequency in a same medium but with a different density due to compression will then be

$$\nu = \frac{\sqrt{1+\frac{\rho_0}{\rho_p}C_a}}{\sqrt{1+\frac{\rho}{\rho_p}C_a}}\nu_0$$

Assuming the same hypothesis required to derive the mathematical framework of special relativity, it is considered that the mass of the particle is negligible compared to the added mass, resulting in $\frac{\rho_0}{\rho_p}C_a \gg 1$ and $\nu \sim \sqrt{\frac{\rho_0}{\rho}}\nu_0$. As noted in section 7.1, $\rho = \left(1+\frac{2GM}{rc^2}\right)\rho_0$, so

$$\nu = \frac{\nu_0}{\sqrt{1+\frac{2GM}{rc^2}}} \sim \sqrt{1-\frac{2GM}{rc^2}}\nu_0$$

which is the gravitational redshift derived from general relativity.

8 Conclusion

Contemporary physics comprises several theories that attempt to explain phenomena such as gravitation, electromagnetism, relativistic or quantum effects. A great challenge is to unify these theories into a single framework and supersede local explanations with a global understanding. This work has been an endeavour to such a theoretical development. Given the existing mathematical frameworks of physical theories and experimental confirmations of predicted effects, we have proposed a single postulate to explain most known physical observations. The universe is assumed to be a four-dimensional ball undergoing accelerating inflation and containing continuous matter near the surface. The continuous hypothesis is central to explaining gravitation, quantum mechanics, electromagnetism and relativistic effects.

We have explained gravitation using the deformation of the material surface and the acceleration of the universe's expansion. Newton's law of universal gravitation has been demonstrated using continuum mechanics. In this theory, this law is not universal and results from the assumption that the material is elastic and thin enough to neglect vertical variations of the fourth directional displacement. In this theory, it is worth noting that when assessing the locations and amounts of dark matter for high masses, it may be unreliable to use Newton's law if any of the assumptions are incorrect.

We have explained quantum mechanics thanks to the well-known pilot-wave alternative interpretation. This approach fits perfectly with the theory presented in the paper. In addition, a relation that provides the Planck constant based on the properties of the assumed 4D material has been found, as predicted. Similarly, a formula for the speed of light has been derived based on the same parameters as the speed of a material wave. This is compatible with our understanding of both quantum mechanics and gravitation.

We have explained electromagnetism as a fluid phenomenon. As a direct consequence of the interpretation of quantum mechanics, and based on the well-known fluid analogy of Maxwell's equations, we have shown the equivalence between Maxwell's equations and the Lorentz force, and the variations of pressure and velocity in a fluid involving lift forces on particles.

We have explained special relativity with the compressible potential flow theory. This theory was particularly studied in aeronautics during the first half of the twentieth century to model supersonic aircraft. It presents a direct analogy with special relativity, which deals with displacement at speeds close to the speed of light, which is a material wave according to the postulate of the paper. It has been recalled that the Lorentz transformation and the Lorentz factor appear naturally in the theory of compressible potential flow. The same increase of mass as in special relativity is obtained. While special relativity attributes this increase to time dilation, here it has been explained by the added mass phenomenon and compressibility.

We have explained general relativity using the same physical phenomenon as special relativity, i.e. the compressibility of the four-dimensional material and the added mass effect. The bending of light and Shapiro time delay in the developed theory result from the modification of the refractive index due to changes in density. Density variations also affect the added mass and, consequently, the natural frequency of physical phenomena, involving gravitational redshift.

Although the proposed theory is rather advanced in each physical sub-theory, it may still require further development. Specifically, it does not address certain relativistic effects. The limitations of Newton's law of gravitation imposed by the assumptions of the developed theory of gravitation could also deserve a deeper study in the future.

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A Tensor calculus in spherindrical coordinates

Let us note $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4})$ the natural basis of $\mathbb{R}^4$ and $(\mathbf{e_r}, \mathbf{e}_\theta, \mathbf{e}_\varphi, \mathbf{e_z})$ the spherindrical basis (Fig. A.1).

