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# Practical Calculations in General Relativity in the Case of Diagonal Metrics

David Pigeon

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# Practical Calculations in General Relativity in the Case of Diagonal Metrics

David Pigeon

May 23, 2023

**Keywords:** Curvature tensors, Diagonal spacetimes, Tetrad method.

## Abstract

This article delves into two methods for computing curvature tensors in diagonal spacetimes, comparing the standard approach with the tetrad method. The analysis is applied to a specific metric, uncovering implications for well-known gravitational models like Schwarzschild and Friedmann-Lemaître-Robertson-Walker.

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## 1 Introduction

The exploration of space-time manifolds, characterized by metrics like  $(M, g)$ , serves as a cornerstone in the understanding of gravitational physics and the structure of the universe. In this intricate landscape, the metric  $g$ , taking on a diagonal form, offers a crucial lens through which to analyze the fundamental properties of spacetime.

In the inaugural chapter of this discourse, an in-depth investigation unfolds, aiming to evaluate and compare two prominent methodologies for computing the components of classical curvature-related tensors within this geometric framework. These tensors, integral to the fabric of general relativity, encompass essential entities such as the curvature tensor, the Riemann tensor, the Ricci tensor, and the Einstein tensor.

The first methodology, adhering to the conventional paradigm, relies on a basis of vector fields  $(\partial_0, \partial_1, \partial_2, \partial_3)$  for its computations. This well-established approach forms the bedrock of many gravitational studies, offering a familiar and robust foundation for theoretical exploration.

In stark contrast, the "tetrad method" emerges as a novel alternative, introducing an alternative basis  $(\theta_0, \theta_1, \theta_2, \theta_3)$  carefully crafted to orthonormalize the metric  $g$ . This distinctive approach not only promises computational efficiency but also fosters intuitive geometric interpretations, thereby enriching our understanding of the underlying spacetime structure.

The comparative analysis undertaken within this chapter extends beyond mere computational efficacy; it delves into the deeper implications of these methodologies within the context of a specific metric  $h$ . This metric, characterized by its diagonal properties and spherical symmetry, serves as a fertile ground for exploration, encompassing well-known metrics such as Schwarzschild, Reissner-Nordström, and Friedmann-Lemaître-Robertson-Walker metrics.

By scrutinizing the interplay between these methodologies and the intricacies of the chosen metric, we aim to shed light on the underlying principles governing gravitational phenomena and deepen our comprehension of the fabric of the cosmos. Through this exploration, we endeavor to unveil new insights into the intricate interplay between geometry and gravity, ultimately enriching our collective understanding of the universe's fundamental nature.

## 2 Generalities and Notations

Throughout this chapter, we consider a differentiable manifold  $M$  of dimension 4. We have a maximal atlas on  $M$ , and at each point  $p$  in  $M$ , we have a chart of the form:

$$(U, \varphi := (x^0, x^1, x^2, x^3))$$

where  $p \in U$  and the calculations will be performed on this chart. Thus, we have a diffeomorphism  $\varphi : U \rightarrow \varphi(U) \subset \mathbb{R}^4$ , and for each  $i \in \{0, 1, 2, 3\}$ , we have an infinitely differentiable function:

$$x^i : U \rightarrow \mathbb{R}$$

called the  **$i$ -th coordinate function**.

We denote  $\mathcal{C}^\infty(M)$  as the set of infinitely differentiable functions  $f$  from  $M$  to  $\mathbb{R}$  (on each chart  $U$ , the function  $f|_U : U \rightarrow \mathbb{R}$  is such that  $f \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^4 \rightarrow \mathbb{R}$  is infinitely differentiable in the usual sense). Throughout this chapter, all considered objects will be infinitely differentiable.

To simplify notation, we will use the symbol  $M$  instead of  $U$  from now on. It will be understood that we are working in a chart with coordinates.

References for the concepts discussed in this chapter are:

$$[2], [6], [8], [9], [11], [17], [20].$$

### 2.1 Tangent and Cotangent Bundles

We begin by defining the notion of vectors and tangent spaces.

#### Definition 2.1.1: Tangent Space at a Point

Let  $p \in M$ .

(i) A **vector at  $p$  in  $M$**  is a function:

$$v : \mathcal{C}^\infty(M) \rightarrow \mathbb{R}$$

such that for all  $f_1, f_2 \in \mathcal{C}^\infty(M)$  and all  $a, b \in \mathbb{R}$ :

- (a)  $v(af_1 + bf_2) = av(f_1) + bv(f_2)$ ;
- (b)  $v(f_1f_2) = f_1(p)v(f_2) + f_2(p)v(f_1)$ .

(ii) The **tangent space  $T_pM$  at  $p$  in  $M$**  is defined as the set of vectors at  $p$  in  $M$ .

We notice that if  $f := a$  is constant on  $M$ , then  $v(f) = 0$  because of point (i.b):

$$v(1) = 2v(1)$$

*i.e.*,  $v(1) = 0$ . Therefore, using point (i.a), we have:

$$\begin{aligned} v(f) &= v(a) \\ &= av(1) \\ &= 0 \end{aligned}$$

#### Notation 2.1.2: Notation for Tangent Space at a Point

Let  $p \in M$ .

(1) We define two operations on  $T_pM$ . For all  $v_1, v_2 \in T_pM$ ,  $a \in \mathbb{R}$ , and  $f \in \mathcal{C}^\infty(M)$ , we have:

- (a)  $(v_1 + v_2)(f) := v_1(f) + v_2(f)$ ;

$$(b) \ v_1(af) := av_1(f).$$

(2) The element  $\partial_{i,p} : \mathcal{C}^\infty(M) \rightarrow \mathbb{R}$  is defined for all  $f \in \mathcal{C}^\infty(M)$  by:

$$\partial_{i,p}(f) := \frac{\partial}{\partial x^i} (f \circ \varphi^{-1})|_{\varphi(p)}.$$

We have the following classical result.

**Proposition 2.1.3: Vector Space Structure of  $T_pM$**

(i) We have:

$$\partial_{i,p} \in T_pM.$$

(ii) The  $\partial_{i,p}$  are linearly independent.

(iii) Let  $p \in U$ . The triplet  $(T_pM, +, \cdot)$  is an  $\mathbb{R}$ -vector space of dimension 4 generated by the  $\partial_{i,p}$ , *i.e.*, we have:

$$T_pM = \text{span}_{\mathbb{R}}(\partial_{i,p}).$$

*Proof.* (i) Let  $f_1, f_2 \in \mathcal{C}^\infty(M)$  and  $a, b \in \mathbb{R}$ . Using linearity of differentiation, composition on the left, and evaluation, we have:

$$\begin{aligned} \partial_{i,p}(af_1 + bf_2) &= \frac{\partial}{\partial x^i} ((af_1 + bf_2) \circ \varphi^{-1})|_{\varphi(p)} \\ &= \frac{\partial}{\partial x^i} (af_1 \circ \varphi^{-1} + bf_2 \circ \varphi^{-1})|_{\varphi(p)} \\ &= a \frac{\partial}{\partial x^i} (f_1 \circ \varphi^{-1})|_{\varphi(p)} + b \frac{\partial}{\partial x^i} (f_2 \circ \varphi^{-1})|_{\varphi(p)} \\ &= a\partial_{i,p}(f_1) + b\partial_{i,p}(f_2) \end{aligned}$$

Using the product rule for differentiation, we have:

$$\begin{aligned} \partial_{i,p}(f_1f_2) &= \frac{\partial}{\partial x^i} ((f_1f_2) \circ \varphi^{-1})|_{\varphi(p)} \\ &= \frac{\partial}{\partial x^i} ((f_1 \circ \varphi^{-1})(f_2 \circ \varphi^{-1}))|_{\varphi(p)} \\ &= f_1(\varphi^{-1}(\varphi(p))) \frac{\partial}{\partial x^i} (f_2 \circ \varphi^{-1})|_{\varphi(p)} + f_2(\varphi^{-1}(\varphi(p))) \frac{\partial}{\partial x^i} (f_1 \circ \varphi^{-1})|_{\varphi(p)} \\ &= f_1(p)\partial_{i,p}(f_2) + f_2(p)\partial_{i,p}(f_1) \end{aligned}$$

Therefore, we have:

$$\partial_{i,p} \in T_pM.$$

(ii) Let  $a^0, a^1, a^2, a^3 \in \mathbb{R}$  such that:

$$\sum_{i=0}^3 a^i \partial_{i,p} = 0$$

*i.e.*, for all  $f \in \mathcal{C}^\infty(M)$ , we have:

$$\sum_{i=0}^3 a^i \partial_{i,p}(f) = 0.$$

Since  $x^j : M \rightarrow \mathbb{R} \in \mathcal{C}^\infty(M)$ , we have:

$$\begin{aligned}
0 &= \sum_{i=0}^3 a^i \partial_{i,p}(x^j) \\
&= \sum_{i=0}^3 a^i \frac{\partial}{\partial x^i} (x^j \circ \varphi^{-1})|_{\varphi(p)} \\
&= \sum_{i=0}^3 a^i \delta_j^i(\varphi^{-1}(\varphi(p))) \\
&= \sum_{i=0}^3 a^i \delta_j^i \\
&= a^j
\end{aligned}$$

Thus, the  $\partial_{i,p}$  are linearly independent.

(iii) Let  $v \in T_p M$ ,  $f \in \mathcal{C}^\infty(M)$ , and  $q \in M$ . Let's define:

$$F := f \circ \varphi^{-1} : \mathbb{R}^4 \rightarrow \mathbb{R}.$$

$$z := (z^0, z^1, z^2, z^3) := \varphi^{-1}(q) \quad y := (y^0, y^1, y^2, y^3) := \varphi^{-1}(p).$$

Let's denote:

$$\begin{aligned}
\lambda &:= (\lambda^0, \lambda^1, \lambda^2, \lambda^3) : [0, 1] \rightarrow \mathbb{R}^4 \\
s &\mapsto (1-s)y + sz
\end{aligned}$$

as the line segment connecting  $y$  and  $z$  in  $\mathbb{R}^4$ . Since  $F$  is differentiable on  $\mathbb{R}^4$  and  $\lambda' = z - y$ , by the fundamental theorem of calculus and the chain rule, we have:

$$\begin{aligned}
f(q) - f(p) &= F(z) - F(y) \\
&= F(\lambda(1)) - F(\lambda(0)) \\
&= [F(\lambda(s))]_0^1 \\
&= \int_0^1 \frac{d}{ds} (F \circ \lambda)(s) ds \\
&= \int_0^1 \sum_{i=0}^3 (\lambda^i)'(s) \partial_i F(\lambda(s)) ds \\
&= \int_0^1 \sum_{i=0}^3 (z^i - y^i) \partial_i F((1-s)y + sz) ds \\
&= \sum_{i=0}^3 (z^i - y^i) \int_0^1 \partial_i F((1-s)y + sz) ds \\
&= \sum_{i=0}^3 (z^i - y^i) G_i(z) \\
&= \sum_{i=0}^3 (x^i \circ \varphi(q) - x^i \circ \varphi(p)) G_i(\varphi(q))
\end{aligned}$$

With:

$$G_i(z) := \int_0^1 \partial_i F((1-s)y + sz) ds.$$

Thus, the function  $f$  is given by:

$$f : q \mapsto f(p) + \sum_{i=0}^3 (x^i \circ \varphi(q) - x^i \circ \varphi(p)) G_i(\varphi(q)).$$

Since  $v$  is linear and zero on constants, and using the properties satisfied by  $v$ , we apply  $v$  to obtain:

$$\begin{aligned} v(f) &= v(f(p)) + \sum_{i=0}^3 v((x^i \circ \varphi(q) - x^i \circ \varphi(p))G_i(\varphi(q))) \\ &= v(f(p)) + \sum_{i=0}^3 (x^i \circ \varphi(q) - x^i \circ \varphi(p))_{q=p} v(G_i(\varphi(p))) + \sum_{i=0}^3 G_i(\varphi(p))v(x^i \circ \varphi - x^i \circ \varphi(p)) \\ &= \sum_{i=0}^3 v(x^i \circ \varphi)\partial_{i,p}(f) \end{aligned}$$

By setting:

$$v^i := v(x^i \circ \varphi)$$

we have:

$$v = v^i \partial_{i,p} \in \text{Vect}_{\mathbb{R}}(\partial_{i,p}).$$

□

#### Definition 2.1.4: Tangent and Cotangent Bundles

(i) (a) The **tangent bundle of  $M$**  is defined as:

$$\begin{aligned} TM &:= \bigcup_{p \in M} \{(p, v), v \in T_p M\} \\ &= \{(p, v), p \in M \wedge v \in T_p M\} \end{aligned}$$

It is naturally equipped with a projection map:

$$\begin{aligned} \pi : TM &\longrightarrow M \\ (p, v) &\longmapsto p \end{aligned}$$

(b) A **vector field** is a section of  $TM$ , *i.e.*, a map  $X : M \longrightarrow TM$  such that  $\pi \circ X = \text{Id}_M$ .

(ii) (a) Let  $p \in M$ . The **cotangent space  $T_p^* M$  at  $p$  of  $M$**  is defined as the dual of  $T_p M$ , *i.e.*, we have:

$$T_p^* M := (T_p M)^*.$$

The elements of  $T_p^* M$  are called **covectors**.

(b) The **cotangent bundle of  $M$**  is defined as:

$$\begin{aligned} T^* M &:= \bigcup_{p \in M} \{(p, \phi), \phi \in T_p^* M\} \\ &= \{(p, \phi), p \in M \wedge \phi \in T_p^* M\} \end{aligned}$$

It is naturally equipped with a projection map:

$$\begin{aligned} \pi^* : T^* M &\longrightarrow M \\ (p, \phi) &\longmapsto p \end{aligned}$$

(c) A **covector field** or **1-form (differential)** is a section of  $T^* M$ , *i.e.*, a map  $\alpha : M \longrightarrow T^* M$  such that  $\pi^* \circ \alpha = \text{Id}_M$ .

**Example 2.1.5: Tangent and Cotangent Bundles**

(1) For every  $i \in \{0, 1, 2, 3\}$ , we denote the section:

$$\begin{aligned} \partial_i : M &\longrightarrow TM \\ p &\longmapsto (p, \partial_{i,p}) \end{aligned}$$

The family  $\mathcal{C} := (\partial_0, \partial_1, \partial_2, \partial_3)$  forms a basis of vector fields, *i.e.*, for every  $p \in M$ ,  $(\partial_{0,p}, \partial_{1,p}, \partial_{2,p}, \partial_{3,p})$  is a basis of  $T_p M$ .

(2) We denote  $(dx^{0,p}, dx^{1,p}, dx^{2,p}, dx^{3,p})$  as the dual basis of  $(\partial_{0,p}, \partial_{1,p}, \partial_{2,p}, \partial_{3,p})$ , *i.e.*, for every  $j \in \{0, 1, 2, 3\}$ , we have a linear form:

$$dx^{j,p} : T_p M \longrightarrow \mathbb{R}$$

such that:

$$dx^{j,p}(\partial_{i,p}) = \delta_i^j.$$

We then define the section:

$$\begin{aligned} dx^j : M &\longrightarrow T^* M \\ p &\longmapsto (p, dx^{j,p}) \end{aligned}$$

The family  $\mathcal{C}^* := (dx^0, dx^1, dx^2, dx^3)$  forms a basis of covector fields, *i.e.*, for every  $p \in M$ ,  $(dx^{0,p}, dx^{1,p}, dx^{2,p}, dx^{3,p})$  is a basis of  $T_p^* M$ .

Throughout, we will omit the subscript "p". Thus, we identify vectors with vector fields and covectors with covector fields. For example, we have:

$$dx^j(\partial_i) = \delta_i^j.$$

**2.2 The Language of Tensors**

Let  $E, F, E_1, \dots, E_k$  be  $\mathbb{R}$ -vector spaces.

- We denote:

$$\mathcal{L}_k(E_1 \times \dots \times E_k, F)$$

the set of  **$k$ -linear maps from  $E_1 \times \dots \times E_k$  to  $F$** .

- We denote:

$$E^* := \mathcal{L}(E, \mathbb{R})$$

the **dual space of  $E$**  and  $E^{**} := (E^*)^*$  the **bidual space of  $E$** . There exists a canonical isomorphism between  $E$  and its bidual  $E^{**}$  given by:

$$\begin{aligned} E &\longrightarrow E^{**} \\ u &\longmapsto (\phi \longmapsto \phi(u)) \end{aligned}$$

From now on, we identify them.

In this subsection, we consider the bases  $\mathcal{B} := (e_1, \dots, e_n)$  and  $\mathcal{B}' := (f_1, \dots, f_m)$  on  $E$  and  $F$ , respectively. We denote  $\mathcal{B}^* := (e^1, \dots, e^n)$  and  $\mathcal{B}'^* := (f^1, \dots, f^m)$  as the bases of  $E^*$  and  $F^*$ , respectively, derived from  $\mathcal{B}$  and  $\mathcal{B}'$  (where  $e^i(e_j) = \delta_j^i$  and  $f^i(f_j) = \delta_j^i$ ).

**Definition 2.2.1: Tensor Product**

The **tensor product of vector spaces**  $E_1, \dots, E_k$  is defined as:

$$E_1 \otimes \dots \otimes E_k := \mathcal{L}_k(E_1^* \times \dots \times E_k^*, \mathbb{R}).$$

The elements of  $E_1 \otimes \dots \otimes E_k$  are called **tensors**.

**Example 2.2.2: Usual Examples**

(1) For any  $T \in E \otimes F$ , we have a decomposition:

$$T = \sum_{i=1}^n \sum_{j=1}^m T_{ij} e_i \otimes f_j = T_{ij} e_i \otimes f_j.$$

The second equality is the **Einstein convention** that we adopt from now on. Thus, we have:

$$T(e^k, f^l) = T_{ij} e_i \otimes f_j(e^k, f^l) = T_{ij} e_i(e^k) f_j(f^l) = T_{ij} \delta_i^k \delta_j^l = T_{kl}.$$

(2) We have:

$$E \otimes E := \mathcal{L}_2(E^* \times E^*, \mathbb{R}), \quad E \otimes E^* := \mathcal{L}_2(E^* \times E, \mathbb{R})$$

For all  $k, l \in \mathbb{N}$ , we have for example:

$$\begin{aligned} E^{\otimes(k+1)} &:= E^{\otimes k} \otimes E & E^{\otimes k} \otimes E^{\otimes l} &:= E^{\otimes(k+l)} \\ E^{\otimes 1} &:= \mathcal{L}(E^*, \mathbb{R}) = E^{**} = E & (E^{\otimes k})^{\otimes l} &:= E^{\otimes(kl)} \\ E^{\otimes 0} &:= \mathbb{R} \end{aligned}$$

There exists a canonical isomorphism  $(E^*)^{\otimes k} = (E^{\otimes k})^*$  which allows us to identify these two vector spaces.

We define:

$$\mathcal{T}^{k,l} E := E^{\otimes k} \otimes (E^*)^{\otimes l}$$

Thus, we construct the set:

$$\mathcal{T} E := \bigoplus_{k,l \geq 0} \mathcal{T}^{k,l} E.$$

- The elements of  $\mathcal{T}^k E := \mathcal{T}^{k,0} E$  (with  $k > 0$ ) are called **contravariant tensors**.
- The elements of  $\mathcal{T}^{0,l} E$  (with  $l > 0$ ) are called **covariant tensors**.
- The elements of  $\mathcal{T}^{k,l} E$  (with  $k, l > 0$ ) are called **mixed tensors**.

Naturally, we define addition and scalar multiplication between tensors of the same type, making the triple  $(\mathcal{T} E, +, \cdot)$  an  $\mathbb{R}$ -vector space. A tensor  $T$  of type  $(k, l)$  admits a unique decomposition of the form:

$$T = T_{j_1 \dots j_k}^{i_1 \dots i_l} e_{i_1} \otimes \dots \otimes e_{i_k} \otimes e^{j_1} \otimes \dots \otimes e^{j_l}$$

We say that this is the **decomposition of  $T$  in the bases  $\mathcal{B}$  and  $\mathcal{B}^*$** .

Below, we define a multiplication operation.

**Definition 2.2.3: Tensor Products**

Let  $x_1, \dots, x_k \in E$  and  $\psi^1, \dots, \psi^l \in E^*$ . We have an element:

$$x_1 \otimes \dots \otimes x_k \otimes \psi^1 \otimes \dots \otimes \psi^l \in \mathcal{T}^{k,l} E := E^{\otimes k} \otimes (E^*)^{\otimes l}$$

defined for all  $\phi^1, \dots, \phi^k \in E^*$  and  $y_1, \dots, y_l \in E$  by:

$$(x_1 \otimes \dots \otimes x_k \otimes \psi^1 \otimes \dots \otimes \psi^l)(\phi^1, \dots, \phi^k, y_1, \dots, y_l) := \prod_{i=1}^k \phi^i(x_i) \times \prod_{j=1}^l \psi^j(y_j).$$

This product generalizes by linearity to arbitrary multilinear forms. Therefore, the product  $S \otimes T$  of tensors of order  $(k_1, l_1)$  and  $(k_2, l_2)$  is of order  $(k_1 + k_2, l_1 + l_2)$ .

#### Example 2.2.4: Simple Examples of Tensor Products

(1) The **product of**  $x, y \in E$  is an element  $x \otimes y \in E \otimes E = \mathcal{L}_2(E^* \times E^*, \mathbb{R})$  defined by:

$$\forall \phi, \psi \in E^*, (x \otimes y)(\phi, \psi) := \phi(x)\psi(y).$$

(2) The **product of**  $\phi, \psi \in E^*$  is an element  $\phi \otimes \psi \in E^* \otimes E^* = \mathcal{L}_2(E \times E, \mathbb{R})$  defined by:

$$\forall x, y \in E, (\phi \otimes \psi)(x, y) := \phi(x)\psi(y).$$

(3) The **product of**  $x \in E$  and  $\psi \in E^*$  is an element  $x \otimes \psi \in E \otimes E^* = \mathcal{L}_2(E^* \times E, \mathbb{R})$  defined by:

$$\forall \phi \in E^*, \forall y \in E, (x \otimes \psi)(\phi, y) := \phi(x)\psi(y).$$

These products are associative, non-commutative, and distributive with respect to  $+$ .

