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Generalized Tauberian theorems for Abel summability with complex coefficients

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Abstract. We prove generalizations of Tauberian results of Szász (1928) or Landau (1913) and Szász (1951), respectively for the converse of Frobenius' (1880) theorem or the inversion of Abel summability on power series. We show that the present Tauberian condition, *i.e.* Weakly-Vanishing Mean Oscillation (*W-VMO*), is not only weaker but it remains necessary and sufficient in the case of complex coefficients. In particular, this means that the usual boundedness assumption can be dropped from the Tauberian condition.

Keywords. Tauberian theory, Abel summability, Cesàro summability, Slow oscillation, Weakly-Vanishing Mean Oscillation (*W-VMO*), Power series.

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1. Introduction and notations

Let us consider a power series $\sum_{n \in \mathbb{N}} a_n x^n$ with real or complex coefficients (a_n) such that the convergence radius is $R = 1$. Its sum is denoted by $f :]-1, 1[\rightarrow \mathbb{C}$. We define the quantities:

$$s_n := \sum_{k=0}^n a_k, \quad \sigma_n := \frac{1}{n+1} \sum_{k=0}^n s_k, \quad v_n := \frac{1}{n+1} \sum_{k=0}^n k a_k, \quad \forall n \in \mathbb{N},$$

for which some calculations provide the equalities:

$$v_n = s_n - \sigma_n, \quad \forall n \in \mathbb{N}. \quad (1)$$

and then also:

$$\sigma_n = \sum_{k=1}^n \frac{v_k}{k}, \quad \forall n \geq 1 \quad (\sigma_0 = s_0 = a_0). \quad (2)$$

Thanks to usual Cauchy products, it is known that the sum $f(x)$ of $\sum_{n \in \mathbb{N}} a_n x^n$ reads in the different forms below, for any $|x| < 1$:

$$f(x) := \sum_{n=0}^{+\infty} a_n x^n = (1-x) \sum_{n=0}^{+\infty} s_n x^n = (1-x)^2 \sum_{n=0}^{+\infty} (n+1) \sigma_n x^n. \quad (3)$$

We recall that the sequence (s_n) (or the series $\sum_{n \in \mathbb{N}} a_n$) is said to be *Abel summable* (A_0) to $s \in \mathbb{C}$ if we have:

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{+\infty} s_n x^n = s \in \mathbb{C}. \quad (4)$$

It is known from Tauber's (1897) 'second theorem' [18] that, if the sequence (s_n) satisfies (4), then the convergence $s_n \rightarrow s$ holds true *if and only if* $v_n = (s_n - \sigma_n) \rightarrow 0$ when $n \rightarrow +\infty$.

Since the pioneering work of Tauber (1897), the Tauberian theory has produced many results in mathematical analysis, Fourier analysis or analytic number theory, see e.g. the monographs by Hardy (1949) [3] and Korevaar (2004) [7]. Among these, E. Landau (1913) proved in [9] that if the two following conditions hold on the sequence (s_n) :

$$\left\{ \begin{array}{l} i) \quad (s_n) \text{ is bounded,} \\ ii) \quad (s_n) \text{ is slowly oscillating (at infinite), i.e.} \\ \qquad \qquad \qquad \lim_{n \rightarrow +\infty} (s_m - s_n) = 0, \quad \text{as } \frac{m}{n} \xrightarrow{n \rightarrow +\infty} 1, \end{array} \right. \quad (5)$$

then, the convergence $s_n \rightarrow s$ follows from its Abel summability (4) to s . The conditions (5) are also clearly necessary for the convergence of (s_n) . Moreover, they can be relaxed in the case of real coefficients to one-sided conditions, e.g. the 'slow decrease (at infinite)' of Schmidt (1925) [14]. Hence, the unilateral Tauberian conditions that have been extensively studied for real-valued sequences (see e.g. [3, 7]) will not be considered further in the sequel.

Next, the following *W-VMO* property of a sequence is introduced and shown in our previous work [1][Proposition 2.1] to be weaker than the 'slow oscillation' condition in (5).

Definition 1 (Weakly-Vanishing Mean Oscillation (W-VMO)).