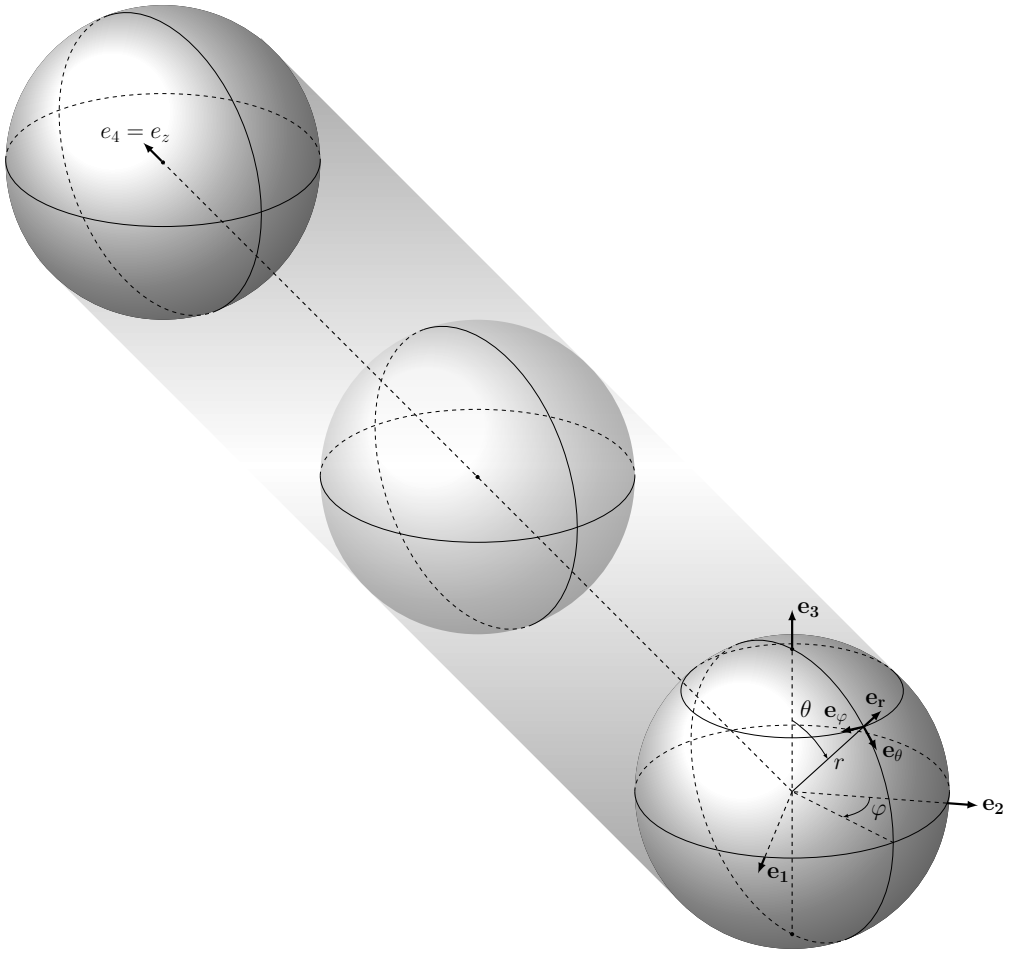

Figure A.1: Example of a spherinder with spherindrical coordinates.

We consider the spherindrical coordinates (r, θ, φ, z) such as $x_1 = r\sin\theta\sin\varphi$, $x_2 = r\sin\theta\cos\varphi$, $x_3 = r\cos\theta$ and $x_4 = z$. The spherindrical basis vectors can be expressed in the natural basis such as:

$$\begin{aligned}
\mathbf{e_r} &= \sin\theta\sin\varphi\ \mathbf{e_1} + \sin\theta\cos\varphi\ \mathbf{e_2} + \cos\theta\ \mathbf{e_3} \\
\mathbf{e}_\theta &= \cos\theta\sin\varphi\ \mathbf{e_1} + \cos\theta\cos\varphi\ \mathbf{e_2} - \sin\theta\ \mathbf{e_3} \\
\mathbf{e}_\varphi &= \cos\varphi\ \mathbf{e_1} - \sin\varphi\ \mathbf{e_2} \\
\mathbf{e_z} &= \mathbf{e_4}
\end{aligned}$$