The quadruple  $(\mathcal{T}E, +, \cdot, \otimes)$  is thus a graded  $\mathbb{R}$ -algebra.

We conclude this subsection with the contraction of tensors.

#### Definition 2.2.5: Contractions of Tensors

Let  $T := x_1 \otimes \dots \otimes x_k \otimes y^1 \otimes \dots \otimes y^l \in E^{\otimes k} \otimes (E^*)^{\otimes l}$  be a tensor of type  $(k, l)$  with  $k, l \geq 1$ ,  $i \in \{1, \dots, k\}$ , and  $j \in \{1, \dots, l\}$ . The **contraction of  $T$  along indices  $i$  and  $j$**  is the tensor:

$$[T]_j^i = y^j(x_i)x_1 \otimes \dots \otimes x_{i-1} \otimes x_{i+1} \otimes \dots \otimes x_k \otimes y^1 \otimes \dots \otimes y^{j-1} \otimes y^{j+1} \otimes \dots \otimes y^l$$

It is a tensor of type  $(k-1, l-1)$ .

These contractions are associative and distributive with respect to  $+$ . It can be shown that they are independent of the chosen basis.

#### Example 2.2.6: Trace of a $(1, 1)$ Tensor

Let  $T \in \mathcal{T}_1^1 E$  be a tensor of type  $(1, 1)$ . Since

$$\mathcal{T}_1^1 E = E \otimes E^* = \mathcal{L}(E)$$

$T$  can also be viewed as a linear map on  $E$ . Let's denote its decomposition in the bases  $\mathcal{B}$  and  $\mathcal{B}^*$  as:

$$T := T_j^i e_i \otimes e^j.$$

The **trace of  $T$**  is defined as:

$$\text{tr}(T) := [T]_1^1.$$

Therefore, we have:

$$\begin{aligned} \text{tr}(T) &= [T]_1^1 \\ &= T_j^i e^j(e_i) \\ &= T_j^i \delta_i^j \\ &= T_j^j \end{aligned}$$

This definition is the same as the standard definition of trace for an endomorphism. The trace is independent of the chosen basis.

### 2.3 Tensor Fields

We denote  $\mathcal{T}^{k,l}M$  the set of **real tensor fields on  $M$  of type  $(k, l)$** , *i.e.*, the set of fields  $\mathcal{A}$  such that at each point  $p \in M$ , we have a tensor  $\mathcal{A}_p$  of type  $(k, l)$ . Thus, we have a  $k$ -linear map:

$$\mathcal{A}_p : (T_p M)^k \times (T_p^* M)^l \longrightarrow \mathbb{R}$$

using the equality:

$$\mathcal{T}^{k,l}T_p M = (T_p M)^{\otimes k} \otimes (T_p^* M)^{\otimes l}.$$

We denote  $\mathcal{T}M$  the set of tensor fields on  $M$ :

$$\mathcal{T}M := \bigsqcup_{k,l \geq 0} \mathcal{T}^{k,l}M.$$

All the operations defined on tensors in the previous subsection extend naturally to tensor fields.

#### Example 2.3.1: Examples of Tensor Fields

- (1) A **scalar field** is a tensor field of type  $(0, 0)$ .
- (2) A **vector field** is a tensor field of type  $(1, 0)$ .
- (3) A **covector field** or **1-form** is a tensor field of type  $(0, 1)$ .
- (4) A **linear map from  $M$  to  $M$**  (*i.e.*, for each  $p \in M$ , there is a linear map from  $T_p M$ ) can be seen as a tensor of type  $(1, 1)$ .

#### Definition 2.3.2: Examples of Tensor Fields

A **spacetime metric (of signature  $(1, 3)$ )** is a tensor field  $g$  of type  $(0, 2)$  such that for every  $p \in M$ :

- (i)  $g_p$  is **symmetric**, *i.e.*, for all vector fields  $X$  and  $Y$  and every  $p \in M$ , we have:

$$g_p(X(p), Y(p)) = g_p(Y(p), X(p)).$$

- (ii)  $g_p$  is **nonsingular**, *i.e.*, for all vector fields  $X$  and  $Y$  and every  $p \in M$ , we have:

$$g_p(X(p), Y(p)) = 0_{\mathbb{R}} \implies X(p) = 0_{T_p M} \vee Y(p) = 0_{T_p M}.$$

- (iii)  $g_p$  has **signature  $(1, 3)$** , *i.e.*, there exists a basis  $\mathcal{B}_p$  of  $T_p M$  such that

$$\mathcal{M}at_{\mathcal{B}_p}(g) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We then say that the pair  $(M, g)$  is a **spacetime manifold** or **Lorentzian manifold of signature  $(1, 3)$** .

From now on, we omit the letter "p". For example, we say that  $g$  is symmetric and write  $g(X, Y) = g(Y, X)$  instead of  $g_p(X(p), Y(p)) = g_p(Y(p), X(p))$ .

### Example 2.3.3: Spherically Symmetric Metrics

(1) **Metric on the unit 2-sphere.** Consider the unit 2-sphere in  $\mathbb{R}^3$ :

$$\mathbb{S}^2 := \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 + z^2 = 1\}.$$

The change of variables between Cartesian and spherical coordinates is given by:

$$\begin{aligned} f: ]0, \pi[ \times ]0, 2\pi[ &\longrightarrow \mathbb{S}^2 \subset \mathbb{R}^3 \\ (\vartheta, \varphi) &\longmapsto (x, y, z) := (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta) \end{aligned}$$

The Jacobian matrix of the change of variables is:

$$J := \begin{pmatrix} \frac{\partial x}{\partial \vartheta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \vartheta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \vartheta} & \frac{\partial z}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \cos \vartheta \cos \varphi & -\sin \vartheta \sin \varphi \\ \cos \vartheta \sin \varphi & \sin \vartheta \cos \varphi \\ -\sin \vartheta & 0 \end{pmatrix}.$$

Let:

$$\delta := dx \otimes dx + dy \otimes dy + dz \otimes dz$$

and  $h_\Omega := f^* \delta$  be the metric on the unit 2-sphere induced by  $\delta$  through the parameterization  $f$ . Since:

$$\mathcal{M}at_{(\partial_x, \partial_y, \partial_z)}(\delta) = I_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we have:

$$\begin{aligned} \mathcal{M}at_{(\partial_\varphi, \partial_\psi)}(h_\Omega) &= {}^t J \mathcal{M}at_{(\partial_x, \partial_y, \partial_z)}(\delta) J \\ &= {}^t J I_3 J \\ &= \begin{pmatrix} \cos \vartheta \cos \varphi & \cos \vartheta \sin \varphi & -\sin \vartheta \\ -\sin \vartheta \sin \varphi & \sin \vartheta \cos \varphi & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \vartheta \cos \varphi & -\sin \vartheta \sin \varphi \\ \cos \vartheta \sin \varphi & \sin \vartheta \cos \varphi \\ -\sin \vartheta & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \vartheta \end{pmatrix} \end{aligned}$$

So we have:

$$h_\Omega := d\vartheta \otimes d\vartheta + \sin^2 \vartheta d\varphi \otimes d\varphi$$

(2) **Transformation from Cartesian to Spherical Coordinates.** The metric  $\eta$  in the  $\mathcal{C}^*$  basis is given by:

$$\eta = dx^0 \otimes dx^0 - dx^1 \otimes dx^1 - dx^2 \otimes dx^2 - dx^3 \otimes dx^3$$

So we have:

$$N_{\mathcal{C}} := \text{Mat}_{\mathcal{C}}(\eta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The change of variables from Cartesian to spherical coordinates is given by:

$$\begin{cases} x^0 := t \\ x^1 := r \sin \vartheta \cos \varphi \\ x^2 := r \sin \vartheta \sin \varphi \\ x^3 := r \cos \vartheta \end{cases}$$

The Jacobian matrix of the coordinate transformation is:

$$J := \begin{pmatrix} \frac{\partial x^0}{\partial t} & \frac{\partial x^0}{\partial r} & \frac{\partial x^0}{\partial \vartheta} & \frac{\partial x^0}{\partial \varphi} \\ \frac{\partial x^1}{\partial t} & \frac{\partial x^1}{\partial r} & \frac{\partial x^1}{\partial \vartheta} & \frac{\partial x^1}{\partial \varphi} \\ \frac{\partial x^2}{\partial t} & \frac{\partial x^2}{\partial r} & \frac{\partial x^2}{\partial \vartheta} & \frac{\partial x^2}{\partial \varphi} \\ \frac{\partial x^3}{\partial t} & \frac{\partial x^3}{\partial r} & \frac{\partial x^3}{\partial \vartheta} & \frac{\partial x^3}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin \vartheta \cos \varphi & r \cos \vartheta \cos \varphi & -r \sin \vartheta \sin \varphi \\ 0 & \sin \vartheta \sin \varphi & r \cos \vartheta \sin \varphi & r \sin \vartheta \cos \varphi \\ 0 & \cos \vartheta & -r \sin \vartheta & 0 \end{pmatrix}$$

Let's define:

$$\mathcal{B} := (\partial_t, \partial_r, \partial_\vartheta, \partial_\varphi) \quad , \quad \mathcal{B}^* := (\partial_t, \partial_r, \partial_\vartheta, \partial_\varphi)$$

For simplicity, let's define:

$$N_{\mathcal{B}} := \text{Mat}_{\mathcal{B}}(\eta), \quad c(\vartheta) := \cos \vartheta, \quad s(\vartheta) := \sin \vartheta, \quad c(\varphi) := \cos \varphi, \quad s(\varphi) := \sin \varphi$$

By change of basis, we have:

$$\begin{aligned} & N_{\mathcal{B}} \\ & = {}^t J N_{\mathcal{B}} J \\ & = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s(\vartheta)c(\varphi) & s(\vartheta)s(\varphi) & c(\vartheta) \\ 0 & rc(\vartheta)c(\varphi) & rc(\vartheta)s(\varphi) & -rs(\vartheta) \\ 0 & -rs(\vartheta)s(\varphi) & rs(\vartheta)c(\varphi) & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s(\vartheta)c(\varphi) & rc(\vartheta)c(\varphi) & -rs(\vartheta)s(\varphi) \\ 0 & s(\vartheta)s(\varphi) & rc(\vartheta)s(\varphi) & rs(\vartheta)c(\varphi) \\ 0 & c(\vartheta) & -rs(\vartheta) & 0 \end{pmatrix} \\ & = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -s(\vartheta)c(\varphi) & -s(\vartheta)s(\varphi) & -c(\vartheta) \\ 0 & -rc(\vartheta)c(\varphi) & -rc(\vartheta)s(\varphi) & rs(\vartheta) \\ 0 & rs(\vartheta)s(\varphi) & -rs(\vartheta)c(\varphi) & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s(\vartheta)c(\varphi) & rc(\vartheta)c(\varphi) & -rs(\vartheta)s(\varphi) \\ 0 & s(\vartheta)s(\varphi) & rc(\vartheta)s(\varphi) & rs(\vartheta)c(\varphi) \\ 0 & c(\vartheta) & -rs(\vartheta) & 0 \end{pmatrix} \\ & = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \vartheta \end{pmatrix} \end{aligned}$$

So we have:

$$\begin{aligned} \eta &= dx^0 \otimes dx^0 - dx^1 \otimes dx^1 - dx^2 \otimes dx^2 - dx^3 \otimes dx^3 \\ &= dt \otimes dt - dr \otimes dr - r^2 d\vartheta \otimes d\vartheta - r^2 \sin^2 \vartheta d\varphi \otimes d\varphi \end{aligned}$$

## 2.4 The diagonal metric of the text

Let's recall what has been defined.

- A differential manifold  $M$  of dimension 4.

- At each point  $p$  of  $M$ , we have a chart of  $M$  of the form:

$$(U, \varphi := (x^0, x^1, x^2, x^3))$$

on which we will perform calculations.

- $\mathcal{C} := (\partial_0, \partial_1, \partial_2, \partial_3)$  is the basis of vector fields on  $U$  associated with the coordinates  $x := (x^0, x^1, x^2, x^3)$ , *i.e.*, we have:

$$\partial_i := \frac{\partial}{\partial x^i}.$$

- $\mathcal{C}^* := (dx^0, dx^1, dx^2, dx^3)$  is the dual basis associated with  $\mathcal{C}$  of covector fields, *i.e.*, we have:

$$dx^i(\partial_j) = \delta_j^i.$$

We omit the "p" in the notations as well as the "U".

We have a metric  $g$  on  $M$  that is an spacetime metric (of signature (1,3)) and is locally given by the form:

$$\begin{aligned} g &= \eta_k \left( e^{u_k(x)} dx^k \right) \otimes \left( e^{u_k(x)} dx^k \right) \\ &= \eta_k e^{2u_k(x)} dx^k \otimes dx^k \\ &= e^{2u_0(x)} dx^0 \otimes dx^0 - e^{2u_1(x)} dx^1 \otimes dx^1 - e^{2u_2(x)} dx^2 \otimes dx^2 - e^{2u_3(x)} dx^3 \otimes dx^3 \end{aligned}$$

where

$$\eta_k := \begin{cases} 1 & \text{if } k := 0 \\ -1 & \text{otherwise} \end{cases}, \quad \eta_{ij} := \delta_j^i \eta_i, \quad \delta_j^i := \delta^{ij} := \delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

We also denote:

- the components of  $g$  as:

$$g_{ij} := g(\partial_i, \partial_j) = \eta_{ij} e^{2u_i(x)}$$

- the metric:

$$\begin{aligned} \eta &= \eta_k dx^k \otimes dx^k \\ &= dx^0 \otimes dx^0 - dx^1 \otimes dx^1 - dx^2 \otimes dx^2 - dx^3 \otimes dx^3 \end{aligned}$$

and thus we have:

$$\eta_{ij} := \eta(\partial_i, \partial_j).$$

- the derivatives:

$$u_{i,k} := \partial_k u_i := \frac{\partial u_i}{\partial x^k}, \quad u_{i,kl} := \partial_{kl}^2 u_i := \frac{\partial^2 u_i}{\partial x^k \partial x^l}.$$

We will omit the variable "x" in the functions  $u_i$  from now on.

#### Example 2.4.1: Example of the metric $h$

In this text, we will consider the following diagonal metric  $h$  as an example:

$$\begin{aligned} h &= e^{2u(t,r)} dt \otimes dt - e^{2v(t,r)} dr \otimes dr - r^2 e^{2b(t)} (d\vartheta \otimes d\vartheta + \sin^2 \vartheta d\varphi \otimes d\varphi) \\ &= e^{2u(t,r)} dt \otimes dt - e^{2v(t,r)} dr \otimes dr - r^2 e^{2b(t)} h_\Omega \end{aligned}$$

where:

$$h_\Omega := d\vartheta \otimes d\vartheta + \sin^2 \vartheta d\varphi \otimes d\varphi$$

is the metric on the unit 2-sphere in spherical coordinates (see point (2) of Example 2.3.3). We are

considering the case where:

$$(x^0, x^1, x^2, x^3) := (t, r, \vartheta, \phi).$$

and

$$\begin{aligned} u_0(t, r) &:= u(t, r) \\ u_1(t, r) &:= v(t, r) \\ u_2(t, r) &:= b(t) + \ln(r) \\ u_3(t, r, \vartheta) &:= b(t) + \ln(r) + \ln \sin(\vartheta) \end{aligned}$$

For indices or exponents, we use the notation:

$$(0, 1, 2, 3) \equiv (t, r, \vartheta, \phi)$$

Derivatives with respect to  $t$  are denoted with a dot (e.g.,  $\dot{u}$  and  $\ddot{u}$ ), and derivatives with respect to  $r$  are denoted with a prime (e.g.,  $u'$  and  $u''$ ).

This metric allows us to consider two common cases in the next chapter.

(a) The **interior and exterior Schwarzschild metrics (SCHW)**, and the **Reissner–Nordström metric (RN)** correspond to the case where:

$$b = 0.$$

(b) The **Friedmann–Lemaître–Robertson–Walker metric (FLRW)** corresponds to the case where:

$$\dot{u} = u' = 0 \quad , \quad v(t, r) = v(r) + b(t).$$

## 2.5 Orthonormalization

The basis  $\mathcal{C}$  is not orthonormal for  $g$  in general, meaning that we generally have:

$$g(\partial_i, \partial_j) = g_{ij} \neq \eta_{ij}.$$

We will construct an orthonormal basis  $\mathcal{C}_\perp := (\theta_0, \theta_1, \theta_2, \theta_3)$  associated with  $\mathcal{C}$ . Let us define:

$$\theta_k := f^k \partial_k.$$

We want to have:

$$\eta_k = g(\theta_k, \theta_k) = g(f^k \partial_k, f^k \partial_k) = (f^k)^2 g(\partial_k, \partial_k) = (f^k)^2 g_{kk}$$

Thus, we have:

$$f^k = \sqrt{\frac{\eta_k}{g_{kk}}} = \frac{1}{\sqrt{|g_{kk}|}} = e^{-u_k}$$

Therefore, we have:

$$\theta_k = f^k \partial_k = \frac{1}{\sqrt{|g_{kk}|}} \partial_k = e^{-u_k} \partial_k \tag{2.5.1}$$

We denote  $\mathcal{C}_\perp^* := (\theta^0, \theta^1, \theta^2, \theta^3)$  as the dual basis associated with  $\mathcal{C}_\perp$ , where:

$$\theta^k := f_k dx^k.$$

Thus, we have:

$$1 = \theta^k(\theta_k) = f^k dx^k(f_k \partial_k) = f^k f_k dx^k(\partial_k) = f^k f_k$$

Therefore, we have:

$$f_k = (f^k)^{-1} = e^{u_k}.$$

Thus, we have:

$$\theta^k = f_k dx^k = e^{u_k} dx^k. \quad (2.5.2)$$

Consequently, we have:

$$g = \theta^0 \otimes \theta^0 - \theta^1 \otimes \theta^1 - \theta^2 \otimes \theta^2 - \theta^3 \otimes \theta^3 = \eta_k \theta^k \otimes \theta^k = \eta_{ij} \theta^i \otimes \theta^j.$$

### Example 2.5.1: Extension 1 of the metric $h$

Let's continue with the example of the metric  $h$  (see 2.4.1) with:

$$\begin{aligned} u_0(t, r) &:= u(t, r) & u_1(t, r) &:= v(t, r) \\ u_2(t, r) &:= b(t) + \ln(r) & u_3(t, r, \vartheta) &:= b(t) + \ln(r) + \ln \sin(\vartheta) \end{aligned}$$

We denote:

$$\mathcal{B}_\perp := (\theta_0, \theta_1, \theta_2, \theta_3) := (\theta_t, \theta_r, \theta_\vartheta, \theta_\phi)$$

the orthonormal basis associated with:

$$\mathcal{B} := (\partial_0, \partial_1, \partial_2, \partial_3) := (\partial_t, \partial_r, \partial_\vartheta, \partial_\phi)$$

and

$$\mathcal{B}_\perp^* := (\theta^0, \theta^1, \theta^2, \theta^3) := (\theta^t, \theta^r, \theta^\vartheta, \theta^\phi)$$

the dual basis associated with  $\mathcal{B}_\perp$ . We have:

$$\begin{aligned} \theta_t = \theta_0 &= e^{-u} \partial_t & \theta_r = \theta_1 &= e^{-v} \partial_r \\ \theta_\vartheta = \theta_2 &= r^{-1} e^{-b} \partial_\vartheta & \theta_\phi = \theta_3 &= r^{-1} \sin^{-1} \vartheta e^{-b} \partial_\phi \\ \theta^t = \theta^0 &= e^u dt & \theta^r = \theta^1 &= e^v dr \\ \theta^\vartheta = \theta^2 &= r e^a d\vartheta & \theta^\phi = \theta^3 &= r \sin \vartheta e^a d\phi \end{aligned}$$

Thus, we have:

$$h = \theta^t \otimes \theta^t - \theta^r \otimes \theta^r - \theta^\vartheta \otimes \theta^\vartheta - \theta^\phi \otimes \theta^\phi.$$

## 2.6 The bases $\mathcal{E}$ and $\mathcal{E}^*$

For the rest of the discussion, we define:

- $\mathcal{E} := (e_0, e_1, e_2, e_3)$  as a basis of vector fields on  $M$ ;
- $\mathcal{E}^* := (e^0, e^1, e^2, e^3)$  as the dual basis associated with  $\mathcal{E}$ , *i.e.*, we have:

$$e^i(e_j) = \delta_j^i.$$

The bases  $\mathcal{E}$  and  $\mathcal{E}^*$  will be used to handle the general case. Let's define:

$$\mathbf{g}_{ij} := g(e_i, e_j)$$

which means that we can write:

$$g = \mathbf{g}_{ij} e^i \otimes e^j.$$

It is important to note that unlike  $\mathcal{C}$  and  $\mathcal{C}_\perp$ , the metric  $g$  is not diagonal in  $\mathcal{E}$ , *i.e.*, we can have:

$$i \neq j \wedge \mathbf{g}_{ij} \neq 0.$$

### 2.7 Tensor Notations

Let  $\mathcal{A}$  be a tensor field of type  $(k, l)$ . We denote:

$$\begin{aligned} \mathbf{A}_{i_1 \dots i_l}^{j_1 \dots j_k} &:= \mathcal{A}(e^{j_1}, \dots, e^{j_k}, e_{i_1}, \dots, e_{i_l}) \\ \mathbb{A}_{i_1 \dots i_l}^{j_1 \dots j_k} &:= \mathcal{A}(dx^{j_1}, \dots, dx^{j_k}, \partial_{i_1}, \dots, \partial_{i_l}) \\ \mathbb{A}_{i_1 \dots i_l}^{j_1 \dots j_k} &:= \mathcal{A}(\theta^{j_1}, \dots, \theta^{j_k}, \theta_{i_1}, \dots, \theta_{i_l}) \end{aligned}$$