Any sequence $(u_n)_{n \in \mathbb{N}}$ of real or complex numbers is said to be of *Weakly-Vanishing Mean Oscillation*, or shortly *W-VMO*, if its mean oscillation $(\omega_n(\lambda))$ satisfies the property:

$$\inf_{0 < \lambda < 1} \left(\limsup_{n \rightarrow +\infty} |\omega_n(\lambda)| \right) = 0, \quad \text{where:} \quad (6)$$

$$\omega_n(\lambda) := \frac{1}{n - \lfloor n\lambda \rfloor} \sum_{k=\lfloor n\lambda \rfloor + 1}^n (u_n - u_k), \quad \forall n \geq 1, \quad \forall \lambda \in]0, 1[.$$

The general result below is proved in [1][Theorem 4.1] using the unified setting proposed for both the Cesàro and Abel summabilities.

Theorem 1 (Tauberian converse of Abel's theorem).

The assumptions *i*) and *ii*) below:

$$\left\{ \begin{array}{l} i) \quad f(1^-) := \lim_{x \rightarrow 1^-} f(x) = s \in \mathbb{C}, \quad \text{i.e. (4) is satisfied,} \\ ii) \quad \text{the sequence } (s_n) \text{ is W-VMO,} \end{array} \right. \quad (7)$$

are necessary and sufficient to get the convergence: $s_n \rightarrow s$ when $n \rightarrow +\infty$.

It is clear that Theorem 1 includes Landau's theorem in [9] since the 'slow oscillation' of (s_n) implies its *W-VMO* property. Moreover, the boundedness assumption in (5) is now no more explicitly required in Theorem 1. Considering also Theorem 3.1 in [1] where the boundedness assumption is not explicitly necessary, a similar observation can be made for the inversion of Cesàro summability. Hence, the objective of the present study is twofold: *i*) explain how the boundedness assumption facilitates the proof of Tauberian results and how to bypass it in the general case of complex-valued sequences, and *ii*) show the link with Tauber's (1897) 'second theorem' [18].

2. Generalization of E. Landau's (1913) Tauberian theorem

As pointed out above, the following Corollary 1 shows that the 'slow oscillation' property of (s_n) alone (without its boundedness) is actually sufficient to ensure that the convergence of (s_n) to s follows from its Abel summability to s . This is new since, even with more sophisticated proofs involving Laplace transforms or Wiener-type Tauberian theorems [19], the boundedness hypothesis is still required for complex-valued sequences; see also [7][Chap. II and III]. However,

we provide in Appendix A an independant proof of Landau's (1913) theorem using elementary arguments to measure how far the boundedness assumption simplifies and shortens the proof.

Corollary 1 (Generalized Landau's Tauberian theorem).

The assumptions *i*) and *ii*) below:

$$\left\{ \begin{array}{l} i) \quad f(1^-) := \lim_{x \rightarrow 1^-} f(x) = s \in \mathbb{C}, \quad \text{i.e. (4) is satisfied,} \\ ii) \quad \text{the sequence } (s_n) \text{ slowly oscillating (at infinite),} \end{array} \right. \quad (8)$$

are necessary and sufficient to get the convergence of (s_n) to s .

Proof. This is a direct consequence of Theorem 1.

First, the conditions (8) are necessary for the desired result. Indeed, if (s_n) converges to s , then (8.i) holds from the Abel (1826) theorem of radial limit on power series. Moreover, the condition (8.ii) clearly holds too since the convergence of (s_n) implies its slow oscillation.

Reciprocally, if (8) holds true, then the sequence (s_n) is slowly oscillating and thus necessarily also W -VMO with [1][Proposition 2.1]. Hence, Theorem 1 does apply and shows that $s_n \rightarrow s$ when $n \rightarrow +\infty$. ■

Now, Corollary 1 (and of course Theorem 1) does include the celebrated Tauberian theorem of Littlewood (1911) [10] (see also [11]), where the 'big \mathcal{O} ' condition of Hardy–Littlewood: $n|a_n| = \mathcal{O}(1)$ is assumed instead of (5); but, the original theorem of Landau (1913) does not. Indeed, it is known that any series $\sum_{n \in \mathbb{N}} a_n$ satisfying the 'big \mathcal{O} ' condition has slowly oscillating sums (s_n) (e.g. see [1][Proposition 2.1]), but not necessarily bounded, e.g. with the harmonic series $\sum_{n \geq 1} 1/n$ that is not Abel summable, we have: $s_n \sim \ln n$. Moreover, the condition: $n|a_n| = \mathcal{O}(1)$ is not necessary for the convergence of the series $\sum_{n \in \mathbb{N}} a_n$, e.g. the alternating Riemann series $\sum_{n \geq 1} (-1)^n / \sqrt{n}$ does converge with Leibniz's criterion but $n|a_n| = \sqrt{n}$.