The nabla operator is given by $\boldsymbol{\nabla} = \mathbf{e_r}\partial_r + \mathbf{e}_\theta\frac{1}{r}\partial_\theta + \mathbf{e}_\varphi\frac{1}{r\sin\theta}\partial_\varphi + \mathbf{e_z}\partial_z$. The symbol ∂_α denotes the partial derivative with respect to the variable α.
Let us calculate the partial derivatives of the spherindrical basis vectors.

$$\begin{aligned}
&\partial_r\mathbf{e}_\alpha = \mathbf{0},\ \forall\ \alpha \in [r, \theta, \varphi, z] \\
&\partial_\theta\mathbf{e_r} = \mathbf{e}_\theta,\ \partial_\theta\ \mathbf{e}_\theta = -\mathbf{e_r},\ \partial_\theta\mathbf{e}_\varphi = \partial_\theta\mathbf{e_z} = \mathbf{0} \\
&\partial_\varphi\mathbf{e_r} = \sin\theta\ \mathbf{e}_\varphi,\ \partial_\varphi\mathbf{e}_\theta = \cos\theta\ \mathbf{e}_\varphi,\ \partial_\varphi\mathbf{e}_\varphi = -\sin\theta\ \mathbf{e_r} - \cos\theta\ \mathbf{e}_\theta,\ \partial_\varphi\mathbf{e_z} = \mathbf{0} \\
&\partial_z\mathbf{e}_\alpha = \mathbf{0},\ \forall\ \alpha \in [r, \theta, \varphi, z]
\end{aligned} \tag{A.1}$$

Let us consider a vector $\mathbf{v} = v_r\ \mathbf{e_r} + v_\theta\ \mathbf{e}_\theta + v_\varphi\ \mathbf{e}_\varphi + v_z\ \mathbf{e_z}$ and calculate the gradient of this vector $\mathbf{G} = \boldsymbol{\nabla}\mathbf{v} = \frac{\partial \mathbf{v}}{\partial \alpha} \otimes \mathbf{e}_\alpha$. Then,

$$\mathbf{G}.\mathbf{e_r} = \frac{\partial \mathbf{v}}{\partial r} = \frac{\partial}{\partial r}(v_r\ \mathbf{e_r} + v_\theta\ \mathbf{e}_\theta + v_\varphi\ \mathbf{e}_\varphi + v_z\ \mathbf{e_z}) = \partial_r v_r\ \mathbf{e_r} + \partial_r v_\theta\ \mathbf{e}_\theta + \partial_r v_\varphi\ \mathbf{e}_\varphi + \partial_r v_z\ \mathbf{e_z}$$

because $\partial_r \mathbf{e}_\alpha = \mathbf{0},\ \forall\ \alpha \in [r, \theta, \varphi, z]$ according to relation (A.1). Then,

$$\begin{array}{l}
\mathbf{G}.\mathbf{e}_\theta = \frac{1}{r}\frac{\partial \mathbf{v}}{\partial \theta} = \frac{1}{r}\left[(\partial_\theta v_r - v_\theta)\,\mathbf{e_r} + (v_r + \partial_\theta v_\theta)\,\mathbf{e}_\theta + \partial_\theta v_\varphi \mathbf{e}_\varphi + \partial_\theta v_z \mathbf{e_z}\right] \\
\mathbf{G}.\mathbf{e}_\varphi = \frac{1}{r\sin\theta}\frac{\partial \mathbf{v}}{\partial \varphi} = \frac{1}{r\sin\theta}\left[(\partial_\varphi v_r - \sin\theta v_\varphi)\,\mathbf{e_r} + (\partial_\varphi v_\theta - \cos\theta v_\varphi)\,\mathbf{e}_\theta + \right. \\
\left.(\partial_\varphi v_\varphi + \sin\theta v_r + \cos\theta v_\theta)\,\mathbf{e}_\varphi + \partial_\varphi v_z \mathbf{e_z}\right] \\
\mathbf{G}.\mathbf{e_z} = \frac{\partial \mathbf{v}}{\partial z} = \partial_z v_r\ \mathbf{e_r} + \partial_z v_\theta\ \mathbf{e}_\theta + \partial_z v_\varphi\ \mathbf{e}_\varphi + \partial_z v_z\ \mathbf{e_z}
\end{array}$$