We will use specific letters depending on the basis used to calculate the components of the tensor fields. Here is a table explaining the letter conventions used:

Tensor Fields	In $\mathcal{E}$ and $\mathcal{E}^*$ Bases	In $\mathcal{C}$ and $\mathcal{C}^*$ Bases	In $\mathcal{C}_\perp$ and $\mathcal{C}_\perp^*$ Bases
$\mathcal{R}, \mathcal{G}, \mathcal{T}, \dots$	<b>R, G, T, ...</b>	R, G, T, ...	$\mathbb{R}, \mathbb{G}, \mathbb{T}, \dots$

Thus, we have the following decompositions:

$$\begin{aligned} \mathcal{A} &= \mathbf{A}_{i_1 \dots i_l}^{j_1 \dots j_k} e_{j_1} \otimes \dots \otimes e_{j_k} \otimes e^{i_1} \otimes \dots \otimes e^{i_l} \\ \mathcal{A} &= \mathbb{A}_{i_1 \dots i_l}^{j_1 \dots j_k} \partial_{j_1} \otimes \dots \otimes \partial_{j_k} \otimes dx^{i_1} \otimes \dots \otimes dx^{i_l} \\ \mathcal{A} &= \mathbb{A}_{i_1 \dots i_l}^{j_1 \dots j_k} \theta_{j_1} \otimes \dots \otimes \theta_{j_k} \otimes \theta^{i_1} \otimes \dots \otimes \theta^{i_l} \end{aligned}$$

**Proposition 2.7.1: Relations between components**

We have:

$$\mathbb{A}_{i_1 \dots i_l}^{j_1 \dots j_k} = \exp\left(\sum_{n=1}^l u_{i_n} - \sum_{n=1}^k u_{j_n}\right) \mathbb{A}_{i_1 \dots i_l}^{j_1 \dots j_k}.$$

*Proof.* We have:

$$\begin{aligned} &\mathbb{A}_{i_1 \dots i_l}^{j_1 \dots j_k} \partial_{j_1} \otimes \dots \otimes \partial_{j_k} \otimes dx^{i_1} \otimes \dots \otimes dx^{i_l} \\ &= \mathcal{A} \\ &= \mathbb{A}_{i_1 \dots i_l}^{j_1 \dots j_k} \theta_{j_1} \otimes \dots \otimes \theta_{j_k} \otimes \theta^{i_1} \otimes \dots \otimes \theta^{i_l} \\ &= \mathbb{A}_{i_1 \dots i_l}^{j_1 \dots j_k} e^{-u_{j_1}} \partial_{j_1} \otimes \dots \otimes e^{-u_{j_k}} \partial_{j_k} \otimes e^{u_{i_1}} dx^{i_1} \otimes \dots \otimes e^{u_{i_l}} dx^{i_l} \\ &= \exp\left(\sum_{n=1}^l u_{i_n} - \sum_{n=1}^k u_{j_n}\right) \mathbb{A}_{i_1 \dots i_l}^{j_1 \dots j_k} \partial_{j_1} \otimes \dots \otimes \partial_{j_k} \otimes dx^{i_1} \otimes \dots \otimes dx^{i_l} \end{aligned}$$

Thus, by the uniqueness of the decomposition of  $\mathcal{A}$  in the  $\mathcal{C}$  and  $\mathcal{C}^*$  bases, we have:

$$\mathbb{A}_{i_1 \dots i_l}^{j_1 \dots j_k} = \exp\left(\sum_{n=1}^l u_{i_n} - \sum_{n=1}^k u_{j_n}\right) \mathbb{A}_{i_1 \dots i_l}^{j_1 \dots j_k}.$$

□

In the following example, we provide the main tensor fields studied in this chapter.

**Example 2.7.2: Common examples**

(1) The curvature tensor decomposes as follows:

$$\begin{aligned}\mathcal{R} &= \mathbf{R}^i{}_{jkl} e_i \otimes e^j \otimes e^k \otimes e^l \\ \mathcal{R} &= \mathbf{R}^i{}_{jkl} \partial_i \otimes dx^j \otimes dx^k \otimes dx^l \\ \mathcal{R} &= \mathbb{R}^i{}_{jkl} \theta_i \otimes \theta^j \otimes \theta^k \otimes \theta^l\end{aligned}$$

Thus, by the previous proposition 2.7.1, we have:

$$\mathbf{R}^i{}_{jkl} = e^{-u_i+u_j+u_k+u_l} \mathbb{R}^i{}_{jkl}.$$

(2) The Riemann tensor decomposes as follows:

$$\begin{aligned}\mathcal{Rm} &= \mathbf{R}_{ijkl} e^i \otimes e^j \otimes e^k \otimes e^l \\ \mathcal{Rm} &= \mathbf{R}_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l \\ \mathcal{Rm} &= \mathbb{R}_{ijkl} \theta^i \otimes \theta^j \otimes \theta^k \otimes \theta^l\end{aligned}$$

Thus, by the previous proposition 2.7.1, we have:

$$\mathbf{R}_{ijkl} = e^{u_i+u_j+u_k+u_l} \mathbb{R}_{ijkl}.$$

(3) The Ricci tensor decomposes as follows:

$$\begin{aligned}\mathcal{Ric} &= \mathbf{R}_{jl} e^j \otimes e^l \\ \mathcal{Ric} &= \mathbf{R}_{jl} dx^j \otimes dx^l \\ \mathcal{Ric} &= \mathbb{R}_{jl} \theta^j \otimes \theta^l\end{aligned}$$

Thus, by the previous proposition 2.7.1, we have:

$$\mathbf{R}_{jl} = e^{u_j+u_l} \mathbb{R}_{jl}.$$

(4) The Einstein tensor decomposes as follows:

$$\begin{aligned}\mathcal{G} &= \mathbf{G}_{jl} e^j \otimes e^l \\ \mathcal{G} &= \mathbf{G}_{jl} dx^j \otimes dx^l \\ \mathcal{G} &= \mathbb{G}_{jl} \theta^j \otimes \theta^l\end{aligned}$$

Thus, by the previous proposition 2.7.1, we have:

$$\mathbf{G}_{jl} = e^{u_j+u_l} \mathbb{G}_{jl}.$$

**2.8 Lowering indices with the tensor  $g$** 

In this subsection, we define an operation that lowers indices. We recall that:

$$g = \mathbf{g}_{ij} e^i \otimes e^j$$

where:

$$\mathbf{g}_{ij} := g(e_i, e_j).$$

**Example 2.8.1: A simple calculation**

Consider two vector fields  $X := X^k e_k$  and  $Y := Y^l e_l$ . Let's calculate  $g(X, Y)$  using the given decomposition of  $g$ . We have:

$$\begin{aligned} g(X, Y) &= \mathbf{g}_{ij} e^i \otimes e^j (X^k e_k, Y^l e_l) \\ &= \mathbf{g}_{ij} e^i (X^k e_k) e^j (Y^l e_l) \\ &= \mathbf{g}_{ij} X^k Y^l e^i (e_k) e^j (e_l) \\ &= \mathbf{g}_{ij} X^k Y^l \delta_k^i \delta_l^j \\ &= \mathbf{g}_{kl} X^k Y^l \end{aligned}$$

**Definition 2.8.2: Lowering of the  $i$ -th index**

The **natural isomorphism associated with  $g$**  is the flat musical application  $\mathbf{a}_g$  defined as:

$$\begin{aligned} \mathbf{a}_g : TM &\longrightarrow T^*M \\ X &\longmapsto Y \longmapsto g(X, Y) \end{aligned}$$

We also denote, for any vector field  $X$ :

$$X^b := \mathbf{a}_g(X).$$

**Notation 2.8.3: Common notations**

Let  $X := \mathbf{X}^i e_i$  be a vector field. We define:

$$\mathbf{X}_i := X^b(e_i)$$

*i.e.*, we have:

$$X^b = \mathbf{X}_i e^i.$$

Since:

$$\mathbf{a}_g(e_i)(e_j) = g(e_i, e_j) = \mathbf{g}_{ij}$$

we have:

$$\mathbf{a}_g(e_i) = \mathbf{g}_{ij} e^j$$

and by linearity, we have:

$$\begin{aligned} X^b &= \mathbf{a}_g(X) \\ &= \mathbf{a}_g(\mathbf{X}^i e_i) \\ &= \mathbf{X}^i \mathbf{a}_g(e_i) \\ &= \mathbf{X}^i \mathbf{g}_{ij} e^j. \end{aligned}$$

Thus, by the uniqueness of the decomposition, we have:

$$\mathbf{X}_j = \mathbf{X}^i \mathbf{g}_{ij}.$$

In matrix form, we have:

$$G := \mathcal{M}at_{\mathcal{E}, \mathcal{E}^*}(\mathbf{a}_g) = \mathcal{M}at_{\mathcal{E}}(g) = (\mathbf{g}_{ij})_{ij}.$$

**Example 2.8.4: Example of  $\mathbf{a}_g$  and  $g$  in the bases  $\mathcal{C}$  and  $\mathcal{C}_1$** (1) In  $\mathcal{C}$ , we have:

$$g_{ij} = \eta_{ij} e^{-u_j}.$$

Thus, we have:

$$\mathcal{M}at_{\mathcal{C}, \mathcal{C}^*}(\mathbf{a}_g) = \mathcal{M}at_{\mathcal{C}}(g) = \begin{pmatrix} e^{u_0} & 0 & 0 & 0 \\ 0 & -e^{u_1} & 0 & 0 \\ 0 & 0 & -e^{u_2} & 0 \\ 0 & 0 & 0 & -e^{u_3} \end{pmatrix}.$$

(2) In  $\mathcal{C}_1$ , we have:

$$\mathcal{M}at_{\mathcal{C}_1, \mathcal{C}_1^*}(\mathbf{a}_g) = \mathcal{M}at_{\mathcal{C}_1}(g) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

**Definition 2.8.5: Flat musical isomorphism**

Let  $\mathcal{A}$  be a tensor field of type  $(k, l)$  with  $k \geq 1$  and  $n \in \{1, \dots, k\}$ . The  $n$ -lowered tensor field of  $\mathcal{A}$  is the tensor field  $[\mathcal{A}]_n$  of type  $(k-1, l+1)$  defined for all covector fields  $\alpha_1, \dots, \alpha_{n-1}, \alpha_{n+1}, \dots, \alpha_k$  and vector fields  $X_0, \dots, X_l$  as:

$$[\mathcal{A}]_n(\alpha_1, \dots, \alpha_{n-1}, \alpha_{n+1}, \dots, \alpha_k, X_0, \dots, X_l) := g(X_0, \mathcal{A}(\alpha_1, \dots, \alpha_{n-1}, \bullet, \alpha_{n+1}, \dots, \alpha_k, X_1, \dots, X_l)).$$

We have the following relation:

$$[\mathcal{A}]_n(\alpha_1, \dots, \alpha_{n-1}, \alpha_{n+1}, \dots, \alpha_k, X_0, \dots, X_l) = \mathbf{a}_g(\mathcal{A}(\alpha_1, \dots, \alpha_{n-1}, \bullet, \alpha_{n+1}, \dots, \alpha_k, X_1, \dots, X_l))(X_0).$$

The following proposition shows why the term "lowered" has been used.

**Proposition 2.8.6: Lowering of the  $n$ -th index**Let  $\mathcal{A}$  be a tensor field of type  $(k, l)$  with  $k \geq 1$  and  $n \in \{1, \dots, k\}$ . Define:

$$\begin{aligned} \mathbf{A}_{i_1 \dots i_l}^{j_1 \dots j_k} &:= \mathcal{A}(e^{j_1}, \dots, e^{j_k}, e_{i_1}, \dots, e_{i_l}) \\ \mathbf{A}_{j_n i_1 \dots i_l}^{j_1 \dots j_{n-1} j_{n+1} \dots j_k} &:= [\mathcal{A}]_n(e^{j_1}, \dots, e^{j_{n-1}}, e^{j_{n+1}}, \dots, e^{j_k}, e_{j_n}, e_{i_1}, \dots, e_{i_l}) \end{aligned}$$

Then we have:

$$\mathbf{A}_{j_n i_1 \dots i_l}^{j_1 \dots j_{n-1} j_{n+1} \dots j_k} = \mathbf{A}_{i_1 \dots i_l}^{j_1 \dots j_{n-1} j_{n+1} \dots j_k} \mathbf{g}_{m j_n}.$$

*Proof.* We have:

$$\begin{aligned} \mathbf{A}_{j_n i_1 \dots i_l}^{j_1 \dots j_{n-1} j_{n+1} \dots j_k} &= [\mathcal{A}]_n(e^{j_1}, \dots, e^{j_{n-1}}, e^{j_{n+1}}, \dots, e^{j_k}, e_{j_n}, e_{i_1}, \dots, e_{i_l}) \\ &= g(e_{j_n}, \mathcal{A}(e^{j_1}, \dots, e^{j_{n-1}}, \bullet, e^{j_{n+1}}, \dots, e^{j_k}, e_{i_1}, \dots, e_{i_l})) \\ &= g(e_{j_n}, \mathbf{A}_{i_1 \dots i_l}^{j_1 \dots j_{n-1} j_{n+1} \dots j_k} e_m) \\ &= \mathbf{A}_{i_1 \dots i_l}^{j_1 \dots j_{n-1} j_{n+1} \dots j_k} g(e_{j_n}, e_m) \\ &= \mathbf{A}_{i_1 \dots i_l}^{j_1 \dots j_{n-1} j_{n+1} \dots j_k} \mathbf{g}_{m j_n} \end{aligned}$$

□

Since  $g$  is diagonal in the bases  $\mathcal{C}$  and  $\mathcal{C}_1$ , we have the following result.

Corollary 2.8.7: Lowering of the  $n$ -th index

Let  $\mathcal{A}$  be a tensor field of type  $(k, l)$ . Then we have:

$$\begin{aligned} \mathbb{A}_{j_n i_1 \dots i_l}^{j_1 \dots j_{n-1} j_{n+1} \dots j_k} &= \eta_{j_n} e^{2u_{j_n}} \mathbb{A}_{i_1 \dots i_l}^{j_1 \dots j_k} \\ \mathbb{A}_{j_n i_1 \dots i_l}^{j_1 \dots j_{n-1} j_{n+1} \dots j_k} &= \eta_{j_n} \mathbb{A}_{i_1 \dots i_l}^{j_1 \dots j_k} \end{aligned}$$

## Example 2.8.8: Two simple examples

(1) Let  $\mathcal{A} := \mathbf{A}^i_j e_i \otimes e^j$  be a tensor field of type  $(1, 1)$ . We denote:

$$\mathbf{A}_{kl} := [\mathcal{A}]_1 (e_k, e_l)$$

which means:

$$[\mathcal{A}]_1 = \mathbf{A}_{kl} e^k \otimes e^l.$$

Since:

$$[\mathcal{A}]_1 (X, Y) = g(X, \mathcal{A}(\bullet, Y))$$

and

$$\begin{aligned} \mathcal{A}(\bullet, e_l) &= \mathbf{A}^i_j e^j (e_l) e_i \\ &= \mathbf{A}^i_j \delta_l^j e_i \\ &= \mathbf{A}^i_l e_i \end{aligned}$$

we have:

$$\begin{aligned} \mathbf{A}_{kl} &= [\mathcal{A}]_1 (e_k, e_l) \\ &= g(e_k, \mathcal{A}(\bullet, e_l)) \\ &= g(e_k, \mathbf{A}^i_l e_i) \\ &= \mathbf{A}^i_l g(e_k, e_i) \\ &= \mathbf{A}^i_l \mathbf{g}_{ki} \end{aligned}$$

For example, we have:

$$\begin{aligned} \mathbb{A}_{kl} &= \eta_{ki} \mathbb{A}^i_l \\ &= \eta_k \mathbb{A}^k_l \\ \mathbf{A}_{kl} &= \mathbf{g}_{ki} \mathbf{A}^i_l \\ &= \mathbf{g}_{kk} \mathbf{A}^k_l \\ &= \eta_k e^{2u_k} \mathbf{A}^k_l \end{aligned}$$

(2) Let  $X := \mathbf{X}^i e_i$  be a vector field, *i.e.*,  $X$  is a tensor field of type  $(1, 0)$ . We denote:

$$\mathbf{X}_k := [X]_1 (e_k)$$

which means:

$$X^b = [X]_1 = \mathbf{X}_k e^k$$

Thus, we have:

$$\begin{aligned} \mathbf{X}_k &= [X]_1 (e_k) \\ &= g(e_k, X) \\ &= g(e_k, \mathbf{X}^i e_i) \\ &= \mathbf{X}^i g(e_k, e_i) \\ &= \mathbf{X}^i \mathbf{g}_{ki} \end{aligned}$$

For example, we have:

$$\begin{aligned}\mathbb{X}_k &= \mathbb{X}^i \eta_{ki} \\ &= \eta_k \mathbb{X}^k \\ X_k &= X^i g_{ki} \\ &= X^k g_{kk} \\ &= \eta_k e^{2u_k} X^k\end{aligned}$$

## 2.9 Raising indices with the tensor $g^*$

In this subsection, we define an operation that raises indices.

**Definition 2.9.1:** The sharp musical isomorphism

The **sharp musical isomorphism associated with  $g$**  is the inverse mapping  $\mathfrak{a}_g^{-1}$  of  $\mathfrak{a}_g$  defined by:

$$\mathfrak{a}_g^{-1}: T^*M \longrightarrow TM$$

We also denote, for any covector field  $\alpha$ :

$$\alpha^\sharp := \mathfrak{a}_g^{-1}(\alpha).$$

**Notation 2.9.2:** Definition of  $\mathfrak{a}_g^{-1}$  and  $g^*$

(1) We denote:

$$\mathbf{g}^{ij} := \mathfrak{a}_g^{-1}(e^i)(e^j)$$

Hence:

$$\mathfrak{a}_g^{-1}(e^i) = \mathbf{g}^{ij} e_j.$$

Since:

$$\mathfrak{a}_g^{-1} \circ \mathfrak{a}_g = \text{Id}_{TM} \quad , \quad \mathfrak{a}_g \circ \mathfrak{a}_g^{-1} = \text{Id}_{T^*M}$$

we have:

$$\begin{aligned}\text{Id}_{TM}(e_k) &= e_k \\ &= \delta_k^j e_j \\ \mathfrak{a}_g^{-1}(\mathfrak{a}_g(e_k)) &= \mathfrak{a}_g^{-1}(\mathbf{g}_{ki} e^i) \\ &= \mathbf{g}_{ki} \mathfrak{a}_g^{-1}(e^i) \\ &= \mathbf{g}_{ki} \mathbf{g}^{ij} e_j\end{aligned}$$

Therefore, by the uniqueness of the decomposition, we have:

$$\mathbf{g}_{ki} \mathbf{g}^{ij} = \delta_k^j.$$

(2) Let  $\alpha := \alpha_i e^i$  be a covector field. We define:

$$\alpha^j := \alpha^\sharp(e_j)$$

Thus:

$$\alpha^\sharp = \alpha^j e_j.$$

Since:

$$\mathbf{a}_g^{-1}(e^i) = \mathbf{g}^{ij} e_j$$

we have, by linearity:

$$\begin{aligned} \alpha^\sharp &= \mathbf{a}_g^{-1}(\alpha) \\ &= \mathbf{a}_g^{-1}(\alpha_i e^i) \\ &= \alpha_i \mathbf{a}_g^{-1}(e^i) \\ &= \alpha_i \mathbf{g}^{ij} e_j. \end{aligned}$$

Hence, by the uniqueness of the decomposition, we have:

$$\alpha^j = \alpha_i \mathbf{g}^{ij}.$$

- (3) We associate the  $(2,0)$ -type tensor  $g^*$  to  $\mathbf{a}_g^{-1}$ , defined for any covector fields  $\alpha := \alpha_i e^i$  and  $\beta := \beta_j e^j$  as:

$$g^*(\alpha, \beta) := \beta(\mathbf{a}_g^{-1}(\alpha)).$$

Thus:

$$\begin{aligned} g^*(e^i, e^j) &= e^j(\mathbf{a}_g^{-1}(e^i)) \\ &= e^j(\mathbf{g}^{ik} e_k) \\ &= \mathbf{g}^{ik} \delta_k^j \\ &= \mathbf{g}^{ij} \end{aligned}$$

and by bilinearity:

$$\begin{aligned} g^*(\alpha, \beta) &= g^*(\alpha_i e^i, \beta_j e^j) \\ &= g^*(e^i, e^j) \alpha_i \beta_j \\ &= \mathbf{g}^{ij} \alpha_i \beta_j \\ &= \alpha^j \beta_j \\ &= \alpha_i \beta^i \end{aligned}$$

with:

$$\alpha^i e_i := \alpha^\sharp, \quad \beta^j e_j := \beta^\sharp.$$

Therefore, we have:

$$g^* = \mathbf{g}^{ij} e_i \otimes e_j$$

Matricially, we have:

$$G^{-1} = \text{Mat}_{\mathcal{C}^*, \mathcal{C}}(\mathbf{a}_g^{-1}) = \text{Mat}_{\mathcal{C}^*}(g^*) = (\mathbf{g}^{ij})_{ij}.$$

**Example 2.9.3:** Example of  $\mathbf{a}_g^{-1}$  and  $g^*$  in the bases  $\mathcal{C}^*$  and  $\mathcal{C}_\perp^*$

(1) In  $\mathcal{C}^*$ , we have:

$$\begin{aligned} \mathbf{g}_{ki} \mathbf{g}^{ij} &= \delta_k^j \\ \mathbf{g}_{ki} &= \eta_{ik} e^{u_k} \end{aligned}$$

So we have:

$$g^{ij} = \eta_{ij} e^{-u_k}.$$

Thus we have:

$$\mathcal{M}at_{\mathcal{C}^*, \mathcal{C}}(\mathbf{a}_g^{-1}) = \mathcal{M}at_{\mathcal{C}^*}(g^*) = \begin{pmatrix} e^{-u_0} & 0 & 0 & 0 \\ 0 & -e^{-u_1} & 0 & 0 \\ 0 & 0 & -e^{-u_2} & 0 \\ 0 & 0 & 0 & -e^{-u_3} \end{pmatrix}.$$

(2) In  $\mathcal{C}_\perp^*$ , we have:

$$\begin{aligned} \mathfrak{G}_{ki} \mathfrak{G}^{ij} &= \delta_k^j \\ \mathfrak{G}_{ki} &= \eta_{ki} \end{aligned}$$

So we have:

$$\mathfrak{g}^{ij} = \eta_{ij}.$$

Thus we have:

$$\mathcal{M}at_{\mathcal{C}_\perp^*, \mathcal{C}_\perp}(\mathbf{a}_g^{-1}) = \mathcal{M}at_{\mathcal{C}_\perp^*}(g^*) = \mathcal{M}at_{\mathcal{C}_\perp}(g) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

#### Definition 2.9.4: Raising the $n$ -th index

Let  $\mathcal{A}$  be a tensor field of type  $(k, l)$  with  $l \geq 1$  and  $n \in \{1, \dots, l\}$ . The  **$n$ -th raised tensor field of  $\mathcal{A}$**  is the tensor field  $[\mathcal{A}]^n$  of type  $(k+1, l-1)$  defined for any covector fields  $\alpha_0, \alpha_1, \dots, \alpha_k$  and vector fields  $X_1, \dots, X_{n-1}, X_{n+1}, \dots, X_l$  by:

$$[\mathcal{A}]^n(\alpha_0, \dots, \alpha_k, X_1, \dots, X_{n-1}, X_{n+1}, \dots, X_l) := g^*(\alpha_0, \mathcal{A}(\alpha_1, \dots, \alpha_k, X_1, \dots, X_{n-1}, \bullet, X_{n+1}, \dots, X_l)).$$

These two concepts are then related by:

$$[\mathcal{A}]^n(\alpha_1, \dots, \alpha_k, X_1, \dots, X_{n-1}, X_{n+1}, \dots, X_l) = \mathbf{a}_g^{-1}(\mathcal{A}(\alpha_1, \dots, \alpha_k, X_1, \dots, X_{n-1}, \bullet, X_{n+1}, \dots, X_l))(\alpha_0).$$

The following proposition shows why the term "raised" has been used.