Despite the bounded character of any convergent sequence, the fact that the boundedness of (s_n) can be explicitly relaxed for the sufficiency of (8) has an interest since there exist unbounded sequences that are however slowly oscillating and thus also W -VMO, e.g. Example 1 below.

Example 1 (Bounded or unbounded W -VMO sequences).

Let us take $u_n = \ln n$ for all $n \geq 1$, such that $u_n \xrightarrow[n \rightarrow +\infty]{} +\infty$, but (u_n) is slowly oscillating since:

$$u_m - u_n = \ln\left(\frac{m}{n}\right) \xrightarrow[n \rightarrow +\infty]{} 0, \quad \text{as } \frac{m}{n} \xrightarrow[n \rightarrow +\infty]{} 1.$$

This implies from [1][Proposition 2.1] that (u_n) is necessarily a W -VMO sequence. By taking $u_n = (-1)^n \ln n$, now (u_n) is neither slowly oscillating nor W -VMO.

It is also possible to find a bounded and slowly oscillating real-valued sequence that is still not a Cauchy sequence, and thus which does not converge, e.g. $u_n = \cos(\ln n)$ or $u_n = \sin(\ln n)$, for all $n \geq 1$.

Similarly, an example of bounded and slowly oscillating complex-valued sequence that does not converge is given by $u_n = e^{i \ln n}$ for all $n \geq 1$.

3. Generalization of Tauberian theorems of O. Szász

3.1. Historical review

O. Szász (1951) proved in [16] that if the two following conditions hold:

$$\left\{ \begin{array}{l} i) \quad \bar{v}_n := \frac{1}{n+1} \sum_{k=0}^n k|a_k| = \mathcal{O}(1), \\ ii) \quad \text{the sequence } (\bar{v}_n) \text{ is slowly oscillating,} \end{array} \right. \quad (9)$$

then, the convergence of (s_n) to s follows from its Abel summability (4) to s . O. Szász (1951) also shows that the above result includes his previous theorem in Szász (1928) [15] where the Tauberian condition (9) is replaced by:

$$\sum_{k=1}^n k^p |a_k|^p = \mathcal{O}(n), \quad \text{for some } p > 1. \quad (10)$$

Indeed, using the Hölder inequality, the condition (10) clearly implies that the sequence (\bar{v}_n) is bounded. Furthermore, Szász (1951) [16] shows that the slow oscillation of (\bar{v}_n) follows from (10). It is also known from a counter-example in Rényi (1946) [13] that the condition (10) with $p = 1$ is no more sufficient to get $s_n \rightarrow s$ from the Abel summability (4). Moreover, it is clear that the above result of Szász (1951) [16] also includes Rényi (1946)'s Theorem in [13] where the condition (9) is replaced by:

$$\bar{v}_n := \frac{1}{n+1} \sum_{k=0}^n k |a_k| \xrightarrow{n \rightarrow +\infty} \bar{v} < +\infty, \quad (11)$$

since (11) implies (9). However, Szász (1928) [15] shows that if the sequence (v_n) , or stronger (\bar{v}_n) , is only bounded, then it is sufficient to get $\sigma_n \rightarrow s$ from (4). Hence, either if (v_n) is bounded or if (s_n) is *W-VMO* with [1][Corollary 4.2], then the convergence of the Cesàro mean $\sigma_n \rightarrow s$ follows from the Abel summability (4) of (s_n) . But none of these latter conditions is necessary as shown by Example 1 given in Remark 3. In the sequel, we prove generalizations of the above results.

Furthermore, Szász (1951) [16] shows two other generalizations of his result that only hold in the case of real coefficients (a_n) , as for the extensions proposed by Rajagopal (1952) [12]. Later, Jakimovski (1954) [5] proposed unilateral Tauberian conditions coming from the one-sided '*slow decrease*' condition of Schmidt's (1925) theorem [14], that are weaker than (9) and also necessary and sufficient to get $s_n \rightarrow s$ from the Abel summability (4), but they are valid only for the real case.

3.2. Generalized Tauberian theorems

We shall need the following technical result due to Zygmund (1926) [20] or Szász (1928) [15], of which a proof is supplied in Appendix B for the sake of self-consistency of the present work.