The gradient of $\mathbf{v}$ can therefore be written

$$\boldsymbol{\nabla}\mathbf{v} = \begin{pmatrix}
\partial_r v_r & \frac{1}{r}\partial_\theta v_r - \frac{v_\theta}{r} & \frac{1}{r\sin\theta}\partial_\varphi v_r - \frac{v_\varphi}{r} & \partial_z v_r \\
\partial_r v_\theta & \frac{1}{r}\partial_\theta v_\theta + \frac{v_r}{r} & \frac{1}{r\sin\theta}\partial_\varphi v_\theta - \cot\theta\frac{v_\varphi}{r} & \partial_z v_\theta \\
\partial_r v_\varphi & \frac{1}{r}\partial_\theta v_\varphi & \frac{1}{r\sin\theta}\partial_\varphi v_\varphi + \frac{v_r}{r} + \cot\theta\frac{v_\theta}{r} & \partial_z v_\varphi \\
\partial_r v_z & \frac{1}{r}\partial_\theta v_z & \frac{1}{r\sin\theta}\partial_\varphi v_z & \partial_z v_z
\end{pmatrix} \tag{A.2}$$

The divergence of the vector $\mathbf{v}$ is given by

$$\boldsymbol{\nabla}.\mathbf{v} = tr(\boldsymbol{\nabla}\mathbf{v}) = \frac{1}{r^2}\partial_r(r^2 v_r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\partial_\varphi v_\varphi + \partial_z v_z$$

Now, let us consider a symmetric second-order tensor field $\boldsymbol{\sigma}$ and calculate its divergence. With ":" denoting the double dot product and $\mathbf{I} = \mathbf{e_r} \otimes \mathbf{e_r} + \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi + \mathbf{e_z} \otimes \mathbf{e_z}$ the second order identity tensor, $\boldsymbol{\nabla}.\boldsymbol{\sigma} = \boldsymbol{\nabla}\boldsymbol{\sigma} : \mathbf{I}$. With $\mathbf{A}$ a third-order tensor field and $\mathbf{b}$ and $\mathbf{c}$ two vectors, we have $\mathbf{A} : (\mathbf{b} \otimes \mathbf{c}) = (\mathbf{A}.\mathbf{b}).\mathbf{c}$. The divergence of $\boldsymbol{\sigma}$ can then be expressed as

$$\begin{aligned}
\boldsymbol{\nabla}.\boldsymbol{\sigma} &= (\boldsymbol{\nabla}\boldsymbol{\sigma}.\mathbf{e_r}).\mathbf{e_r} + (\boldsymbol{\nabla}\boldsymbol{\sigma}.\mathbf{e}_\theta).\mathbf{e}_\theta + (\boldsymbol{\nabla}\boldsymbol{\sigma}.\mathbf{e}_\varphi).\mathbf{e}_\varphi + (\boldsymbol{\nabla}\boldsymbol{\sigma}.\mathbf{e_z}).\mathbf{e_z} \\
&= (\partial_r\boldsymbol{\sigma}).\mathbf{e_r} + \frac{1}{r}(\partial_\theta\boldsymbol{\sigma}).\mathbf{e}_\theta + \frac{1}{r\sin\theta}(\partial_\varphi\boldsymbol{\sigma}).\mathbf{e}_\varphi + (\partial_z\boldsymbol{\sigma}).\mathbf{e_z}
\end{aligned}$$