#### Proposition 2.9.5: Raising the $n$ -th index

Let  $\mathcal{A}$  be a tensor field of type  $(k, l)$  and  $n \in \{1, \dots, l\}$ . Let's define:

$$\begin{aligned} \mathbf{A}_{i_1 \dots i_l}^{j_1 \dots j_k} &:= \mathcal{A}(e^{j_1}, \dots, e^{j_k}, e_{i_1}, \dots, e_{i_l}) \\ \mathbf{A}_{i_1 \dots i_{n-1} i_{n+1} \dots i_l}^{i_n j_1 \dots j_k} &:= [\mathcal{A}]^n(e^{i_n}, e^{j_1}, \dots, e^{j_k}, e_{i_1}, \dots, e_{i_{n-1}}, e_{i_{n+1}}, \dots, e_{i_l}) \end{aligned}$$

Then we have:

$$\mathbf{A}_{i_1 \dots i_{n-1} i_{n+1} \dots i_l}^{i_n j_1 \dots j_k} = \mathbf{A}_{i_1 \dots i_{n-1} m i_{n+1} \dots i_l}^{j_1 \dots j_k} \mathfrak{g}^{m i_n}.$$

*Proof.* We have:

$$\begin{aligned} \mathbf{A}_{i_1 \dots i_{n-1} i_{n+1} \dots i_l}^{i_n j_1 \dots j_k} &= [\mathcal{A}]^n(e^{i_n}, e^{j_1}, \dots, e^{j_k}, e_{i_1}, \dots, e_{i_{n-1}}, e_{i_{n+1}}, \dots, e_{i_l}) \\ &= g^*(e^{i_n}, \mathcal{A}(e^{j_1}, \dots, e^{j_k}, e_{i_1}, \dots, e_{i_{n-1}}, \bullet, e_{i_{n+1}}, \dots, e_{i_l})) \\ &= g^*(e^{i_n}, \mathbf{A}_{i_1 \dots i_{n-1} m i_{n+1} \dots i_l}^{j_1 \dots j_k} e^m) \end{aligned}$$

$$\begin{aligned}
&= \mathbf{A}_{i_1 \dots i_{n-1} m i_{n+1} \dots i_l}^{j_1 \dots j_k} g^*(e^{i_n}, e^m) \\
&= \mathbf{A}_{i_1 \dots i_{n-1} m i_{n+1} \dots i_l}^{j_1 \dots j_k} \mathbf{g}^{m i_n}
\end{aligned}$$

□

As  $g^*$  is diagonal in  $\mathcal{C}^*$  and  $\mathcal{C}_\perp^*$ , we have the following result.

### Corollary 2.9.6: Raising the $i$ -th index

Let  $\mathcal{A}$  be a tensor field of type  $(k, l)$  with  $l \geq 1$ . Then we have:

$$\begin{aligned}
\mathbb{A}_{i_1 \dots i_{n-1} i_{n+1} \dots i_l}^{i_n j_1 \dots j_k} &= \mathbb{A}_{i_1 \dots i_l}^{j_1 \dots j_k} \eta_{i_n} e^{-2u_{i_n}} \\
\mathbb{A}_{i_1 \dots i_{n-1} i_{n+1} \dots i_l}^{i_n j_1 \dots j_k} &= \mathbb{A}_{i_1 \dots i_l}^{j_1 \dots j_k} \eta_{i_n} e^{-2u_{i_n}}
\end{aligned}$$

Therefore, we see that the defined operation raises the indices of a tensor field. Let's continue the calculations with two simple examples.

### Example 2.9.7: Two simple examples

(1) Let  $\mathcal{A} := \mathbf{A}^i_j e_i \otimes e^j$  be a tensor field of type  $(1, 1)$ . We denote:

$$\mathbf{A}^{kl} := [\mathcal{A}]^1(e^k, e^l).$$

Since:

$$\begin{aligned}
[\mathcal{A}]^1(\alpha, \beta) &= g^*(\alpha, \mathcal{A}(\beta, \bullet)) \\
\mathcal{A}(e^l, \bullet) &= \mathbf{A}^i_j e_i(e^l) e^j \\
&= \mathbf{A}^i_j \delta_i^l e^j \\
&= \mathbf{A}^l_j e^j
\end{aligned}$$

we have:

$$\begin{aligned}
\mathbf{A}^{kl} &= [\mathcal{A}]^1(e^k, e^l) \\
&= g^*(e^k, \mathcal{A}(e^l, \bullet)) \\
&= g^*(e^k, \mathbf{A}^l_j e^j) \\
&= \mathbf{A}^l_j g^*(e^k, e^j) \\
&= \mathbf{A}^l_j \mathbf{g}^{kj}
\end{aligned}$$

Therefore, for example:

$$\begin{aligned}
\mathbb{A}^{kl} &= \eta^{kj} \mathbb{A}^l_j \\
&= \eta_k \mathbb{A}^l_k \\
\mathbb{A}^{kl} &= \mathbf{g}^{kj} \mathbb{A}^l_j \\
&= \eta_k e^{-2u_k} \mathbb{A}^l_k
\end{aligned}$$

(2) Let  $\alpha := \alpha_i e^i$  be a covector field, *i.e.*,  $\alpha$  is a tensor of type  $(0, 1)$ . We denote:

$$\alpha^k := [\alpha]^1(e^k)$$

Hence, we have:

$$\alpha^\sharp = [\alpha]_1 = \alpha^k e_k$$

Thus, we have:

$$\begin{aligned}
 \alpha^k &= [\alpha]^1(e^k) \\
 &= g^*(e^k, \alpha) \\
 &= g^*(e^k, \alpha_i e^i) \\
 &= \alpha_i g^*(e^k, e^i) \\
 &= \alpha_i \mathbf{g}^{ki}
 \end{aligned}$$

Therefore, for example:

$$\begin{aligned}
 \alpha^k &= \alpha_i \eta^{ki} \\
 &= \eta^{kk} \alpha_k \\
 \alpha^k &= \alpha_i \mathbf{g}^{ki} \\
 &= \alpha_k \mathbf{g}^{kk} \\
 &= \eta^{kk} e^{-2u_k} \alpha^k
 \end{aligned}$$

### 3 Connections associated with curvature

#### 3.1 Lie Derivative and Lie Bracket

We start by recalling the definition of the Lie derivative and the Lie bracket along a vector field. They are useful for defining the Levi-Civita connection. We will focus on the derivative of scalar fields. By the definition of the tangent vector (see 2.1.1), we have defined the object "X(f)," which can be seen as an action of X on f. This is also known as the Lie derivative of f along X.

##### Definition 3.1.1: Lie Derivative

Let X be a vector field given by  $X := X^j \partial_j$ , and let f be a scalar field. The **Lie derivative along X** is defined as:

$$X(f) := X^i \partial_i(f).$$

##### Remark 3.1.2: Remark on the Lie derivative

We can extend the definition of the Lie derivative along X to  $\Omega M$  (the algebra of differential forms on M) as the unique linear map:

$$\mathcal{L}_X : \Omega M \rightarrow \Omega M$$

such that:

(i) for any scalar field f, we have:

$$\mathcal{L}_X(f) := X(f) := df(X);$$

(ii)  $\mathcal{L}_X$  is a derivation of the algebra  $\Omega M$ , i.e.:

$$\forall \alpha, \beta \in \Omega M, \mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X(\alpha) \wedge \beta + \alpha \wedge \mathcal{L}_X(\beta);$$

(iii) the maps  $\mathcal{L}_X$  and d commute.

We will only use the definition for scalar fields in the following.

**Lemma 3.1.3: Alternative expression for the Lie derivative**

Let  $X$  be a vector field. We have:

$$X(f) = df(X).$$

*Proof.* Since:

$$df = \partial_i(f)dx^i$$

we have:

$$\begin{aligned} df(X) &= df(X^j\partial_j) \\ &= X^j df(\partial_j) \\ &= X^j \partial_i(f) dx^i(\partial_j) \\ &= X^j \partial_i(f) \delta_j^i \\ &= X^i \partial_i(f) \\ &= X(f) \end{aligned}$$

□

This leads us to the definition of the Lie bracket.

**Definition 3.1.4: Lie Bracket**

Let  $X$  and  $Y$  be two vector fields. The **Lie bracket of  $X$  and  $Y$**  is the map  $[X, Y]$  that associates to any scalar field  $f$  the scalar field:

$$[X, Y](f) := X(Y(f)) - Y(X(f)).$$

In particular, by the Schwartz theorem, we have:

$$\begin{aligned} [\partial_i, \partial_j](f) &= \partial_i(\partial_j(f)) - \partial_j(\partial_i(f)) \\ &= \partial_{i,j}(f) - \partial_{j,i}(f) \\ &= 0 \end{aligned}$$

*i.e.*, we have:

$$[\partial_i, \partial_j] = 0. \quad (3.1.1)$$

**Lemma 3.1.5: Local descriptions**

Let  $X$  and  $Y$  be two vector fields and  $f$  be a scalar field.

(i) We have:

$$Y(X(f)) = Y^j \partial_j X^i \partial_i f + Y^j X^i \partial_{ji}^2 f$$

(ii) We have:

$$[X, Y](f) = (X^j \partial_j Y^i - Y^j \partial_j X^i) \partial_i f.$$

*Proof.* (i) Since:

$$X(f) = X^i \partial_i f$$

we have:

$$\begin{aligned} Y(X(f)) &= Y^j \partial_j (X^i \partial_i f) \\ &= Y^j \partial_j X^i \partial_i f + Y^j X^i \partial_{ji}^2 f \end{aligned}$$

(ii) Since:

$$\partial_{ji}^2 f - \partial_{ij}^2 f = 0$$

we have:

$$\begin{aligned} [X, Y](f) &= X(Y(f)) - Y(X(f)) \\ &= X^j \partial_j Y^i \partial_i f + X^j Y^i \partial_{ji}^2 f - Y^j \partial_j X^i \partial_i f - Y^j X^i \partial_{ji}^2 f \\ &= (X^j \partial_j Y^i - Y^j \partial_j X^i) \partial_i f + X^j Y^i (\partial_{ji}^2 f - \partial_{ij}^2 f) \\ &= (X^j \partial_j Y^i - Y^j \partial_j X^i) \partial_i f \end{aligned}$$

□

Finally, we have the Jacobi identity.

### Proposition 3.1.6: Jacobi Identity

Let  $X$ ,  $Y$ , and  $Z$  be three vector fields. We have:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

*Proof.* We have:

$$\begin{aligned} [X, [Y, Z]](f) &= X((YZ - ZY)(f)) - (YZ - ZY)(X(f)) \\ &= X(Y(Z(f))) - X(Z(Y(f))) - Y(Z(X(f))) + Z(Y(X(f))) \end{aligned}$$

By removing the parentheses and the  $f$ , and permuting the variables  $X$ ,  $Y$ , and  $Z$ , we have:

$$\begin{aligned} [X, [Y, Z]] &= XYZ - XZY - YZX + ZYX \\ [Y, [Z, X]] &= YZX - YXZ - ZXY + XZY \\ [Z, [X, Y]] &= ZXY - ZYX - XYZ + YXZ \end{aligned}$$

By summing the three equations, we get:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

□

## 3.2 Levi-Civita Connection

We start with a simplified definition of a connection that highlights the two essential properties that will be used in calculations.

### Definition 3.2.1: Koszul Connection

A **connection**  $\nabla$  is a linear map that associates a vector field  $\nabla_X Y$  to every pair of vector fields  $X$  and  $Y$ , satisfying the following two properties:

(i) (**Linearity in  $X$** ) For any scalar field  $f$  and vector fields  $X$  and  $Y$ , we have:

$$\nabla_{fX} Y = f \nabla_X Y.$$

(ii) (**Leibniz Rule**) For any scalar field  $f$  and vector fields  $X$  and  $Y$ , we have:

$$\nabla_X (fY) = df(X)Y + f \nabla_X Y = df(X)Y + \nabla_{fX} Y.$$

We then define the Levi-Civita connection as the unique torsion-free and  $g$ -parallel connection.

**Proposition 3.2.2:** The connection associated with  $g$

The Levi-Civita connection is the unique connection  $\nabla$  on  $TM$  such that:

- (i)  $\nabla$  is **torsion-free**, *i.e.*, for any vector fields  $X$  and  $Y$ , we have:

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

- (ii)  $g$  is **parallel**, *i.e.*, for any vector fields  $X$ ,  $Y$ , and  $Z$ , we have:

$$Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

*Proof.* We present a proof by analysis-synthesis.

- **Analysis-Uniqueness:** Let  $X$  and  $Y$  be two vector fields on  $M$ . Since the metric  $g$  is a symmetric non-degenerate bilinear form, to compute  $\nabla_X Y$ , it suffices to have an expression for  $2g(\nabla_X Y, Z)$  for all vector fields  $Z$ . Using bilinearity and symmetry of  $g$ , for any vector field  $Z$ , we have:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= g(2\nabla_X Y, Z) \\ &= g(\nabla_X Y, Z) + g(\nabla_X Z, Y) + g(\nabla_Y Z, X) + g(\nabla_Y X, Z) - g(\nabla_Z X, Y) - g(\nabla_Z Y, X) \\ &\quad + g(\nabla_X Y - \nabla_Y X, Z) + g(\nabla_Z X - \nabla_X Z, Y) - g(\nabla_Y Z - \nabla_Z Y, X) \\ &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + g(\nabla_Y Z, X) + g(Z, \nabla_Y X) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) \\ &\quad + g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X) \\ &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) + g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X) \end{aligned}$$

Thus, the value of  $\nabla_X Y$  is uniquely determined.

- **Synthesis-Existence:** For any vector fields  $X$  and  $Y$ , we define  $\nabla_X Y$  as the unique vector field on  $M$  such that for any vector field  $Z$ , we have:

$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) + g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X).$$

Let us show that  $\nabla$  is indeed a connection on  $M$ . For any vector field  $Z$ , we have:

- (i) For any vector field  $Z$  and any scalar field  $f \in \mathcal{C}^p(X, \mathbb{R})$ , we have:

$$\begin{aligned} 2g(\nabla_X(fY), Z) &= X(g(fY, Z)) + (fY)(g(Z, X)) - Z(g(X, fY)) \\ &\quad + g([X, fY], Z) + g([Z, X], fY) - g([fY, Z], X) \\ &= df(X)g(Y, Z) - df(Z)g(X, Y) + df(X)g(Y, Z) \\ &\quad + df(Z)g(X, Y) + f2g(\nabla_X Y, Z) \\ &= 2g(df(X)Y + f\nabla_X Y, Z), \end{aligned}$$

and we also have:

$$\begin{aligned} 2g(\nabla_{fX} Y, Z) &= (fX)(g(Y, Z)) + Y(g(Z, fX)) - Z(g(fX, Y)) \\ &\quad + g([fX, Y], Z) + g([Z, fX], Y) - g([Y, Z], fX) \\ &= df(Y).g(Z, X) - df(Z).g(X, Y) - df(Y).g(X, Z) \\ &\quad + df(Z).g(X, Y) + f.2g(\nabla_X Y, Z) \\ &= 2g(f\nabla_X Y, Z). \end{aligned}$$

Hence, the result follows from the degeneracy of  $g$ .

(ii) Let's show that  $\nabla$  is torsion-free. For any vector field  $Z$ , we have:

$$\begin{aligned} 2g(\nabla_X Y - \nabla_Y X, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) + g([X, Y], Z) \\ &\quad + g([Z, X], Y) - g([Y, Z], X) - Y(g(X, Z)) - X(g(Z, Y)) \\ &\quad + Z(g(Y, X)) - g([Y, X], Z) - g([Z, Y], X) + g([X, Z], Y) \\ &= 2g([X, Y], Z). \end{aligned}$$

Hence, the result follows from the degeneracy of  $g$ .

(iii) Let's show that  $g$  is parallel with respect to  $\nabla$ . For any vector field  $Z$ , we have:

$$\begin{aligned} 2g(\nabla_X Y, Z) + 2g(Y, \nabla_X Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) + g([X, Y], Z) \\ &\quad + g([Z, X], Y) - g([Y, Z], X) + X(g(Z, Y)) + Z(g(Y, X)) \\ &\quad - Y(g(X, Z)) + g([X, Z], Y) + g([Y, X], Z) - g([Z, Y], X) \\ &= 2X(g(Y, Z)). \end{aligned}$$

Hence, the result follows from the degeneracy of  $g$ .

Thus,  $\nabla$  is the unique Koszul connection satisfying the theorem.  $\square$

Therefore, we have shown that  $\nabla$  satisfies:

$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) + g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X).$$

We extend the definition of the connection to arbitrary tensor fields. For this purpose, we define the action of  $\nabla$  on 1-forms and with respect to the operations  $+$  and  $\otimes$ .

### Definition 3.2.3: Connection on tensors

Let  $f$  be a scalar field,  $A$  and  $B$  be tensor fields,  $X$  and  $Y$  be vector fields, and  $\alpha$  be a covector field.

(i) We denote:

$$\nabla_X f := X(f).$$

(ii) We denote:

$$(\nabla_X \alpha)(Y) := X(\alpha(Y)) - \alpha(\nabla_X Y).$$

(iii) We denote:

$$\nabla_X (A \otimes B) := (\nabla_X A) \otimes B + A \otimes (\nabla_X B).$$

(iv) Suppose  $A$  and  $B$  are of the same type. We denote:

$$\nabla_X (A + B) := \nabla_X A + \nabla_X B.$$

We then have the following result, which gives an explicit formula for the action of  $\nabla$  on tensor fields.

### Proposition 3.2.4: Formula for the connection on tensors

Let  $A$  be a tensor field of type  $(k, l)$ . For any covector fields  $\alpha_1, \dots, \alpha_k$  and any vector fields  $X_1, \dots, X_l, Y$ , we have:

$$\begin{aligned} &(\nabla_Y A)(\alpha_1, \dots, \alpha_k, X_1, \dots, X_l) \\ &= Y(A(\alpha_1, \dots, \alpha_k, X_1, \dots, X_l)) \\ &\quad - A(\nabla_Y \alpha_1, \dots, \alpha_k, X_1, \dots, X_l) - \dots - A(\alpha_1, \dots, \nabla_Y \alpha_k, X_1, \dots, X_l) \\ &\quad - A(\alpha_1, \dots, \alpha_k, \nabla_Y X_1, \dots, X_l) - \dots - A(\alpha_1, \dots, \alpha_k, X_1, \dots, \nabla_Y X_l) \end{aligned}$$

*Proof.* We fix  $Y$  as a vector field. We prove the result by a double induction on  $(k, l) \in \mathbb{N}^2$ . Let  $\mathcal{Q}(k, l)$  be the proposition.