Lemma 1 (Zygmund (1926) – Szász (1928)).

We have the following assertion:

$$f(1^-) := \lim_{x \rightarrow 1^-} f(x) = s \in \mathbb{C} \quad \implies \quad \lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{+\infty} \sigma_n x^n = s.$$

In other words, if the sequence (s_n) is Abel summable to $s \in \mathbb{C}$, then its Cesàro mean (σ_n) is also Abel summable to the same limit s .

The result below, that provides a Tauberian converse for the theorem of radial limit of Frobenius (1880) [2], generalizes a theorem of Szász (1928) [15] where (v_n) is supposed to be bounded.

Theorem 2 (Tauberian converse of Frobenius' theorem).

The assumptions i) and ii) below:

$$\left\{ \begin{array}{l} i) \quad f(1^-) := \lim_{x \rightarrow 1^-} f(x) = s \in \mathbb{C}, \quad \text{i.e. (4) is satisfied,} \\ ii) \quad \text{the sequence } (\sigma_n) \text{ is } W\text{-VMO,} \end{array} \right. \quad (12)$$

are necessary and sufficient to get the convergence of (σ_n) to s .

Proof. First, if $\sigma_n \rightarrow s$ when $n \rightarrow +\infty$, the Frobenius (1880) theorem shows that (12.i) is satisfied. Moreover, since (σ_n) converges, it clearly implies that (σ_n) is slowly oscillating and thus also *W-VMO* with [1][Proposition 2.1], i.e. (12.ii) is verified.

Conversely, if (12.i) is satisfied, the Zygmund–Szász Lemma 1 ensures that the sequence (σ_n) is also Abel summable to s , i.e.

$$\lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{+\infty} \sigma_n x^n = s. \quad (13)$$

Then, Theorem 1 shows that the convergence of (σ_n) to s follows from (13) if and only if the sequence (σ_n) is W -VMO, and thus if the condition (12.ii) is satisfied. ■

Remark 1 (Generalization of Szász's (1928) theorem).

Let us observe that, if (v_n) is bounded, then the sequence (σ_n) is slowly oscillating (at infinite) and thus W -VMO. Indeed, we have using (2) for integers $m > n$:

$$\begin{aligned} |\sigma_m - \sigma_n| &\leq \sum_{k=n+1}^m \frac{|v_k|}{k} \leq \frac{1}{n+1} \left(1 + \frac{n+1}{n+2} + \dots + \frac{n+1}{m} \right) \sup_{n \in \mathbb{N}} |v_n| \\ &\leq \frac{n}{n+1} \left(\frac{m}{n} - 1 \right) \sup_{n \in \mathbb{N}} |v_n| \xrightarrow{n \rightarrow +\infty} 0, \quad \text{as } \frac{m}{n} \xrightarrow{n \rightarrow +\infty} 1. \end{aligned}$$

This shows that Theorem 2 actually includes the theorem of Szász (1928) [15].

The result below includes the theorem of Szász (1951) in [16] assuming the Tauberian conditions (9) and makes the connection with the 'second theorem' of Tauber (1897) [18].

Theorem 3 (Tauberian converse for Abel summability).

The assumptions *i*) and *ii*) below:

$$\begin{cases} i) & f(1^-) := \lim_{x \rightarrow 1^-} f(x) = s \in \mathbb{C}, \quad \text{i.e. (4) is satisfied,} \\ ii) & \text{the sequence } (v_n) \text{ is } W\text{-VMO,} \end{cases} \quad (14)$$

are necessary and sufficient to get the convergence of (s_n) to s .

Proof. First, the assumptions *i*) et *ii*) are necessary to get the convergence of (s_n) to s . Indeed, if $s_n \rightarrow s$ when $n \rightarrow +\infty$, the Abel (1826) theorem shows that the condition *i*) holds. Moreover, since $s_n \rightarrow s$, we have also $\sigma_n \rightarrow s$ from the usual lemma of Cauchy–Cesàro's arithmetic mean and then, Eq. (1) implies that $v_n \rightarrow 0$. Thus, it clearly results that the sequence (v_n) is slowly oscillating and then also W -VMO from [1][Proposition 2.1], i.e. the condition *ii*) holds.