The calculation of $\partial_\gamma\boldsymbol{\sigma}$ can be divided into two parts:

$$\begin{aligned}
\partial_\gamma\boldsymbol{\sigma} &= \partial_\gamma \sum_{\alpha=r,\theta,\varphi,z}\ \sum_{\beta=r,\theta,\varphi,z} \sigma_{\alpha\beta}\ \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \\
&= \sum_{\alpha=r,\theta,\varphi,z}\ \sum_{\beta=r,\theta,\varphi,z} (\partial_\gamma\sigma_{\alpha\beta})\mathbf{e}_\alpha \otimes \mathbf{e}_\beta + \sum_{\alpha=r,\theta,\varphi,z}\ \sum_{\beta=r,\theta,\varphi,z} \sigma_{\alpha\beta}[(\partial_\gamma\mathbf{e}_\alpha) \otimes \mathbf{e}_\beta + \mathbf{e}_\alpha \otimes (\partial_\gamma\mathbf{e}_\beta)]
\end{aligned}$$

The derivatives of the basis vectors are given by relations (A.1).
Since $\partial_r\mathbf{e}_\alpha = \mathbf{0},\ \forall\ \alpha \in [r, \theta, \varphi, z]$, $\gamma = r$ yields $\partial_r\boldsymbol{\sigma} = \sum_{\alpha=r,\theta,\varphi,z}\ \sum_{\beta=r,\theta,\varphi,z} (\partial_r\sigma_{\alpha\beta})\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$.
So,

$$(\partial_r\boldsymbol{\sigma}).\mathbf{e_r} = \sum_{\beta=r,\theta,\varphi,z} (\partial_r\sigma_{r\beta})\mathbf{e}_\beta = (\partial_r\sigma_{rr})\mathbf{e_r} + (\partial_r\sigma_{r\theta})\mathbf{e}_\theta + (\partial_r\sigma_{r\varphi})\mathbf{e}_\varphi + (\partial_r\sigma_{rz})\mathbf{e_z}$$

If $\gamma = \theta$,

$$\begin{aligned}
&\sum_{\alpha=r,\theta,\varphi,z}\ \sum_{\beta=r,\theta,\varphi,z} \sigma_{\alpha\beta}[(\partial_\theta\mathbf{e}_\alpha) \otimes \mathbf{e}_\beta + \mathbf{e}_\alpha \otimes (\partial_\theta\mathbf{e}_\beta)] = \\
&\sigma_{rr}(\mathbf{e}_\theta \otimes \mathbf{e_r} + \mathbf{e_r} \otimes \mathbf{e}_\theta) + \sigma_{r\theta}(\mathbf{e}_\theta \otimes \mathbf{e}_\theta - \mathbf{e_r} \otimes \mathbf{e_r} - \mathbf{e_r} \otimes \mathbf{e_r} + \mathbf{e}_\theta \otimes \mathbf{e}_\theta) \\
&+ \sigma_{r\varphi}(\mathbf{e}_\theta \otimes \mathbf{e}_\varphi + \mathbf{e}_\varphi \otimes \mathbf{e}_\theta) + \sigma_{rz}(\mathbf{e}_\theta \otimes \mathbf{e_z} + \mathbf{e_z} \otimes \mathbf{e}_\theta) - \sigma_{\theta\theta}(\mathbf{e_r} \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e_r}) \\
&- \sigma_{\theta\varphi}(\mathbf{e_r} \otimes \mathbf{e}_\varphi + \mathbf{e}_\varphi \otimes \mathbf{e_r}) - \sigma_{\theta z}(\mathbf{e_r} \otimes \mathbf{e_z} + \mathbf{e_z} \otimes \mathbf{e_r})
\end{aligned}$$