- (i) **Case  $\mathcal{Q}(0, 0)$  true:** The result follows directly from the definition of  $\nabla$  on vector fields (see point (i) of Definition 3.2.3).
- (ii) **Case  $\mathcal{Q}(0, l)$  true  $\Rightarrow \mathcal{Q}(0, l+1)$  true:** Let  $l \in \mathbb{N}$  and  $A$  be a tensor field of type  $(0, l+1)$ . Assume that  $\mathcal{Q}(k, l)$  is true. Then, by linearity of  $\nabla$  (see point (iv) of Definition 3.2.3), we can assume that  $A$  contains only one term of the form:

$$A := \mathbf{A}e^{i_1} \otimes \dots \otimes e^{i_{l+1}}.$$

Let's define:

$$B := \mathbf{A}e^{i_1} \otimes \dots \otimes e^{i_l}.$$

Therefore, we have:

$$A = B \otimes e^{i_{l+1}}.$$

Then, by point (iii) of Definition 3.2.3, we have:

$$\begin{aligned} \nabla_Y A &= \nabla_Y (B \otimes e^{i_{l+1}}) \\ &= (\nabla_Y B) \otimes e^{i_{l+1}} + B \otimes (\nabla_Y e^{i_{l+1}}) \end{aligned}$$

Using the Leibniz formula, we have:

$$\begin{aligned} Y(A(X_1, \dots, X_{l+1})) &= Y(B(X_1, \dots, X_l) e^{i_{l+1}}(X_{l+1})) \\ &= Y(B(X_1, \dots, X_l)) e^{i_{l+1}}(X_{l+1}) + B(X_1, \dots, X_l) Y(e^{i_{l+1}}(X_{l+1})). \end{aligned}$$

Hence, by  $\mathcal{Q}(0, l)$  and point (iii) of Definition 3.2.3, we have:

$$\begin{aligned} &(\nabla_Y A)(X_1, \dots, X_{l+1}) \\ &= (\nabla_Y B) \otimes e^{i_{l+1}}(X_1, \dots, X_{l+1}) + B \otimes (\nabla_Y e^{i_{l+1}})(X_1, \dots, X_{l+1}) \\ &= (\nabla_Y B)(X_1, \dots, X_l) e^{i_{l+1}}(X_{l+1}) + B(X_1, \dots, X_l) (\nabla_Y e^{i_{l+1}})(X_{l+1}) \\ &= Y(B(X_1, \dots, X_l)) e^{i_{l+1}}(X_{l+1}) - B(\nabla_Y X_1, \dots, X_l) e^{i_{l+1}}(X_{l+1}) - \dots - B(X_1, \dots, \nabla_Y X_l) e^{i_{l+1}}(X_{l+1}) \\ &\quad + B(X_1, \dots, X_l) (Y(e^{i_{l+1}}(X_{l+1})) - e^{i_{l+1}}(\nabla_Y X_{l+1})) \\ &= Y(A(X_1, \dots, X_{l+1})) - A(\nabla_Y X_1, \dots, X_{l+1}) - \dots - A(X_1, \dots, \nabla_Y X_{l+1}) \end{aligned}$$

Thus,  $\mathcal{Q}(0, l+1)$  is true.

- (iii) **Cas  $\mathcal{Q}(k, l)$  vraie  $\Rightarrow \mathcal{Q}(k+1, l)$  vraie.** Soit  $(k, l) \in \mathbb{N}^2$  et un champ tensoriel  $A$  de type  $(k, l)$ . Supposons que  $\mathcal{Q}(k, l)$  soit vraie. Alors par linéarité de  $\nabla$  (voir le point (iv) de la définition 3.2.3), on peut supposer que  $A$  ne contient qu'un seul terme du type :

$$A := \mathbf{A}e_{j_0} \otimes \dots \otimes e^{j_k} \otimes e^{i_1} \otimes \dots \otimes e^{i_l}.$$

Posons :

$$B := \mathbf{A}e_{j_1} \otimes \dots \otimes e^{j_k} \otimes e^{i_1} \otimes \dots \otimes e^{i_l}.$$

Donc on a :

$$A = e_{j_0} \otimes B.$$

On a alors par le point (iii) de la définition 3.2.3 :

$$\begin{aligned} \nabla_Y A &= \nabla_Y (e_{j_0} \otimes B) \\ &= (\nabla_Y e_{j_0}) \otimes B + e_{j_0} \otimes (\nabla_Y B) \end{aligned}$$

On a par la formule de Leibniz :

$$\begin{aligned} & Y(A(\alpha_0, \dots, \alpha_k, X_1, \dots, X_l)) \\ &= Y(e_{j_0}(\alpha_0)B(\alpha_1, \dots, \alpha_k, X_1, \dots, X_l)) \\ &= Y(e_{j_0}(\alpha_0))B(\alpha_1, \dots, \alpha_k, X_1, \dots, X_l) + Y(B(\alpha_1, \dots, \alpha_k, X_1, \dots, X_l))e_{j_0}(\alpha_0). \end{aligned}$$

Ainsi on a par  $\mathcal{Q}(k, l)$  et par le point (iii) de la définition 3.2.3 :

$$\begin{aligned} & (\nabla_Y A)(\alpha_0, \dots, \alpha_k, X_1, \dots, X_l) \\ &= (\nabla_Y e_{j_0}) \otimes B(\alpha_0, \dots, \alpha_k, X_1, \dots, X_l) + e_{j_0} \otimes (\nabla_Y B)(\alpha_0, \dots, \alpha_k, X_1, \dots, X_l) \\ &= (\nabla_Y e_{j_0})(\alpha_0)B(\alpha_1, \dots, \alpha_k, X_1, \dots, X_l) + e_{j_0}(\alpha_0)(\nabla_Y B)(\alpha_1, \dots, \alpha_k, X_1, \dots, X_l) \\ &= Y(e_{j_0}(\alpha_0))B(\alpha_1, \dots, \alpha_k, X_1, \dots, X_l) + e_{j_0}(\alpha_0)[Y(B(\alpha_1, \dots, \alpha_k, X_1, \dots, X_l)) \\ &\quad - B(\nabla_Y \alpha_1, \dots, \alpha_k, X_1, \dots, X_l) - \dots - B(\alpha_1, \dots, \nabla_Y \alpha_k, X_1, \dots, X_l) \\ &\quad - B(\alpha_1, \dots, \alpha_k, \nabla_Y X_1, \dots, X_l) - \dots - B(\alpha_1, \dots, \alpha_k, X_1, \dots, \nabla_Y X_l)] \\ &= Y(A(\alpha_0, \dots, \alpha_k, X_1, \dots, X_l)) \\ &\quad - A(\nabla_Y \alpha_0, \dots, \alpha_k, X_1, \dots, X_l) - \dots - A(\alpha_0, \dots, \nabla_Y \alpha_k, X_1, \dots, X_l) \\ &\quad - A(\alpha_0, \dots, \alpha_k, \nabla_Y X_1, \dots, X_l) - \dots - A(\alpha_0, \dots, \alpha_k, X_1, \dots, \nabla_Y X_l) \end{aligned}$$

So  $\mathcal{Q}(k+1, l)$  is true.

(iv) By the double induction principle,  $\mathcal{Q}(k, l)$  is true for all  $(k, l) \in \mathbb{N}^2$ .

□

### Notation 3.2.5: Notations for components

Let  $\mathcal{A}$  be a tensor field of type  $(k, l)$ . We will use a component notation for the action of  $\nabla$  on  $\mathcal{A}$ .

(1) Suppose the local description of  $\mathcal{A}$  is given by:

$$\mathcal{A} = \mathbf{A}_{i_1 \dots i_l}^{j_1 \dots j_k} e_{j_1} \otimes \dots \otimes e_{j_k} \otimes e^{i_1} \otimes \dots \otimes e^{i_l}.$$

We then write:

$$\nabla_m \mathbf{A}_{i_1 \dots i_l}^{j_1 \dots j_k} := \nabla_{e_m} \mathcal{A}(e^{j_1}, \dots, e^{j_k}, e_{i_1}, \dots, e_{i_l}).$$

(2) Suppose the local description of  $\mathcal{A}$  is given by:

$$\mathcal{A} = \mathbf{A}_{i_1 \dots i_l}^{j_1 \dots j_k} \partial_{j_1} \otimes \dots \otimes \partial_{j_k} \otimes dx^{i_1} \otimes \dots \otimes dx^{i_l}.$$

We then write:

$$\nabla_m \mathbf{A}_{i_1 \dots i_l}^{j_1 \dots j_k} := \nabla_{\partial_m} \mathcal{A}(dx^{j_1}, \dots, dx^{j_k}, \partial_{i_1}, \dots, \partial_{i_l}).$$

(3) Suppose the local description of  $\mathcal{A}$  is given by:

$$\mathcal{A} = \mathbf{A}_{i_1 \dots i_l}^{j_1 \dots j_k} \theta_{j_1} \otimes \dots \otimes \theta_{j_k} \otimes \theta^{i_1} \otimes \dots \otimes \theta^{i_l}.$$

We then write:

$$\nabla_m \mathbf{A}_{i_1 \dots i_l}^{j_1 \dots j_k} := \nabla_{\theta_m} \mathcal{A}(\theta^{j_1}, \dots, \theta^{j_k}, \theta_{i_1}, \dots, \theta_{i_l}).$$

Thus, the writing style of the component of  $\mathcal{A}$  indicates whether  $m$  refers to  $e_m$ ,  $\partial_m$ , or  $\theta_m$ .

### 3.3 Trace and divergence

Throughout this subsection, we fix a tensor field  $\mathcal{A}$  of type  $(k, l)$ . We use the notations introduced in tensor field contraction (see definition 2.2.5). We begin with a standard notation for the action of nabla on tensor fields.

#### Notation 3.3.1: The tensor field $\nabla\mathcal{A}$

We denote by  $\nabla\mathcal{A}$  the tensor field of type  $(k, l + 1)$  defined by:

$$\nabla\mathcal{A}(\alpha_1, \dots, \alpha_k, X_1, \dots, X_{l+1}) := (\nabla_{X_{l+1}}\mathcal{A})(\alpha_1, \dots, \alpha_k, X_1, \dots, X_l)$$

#### Definition 3.3.2: Trace of a tensor field

(i) Suppose  $k, l \geq 1$ . Let  $m \in \{1, \dots, k\}$  and  $n \in \{1, \dots, l\}$ . The **trace of  $\mathcal{A}$  along  $(n, m)$**  is given by:

$$\text{tr}_{(m,n)}\mathcal{A} := [\mathcal{A}]_n^m.$$

(a) If  $k := 1$  (thus  $m := 1$ ), it is simply denoted as:

$$\text{tr}_{(n)}\mathcal{A} := [\mathcal{A}]_n^1.$$

(b) If  $l := 1$  (thus  $n := 1$ ), it is simply denoted as:

$$\text{tr}_{(m)}\mathcal{A} := [\mathcal{A}]_1^m.$$

(ii) Suppose  $l \geq 2$  and  $k := 0$ . Let  $m \in \{1, \dots, l - 1\}$  and  $n \in \{1, \dots, l\}$ . The  **$g$ -trace of  $\mathcal{A}$  along  $(m, n)$**  is given by:

$$\text{tr}_{g,(m,n)}\mathcal{A} := [[\mathcal{A}]_m^n]_1^1.$$

Let's examine the type of tensor fields appearing in the definitions.

(i) We have:

- $\mathcal{A}$  is a tensor field of type  $(k, l)$ .
- $\text{tr}_{(m,n)}\mathcal{A}$  is a tensor field of type  $(k - 1, l - 1)$ .

(ii) We have:

- $\mathcal{A}$  is a tensor field of type  $(0, l)$ .
- $[\mathcal{A}]_n^m$  is a tensor field of type  $(1, l - 1)$ .
- $\text{tr}_{g,(m,n)}\mathcal{A}$  is a tensor field of type  $(0, l - 2)$ .

#### Example 3.3.3: Example of the trace of a tensor field of type $(1, 1)$

Let  $\mathcal{A}$  be a tensor field of type  $(0, 2)$ . Suppose that  $\mathcal{A}$  is **symmetric**, meaning that for all vector fields  $X$  and  $Y$ , we have  $\mathcal{A}(X, Y) = \mathcal{A}(Y, X)$ . Then we have:

$$[\mathcal{A}]^1 = [\mathcal{A}]^2.$$

Therefore, we have:

$$\text{tr}_{g,(1,1)}\mathcal{A} = \text{tr}_{g,(2,1)}\mathcal{A}.$$

In this case, we simply write:

$$\text{tr}_g\mathcal{A} := \text{tr}_{g,(i,1)}\mathcal{A}.$$

## Definition 3.3.4: Divergence of a tensor field

(i) Suppose  $k \geq 1$ . Let  $n \in \{1, \dots, k\}$ . The **divergence of  $\mathcal{A}$  along  $n$**  is given by:

$$\operatorname{div}_{(n)} \mathcal{A} := [\nabla \mathcal{A}]_{l+1}^n.$$

If  $k := 1$  (thus  $n := 1$ ), it is simply denoted as:

$$\operatorname{div} \mathcal{A} := \operatorname{div}_{(1)} \mathcal{A} := [\nabla \mathcal{A}]_{l+1}^1.$$

(ii) Suppose  $l \geq 1$ . Let  $n \in \{1, \dots, l\}$ . The  **$g$ -divergence of  $\mathcal{A}$  along  $n$**  is given by:

$$\operatorname{div}_{g,(n)} \mathcal{A} := [\nabla [\mathcal{A}]^n]_{l+1}^1.$$

If  $l := 1$  (thus  $n := 1$ ), it is simply denoted as:

$$\operatorname{div}_g \mathcal{A} := \operatorname{div}_{g,(1)} \mathcal{A} := [\nabla \mathcal{A}]_{l+1}^1.$$

Let's examine the type of tensor fields appearing in the definitions.

(i) We have:

- $\mathcal{A}$  is a tensor field of type  $(k, l)$ .
- $\nabla \mathcal{A}$  is a tensor field of type  $(k, l + 1)$ .
- $\operatorname{div}_{(n)} \mathcal{A}$  is a tensor field of type  $(k - 1, l)$ .

(ii) We have:

- $\mathcal{A}$  is a tensor field of type  $(k, l)$ .
- $[\mathcal{A}]^n$  is a tensor field of type  $(k + 1, l - 1)$ .
- $\nabla [\mathcal{A}]^n$  is a tensor field of type  $(k + 1, l)$ .
- $\operatorname{div}_{g,(n)} \mathcal{A}$  is a tensor field of type  $(k, l - 1)$ .

Example 3.3.5: Divergences of a vector field and covector field in  $\mathcal{C}$  and  $\mathcal{C}^*$ 

(1) Let  $X$  be a vector field given by  $X := X^j \partial_j$ . By the definition of a connection, we have:

$$\begin{aligned} \operatorname{div}(X) &= [\nabla X]_1^1 \\ &= dx^i (\nabla_{\partial_i} X) \\ &= dx^i (\nabla_{\partial_i} (X^j \partial_j)) \\ &= dx^i (dX^j (\partial_i) \partial_j + X^j \nabla_{\partial_i} \partial_j) \\ &= dx^i (\partial_i X^j \partial_j + X^j \nabla_{\partial_i} \partial_j) \\ &= \partial_i X^j \delta_j^i + X^j \Gamma_{ij}^i \\ &= \partial_i X^i + X^j \Gamma_{ij}^i \end{aligned}$$

(2) Let  $\alpha$  be a covector field given by  $\alpha := \alpha_j dx^j$ . The **divergence of  $\alpha$**  is defined as:

$$\operatorname{div}_g \alpha := \operatorname{div} \alpha^\sharp := \operatorname{div} [\alpha]^1.$$

(3) Let  $\mathcal{A}$  be a tensor field of type  $(0, 2)$ . Suppose that  $\mathcal{A}$  is **symmetric**, meaning that for all vector fields  $X$  and  $Y$ , we have  $\mathcal{A}(X, Y) = \mathcal{A}(Y, X)$ . Then we have:

$$[\mathcal{A}]^1 = [\mathcal{A}]^2$$

Therefore, we have:

$$\operatorname{div}_{g,(1)}\mathcal{A} = \operatorname{div}_{g,(2)}\mathcal{A}$$

In this case, we simply write:

$$\operatorname{div}_g\mathcal{A} := \operatorname{div}_{g,(i)}\mathcal{A}.$$

### 3.4 Definition of the connection in the bases $\mathcal{E}$ and $\mathcal{E}^*$

The elements  $\nabla_{e_k}e_j$  are vector fields and thus they can be expressed in the  $\mathcal{E}$  basis. Hence the following definitions.

#### Definition 3.4.1: Rotation coefficients

Let  $i, j, k \in \{0, 1, 2, 3\}$ .

(i) The **rotation coefficients**  $\mathbf{\Gamma}^i_{kj}$  are defined as:

$$\mathbf{\Gamma}^i_{kj} := e^i(\nabla_{e_k}e_j).$$

(ii) The **matrix of 1-forms of rotation**

$$\boldsymbol{\omega} := (\boldsymbol{\omega}^i_j)_{ij} \in \mathcal{M}_4(\Lambda^1(U))$$

is defined as:

$$\boldsymbol{\omega}^i_j := \mathbf{\Gamma}^i_{kj}e^k.$$

Thus, we have:

$$\begin{aligned} \boldsymbol{\omega}^i_j(e_k) &= \mathbf{\Gamma}^i_{kj} \\ \mathbf{\Gamma}^i_{kj}e_i &= \nabla_{e_k}e_j = \boldsymbol{\omega}^i_j(e_k)e_i \end{aligned}$$

#### Example 3.4.2: Christoffel symbols and tetradic coefficients

Let  $i, j, k \in \{0, 1, 2, 3\}$ .

(1) The elements  $\nabla_{\partial_k}\partial_j$  are vector fields and thus they can be expressed in the  $\mathcal{E}$  basis.

(a) The **Christoffel symbols**  $\Gamma^i_{kj}$  are defined as:

$$\Gamma^i_{kj} := dx^i(\nabla_{\partial_k}\partial_j).$$

(b) The **matrix of 1-forms of Christoffel**

$$\omega := (\omega^i_j)_{ij} \in \mathcal{M}_4(\Lambda^1(U))$$

is defined as:

$$\omega^i_j := \Gamma^i_{kj}dx^k.$$

Thus, we have:

$$\begin{aligned} \omega^i_j(\partial_k) &= \Gamma^i_{kj} \\ \Gamma^i_{kj}\partial_i &= \nabla_{\partial_k}\partial_j = \omega^i_j(\partial_k)\theta_i \end{aligned}$$

(2) The elements  $\nabla_{\theta_k}\theta_j$  are vector fields and thus they can be expressed in the  $\mathcal{E}_\perp$  basis.

(a) The **tetradic coefficients** of  $\mathbb{F}^i_{kj}$  are defined as:

$$\mathbb{F}^i_{kj} := \theta^i (\nabla_{\theta_k} \theta_j).$$

(b) The **matrix of tetradic 1-forms**

$$\varpi := (\varpi^i_j)_{ij} \in \mathcal{M}_4(\Lambda^1(U))$$

is defined as:

$$\varpi^i_j := \mathbb{F}^i_{kj} \theta^k.$$

Thus, we have:

$$\begin{aligned} \varpi^i_j(\theta_k) &= \mathbb{F}^i_{kj} \\ \mathbb{F}^i_{kj} \theta_i &= \nabla_{\theta_k} \theta_j = \varpi^i_j(\theta_k) \theta_i \end{aligned}$$

The following lemma establishes the relationship between  $\Gamma^j_{ki}$  and  $\mathbb{F}^j_{ki}$ .

**Lemma 3.4.3: Relationship between  $\Gamma^j_{ki}$  and  $\mathbb{F}^j_{ki}$**

Let  $i, j, k \in \{0, 1, 2, 3\}$  such that  $i \neq j$ .

(i) (a) We have:

$$\mathbb{F}^j_{kj} = e^{-u_k} (-u_{j,k} + \Gamma^j_{kj}) \qquad \Gamma^j_{kj} = u_{j,k} + e^{u_k} \mathbb{F}^j_{kj}$$

(b) We have:

$$\mathbb{F}^i_{kj} = e^{u_i - u_j - u_k} \Gamma^i_{kj} \qquad \Gamma^i_{kj} = e^{-u_i + u_j + u_k} \mathbb{F}^i_{kj}$$

(ii) (a) We have:

$$\varpi^j_j = \omega^j_j - u_{j,k} e^{-u_k} \theta^k \qquad \omega^j_j = \varpi^j_j + u_{j,k} dx^k$$

(b) We have:

$$\varpi^i_j = e^{u_i - u_j} \omega^i_j \qquad \omega^i_j = e^{-u_i + u_j} \varpi^i_j$$

*Proof.* (i) For all scalar fields  $f$  and  $g$ , and vector fields  $X$  and  $Y$ , we have:

$$\nabla_{gX}(fY) = g\nabla_X(fY) = gdf(X)Y + fg\nabla_X Y.$$

Thus, we have:

$$\begin{aligned} \nabla_{\theta_k} \theta_j &= \nabla_{e^{-u_k} \partial_k} (e^{-u_j} \partial_j) \\ &= e^{-u_k} d(e^{-u_j}) (\partial_k) \partial_j + e^{-u_j - u_k} \nabla_{\partial_k} \partial_j \\ &= -u_{j,k} e^{-u_j - u_k} \partial_j + e^{-u_j - u_k} \Gamma^i_{kj} \partial_i \\ &= e^{-u_j - u_k} \left[ (-u_{j,k} + \Gamma^j_{kj}) \partial_j + \sum_{i \neq j} \Gamma^i_{kj} \partial_i \right] \end{aligned}$$

On the other hand, we have:

$$\begin{aligned} \nabla_{\theta_k} \theta_j &= \mathbb{F}^i_{kj} \theta_i \\ &= \mathbb{F}^i_{kj} e^{-u_i} \partial_i \end{aligned}$$

By uniqueness of the decomposition, we have for  $i \neq j$ :

$$\begin{aligned}\mathbb{F}^j_{kj}e^{-u_j} &= e^{-u_j-u_k}(-u_{j,k} + \Gamma^j_{kj}) \\ \mathbb{F}^i_{kj}e^{-u_i} &= e^{-u_j-u_k}\Gamma^i_{kj}\end{aligned}$$

and also:

$$\begin{aligned}\Gamma^j_{kj} &= u_{j,k} + e^{u_k}\mathbb{F}^j_{kj} \\ \Gamma^i_{kj} &= e^{-u_i+u_j+u_k}\mathbb{F}^i_{kj}\end{aligned}$$

Hence, the points (a) and (b).