Conversely, if the hypothesis *i*) is verified, this implies from the Zygmund–Szász Lemma 1 that:

$$\lim_{x \rightarrow 1^-} g(x) = s, \quad \text{where } g(x) := (1-x) \sum_{n=0}^{+\infty} \sigma_n x^n. \quad (15)$$

Then, by taking the difference between Eqs (4) and (15) using (1), it comes:

$$\lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{+\infty} v_n x^n = 0, \quad (16)$$

which means that the sequence (v_n) is Abel summable to 0. Then, assuming also that the condition *ii*) is fulfilled, it results with (16) from Theorem 1 that: $v_n \rightarrow 0$ when $n \rightarrow +\infty$.

Now, it is known by the 'second theorem' of Tauber (1897) [18] that the convergence of (s_n) to s follows from the hypothesis *i*) if and only if we have: $v_n = (s_n - \sigma_n) \rightarrow 0$ when $n \rightarrow +\infty$. The usual and original proof of that result uses Tauber's (1897) 'first theorem' [18]. However, let us observe that Tauber's 'second theorem' now falls in a few lines using the present setting. Indeed, since (v_n) is bounded because it converges, then Remark 1 shows that the sequence (σ_n) is slowly oscillating (at infinite) and thus W -VMO. Then, Theorem 2 ensures with *i*) that $\sigma_n \rightarrow s$ when $n \rightarrow +\infty$. Finally, using Eq. (1) with $v_n \rightarrow 0$, this implies the convergence $s_n \rightarrow s$, which concludes the proof. ■

Remark 2 (Generalization of Szász's (1951) theorem).

Let us observe that if the conditions (9) are fulfilled, then the sequence (v_n) is also bounded and slowly oscillating, and thus W -VMO. Indeed, since the positive sequence (\bar{v}_n) is bounded, we have:

$$\limsup_{n \rightarrow +\infty} |v_n| \leq \limsup_{n \rightarrow +\infty} \bar{v}_n < +\infty, \quad (17)$$

and (v_n) is bounded too (the converse being clearly not true, see e.g. Example 3 in Remark 3). Besides, it follows for integers $m > n$:

$$\begin{aligned} v_m - v_n &= \frac{1}{m+1} \sum_{k=0}^n k a_k + \frac{1}{m+1} \sum_{k=n+1}^m k a_k - v_n \\ &= \left(\frac{n+1}{m+1} - 1 \right) v_n + \frac{1}{m+1} \sum_{k=n+1}^m k a_k. \end{aligned}$$

Then, it follows the bound:

$$\begin{aligned} |v_m - v_n| &\leq \left| \frac{n+1}{m+1} - 1 \right| |v_n| + \frac{1}{m+1} \left(\sum_{k=0}^m k |a_k| - \sum_{k=0}^n k |a_k| \right) \\ &\leq \left| \frac{n+1}{m+1} - 1 \right| \sup_{n \in \mathbb{N}} |v_n| + \left(\bar{v}_m - \frac{n+1}{m+1} \bar{v}_n \right). \end{aligned}$$

Now, since (v_n) is bounded and (\bar{v}_n) is slowly oscillating, we get:

$$|v_m - v_n| \xrightarrow{n \rightarrow +\infty} 0, \quad \text{as } \frac{m}{n} \xrightarrow{n \rightarrow +\infty} 1,$$

which means that (v_n) is also slowly oscillating. This shows that Theorem 3 actually includes the theorem of Szász (1951) [16].

We now provide several examples in Remark 3 to show that Theorems 2 and 3 are actually far more general than Szász's results in [15, 16]. Indeed, the W -VMO Tauberian condition is not only weaker but also necessary.

Remark 3 (Examples).

- (1) Let us take the positive and unbounded sequence $(s_n)_{n \geq 1}$ defined by:

$$s_n := \begin{cases} \ln n, & \text{if } n = 2^p, p \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$

which gives: $\liminf_{n \rightarrow +\infty} s_n = 0$ and $\limsup_{n \rightarrow +\infty} s_n = +\infty$.