We replaced the derivatives directly and used the symmetry of $\boldsymbol{\sigma}$. Then,

$$\begin{aligned}(\partial_\theta \boldsymbol{\sigma}).\mathbf{e}_\theta &= \sum_{\beta=r,\theta,\varphi,z} (\partial_\theta \sigma_{\theta\beta})\mathbf{e}_\beta + (\sigma_{rr}-\sigma_{\theta\theta})\mathbf{e_r} + 2\sigma_{r\theta}\mathbf{e}_\theta + \sigma_{r\varphi}\mathbf{e}_\varphi + \sigma_{rz}\mathbf{e_z} \\ &= (\sigma_{rr}-\sigma_{\theta\theta}+\partial_\theta\sigma_{r\theta})\mathbf{e_r} + (2\sigma_{r\theta}+\partial_\theta\sigma_{\theta\theta})\mathbf{e}_\theta + (\sigma_{r\varphi}+\partial_\theta\sigma_{\theta\varphi})\mathbf{e}_\varphi + (\sigma_{rz}+\partial_z\sigma_{\theta z})\mathbf{e_z}\end{aligned}$$

If $\gamma = \varphi$,

$$\begin{aligned}&\sum_{\alpha=r,\theta,\varphi,z}\sum_{\beta=r,\theta,\varphi,z} \sigma_{\alpha\beta}[(\partial_\varphi \mathbf{e}_\alpha)\otimes\mathbf{e}_\beta + \mathbf{e}_\alpha\otimes(\partial_\varphi\mathbf{e}_\beta)] = \\ &\sigma_{rr}\sin\theta(\mathbf{e_r}\otimes\mathbf{e}_\varphi + \mathbf{e}_\varphi\otimes\mathbf{e_r}) + \sigma_{r\theta}(\cos\theta\mathbf{e_r}\otimes\mathbf{e}_\varphi + \sin\theta\mathbf{e}_\varphi\otimes\mathbf{e}_\theta + \cos\theta\mathbf{e}_\varphi\otimes\mathbf{e_r} + \sin\theta\mathbf{e}_\theta\otimes\mathbf{e}_\varphi) \\ &+ \sigma_{r\varphi}(2\sin\theta\mathbf{e}_\varphi\otimes\mathbf{e}_\varphi - 2\sin\theta\mathbf{e_r}\otimes\mathbf{e_r} - \cos\theta(\mathbf{e_r}\otimes\mathbf{e}_\theta + \mathbf{e}_\theta\otimes\mathbf{e_r})) + \sigma_{rz}\sin\theta(\mathbf{e}_\varphi\otimes\mathbf{e_z} + \mathbf{e_z}\otimes\mathbf{e}_\varphi) \\ &+ \sigma_{\theta\theta}\cos\theta(\mathbf{e}_\theta\otimes\mathbf{e}_\varphi + \mathbf{e}_\varphi\otimes\mathbf{e}_\theta) + \sigma_{\theta\varphi}(2\cos\theta\mathbf{e}_\varphi\otimes\mathbf{e}_\varphi - 2\cos\theta\mathbf{e}_\theta\otimes\mathbf{e}_\theta - \sin\theta(\mathbf{e}_\theta\otimes\mathbf{e_r} + \mathbf{e_r}\otimes\mathbf{e}_\theta)) \\ &+ \sigma_{\theta z}\cos\theta(\mathbf{e}_\varphi\otimes\mathbf{e_z} + \mathbf{e_z}\otimes\mathbf{e}_\varphi) - \sigma_{\varphi\varphi}(\sin\theta(\mathbf{e}_\varphi\otimes\mathbf{e_r} + \mathbf{e_r}\otimes\mathbf{e}_\varphi) + \cos\theta(\mathbf{e}_\varphi\otimes\mathbf{e}_\theta + \mathbf{e}_\theta\otimes\mathbf{e}_\varphi)) \\ &- \sigma_{\varphi z}(\sin\theta(\mathbf{e_r}\otimes\mathbf{e_z} + \mathbf{e_z}\otimes\mathbf{e_r}) + \cos\theta(\mathbf{e}_\theta\otimes\mathbf{e_z} + \mathbf{e_z}\otimes\mathbf{e}_\theta))\end{aligned}$$