(ii) (a) Using (i.a), we have:

$$\begin{aligned}\omega^j_j &= \Gamma^j_{kj}dx^k \\ &= (u_{j,k} + e^{u_k}\mathbb{F}^j_{kj})dx^k \\ &= u_{j,k}dx^k + \mathbb{F}^j_{kj}e^{u_k}dx^k \\ &= u_{j,k}dx^k + \mathbb{F}^j_{kj}\theta^k \\ &= \varpi^j_j + u_{j,k}dx^k\end{aligned}$$

Thus, we have:

$$\varpi^j_j = \omega^j_j - u_{j,k}e^{-u_k}\theta^k \qquad \omega^j_j = \varpi^j_j + u_{j,k}dx^k$$

(b) We have:

$$\begin{aligned}\omega^i_j &= \Gamma^i_{kj}dx^k \\ &= e^{-u_i+u_j+u_k}\mathbb{F}^i_{kj}e^{-u_k}\theta^k \\ &= e^{-u_i+u_j}\varpi^i_j\end{aligned}$$

Thus, we have:

$$\varpi^i_j = e^{u_i-u_j}\omega^i_j \qquad \omega^i_j = e^{-u_i+u_j}\varpi^i_j$$

□

### 3.5 Practical Calculations of Connection in $\mathcal{C}$ and $\mathcal{C}^*$ bases

In this subsection, we calculate the Christoffel symbols using a classical approach that relates  $\Gamma^i_{jk}$  to the tensor fields  $g$  and  $g^*$ . We begin with a proposition that provides an explicit formula for the calculation of  $\nabla$  on a tensor field using the Christoffel symbols.

Let's start with an example.

#### Example 3.5.1: Calculation Example

(1) We have:

$$\begin{aligned}(\nabla_{\partial_k} dx^i)(\partial_j) &= \partial_k(dx^i(\partial_j)) - dx^i(\nabla_{\partial_k}\partial_j) \\ &= \partial_k(\delta^i_j) - \Gamma^i_{kj} \\ &= -\Gamma^i_{kj}\end{aligned}$$

(2) Let  $\alpha$  be a covector field given by  $\alpha := \alpha_i dx^i$ . Then, using (1), we have:

$$\begin{aligned} (\nabla_{\partial_k} \alpha) (\partial_j) &= (\nabla_{\partial_k} (\alpha_i dx^i)) (\partial_j) \\ &= \partial_k (\alpha_i dx^i (\partial_j)) - \alpha_i dx^i (\nabla_{\partial_k} \partial_j) \\ &= \partial_k \alpha_j - \alpha_i \Gamma^i_{kj} \end{aligned}$$

(3) Let  $A$  be a tensor field given by  $A := A_{il} dx^i \otimes dx^l$ . Then, using (1) and (2), we have:

$$\begin{aligned} (\nabla_{\partial_k} A) (\partial_j, \partial_p) &= \nabla_{\partial_k} (A_{il} dx^i \otimes dx^l) (\partial_j, \partial_p) \\ &= (\nabla_{\partial_k} (A_{il} dx^i) \otimes dx^l) (\partial_j, \partial_p) + (A_{il} dx^i \otimes \nabla_{\partial_k} (dx^l)) (\partial_j, \partial_p) \\ &= \nabla_{\partial_k} (A_{il} dx^i) (\partial_j) dx^l (\partial_p) + A_{il} dx^i (\partial_j) \nabla_{\partial_k} (dx^l) (\partial_p) \\ &= (\partial_k A_{jl} - A_{il} \Gamma^i_{kj}) \delta_p^l - A_{il} \delta_j^i \Gamma^l_{kp} \\ &= \partial_k A_{jp} - A_{ip} \Gamma^i_{kj} - A_{jl} \Gamma^l_{kp} \end{aligned}$$

**Proposition 3.5.2: Formula for the Connection on Tensors**

Let  $\mathcal{A}$  be a tensor field of type  $(k, l)$  with the following local coordinate representation:

$$\mathcal{A} = A_{i_1 \dots i_l}^{j_1 \dots j_k} \partial_{j_1} \otimes \dots \otimes \partial_{j_k} \otimes dx^{i_1} \otimes \dots \otimes dx^{i_l}.$$

Then we have:

$$\begin{aligned} \nabla_m A_{i_1 \dots i_l}^{j_1 \dots j_k} &= \partial_m A_{i_1 \dots i_l}^{j_1 \dots j_k} \\ &\quad + \Gamma^{j_1}_{nm} A_{i_1 \dots i_l}^{n j_2 \dots j_k} + \dots + \Gamma^{j_k}_{nm} A_{i_1 \dots i_l}^{j_1 \dots j_{k-1} n} \\ &\quad - \Gamma^n_{i_1 m} A_{n i_2 \dots i_l}^{j_1 \dots j_k} - \dots - \Gamma^n_{i_l m} A_{i_1 \dots i_{l-1} n}^{j_1 \dots j_k} \end{aligned}$$

*Proof.* Using the notations 3.2.5, we have:

$$\nabla_m A_{i_1 \dots i_l}^{j_1 \dots j_k} = \nabla_{\partial_m} \mathcal{A} (dx^{j_1}, \dots, dx^{j_k}, \partial_{i_1}, \dots, \partial_{i_l}).$$

We have three relations:

- For any scalar field  $f$ , we have:

$$\nabla_{\partial_m} f = \partial_m f$$

- By the definition of  $\Gamma^n_{im}$ , we have:

$$\nabla_{\partial_m} \partial_i = \Gamma^n_{im} \partial_n$$

- From point (1) of Example 3.5.1, we have:

$$\nabla_{\partial_m} dx^j = -\Gamma^j_{nm} dx^n.$$

Therefore, using Proposition 3.2.4, we obtain:

$$\begin{aligned}
 & \nabla_m A_{i_1 \dots i_l}^{j_1 \dots j_k} \\
 &= (\nabla_{\partial_m} A) (dx^{j_1}, \dots, dx^{j_k}, \partial_{i_1}, \dots, \partial_{i_l}) \\
 &= \nabla_{\partial_m} (A (dx^{j_1}, \dots, dx^{j_k}, \partial_{i_1}, \dots, \partial_{i_l})) \\
 &\quad - A (\nabla_{\partial_m} dx^{j_1}, \dots, dx^{j_k}, \partial_{i_1}, \dots, \partial_{i_l}) - \dots - A (dx^{j_1}, \dots, \nabla_{\partial_m} dx^{j_k}, \partial_{i_1}, \dots, \partial_{i_l}) \\
 &\quad - A (dx^{j_1}, \dots, dx^{j_k}, \nabla_{\partial_m} \partial_{i_1}, \dots, \partial_{i_l}) - \dots - A (dx^{j_1}, \dots, dx^{j_k}, \partial_{i_1}, \dots, \nabla_{\partial_m} \partial_{i_l}) \\
 &= \nabla_{\partial_m} A_{i_1 \dots i_l}^{j_1 \dots j_k} \\
 &\quad - A (-\Gamma_{nm}^{j_1} dx^n, \dots, dx^{j_k}, \partial_{i_1}, \dots, \partial_{i_l}) - \dots - A (dx^{j_1}, \dots, -\Gamma_{nm}^{j_k} dx^n, \partial_{i_1}, \dots, \partial_{i_l}) \\
 &\quad - A (dx^{j_1}, \dots, dx^{j_k}, \Gamma_{i_1 m}^n \partial_n, \dots, \partial_{i_l}) - \dots - A (dx^{j_1}, \dots, dx^{j_k}, \partial_{i_1}, \dots, \Gamma_{i_l m}^n \partial_n) \\
 &= \nabla_{\partial_m} A_{i_1 \dots i_l}^{j_1 \dots j_k} \\
 &\quad + \Gamma_{nm}^{j_1} A (dx^n, \dots, dx^{j_k}, \partial_{i_1}, \dots, \partial_{i_l}) + \dots + \Gamma_{nm}^{j_k} A (dx^{j_1}, \dots, dx^n, \partial_{i_1}, \dots, \partial_{i_l}) \\
 &\quad - \Gamma_{i_1 m}^n A (dx^{j_1}, \dots, dx^{j_k}, \partial_n, \dots, \partial_{i_l}) - \dots - \Gamma_{i_l m}^n A (dx^{j_1}, \dots, dx^{j_k}, \partial_{i_1}, \dots, \partial_n) \\
 &= \partial_m A_{i_1 \dots i_l}^{j_1 \dots j_k} \\
 &\quad + \Gamma_{nm}^{j_1} A_{i_1 \dots i_l}^{nj_2 \dots j_k} + \dots + \Gamma_{nm}^{j_k} A_{i_1 \dots i_l}^{j_1 \dots j_{k-1} n} \\
 &\quad - \Gamma_{i_1 m}^n A_{ni_2 \dots i_l}^{j_1 \dots j_k} - \dots - \Gamma_{i_l m}^n A_{i_1 \dots i_{l-1} n}^{j_1 \dots j_k}
 \end{aligned}$$

□

We could generalize this result in the bases  $\mathcal{E}$  and  $\mathcal{E}^*$ , but the result is not practical. The connection  $\nabla$  is  $g$ -parallel, *i.e.*,

$$\nabla_{\partial_j} g = 0.$$

From this, we deduce the following usual result.

**Lemma 3.5.3: Expression of Christoffel symbols in terms of the metric**

We have:

$$\Gamma_{kl}^i = \frac{1}{2} g^{im} (g_{mk,l} + g_{ml,k} - g_{kl,m}).$$

*Proof.* Using point (3) of Example 3.5.1, we have:

$$\begin{aligned}
 0 &= \nabla_{\partial_i} g (\partial_m, \partial_n) \\
 &= g_{mn,i} - g_{jn} \Gamma_{im}^j - g_{ml} \Gamma_{in}^l \\
 &= g_{mn,i} - g_{pn} \Gamma_{im}^p - g_{mp} \Gamma_{in}^p
 \end{aligned}$$

Hence, we have:

$$g_{mn,i} = g_{pn} \Gamma_{im}^p + g_{mp} \Gamma_{in}^p.$$

By permuting the indices and exponents, we obtain:

$$\begin{aligned}
 (1): \quad & g_{mk,l} = g_{pk} \Gamma_{lm}^p + g_{mp} \Gamma_{lk}^p \\
 (2): \quad & g_{ml,k} = g_{pl} \Gamma_{km}^p + g_{mp} \Gamma_{kl}^p \\
 (3): \quad & g_{kl,m} = g_{pl} \Gamma_{mk}^p + g_{kp} \Gamma_{ml}^p
 \end{aligned}$$

By taking (1) + (2) - (3), we have:

$$\begin{aligned} \frac{1}{2}g^{im}(g_{mk,l} + g_{ml,k} - g_{kl,m}) &= \frac{1}{2}g^{im}(g_{pk}\Gamma_{lm}^p + g_{mp}\Gamma_{lk}^p + g_{pl}\Gamma_{km}^p + g_{mp}\Gamma_{kl}^p - g_{pl}\Gamma_{mk}^p - g_{kp}\Gamma_{ml}^p) \\ &= \frac{1}{2}g^{im}2\Gamma_{lk}^p g_{mp} \\ &= \Gamma_{lk}^p g^{im} g_{mp} \\ &= \Gamma_{lk}^p \delta_p^i \\ &= \Gamma_{kl}^i \end{aligned}$$

□

From this, we obtain the exact values of  $\Gamma_{ki}^j$ .

**Theorem 3.5.4: Exact values of  $\Gamma_{ki}^j$**

Let  $i, j, k \in \{0, 1, 2, 3\}$  **distinct**, and  $l$  arbitrary.

(i) We have:

$$\Gamma_{li}^i = \Gamma_{il}^i = u_{i,l}.$$

(ii) We have:

$$\Gamma_{ii}^j = -\eta_i \eta_j u_{i,j} e^{2u_i - 2u_j}.$$

(iii) We have:

$$\Gamma_{ki}^j = 0.$$

*Proof.* (i) Since  $g$  is symmetric, we have:

$$\begin{aligned} \Gamma_{li}^i &= \frac{1}{2}g^{im}(g_{ml,i} + g_{mi,j} - g_{li,m}) \\ &= \frac{1}{2}g^{im}(g_{mi,l} + g_{ml,i} - g_{il,m}) \\ &= \Gamma_{il}^i \end{aligned}$$

Furthermore, we have:

$$\begin{aligned} \Gamma_{li}^i &= \frac{1}{2}g^{im}(g_{ml,i} + g_{mi,l} - g_{li,m}) \\ &= \frac{1}{2}g^{ii}(g_{il,i} + g_{ii,l} - g_{li,i}) \\ &= \frac{1}{2}g^{ii}g_{ii,l} \\ &= u_{i,l} \end{aligned}$$

(ii) We have:

$$\begin{aligned} \Gamma_{ii}^j &= \frac{1}{2}g^{jm}(g_{mi,i} + g_{mi,i} - g_{ii,m}) \\ &= \frac{1}{2}g^{jj}(g_{ji,i} + g_{ji,i} - g_{ii,j}) \\ &= -\frac{1}{2}g^{jj}g_{ii,j} \\ &= -\eta_i \eta_j u_{i,j} e^{2u_i - 2u_j} \end{aligned}$$

(iii) Since  $g$  is diagonal, we directly have:

$$\begin{aligned}\Gamma^j_{ki} &= \frac{1}{2}g^{jm}(g_{mk,i} + g_{mi,k} - g_{ki,m}) \\ &= \frac{1}{2}g^{jj}(g_{jk,i} + g_{ji,k} - 0) \\ &= \frac{1}{2}g^{jj}(0 + 0) \\ &= 0\end{aligned}$$

□

We denote:

$$\Gamma^i := \begin{pmatrix} \Gamma^i_{00} & \Gamma^i_{01} & \Gamma^i_{02} & \Gamma^i_{03} \\ \Gamma^i_{10} & \Gamma^i_{11} & \Gamma^i_{12} & \Gamma^i_{13} \\ \Gamma^i_{20} & \Gamma^i_{21} & \Gamma^i_{22} & \Gamma^i_{23} \\ \Gamma^i_{30} & \Gamma^i_{31} & \Gamma^i_{32} & \Gamma^i_{33} \end{pmatrix}.$$

• Case  $i := 0$ . We have:

$$\Gamma^0 = \begin{pmatrix} u_{0,0} & u_{0,1} & u_{0,2} & u_{0,3} \\ u_{0,1} & u_{1,0}e^{-2u_0+2u_1} & 0 & 0 \\ u_{0,2} & 0 & u_{2,0}e^{-2u_0+2u_2} & 0 \\ u_{0,3} & 0 & 0 & u_{3,0}e^{-2u_0+2u_3} \end{pmatrix}.$$

• Case  $i := 1$ . We have:

$$\Gamma^1 = \begin{pmatrix} u_{0,1}e^{-2u_1+2u_0} & u_{1,0} & 0 & 0 \\ u_{1,0} & u_{1,1} & u_{1,2} & u_{1,3} \\ 0 & u_{1,2} & -u_{2,1}e^{-2u_1+2u_2} & 0 \\ 0 & u_{1,3} & 0 & -u_{3,1}e^{-2u_1+2u_3} \end{pmatrix}.$$

• Case  $i := 2$ . We have:

$$\Gamma^2 = \begin{pmatrix} u_{0,2}e^{-2u_2+2u_0} & 0 & u_{2,0} & 0 \\ 0 & -u_{1,2}e^{-2u_2+2u_1} & u_{2,1} & 0 \\ u_{2,0} & u_{2,1} & u_{2,2} & u_{2,3} \\ 0 & 0 & u_{2,3} & -u_{3,2}e^{-2u_2+2u_3} \end{pmatrix}.$$

• Case  $i := 3$ . We have:

$$\Gamma^3 = \begin{pmatrix} u_{0,3}e^{-2u_3+2u_0} & 0 & 0 & u_{3,0} \\ 0 & -u_{1,3}e^{-2u_3+2u_1} & 0 & u_{3,1} \\ 0 & 0 & -u_{2,3}e^{-2u_3+2u_2} & u_{3,2} \\ u_{3,0} & u_{3,1} & u_{3,2} & u_{3,3} \end{pmatrix}.$$

**Example 3.5.5: Sequence 2 of the metric  $h$**

We resume the example with the metric  $h$  (see 2.4.1 and 2.5.1) with:

$$\begin{aligned}u_0(t, r) &:= u(t, r) & u_1(t, r) &:= v(t, r) \\ u_2(t, r) &:= b(t) + \ln(r) & u_3(t, r, \vartheta) &:= b(t) + \ln(r) + \ln \sin(\vartheta)\end{aligned}$$

We have:

$$\begin{aligned}\Gamma^0 &= \begin{pmatrix} \dot{u} & u' & 0 & 0 \\ u' & \dot{v}e^{-2u+2v} & 0 & 0 \\ 0 & 0 & r^2\dot{b}e^{2b-2u} & 0 \\ 0 & 0 & 0 & r^2\dot{b}\sin^2\vartheta e^{2b-2u} \end{pmatrix} \\ \Gamma^1 &= \begin{pmatrix} u'e^{2u-2v} & \dot{v} & 0 & 0 \\ \dot{v} & v' & 0 & 0 \\ 0 & 0 & -re^{2b-2v} & 0 \\ 0 & 0 & 0 & -r\sin^2\vartheta e^{2b-2v} \end{pmatrix} \\ \Gamma^2 &= \begin{pmatrix} 0 & 0 & \dot{b} & 0 \\ 0 & 0 & r^{-1} & 0 \\ \dot{b} & r^{-1} & 0 & 0 \\ 0 & 0 & 0 & -\cos\vartheta\sin\vartheta \end{pmatrix} \\ \Gamma^3 &= \begin{pmatrix} 0 & 0 & 0 & \dot{b} \\ 0 & 0 & 0 & r^{-1} \\ 0 & 0 & 0 & \cot\vartheta \\ \dot{b} & r^{-1} & \cot\vartheta & 0 \end{pmatrix}\end{aligned}$$

Thus, we have:

$$\omega = \begin{pmatrix} \dot{u}dx^0 + u'dx^1 & u'dx^0 + \dot{v}e^{-2u+2v}dx^1 & r^2\dot{b}e^{2b-2u}dx^2 & r^2\dot{b}\sin^2\vartheta e^{-2b-2u}dx^3 \\ u'e^{2u-2v}dx^0 + \dot{v}dx^1 & \dot{v}dx^0 + v'dx^1 & -re^{2b-2v}dx^2 & -r\sin^2\vartheta e^{2b-2v}dx^3 \\ \dot{b}dx^2 & r^{-1}dx^2 & \dot{b}dx^0 + r^{-1}dx^1 & -\cos\vartheta\sin\vartheta dx^3 \\ \dot{b}dx^3 & r^{-1}dx^3 & \cot\vartheta dx^3 & \dot{b}dx^0 + r^{-1}dx^1 + \cot\vartheta dx^2 \end{pmatrix}.$$

We conclude this subsection with another formula for computing the sum  $\sum_{i=0}^3 \Gamma_{ij}^i$ .