We get $\sigma_n = \sigma_{2^p}$ for any integer n with $2^p \leq n < 2^{p+1}$ ($p \in \mathbb{N}$), where:

$$\sigma_{2^p} = \frac{\ln 2}{2^p} (1 + 2 + \dots + p) = \frac{p(p+1)}{2^{p+1}} \ln 2 \xrightarrow{p \rightarrow +\infty} 0.$$

Then, (s_n) is Cesàro summable with $\sigma_n \rightarrow 0$, and thus Abel summable to 0 from the Frobenius (1880) theorem. Since (σ_n) converges, it has a slow oscillation and it is thus W -VMO. But since (s_n) is not bounded, Eq. (1) with (σ_n) bounded shows that (v_n) is unbounded too. Hence, Theorem 2 does apply, whereas the theorem in Szász (1928) [15] cannot be applied. Moreover, (s_n) cannot be a W -VMO (or slowly oscillating) sequence by contradiction with Theorem 3 using Eq. (1). Thus, the result in [1][Corollary 4.2] cannot be applied too.

However, since $(s_n) \geq 0$ in this case, Lemma 2.4 in [1] or the theorem of Hardy–Littlewood (1914) [4] (or also Karamata's (1930) lemma [6]) can still be applied. But, that will not be anymore the case in the next Example 2.

- (2) Let us take the unbounded sequence $(s_n)_{n \geq 1}$ defined by:

$$s_n := \begin{cases} (-1)^p p \ln 2, & \text{if } n = 2^p, p \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$

where now, $\liminf_{n \rightarrow +\infty} s_n = -\infty$ and $\limsup_{n \rightarrow +\infty} s_n = +\infty$.

Hence in this case, only Theorem 2 does still apply from what precedes in Example 1.

- (3) Let us consider the case with $a_n = (-1)^n \frac{\ln n}{n}$ for all $n \geq 1$, where the alternating series $\sum_{n \geq 1} a_n$ converges to some $s \in \mathbb{R}$ from the Leibniz criterion. Then, the Kronecker lemma [8] ensures that:

$$v_n = \frac{1}{n} \sum_{k=1}^n (-1)^k \ln k \xrightarrow{n \rightarrow +\infty} 0,$$

and (v_n) is slowly oscillating and thus also W -VMO. But we have:

$$\bar{v}_n = \frac{1}{n} \sum_{k=1}^n \ln k = \frac{\ln(n!)}{n} \underset{n \rightarrow +\infty}{\sim} \ln n \underset{n \rightarrow +\infty}{\longrightarrow} +\infty.$$

Moreover, the theorem of Abel (1826) shows that the series $\sum_{n \geq 1} a_n$ is Abel summable to its sum s . Hence in that case, Theorem 3 does apply as well as Theorem 1 or Corollary 1. But, the Tauberian condition (9) of [16] does not hold, neither the 'big- O ' condition of Littlewood (1911) [10] because $n|a_n| = \ln n$ is not bounded.

Appendix A. Alternative proof of E. Landau's (1913) Tauberian theorem

Here, we give an independant proof of Landau's theorem, without using [1], that is simpler than the original proof of Landau (1913) in [9] that used differentiation techniques of Hardy-Littlewood. Moreover, it has the interest to make the connection with the inversion of Cesàro summability. Indeed, we show that the boundedness of (s_n) provides the Tauberian result in Lemma 2. Next, the issue amounts to solve the inversion of Cesàro summability with both the boundedness and 'slow oscillation' Tauberian condition on (s_n) that is supplied by Lemma 3.

Lemma 2 (Weak Tauberian converse of Frobenius' theorem).

We assume both *i*) and *ii*) below:

$$\begin{cases} i) & f(1^-) := \lim_{x \rightarrow 1^-} f(x) = s \in \mathbb{C}, \quad \text{i.e. (4) is satisfied,} \\ ii) & \text{the sequence } (s_n) \text{ is bounded.} \end{cases} \quad (18)$$

Then, we have the convergence: $\sigma_n \rightarrow s$ when $n \rightarrow +\infty$.

The assumption *ii*) is not necessary to get the convergence of (σ_n) .

Proof. With the assumption *ii*) of boundedness of (s_n) , it is clear that (σ_n) is bounded. This implies with Eq. (1) that the sequence (v_n) is also bounded.