Then,

$$\begin{aligned}(\partial_\varphi\boldsymbol{\sigma}).\mathbf{e}_\varphi &= \sum_{\beta=r,\theta,\varphi,z}(\partial_\varphi\sigma_{\varphi\beta})\mathbf{e}_\beta + (\sin\theta\sigma_{rr} + \cos\theta\sigma_{r\theta} - \sin\theta\sigma_{\varphi\varphi})\mathbf{e_r} \\ &+ (\sin\theta\sigma_{r\theta} + \cos\theta\sigma_{\theta\theta} - \cos\theta\sigma_{\varphi\varphi})\mathbf{e}_\theta + (2\sin\theta\sigma_{r\varphi} + 2\cos\theta\sigma_{\theta\varphi})\mathbf{e}_\varphi + (\sin\theta\sigma_{rz} + \cos\theta\sigma_{\theta z})\mathbf{e_z} \\ &= (\sin\theta\sigma_{rr} + \cos\theta\sigma_{r\theta} - \sin\theta\sigma_{\varphi\varphi} + \partial_\varphi\sigma_{r\varphi})\mathbf{e_r} + (\sin\theta\sigma_{r\theta} + \cos\theta\sigma_{\theta\theta} - \cos\theta\sigma_{\varphi\varphi} + \partial_\varphi\sigma_{\theta\varphi})\mathbf{e}_\theta \\ &+ (2\sin\theta\sigma_{r\varphi} + 2\cos\theta\sigma_{\theta\varphi} + \partial_\varphi\sigma_{\varphi\varphi})\mathbf{e}_\varphi + (\sin\theta\sigma_{rz} + \cos\theta\sigma_{\theta z} + \partial_\varphi\sigma_{\varphi z})\mathbf{e_z}\end{aligned}$$

If $\gamma = z$,

$$(\partial_z\boldsymbol{\sigma}).\mathbf{e_z} = (\partial_z\sigma_{rz})\mathbf{e_r} + (\partial_z\sigma_{\theta z})\mathbf{e}_\theta + (\partial_z\sigma_{\varphi z})\mathbf{e}_\varphi + (\partial_z\sigma_{zz})\mathbf{e_z}$$

Finally, the divergence of $\boldsymbol{\sigma}$ given in the spherindrical basis is

$$\boldsymbol{\nabla}.\boldsymbol{\sigma} = \begin{pmatrix} \partial_r\sigma_{rr} + \frac{1}{r}\partial_\theta\sigma_{r\theta} + \frac{1}{r\sin\theta}\partial_\varphi\sigma_{r\varphi} + \partial_z\sigma_{rz} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + \frac{\sigma_{rr}-\sigma_{\varphi\varphi}}{r} + \frac{\cot\theta}{r}\sigma_{r\theta} \\ \partial_r\sigma_{r\theta} + \frac{1}{r}\partial_\theta\sigma_{\theta\theta} + \frac{1}{r\sin\theta}\partial_\varphi\sigma_{\theta\varphi} + \partial_z\sigma_{\theta z} + 3\frac{\sigma_{r\theta}}{r} + \cot\theta\frac{\sigma_{\theta\theta}-\sigma_{\varphi\varphi}}{r} \\ \partial_r\sigma_{r\varphi} + \frac{1}{r}\partial_\theta\sigma_{\theta\varphi} + \frac{1}{r\sin\theta}\partial_\varphi\sigma_{\varphi\varphi} + \partial_z\sigma_{\varphi z} + 3\frac{\sigma_{r\varphi}}{r} + 2\frac{\cot\theta}{r}\sigma_{\theta\varphi} \\ \partial_r\sigma_{rz} + \frac{1}{r}\partial_\theta\sigma_{\theta z} + \frac{1}{r\sin\theta}\partial_\varphi\sigma_{\varphi z} + \partial_z\sigma_{zz} + 2\frac{\sigma_{rz}}{r} + \frac{\cot\theta}{r}\sigma_{\theta z} \end{pmatrix} \tag{A.3}$$

B Compressible potential flow equation

The governing equations are (6.1), equivalent to

$$\begin{array}{l} \frac{1}{\rho}\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{u} + \frac{1}{\rho}\mathbf{u}\cdot\boldsymbol{\nabla}\rho = 0 \\ \frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{u}) = -\frac{\boldsymbol{\nabla}P}{\rho} \end{array} \tag{B.1}$$