**Proposition 3.5.6:** Another formula for  $\Gamma_{ij}^i$

For  $j \in \{0, 1, 2, 3\}$ , we have:

$$\sum_{i=0}^3 \Gamma_{ij}^i = \partial_j \left( \ln \sqrt{|\det g|} \right).$$

*Proof.* We prove the result in two steps:

(1) For any  $A \in \text{GL}_n(\mathbb{R})$ , we have:

$$d_A \det : H \mapsto \det(A) \text{tr}(A^{-1}H).$$

(2) We have:

$$\sum_{i=0}^3 \Gamma_{ij}^i = \partial_j \left( \ln \sqrt{|\det g|} \right).$$

(1) We start with a result from linear algebra. We equip  $\mathcal{M}_n(\mathbb{R})$  with the **Frobenius norm**:

$$\forall M \in \mathcal{M}_n(\mathbb{R}), \|M\| := \sqrt{\text{tr}({}^t M M)}.$$

This norm is a sub-multiplicative norm on  $\mathcal{M}_n(\mathbb{R})$ , *i.e.*, we have:

$$\forall A, B \in \mathcal{M}_n(\mathbb{R}), \|AB\| \leq \|A\| \cdot \|B\|.$$

Let us show that for any  $A \in \text{GL}_n(\mathbb{R})$ , we have:

$$d_A \det : H \mapsto \det(A) \text{tr}(A^{-1}H).$$

To do this, let us first calculate the differential of  $\det$  at  $I_n$ . Since  $\det$  is of class  $\mathcal{C}^\infty$ , it suffices to calculate the derivative of  $\det$  at  $I_n$  in an arbitrary direction  $H \in \mathcal{M}_n(\mathbb{R})$ . Let  $H$  be a real matrix  $H \in \mathcal{M}_n(\mathbb{R})$  with eigenvalues  $\lambda_1, \dots, \lambda_n$ . For any real number  $t$  in the vicinity of 0, we have:

$$\det(I_n + tH) = \prod_{i=1}^n (1 + t\lambda_i) = 1 + t \sum_{i=1}^n \lambda_i + O(t^2) = 1 + t \operatorname{tr}(H) + O(t^2).$$

Since the map  $H \mapsto \operatorname{tr}(H)$  is linear, we have:

$$d_{I_n} \det : H \mapsto \operatorname{tr}(H).$$

Let  $A \in \operatorname{GL}_n(\mathbb{R})$ . We deduce that for any matrix  $H \in \mathcal{M}_n(\mathbb{R})$ :

$$\begin{aligned} \det(A + H) &= \det(A(I_n + A^{-1}H)) \\ &= \det(A) \det(I_n + A^{-1}H) \\ &= \det(A) (\det(I_n) + d_{I_n}(A^{-1}H) + o(\|A^{-1}H\|)) \\ &= \det(A) + \det(A) \operatorname{tr}(A^{-1}H) + o(\|A^{-1}H\|) \det(A) \end{aligned}$$

Now,  $\|\cdot\|$  is a sub-matrix norm, so we have  $\|A^{-1}H\| \leq \|A^{-1}\| \|H\|$ . Thus:

$$o(\|A^{-1}H\|) \det(A) = o(\|H\|).$$

Since the map  $H \mapsto \det(A) \operatorname{tr}(A^{-1}H)$  is linear, we have:

$$d_A \det : H \mapsto \det(A) \operatorname{tr}(A^{-1}H).$$

(2) Since:

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (g_{lj,i} + g_{li,j} - g_{ij,l})$$

we have:

$$\begin{aligned} \sum_{i=0}^3 \Gamma_{ij}^i &= \sum_{i,l=0}^3 \frac{1}{2} (g^{il} g_{lj,i} + g^{il} g_{il,j} - g^{il} g_{ij,l}) \\ &= \frac{1}{2} \left[ \sum_{i,l=0}^3 g^{il} g_{lj,i} + \sum_{i,l=0}^3 g^{il} g_{il,j} - \sum_{i,l=0}^3 g^{il} g_{ij,l} \right] \\ &= \frac{1}{2} \left[ \sum_{i,l=0}^3 g^{il} g_{lj,i} + \sum_{i,l=0}^3 g^{il} g_{il,j} - \sum_{i,l=0}^3 g^{il} g_{lj,i} \right] \\ &= \frac{1}{2} \sum_{i,l=0}^3 g^{il} g_{il,j} \end{aligned}$$

Using (1), we have:

$$\begin{aligned} \partial_j (\det g) dx^j &= d(\det g) \\ &= (\det g) g^{il} dg_{il} \\ &= (\det g) g^{il} g_{il,j} dx^j \end{aligned}$$

Thus:

$$g^{il} g_{il,j} = \frac{1}{\det g} \partial_j (\det g)$$

Therefore, we have:

$$\begin{aligned} \sum_{i=0}^3 \Gamma_{ij}^i &= \frac{1}{2 \det g} \partial_j (\det g) \\ &= \frac{1}{\sqrt{|\det g|}} \partial_j \sqrt{|\det g|} \\ &= \partial_j (\ln \sqrt{|\det g|}). \end{aligned}$$

□

**Corollary 3.5.7: Alternative formulas for  $\Gamma_{ij}^i$**

For  $j \in \{0, 1, 2, 3\}$ , we have:

$$\sum_{i=0}^3 \Gamma_{ij}^i = \sum_{i=0}^3 u_{i,j}.$$

*Proof.* Since  $g$  is diagonal, we have:

$$\begin{aligned} \det g &= \prod_{i=0}^3 g_{ii} \\ &= -e^{2u_0+2u_1+2u_2+2u_3} \end{aligned}$$

Thus, we have:

$$\ln \sqrt{|\det g|} = u_0 + u_1 + u_2 + u_3.$$

And therefore we have:

$$\sum_{i=0}^3 \Gamma_{ij}^i = \sum_{i=0}^3 u_{i,j}.$$

□

### 3.6 Practical calculations of the connection in the $\mathcal{C}_\perp$ and $\mathcal{C}_\perp^*$ bases

In this subsection, we calculate the tetradic coefficients  $\mathbb{F}_{ki}^j$  using the tetrad method. We start with some simple properties of the symbols  $\mathbb{F}_{ki}^j$  and the 1-forms  $\varpi^i_j$ .

**Lemma 3.6.1: Symmetries of the symbols  $\mathbb{F}_{ki}^j$  and  $\varpi^i_j$**

For  $i, j, k, l \in \{0, 1, 2, 3\}$ , we have:

(i) We have:

$$\varpi^l_k = -\eta_l \eta_k \varpi^k_l.$$

(ii) We have:

$$\mathbb{F}_{ki}^j = -\eta_j \eta_i \mathbb{F}_{kj}^i.$$

(iii) We have:

$$\varpi^j_j = 0.$$

(iv) We have:

$$\mathbb{F}_{kj}^j = 0.$$

*Proof.* (i) Since  $\eta_{kl}$  is constant, for any vector field  $X$ , we have:

$$\begin{aligned}
 0 &= \nabla_X \eta_{kl} \\
 &= \nabla_X g(\theta_k, \theta_l) \\
 &= g(\nabla_X \theta_k, \theta_l) + g(\theta_k, \nabla_X \theta_l) \\
 &= g(\varpi^i_k(X)\theta_i, \theta_l) + g(\theta_k, \varpi^i_l(X)\theta_i) \\
 &= \varpi^i_k(X)g(\theta_i, \theta_l) + \varpi^i_l(X)g(\theta_k, \theta_i) \\
 &= \varpi^i_k(X)\eta_{il} + \varpi^i_l(X)\eta_{ki} \\
 &= \varpi^l_k(X)\eta_l + \varpi^k_l(X)\eta_k \\
 &= (\eta_l \varpi^l_k + \eta_k \varpi^k_l)(X)
 \end{aligned}$$

Since this is true for any vector field  $X$ , we have:

$$\eta_l \varpi^l_k + \eta_k \varpi^k_l = 0$$

Thus, we have:

$$\varpi^l_k = -\eta_l \eta_k \varpi^k_l.$$

(ii) We have by (i):

$$\mathbb{F}^i_{kj} = \varpi^i_j(\theta_k) = -\eta_i \eta_j \varpi^j_i(\theta_k) = -\eta_i \eta_j \mathbb{F}^j_{ki}.$$

(iii) We have by (i):

$$\varpi^j_j = -\eta_j \eta_j \varpi^j_j = -\varpi^j_j.$$

Therefore, we have  $\varpi^j_j = 0$ .

(iv) We have by (iii):

$$\mathbb{F}^j_{kj} = \varpi^j_j(\theta_k) = 0.$$

□

The following fundamental result establishes the connection between the 1-forms  $\varpi^i_j$  and the functions  $u_i$  appearing in the metric. This will allow us to compute the symbols  $\mathbb{F}^j_{ki}$  in terms of these functions. To do this, we will calculate the 2-forms in two different ways:

$$d\theta^i \in \Lambda^2(U)$$

We denote:

$$\theta^{j,i} := \theta^j \wedge \theta^i$$

Thus, we have:

$$\theta^{j,i} = -\theta^{i,j}$$

and:

$$\theta^{i,j}(\theta_k, \theta_l) = \theta^i \wedge \theta^j(\theta_k, \theta_l) = \theta^i \otimes \theta^j(\theta_k, \theta_l) - \theta^j \otimes \theta^i(\theta_k, \theta_l) = \delta_k^i \delta_l^j - \delta_k^j \delta_l^i \tag{3.6.1}$$

**Proposition 3.6.2: Calculation of  $d\theta^i$**

For  $i \in \{0, 1, 2, 3\}$ , we have:

(i) We have:

$$d\theta^i = u_{i,j} e^{-u_j} \theta^{j,i}.$$

(ii) We have:

$$d\theta^i = -\varpi^i_j \wedge \theta^j.$$

*Proof.* (i) We have:

$$\begin{aligned}
 d\theta^i &= d(f_i dx^i) \\
 &= \partial_j (f_i) dx^j \wedge dx^i \\
 &= \partial_j (f_i) f^j \theta^j \wedge f^i \theta^i \\
 &= \partial_j (e^{u_i}) f^i f^j \theta^j \wedge \theta^i \\
 &= u_{i,j} e^{u_i} e^{-u_j - u_i} \theta^j \wedge \theta^i \\
 &= u_{i,j} e^{-u_j} \theta^j \wedge \theta^i \\
 &= u_{i,j} e^{-u_j} \theta^{j,i}
 \end{aligned}$$

(ii) We have  $f_k f^k = 1$ , which implies:

$$f^k df_k + f^k df_k = 0$$

Thus, we have  $f^k df_k = -f_k df^k$ . Therefore, we have:

$$\begin{aligned}
 d\theta^i &= d(f_i dx^i) = df_i \wedge dx^i \\
 &= df_i \wedge f^i \theta^i \\
 &= f^i df_i \wedge \theta^i \\
 &= -f_i df^i \wedge \theta^i \\
 &= -f_i \partial_j (f^i) dx^j \wedge \theta^i \\
 &= -f_i \partial_j (f^i) f^j \theta^j \wedge \theta^i
 \end{aligned}$$

Using equation (3.6.1), we have:

$$\begin{aligned}
 d\theta^i (\theta_k, \theta_l) \theta_i &= f_i f^j \partial_j (f^i) \theta^j \wedge \theta^i (\theta_k, \theta_l) \theta_i \\
 &= f_i f^j \partial_j (f^i) (\delta_k^j \delta_l^i + \delta_l^j \delta_k^i) \theta_i \\
 &= f_l f^k \partial_k (f^l) \theta_l - f_k f^l \partial_l (f^k) \theta_k
 \end{aligned}$$

Using the identity:

$$\begin{aligned}
 \nabla_{\theta_k} \theta_l &= \nabla_{f^k \partial_k} (f^l \partial_l) \\
 &= f^k \nabla_{\partial_k} (f^l \partial_l) \\
 &= f^k d(f^l) (\partial_k) \partial_l + f^k f^l \nabla_{\partial_k} (\partial_l)
 \end{aligned}$$

and since  $[\partial_k, \partial_l] = 0$ , we have:

$$\begin{aligned}
 \varpi^i_j \wedge \theta^j (\theta_k, \theta_l) \theta_i &= \varpi^i_j \otimes \theta^j (\theta_k, \theta_l) \theta_i - \theta^j \otimes \varpi^i_j (\theta_k, \theta_l) \theta_i \\
 &= \theta^j (\theta_l) \varpi^i_j (\theta_k) \theta_i - \theta^j (\theta_k) \varpi^i_j (\theta_l) \theta_i \\
 &= \delta_l^j \nabla_{\theta_k} \theta_j - \delta_k^j \nabla_{\theta_l} \theta_j \\
 &= \nabla_{\theta_k} \theta_l - \nabla_{\theta_l} \theta_k \\
 &= f^k d(f^l) (\partial_k) \partial_l + f^k f^l \nabla_{\partial_k} (\partial_l) - f^l d(f^k) (\partial_l) \partial_k - f^k f^l \nabla_{\partial_l} (\partial_k) \\
 &= f^k \partial_k (f^l) \partial_l - f^l \partial_l (f^k) \partial_k + f^k f^l [\partial_k, \partial_l] \\
 &= f^k \partial_k (f^l) \partial_l - f^l \partial_l (f^k) \partial_k \\
 &= -d\theta^i (\theta_k, \theta_l) \theta_i
 \end{aligned}$$

Hence the result, as this holds for any vector fields  $\theta_k, \theta_l, \theta_i$ .

□

So we have:

$$\begin{aligned} d\theta^0 &= u_{0,j}e^{-u_j}\theta^{j,0} = u_{0,1}e^{-u_1}\theta^{1,0} + u_{0,2}e^{-u_2}\theta^{2,0} + u_{0,3}e^{-u_3}\theta^{3,0} \\ d\theta^1 &= u_{1,j}e^{-u_j}\theta^{j,1} = u_{1,0}e^{-u_0}\theta^{0,1} + u_{1,2}e^{-u_2}\theta^{2,1} + u_{1,3}e^{-u_3}\theta^{3,1} \\ d\theta^2 &= u_{2,j}e^{-u_j}\theta^{j,2} = u_{2,0}e^{-u_0}\theta^{0,2} + u_{2,1}e^{-u_1}\theta^{1,2} + u_{2,3}e^{-u_3}\theta^{3,2} \\ d\theta^3 &= u_{3,j}e^{-u_j}\theta^{j,3} = u_{3,0}e^{-u_0}\theta^{0,3} + u_{3,1}e^{-u_1}\theta^{1,3} + u_{3,2}e^{-u_2}\theta^{2,3} \end{aligned}$$

**Example 3.6.3: Sequence 3 of the metric  $h$**

We revisit the example with the metric  $h$  (see 2.4.1, 2.5.1, and 3.5.5) with:

$$\begin{aligned} u_0(t, r) &:= u(t, r) & u_1(t, r) &:= v(t, r) \\ u_2(t, r) &:= b(t) + \ln(r) & u_3(t, r, \vartheta) &:= b(t) + \ln(r) + \ln \sin(\vartheta) \end{aligned}$$

We have:

$$\begin{aligned} d\theta^t &= u_{0,1}e^{-u_1}\theta^{1,0} \\ &= u'e^v\theta^{r,t} \\ d\theta^r &= u_{1,0}e^{-u_0}\theta^{0,1} \\ &= \dot{v}e^{-u}\theta^{t,r} \\ d\theta^\vartheta &= u_{2,0}e^{-u_0}\theta^{0,2} + u_{2,1}e^{-u_1}\theta^{1,2} \\ &= \dot{b}e^{-u}\theta^{t,\vartheta} + r^{-1}e^{-v}\theta^{r,\vartheta} \\ d\theta^\phi &= u_{3,0}e^{-u_0}\theta^{0,3} + u_{3,1}e^{-u_1}\theta^{1,3} + u_{3,2}e^{-u_2}\theta^{2,3} \\ &= \dot{b}e^{-u}\theta^{t,\phi} + r^{-1}e^{-v}\theta^{r,\phi} + r^{-1}\cot\vartheta e^{-b}\theta^{\vartheta,\phi} \end{aligned}$$

We can then deduce the exact values of the symbols  $\mathbb{F}^j_{ki}$  and  $\varpi^i_j$  in terms of the elements of the functions  $u_i$  appearing in the metric.

**Theorem 3.6.4: Exact values of  $\mathbb{F}^j_{ki}$  and  $\varpi^i_j$**

Let  $i, j, k \in \{0, 1, 2, 3\}$  **distinct**.

(i) We have:

$$\begin{aligned} \mathbb{F}^i_{ij} &= u_{i,j}e^{-u_j} \\ \mathbb{F}^j_{ii} &= -\eta_i\eta_j u_{i,j}e^{-u_j} \\ \mathbb{F}^j_{ki} &= 0 \end{aligned}$$

(ii) We have:

$$\begin{aligned} \varpi^j_j &= 0 \\ \varpi^i_j &= u_{i,j}e^{-u_j}\theta^i - \eta_i\eta_j u_{j,i}e^{-u_j}\theta^j \end{aligned}$$

*Proof.* Let's define the following for the proof:

$$a^i_{kj} := \mathbb{F}^i_{kj} - \mathbb{F}^i_{jk} \quad , \quad b_{ji} := u_{i,j}e^{-u_j}.$$

Using the notations and the previous proposition 3.6.2, we have:

$$\begin{aligned}
 b_{ji}\theta^{ji} &= u_{i,j}e^{-u_j}\theta^{j,i} \\
 &= d\theta^i \\
 &= -\varpi^i_j \wedge \theta^j \\
 &= -\mathbb{F}^i_{kj}\theta^k \wedge \theta^j \\
 &= \sum_{0 \leq j < k \leq 3} (\mathbb{F}^i_{kj} - \mathbb{F}^i_{jk})\theta^j \wedge \theta^k \\
 &= \sum_{0 \leq j < k \leq 3} a^i_{kj}\theta^{j,k}.
 \end{aligned}$$

Thus, we have:

(I) for all distinct  $i, j, k$ :

$$0 = a^i_{kj} = \mathbb{F}^i_{kj} - \mathbb{F}^i_{jk} = 0 \Rightarrow \mathbb{F}^i_{kj} = \mathbb{F}^i_{jk}.$$

(II) for all distinct  $i, j$ :

$$b_{ji} = -a^i_{ji} = a^i_{ij}.$$

Let's prove the theorem.

(i) We have the symmetries:

$$\mathbb{F}^j_{ki} = \mathbb{F}^j_{ik}, \mathbb{F}^i_{kj} = \mathbb{F}^i_{jk}, \mathbb{F}^k_{ij} = \mathbb{F}^k_{ji}$$

We will show that  $\mathbb{F}^j_{ki} = 0$  by demonstrating that  $\mathbb{F}^j_{ki} = -\mathbb{F}^j_{ki}$ . Using (I) and point (ii) of lemma 3.6.1, we have:

$$\begin{aligned}
 \mathbb{F}^j_{ki} &= -\eta_i\eta_j\mathbb{F}^i_{kj} \\
 &= -\eta_i\eta_j\mathbb{F}^i_{jk} \\
 &= \eta_i\eta_j\eta_k\eta_i\mathbb{F}^k_{ji} \\
 &= \eta_j\eta_k\mathbb{F}^k_{ij} \\
 &= -\mathbb{F}^j_{ik} = -\mathbb{F}^j_{ki}
 \end{aligned}$$

Hence, the result follows from symmetry.

(ii) Since  $\mathbb{F}^i_{ji} = 0$ , we have:

$$u_{i,j}e^{-u_j} = b_{ji} = -a^i_{ji} = -\mathbb{F}^i_{ji} + \mathbb{F}^i_{ij} = \mathbb{F}^i_{ij}$$

Therefore, we have:

$$\mathbb{F}^i_{ij} = u_{i,j}e^{-u_j}.$$

(iii) We have by (ii):

$$\begin{aligned}
 \mathbb{F}^j_{ii} &= -\eta_j\eta_i\mathbb{F}^i_{ij} \\
 &= -\eta_i\eta_ju_{i,j}e^{-u_j}.
 \end{aligned}$$

(iv) We have:

$$\begin{aligned}
 \varpi^i_j &= \mathbb{F}^i_{kj}\theta^k \\
 &= \mathbb{F}^i_{ij}\theta^i + \mathbb{F}^i_{jj}\theta^j \\
 &= u_{i,j}e^{-u_j}\theta^i - \eta_i\eta_ju_{j,i}e^{-u_i}\theta^j
 \end{aligned}$$

□

We denote:

$$\mathbb{F}^i = \begin{pmatrix} \mathbb{F}^i_{00} & \mathbb{F}^i_{01} & \mathbb{F}^i_{02} & \mathbb{F}^i_{03} \\ \mathbb{F}^i_{10} & \mathbb{F}^i_{11} & \mathbb{F}^i_{12} & \mathbb{F}^i_{13} \\ \mathbb{F}^i_{20} & \mathbb{F}^i_{21} & \mathbb{F}^i_{22} & \mathbb{F}^i_{23} \\ \mathbb{F}^i_{30} & \mathbb{F}^i_{31} & \mathbb{F}^i_{32} & \mathbb{F}^i_{33} \end{pmatrix}$$

- Case  $i := 0$ . We have:

$$\mathbb{F}^0 = \begin{pmatrix} 0 & u_{0,1}e^{-u_1} & u_{0,2}e^{-u_2} & u_{0,3}e^{-u_3} \\ 0 & u_{1,0}e^{-u_0} & 0 & 0 \\ 0 & 0 & u_{2,0}e^{-u_0} & 0 \\ 0 & 0 & 0 & u_{3,0}e^{-u_0} \end{pmatrix}.$$

- Case  $i := 1$ . We have:

$$\mathbb{F}^1 = \begin{pmatrix} u_{0,1}e^{-u_1} & 0 & 0 & 0 \\ u_{1,0}e^{-u_0} & 0 & u_{1,2}e^{-u_2} & u_{1,3}e^{-u_3} \\ 0 & 0 & -u_{2,1}e^{-u_1} & 0 \\ 0 & 0 & 0 & -u_{3,1}e^{-u_1} \end{pmatrix}.$$

- Case  $i := 2$ . We have:

$$\mathbb{F}^2 = \begin{pmatrix} u_{0,2}e^{-u_2} & 0 & 0 & 0 \\ 0 & -u_{1,2}e^{-u_2} & 0 & 0 \\ u_{2,0}e^{-u_0} & u_{2,1}e^{-u_1} & 0 & u_{2,3}e^{-u_3} \\ 0 & 0 & 0 & -u_{3,2}e^{-u_2} \end{pmatrix}.$$

- Case  $i := 3$ . We have:

$$\mathbb{F}^3 = \begin{pmatrix} u_{0,3}e^{-u_3} & 0 & 0 & 0 \\ 0 & -u_{1,3}e^{-u_3} & 0 & 0 \\ 0 & 0 & -u_{2,3}e^{-u_3} & 0 \\ u_{3,0}e^{-u_0} & u_{3,1}e^{-u_1} & u_{3,2}e^{-u_2} & 0 \end{pmatrix}.$$

By Lemma 3.4.3, we obtain the values  $\Gamma^i_{jk}$  found in the previous subsection.