Then, it amounts to show Szász's (1928) theorem [15] that gives a Tauberian converse of Frobenius' theorem. Indeed, the assumption *i*) of Abel summability of (s_n) implies using Zygmund–Szász's Lemma 1 that:

$$\lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{+\infty} \sigma_n x^n = s, \quad (19)$$

which means that the sequence (σ_n) is also Abel summable to s . Besides, Eq. (2) shows with a Cauchy product that (19) also reads:

$$\lim_{x \rightarrow 1^-} \sum_{n=1}^{+\infty} \frac{v_n}{n} x^n = s. \quad (20)$$

Now, we observe that the coefficients $b_n := v_n/n$ ($n \geq 1$) of the above power series verify: $n|b_n| = |v_n| = \mathcal{O}(1)$. Hence, the Tauberian theorem of Littlewood (1911) [10] ensures finally with (2) that:

$$\lim_{n \rightarrow +\infty} \sigma_n = \sum_{n=1}^{+\infty} \frac{v_n}{n} = s. \quad (21)$$

The present result also falls in a few lines by observing that the boundedness of (s_n) implies that its Cesàro's arithmetic mean (σ_n) is slowly oscillating (and bounded too), thus also *W-VMO*; see Remark 1 since (v_n) is then bounded with (1). Then, the convergence of (σ_n) to s follows from Theorem 2.

Moreover, Example 1 in Remark 3 shows that the boundedness of (s_n) is not necessary to get the convergence of (σ_n) . ■

Lemma 3 (Weak Tauberian converse of Cauchy–Cesàro's lemma).

We assume that a real or complex sequence (s_n) is Cesàro summable to some $s \in \mathbb{C}$, i.e. $\sigma_n \rightarrow s$ when $n \rightarrow +\infty$.

Then, we have the convergence of (s_n) to s if and only if the sequence (s_n) is bounded and slowly oscillating.

Proof. First, the convergence of (s_n) implies both its boundedness and its slow oscillation. Thus, these latter conditions are necessary.

Reciprocally, we assume that the sequence (s_n) is bounded, slowly oscillating and such that $\sigma_n \rightarrow s$ when $n \rightarrow +\infty$. Since (s_n) is bounded, it suffices using Bolzano–Weierstrass' theorem to show that s is the single subsequential limit of (s_n) . Let us set:

$$C := \sup_{n \in \mathbb{N}} |s_n| < +\infty,$$

and consider any subsequential limit $\ell \in \mathbb{C}$ of (s_n) , i.e. $s_{\varphi(n)} \rightarrow \ell$ when $n \rightarrow +\infty$, where $\varphi: \mathbb{N} \rightarrow \mathbb{N}$ is a strictly increasing extraction function of integers ($\varphi(n) \geq n$). By considering the Cesàro arithmetic mean $(\tilde{\sigma}_n)$ of $(s_{\varphi(n)})$ defined by:

$$\tilde{\sigma}_n := \frac{1}{n+1} \sum_{k=0}^n s_{\varphi(k)}, \quad \forall n \in \mathbb{N},$$

it is clear with Cauchy–Cesàro’s lemma that $\tilde{\sigma}_n \rightarrow \ell$ when $n \rightarrow +\infty$. Moreover, we have:

$$\tilde{\sigma}_n - \sigma_n = \frac{1}{n+1} \sum_{k=0}^n (s_{\varphi(k)} - s_k).$$

Then, it follows for all $p, n \in \mathbb{N}$ such that $p < n$:

$$\begin{aligned} |\tilde{\sigma}_n - \sigma_n| &\leq \frac{1}{n+1} \sum_{k=0}^p |s_{\varphi(k)} - s_k| + \frac{1}{n+1} \sum_{k=p+1}^n |s_{\varphi(k)} - s_k| \\ &\leq \frac{2(p+1)}{n+1} C + \frac{n-p}{n+1} \sup_{k \geq p} |s_{\varphi(k)} - s_k|. \end{aligned}$$

Now, passing to the upper limit when $n \rightarrow +\infty$ for a fixed value of p , we get:

$$\limsup_{n \rightarrow +\infty} |\tilde{\sigma}_n - \sigma_n| \leq \sup_{k \geq p} |s_{\varphi(k)} - s_k|,$$

and then taking the limit when $p \rightarrow +\infty$, it yields:

$$\limsup_{n \rightarrow +\infty} |\tilde{\sigma}_n - \sigma_n| \leq \limsup_{p \rightarrow +\infty} |s_{\varphi(p)} - s_p|. \quad (22)$$

Thus, for any φ satisfying $\varphi(p)/p \rightarrow 1$ when $p \rightarrow +\infty$ (e.g. $\varphi(p) = p+1$), the ‘slow oscillation’ of (s_n) reads:

$$\limsup_{p \rightarrow +\infty} |s_{\varphi(p)} - s_p| = 0, \quad \text{as } 1 \leq \frac{\varphi(p)}{p} \xrightarrow{p \rightarrow +\infty} 1.$$