With $\mathbf{u} = \boldsymbol{\nabla}\phi$, the momentum equation becomes:

$$\begin{aligned} &\frac{\partial \boldsymbol{\nabla}\phi}{\partial t} + \frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi) = -\frac{\boldsymbol{\nabla}P}{\rho} = -\boldsymbol{\nabla}\int_{P_0}^{P}\frac{d\tilde{P}}{\rho} \\ \Rightarrow &\frac{\partial \phi}{\partial t} + \frac{1}{2}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi) + \int_{P_0}^{P}\frac{d\tilde{P}}{\rho} = C \\ \Rightarrow &\frac{\partial^2 \phi}{\partial t^2} + \frac{1}{2}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi) + \frac{\partial}{\partial t}\int_{P_0}^{P}\frac{d\tilde{P}}{\rho} = 0 \\ \Rightarrow &\frac{\partial^2 \phi}{\partial t^2} + \frac{1}{2}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi) + \frac{1}{\rho}\frac{\partial P}{\partial t} = 0 \end{aligned} \tag{B.2}$$

The definition of the speed of the material wave (here the speed of light) yields $c^2 = \frac{dP}{d\rho}$. Assuming constant the speed of wave, $c^2\frac{\partial \rho}{\partial t} = \frac{\partial P}{\partial t}$. The momentum equation therefore becomes

$$\frac{\partial^2 \phi}{\partial t^2} + \frac{1}{2}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi) + \frac{c^2}{\rho}\frac{\partial \rho}{\partial t} = 0 \Rightarrow \frac{1}{\rho}\frac{\partial \rho}{\partial t} = -\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi)\right)$$

From equation (B.2),

$$\frac{1}{\rho}\mathbf{u}\cdot\boldsymbol{\nabla}\rho = \frac{1}{\rho c^2}\mathbf{u}\cdot\boldsymbol{\nabla}P = -\frac{1}{c^2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi)\right)$$

As $\boldsymbol{\nabla}\cdot\mathbf{u} = \Delta\phi$, the continuity equation of (B.1) yields

$$\begin{aligned} &\Delta\phi - \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi)\right) - \frac{1}{c^2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi)\right) = 0 \\ \Rightarrow &\Delta\phi - \frac{1}{c^2}\left(\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi) + \frac{1}{2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi)\right) = 0 \end{aligned}$$

Let us note x_i with $i \in [0,2]$ representing coordinates x, y and z and $j \equiv i+1 \pmod 3$. Then the general form of the equation is

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \sum_{i=0}^{2}\left[\left(1 - \left(\frac{\frac{\partial \phi}{\partial x_i}}{c}\right)^2\right)\frac{\partial^2 \phi}{\partial x_i^2} - \frac{2}{c^2}\left(\frac{\partial \phi}{\partial x_i}\frac{\partial^2 \phi}{\partial x_i \partial t} + \frac{\partial \phi}{\partial x_i}\frac{\partial \phi}{\partial x_j}\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right)\right] \tag{B.3}$$

Let us assume small perturbations in the flow, neglecting third order terms. By introducing the Mach number in each direction $Ma_i = \frac{u_i}{c} = \frac{\frac{\partial \phi}{\partial x_i}}{c}$, equation (B.3) looks like

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \sum_{i=0}^{2}\left(1 - Ma_i^2\right)^2\frac{\partial^2 \phi}{\partial x_i^2} + \frac{2}{c}\sum_{i=0}^{2}Ma_i\frac{\partial^2 \phi}{\partial x_i \partial t} = 0 \tag{B.4}$$

Finally, if the fluid velocity is assumed to be mainly along one direction such that $\mathbf{u} = (V_\infty + u_x)\mathbf{e_x} + u_y\mathbf{e_y} + u_z\mathbf{e_z}$ with u_x, u_y and u_z small velocity disturbances of the flow far smaller than V_∞, then the second-order terms in u_{x_i} can be neglected in equation (B.4):

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \left(1 - Ma^2\right)^2\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} + \frac{2}{c}Ma\frac{\partial^2 \phi}{\partial x \partial t} = 0 \tag{B.5}$$