This implies with (22) and the convergence of (σ_n) to s that:

$$\lim_{n \rightarrow +\infty} \tilde{\sigma}_n = \lim_{n \rightarrow +\infty} \sigma_n = s, \quad \text{and thus: } \ell = s.$$

In particular, this conclusion holds if we choose $\varphi(n) := n+1$, which shows that $s_{n+1} \rightarrow s$ when $n \rightarrow +\infty$, i.e. the convergence of (s_n) to s . ■

Appendix B. Proof of Lemma 1 of Zygmund (1926) and Szász (1928)

O. Szász (1952) claims in [17] that Lemma 1 in Szász (1928) [15] is also due to A. Zygmund (1926) [20]. Here we give a proof, inspired from Szász (1952) [17].

Proof. We define:

$$g(x) := (1-x) \sum_{n=0}^{+\infty} \sigma_n x^n, \quad |x| < 1.$$

By putting

$$S_n := (n+1)\sigma_n = \sum_{k=0}^n s_k, \quad \forall n \in \mathbb{N},$$

such that: $\sigma_n = \frac{S_n}{n+1}$, we observe that: $\frac{1}{n+1} = \int_0^1 \rho^n d\rho$.

In these conditions and by using (3), we have for any $|x| < 1$:

$$\begin{aligned} \sum_{n=0}^{+\infty} \sigma_n x^n &= \sum_{n=0}^{+\infty} S_n \left(\int_0^1 \rho^n d\rho \right) x^n = \int_0^1 \left(\sum_{n=0}^{+\infty} S_n (\rho x)^n \right) d\rho \\ &= \int_0^1 \frac{f(\rho x)}{(1-\rho x)^2} d\rho. \end{aligned}$$

We then calculate for $0 < x < 1$ by doing the change of variable: $\rho x = 1 - \frac{1}{t}$, i.e. $t = \frac{1}{1 - \rho x}$ such that: $x d\rho = \frac{1}{t^2} dt = (1 - \rho x)^2 dt$. It comes:

$$\frac{g(x)}{(1-x)} := \sum_{n=0}^{+\infty} \sigma_n x^n = \frac{1}{x} \int_1^{1/(1-x)} f\left(1 - \frac{1}{t}\right) dt, \quad 0 < x < 1.$$

Now for $X > 0$, we put $x = 1 - \frac{1}{1+X}$ such that $\frac{1}{1-x} = 1+X$ and $\frac{1}{x} = \frac{1+X}{X}$. Moreover, we have $x \rightarrow 1^-$ when $X \rightarrow +\infty$. Then it follows:

$$g\left(1 - \frac{1}{1+X}\right) = \frac{1}{X} \int_1^{1+X} f\left(1 - \frac{1}{t}\right) dt, \quad \forall X > 0.$$

Finally, with the change of variable $t = z + 1$, we get:

$$g\left(1 - \frac{1}{1+X}\right) = \frac{1}{X} \int_0^X f\left(1 - \frac{1}{z+1}\right) dz, \quad \forall X > 0. \quad (23)$$

Thus, the right-hand side term appears to be the Cesàro mean of the continuous function $\phi : [0, +\infty[\rightarrow \mathbb{C}$ defined by:

$$\phi(z) := f\left(1 - \frac{1}{z+1}\right), \quad \forall z \geq 0.$$

Moreover, since $f(x) \rightarrow s \in \mathbb{C}$ when $x \rightarrow 1^-$ by hypothesis, we have: $\phi(z) \rightarrow s$ when $z \rightarrow +\infty$. Hence, we can conclude with Eq. (23) and the lemma of Cesàro mean for a function that:

$$\lim_{x \rightarrow 1^-} g(x) = \lim_{X \rightarrow +\infty} g\left(1 - \frac{1}{1+X}\right) = s.$$

Indeed, an easy calculation shows the inequality:

$$\limsup_{x \rightarrow +\infty} |\sigma(x) - s| \leq \limsup_{z \rightarrow +\infty} |\phi(z) - s|, \quad \text{where } \sigma(x) := \frac{1}{x} \int_0^x \phi(z) dz.$$

which implies the desired result if $|\phi(z) - s| \rightarrow 0$ when $z \rightarrow +\infty$. ■

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