**Example 3.6.5: Sequence 4 of the metric  $h$**

We revisit the example with the metric  $h$  (see 2.4.1, 2.5.1, 3.5.5, and 3.6.3) with:

$$\begin{aligned} u_0(t, r) &:= u(t, r) & u_1(t, r) &:= v(t, r) \\ u_2(t, r) &:= b(t) + \ln(r) & u_3(t, r, \vartheta) &:= b(t) + \ln(r) + \ln \sin(\vartheta) \end{aligned}$$

Therefore, we have:

$$\begin{aligned} \mathbb{F}^0 &= \begin{pmatrix} 0 & u'e^{-v} & 0 & 0 \\ 0 & \dot{v}e^{-u} & 0 & 0 \\ 0 & 0 & \dot{b}e^{-u} & 0 \\ 0 & 0 & 0 & \dot{b}e^{-u} \end{pmatrix} \\ \mathbb{F}^1 &= \begin{pmatrix} u'e^{-v} & 0 & 0 & 0 \\ \dot{v}e^{-u} & 0 & 0 & 0 \\ 0 & 0 & -r^{-1}e^{-v} & 0 \\ 0 & 0 & 0 & -r^{-1}e^{-v} \end{pmatrix} \\ \mathbb{F}^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dot{b}e^{-u} & r^{-1}e^{-v} & 0 & 0 \\ 0 & 0 & 0 & -r^{-1} \cot \vartheta e^{-b} \end{pmatrix} \end{aligned}$$

$$\mathbb{F}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dot{b}e^{-u} & r^{-1}e^{-v} & r^{-1} \cot \vartheta e^{-b} & 0 \end{pmatrix}$$

Thus, we have:

$$\varpi = \begin{pmatrix} 0 & u'e^{-v}\theta^0 + \dot{v}e^{-u}\theta^1 & \dot{b}e^{-u}\theta^2 & \dot{b}e^{-u}\theta^3 \\ u'e^{-v}\theta^0 + \dot{v}e^{-u}\theta^1 & 0 & -r^{-1}e^{-v}\theta^2 & -r^{-1}e^{-v}\theta^3 \\ \dot{b}e^{-u}\theta^2 & r^{-1}e^{-v}\theta^2 & 0 & -r^{-1} \cot \vartheta e^{-b}\theta^3 \\ \dot{b}e^{-u}\theta^3 & r^{-1}e^{-v}\theta^3 & r^{-1} \cot \vartheta e^{-b}\theta^3 & 0 \end{pmatrix}.$$

## 4 Curvature Tensor and Riemann Tensor

### 4.1 Generalities

#### Definition 4.1.1: Curvature mappings

- (i) Let  $X$  and  $Y$  be two vector fields. The **curvature mapping associated with  $X$  and  $Y$**  is the mapping  $\mathcal{R}(X, Y)$  defined by:

$$\mathcal{R}(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$$

That is, for any vector field  $Z$ , we associate the vector field:

$$\mathcal{R}(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

- (ii) The **Riemann mapping** is the mapping that associates a scalar field to all vector fields  $X, Y, Z$ , and  $W$ :

$$\mathcal{R}m(W, Z, X, Y) := g(W, \mathcal{R}(X, Y)Z).$$

The mapping  $\mathcal{R}$  is a tensor field of type  $(1, 3)$  and the mapping  $\mathcal{R}m$  is a tensor field of type  $(0, 4)$ .

#### Proposition 4.1.2: Usual properties

Let  $X, Y, Z, W$ , and  $T$  be vector fields.

- (i) (**First Bianchi identity**) We have:

$$\begin{aligned} \mathcal{R}(X, Y)Z + \mathcal{R}(Y, Z)X + \mathcal{R}(Z, X)Y &= 0 \\ \mathcal{R}m(W, Z, X, Y) + \mathcal{R}m(W, X, Y, Z) + \mathcal{R}m(W, Y, Z, X) &= 0 \end{aligned}$$

- (ii) We have:

$$\begin{aligned} \mathcal{R}(X, Y)Z &= -\mathcal{R}(Y, X)Z \\ \mathcal{R}m(W, Z, X, Y) &= -\mathcal{R}m(W, Z, Y, X) \end{aligned}$$

- (iii) We have:

$$\mathcal{R}m(W, Z, X, Y) = -\mathcal{R}m(Z, W, X, Y).$$

- (iv) We have:

$$\mathcal{R}m(W, Z, X, Y) = \mathcal{R}m(X, Y, W, Z).$$

(v) **(Second Bianchi identity)** We have:

$$\begin{aligned} & (\nabla_X \mathcal{R})(Y, Z) + (\nabla_Y \mathcal{R})(Z, X) + (\nabla_Z \mathcal{R})(X, Y) = 0 \\ & \nabla_X \mathcal{R}m(T, W, Z, Y) + \nabla_Y \mathcal{R}m(T, W, X, Z) + \nabla_Z \mathcal{R}m(T, W, Y, X) = 0 \\ & \nabla \mathcal{R}m(T, W, Z, Y, X) + \nabla \mathcal{R}m(T, W, X, Z, Y) + \nabla \mathcal{R}m(T, W, Y, X, Z) = 0 \end{aligned}$$

*Proof.* (i) By the Jacobi identity (see Proposition 3.1.6), we have:

$$\begin{aligned} & \mathcal{R}(X, Y)Z + \mathcal{R}(Y, Z)X + \mathcal{R}(Z, X)Y \\ &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \\ & \quad + \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y, Z]} X \\ & \quad + \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z, X]} Y \\ &= \nabla_X (\nabla_Y Z - \nabla_Z Y) + \nabla_Y (\nabla_Z X - \nabla_X Z) + \nabla_Z (\nabla_X Y - \nabla_Y X) \\ & \quad - (\nabla_{[X, Y]} Z + \nabla_{[Y, Z]} X + \nabla_{[Z, X]} Y) \\ &= (\nabla_X [Y, Z] - \nabla_{[Y, Z]} X) + (\nabla_Y [Z, X] - \nabla_{[Z, X]} Y) + (\nabla_Z [X, Y] - \nabla_{[X, Y]} Z) \\ &= [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] \\ &= 0 \end{aligned}$$

Thus, by linearity of  $g$ , we have:

$$\begin{aligned} & \mathcal{R}m(W, Z, X, Y) + \mathcal{R}m(W, X, Y, Z) + \mathcal{R}m(W, Y, Z, X) \\ &= g(W, \mathcal{R}(X, Y)Z) + g(W, \mathcal{R}(Y, Z)X) + g(W, \mathcal{R}(Z, X)Y) \\ &= g(W, \mathcal{R}(X, Y)Z + \mathcal{R}(Y, Z)X + \mathcal{R}(Z, X)Y) \\ &= g(W, 0) \\ &= 0 \end{aligned}$$

(ii) As:

$$\begin{aligned} -\nabla_{[X, Y]} Z &= \nabla_{-[X, Y]} Z \\ &= \nabla_{[Y, X]} Z \end{aligned}$$

we have:

$$\begin{aligned} \mathcal{R}(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \\ &= -(\nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y, X]} Z) \\ &= -\mathcal{R}(Y, X)Z \end{aligned}$$

And thus we have:

$$\begin{aligned} \mathcal{R}m(W, Z, X, Y) &= g(W, \mathcal{R}(X, Y)Z) \\ &= g(W, -\mathcal{R}(Y, X)Z) \\ &= -g(W, \mathcal{R}(Y, X)Z) \\ &= -\mathcal{R}m(W, Z, Y, X) \end{aligned}$$

(iii) We have:

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

*i.e.*, we have:

$$g(\nabla_X Y, Z) = X(g(Y, Z)) - g(Y, \nabla_X Z).$$

Thus, we have:

$$\begin{aligned} \mathcal{R}m(W, Z, X, Y) &= g(W, \mathcal{R}(X, Y)Z) \\ &= g(W, \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z) \\ &= g(W, \nabla_X \nabla_Y Z) - g(W, \nabla_Y \nabla_X Z) - g(W, \nabla_{[X, Y]}Z) \\ &= [X(g(W, \nabla_Y Z)) - g(\nabla_X W, \nabla_Y Z)] \\ &\quad - [Y(g(W, \nabla_X Z)) - g(\nabla_Y W, \nabla_X Z)] \\ &\quad - [X, Y](g(W, Z)) - g(\nabla_{[X, Y]}W, Z) \\ &= [X(Y(g(W, Z))) - X(g(\nabla_Y W, Z)) - g(\nabla_X W, \nabla_Y Z)] \\ &\quad - [Y(X(g(W, Z))) - Y(g(\nabla_X W, Z)) - g(\nabla_Y W, \nabla_X Z)] \\ &\quad - [X, Y](g(W, Z)) - g(\nabla_{[X, Y]}W, Z) \\ &= (XY - YX)(g(W, Z)) - X(g(\nabla_Y W, Z)) - g(\nabla_X W, \nabla_Y Z) \\ &\quad + Y(g(\nabla_X W, Z)) + g(\nabla_Y W, \nabla_X Z) - [X, Y](g(W, Z)) + g(\nabla_{[X, Y]}W, Z) \\ &= [X, Y](g(W, Z)) - g(\nabla_X \nabla_Y W, Z) - g(\nabla_X W, \nabla_Y Z) \\ &\quad + g(\nabla_Y \nabla_X W, Z) + g(\nabla_Y W, \nabla_X Z) - [X, Y](g(W, Z)) + g(\nabla_{[X, Y]}W, Z) \\ &= -g(\nabla_X \nabla_Y W, Z) - g(\nabla_X W, \nabla_Y Z) + g(\nabla_Y \nabla_X W, Z) \\ &\quad + g(\nabla_Y W, \nabla_X Z) + g(\nabla_{[X, Y]}W, Z) \\ &= -[g(\nabla_X \nabla_Y W, Z) - g(\nabla_Y \nabla_X W, Z) - g(\nabla_{[X, Y]}W, Z)] \\ &= -g(\nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \nabla_{[X, Y]}W, Z) \\ &= -g(Z, \mathcal{R}(X, Y)W) \\ &= -\mathcal{R}m(Z, W, X, Y) \end{aligned}$$

(iv) We have by (i, ii, iii):

$$\begin{aligned} &2\mathcal{R}m(Z, W, X, Y) - 2\mathcal{R}m(X, Y, Z, W) \\ &= \mathcal{R}m(Z, W, X, Y) + \mathcal{R}m(W, Z, Y, X) + \mathcal{R}m(X, Y, W, Z) + \mathcal{R}m(Y, X, Z, W) \\ &= \mathcal{R}m(Z, W, X, Y) + \mathcal{R}m(Z, X, Y, W) + \mathcal{R}m(Z, Y, W, X) \\ &\quad + \mathcal{R}m(W, Z, Y, X) + \mathcal{R}m(W, Y, X, Z) + \mathcal{R}m(W, X, Z, Y) \\ &\quad + \mathcal{R}m(X, Y, W, Z) + \mathcal{R}m(X, W, Z, Y) + \mathcal{R}m(X, Z, Y, W) \\ &\quad + \mathcal{R}m(Y, X, Z, W) + \mathcal{R}m(Y, Z, W, X) + \mathcal{R}m(Y, W, X, Z) \\ &\quad - (\mathcal{R}m(Z, X, Y, W) + \mathcal{R}m(Z, Y, W, X)) - (\mathcal{R}m(W, Y, X, Z) + \mathcal{R}m(W, X, Z, Y)) \\ &\quad - (\mathcal{R}m(X, W, Z, Y) + \mathcal{R}m(X, Z, Y, W)) \\ &\quad - (\mathcal{R}m(Y, Z, W, X) + \mathcal{R}m(Y, W, X, Z)) \\ &= 0 + 0 + 0 + 0 \\ &\quad - (\mathcal{R}m(Z, X, Y, W) + \mathcal{R}m(X, Z, Y, W)) - (\mathcal{R}m(W, Y, X, Z) + \mathcal{R}m(Y, W, X, Z)) \\ &\quad - (\mathcal{R}m(X, W, Z, Y) + \mathcal{R}m(W, X, Z, Y)) \\ &\quad - (\mathcal{R}m(Y, Z, W, X) + \mathcal{R}m(Z, Y, W, X)) \\ &= 0 - 0 - 0 - 0 \\ &= 0 \end{aligned}$$

*i.e.* we have:

$$\mathcal{R}m(Z, W, X, Y) = \mathcal{R}m(X, Y, Z, W).$$

(v) We have by the action of  $\nabla$  on tensor fields (see Proposition 3.2.4):

$$\begin{aligned}
& \nabla_X \mathcal{R}(Y, Z)W + \nabla_Y \mathcal{R}(Z, X)W + \nabla_Z \mathcal{R}(X, Y)W \\
&= \nabla_X (\mathcal{R}(Y, Z)W) + \nabla_Y (\mathcal{R}(Z, X)W) + \nabla_Z (\mathcal{R}(X, Y)W) \\
&\quad - \mathcal{R}(\nabla_X Y, Z)W - \mathcal{R}(Y, \nabla_X Z)W - \mathcal{R}(\nabla_Y Z, X)W - \mathcal{R}(Z, \nabla_Y X)W - \mathcal{R}(\nabla_Z X, Y)W - \mathcal{R}(X, \nabla_Z Y)W \\
&\quad - \mathcal{R}(Y, Z)\nabla_X W - \mathcal{R}(Z, X)\nabla_Y W - \mathcal{R}(X, Y)\nabla_Z W \\
&=: \text{(I)} \\
&\quad - \text{(II)} \\
&\quad - \text{(III)}
\end{aligned}$$

Let's calculate elements (II) and (I) separately. We have:

$$\begin{aligned}
\text{(II)} &= \mathcal{R}(\nabla_X Y, Z) + \mathcal{R}(Y, \nabla_X Z) + \mathcal{R}(\nabla_Y Z, X) + \mathcal{R}(Z, \nabla_Y X) \\
&\quad + \mathcal{R}(\nabla_Z X, Y) + \mathcal{R}(X, \nabla_Z Y) \\
&= \mathcal{R}(\nabla_X Y - \nabla_Y X, Z) + \mathcal{R}(\nabla_Y Z - \nabla_Z Y, X) \\
&\quad + \mathcal{R}(\nabla_Z X - \nabla_X Z, Y) \\
&= \mathcal{R}([X, Y], Z) + \mathcal{R}([Y, Z], X) + \mathcal{R}([Z, X], Y) \\
&=: \text{(IV)}
\end{aligned}$$

By the Jacobi identity, we have:

$$\begin{aligned}
\nabla_{[X, [Y, Z]]}W + \nabla_{[Y, [Z, X]]}W + \nabla_{[Z, [X, Y]]}W &= \nabla_{[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]}W \\
&= \nabla_0 W \\
&= 0
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
\text{(I)} &= \nabla_X (\mathcal{R}(Y, Z)W) + \nabla_Y (\mathcal{R}(Z, X)W) + \nabla_Z (\mathcal{R}(X, Y)W) \\
&= \nabla_X \nabla_Y \nabla_Z W - \nabla_X \nabla_Z \nabla_Y W + \nabla_Y \nabla_Z \nabla_X W \\
&\quad - \nabla_Y \nabla_X \nabla_Z W + \nabla_Z \nabla_X \nabla_Y W - \nabla_Z \nabla_Y \nabla_X W \\
&\quad - \nabla_X \nabla_{[Y, Z]}W - \nabla_Y \nabla_{[Z, X]}W - \nabla_Z \nabla_{[X, Y]}W \\
&= \nabla_Y \nabla_Z \nabla_X W - \nabla_Z \nabla_Y \nabla_X W + \nabla_Z \nabla_X \nabla_Y W \\
&\quad - \nabla_X \nabla_Z \nabla_Y W + \nabla_X \nabla_Y \nabla_Z W - \nabla_Y \nabla_X \nabla_Z W \\
&\quad - \nabla_{[Y, Z]} \nabla_X W - \mathcal{R}(X, [Y, Z]) - \nabla_{[X, [Y, Z]]}W \\
&\quad - \nabla_{[Z, X]} \nabla_Y W - \mathcal{R}(Y, [Z, X]) - \nabla_{[Y, [Z, X]]}W \\
&\quad - \nabla_{[X, Y]} \nabla_Z W - \mathcal{R}(Z, [X, Y]) - \nabla_{[Z, [X, Y]]}W \\
&= \mathcal{R}(Y, Z)\nabla_X W + \mathcal{R}(Z, X)\nabla_Y W + \mathcal{R}(X, Y)\nabla_Z W \\
&\quad - \mathcal{R}(X, [Y, Z])W - \mathcal{R}(Y, [Z, X])W - \mathcal{R}(Z, [X, Y])W \\
&= \text{(III)} + \mathcal{R}([Y, Z], X)W + \mathcal{R}([Z, X], Y)W + \mathcal{R}([X, Y], Z)W \\
&= \text{(III)} + \text{(IV)}
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
\nabla_X \mathcal{R}(Y, Z) + \nabla_Y \mathcal{R}(Z, X) + \nabla_Z \mathcal{R}(X, Y) &= \text{(I)} - \text{(II)} - \text{(III)} \\
&= \text{(III)} + \text{(IV)} - \text{(IV)} - \text{(III)} \\
&= 0
\end{aligned}$$

We also have:

$$\begin{aligned}
 \nabla_X \mathcal{R}m(T, W, Z, Y) &= X(\mathcal{R}m(T, W, Z, Y)) - \mathcal{R}m(\nabla_X T, W, Z, Y) - \mathcal{R}m(T, \nabla_X W, Z, Y) \\
 &\quad - \mathcal{R}m(T, W, \nabla_X Z, Y) - \mathcal{R}m(T, W, Z, \nabla_X Y) \\
 &= X(g(T, \mathcal{R}(Z, Y)W)) - \mathcal{R}m(\nabla_X T, W, Z, Y) - \mathcal{R}m(T, \nabla_X W, Z, Y) \\
 &\quad - \mathcal{R}m(T, W, \nabla_X Z, Y) - \mathcal{R}m(T, W, Z, \nabla_X Y) \\
 &= g(\nabla_X T, \mathcal{R}(Z, Y)W) + g(T, \nabla_X(\mathcal{R}(Z, Y)W)) - g(\nabla_X T, \mathcal{R}(Z, Y)W) \\
 &\quad - g(T, \mathcal{R}(Z, Y)\nabla_X W) - g(T, \mathcal{R}(\nabla_X Z, Y)W) - g(T, \mathcal{R}(Z, \nabla_X Y)W) \\
 &= g(T, \nabla_X(\mathcal{R}(Z, Y)W)) - \mathcal{R}(Z, Y)\nabla_X W - \mathcal{R}(\nabla_X Z, Y)W - \mathcal{R}(Z, \nabla_X Y)W \\
 &= g(T, \nabla_X \mathcal{R}(Z, Y)W)
 \end{aligned}$$

Thus, by permuting  $X, Z,$  and  $Y,$  we have:

$$\begin{aligned}
 &\nabla_X \mathcal{R}m(T, W, Z, Y) + \nabla_Y \mathcal{R}m(T, W, X, Z) + \nabla_Z \mathcal{R}m(T, W, Y, X) \\
 &= g(T, \nabla_X \mathcal{R}(Z, Y)W) + g(T, \nabla_Y \mathcal{R}(X, Z)W) + g(T, \nabla_Z \mathcal{R}(Y, X)W) \\
 &= g(T, \nabla_X \mathcal{R}(Z, Y)W + \nabla_Y \mathcal{R}(Z, X)W + \nabla_Z \mathcal{R}(Y, X)W) \\
 &= g(T, 0) \\
 &= 0
 \end{aligned}$$

□

## 4.2 Computation of components in $\mathcal{E}$ and $\mathcal{E}^*$

### Definition 4.2.1: Curvature and Riemann tensors

(i) The **components of the curvature tensor in  $\mathcal{E}$  and  $\mathcal{E}^*$**  are given by:

$$\mathbf{R}^i{}_{jkl} := e^i(\mathcal{R}(e_k, e_l)e_j).$$

(ii) The **components of the Riemann tensor in  $\mathcal{E}$  and  $\mathcal{E}^*$**  are given by:

$$\mathbf{R}_{ijkl} := \mathcal{R}m(e_i, e_j, e_k, e_l).$$

We have the following decompositions:

$$\begin{aligned}
 \mathcal{R} &= \mathbf{R}^i{}_{jkl} e_i \otimes e^j \otimes e^k \otimes e^l \\
 \mathcal{R}m &= \mathbf{R}_{ijkl} e^i \otimes e^j \otimes e^k \otimes e^l
 \end{aligned}$$

There is a simple relation between these two components.

### Lemma 4.2.2: Relation between the two components

We have:

$$\mathbf{R}_{ijkl} = \mathbf{g}_{in} \mathbf{R}^n{}_{jkl}.$$

*Proof.* This follows directly from proposition 2.8.6. We provide the proof here for completeness. We have:

$$\begin{aligned}
 \mathbf{R}_{ijkl} &= \mathcal{R}m(e_i, e_j, e_k, e_l) \\
 &= g(e_i, \mathcal{R}(e_j, e_k)e_l) \\
 &= g(e_i, \mathbf{R}^n{}_{jkl} e_n) \\
 &= \mathbf{R}^n{}_{jkl} g(e_i, e_n) \\
 &= \mathbf{g}_{in} \mathbf{R}^n{}_{jkl}
 \end{aligned}$$

□

From proposition 4.2.3, we deduce the following proposition.

**Proposition 4.2.3: Usual properties**

Let  $i, j, k, l \in \{0, 1, 2, 3\}$ .

(i) **(First Bianchi identity)** We have:

$$\begin{aligned} \mathbf{R}^i{}_{jkl} + \mathbf{R}^i{}_{klj} + \mathbf{R}^i{}_{ljk} &= 0 \\ \mathbf{R}_{ijkl} + \mathbf{R}_{iklj} + \mathbf{R}_{iljk} &= 0 \end{aligned}$$

(ii) We have:

$$\begin{aligned} \mathbf{R}^i{}_{jkl} &= -\mathbf{R}^i{}_{jlk} \\ \mathbf{R}_{ijkl} &= -\mathbf{R}_{ijlk} \end{aligned}$$

(iii) We have:

$$\mathbf{R}_{ijkl} = -\mathbf{R}_{jikl}.$$

(iv) We have:

$$\mathbf{R}_{ijkl} = \mathbf{R}_{klij}.$$

(v) **(Second Bianchi identity)** We have:

$$\nabla_n \mathbf{R}_{ijkl} + \nabla_k \mathbf{R}_{ijln} + \nabla_l \mathbf{R}_{ijnk} = 0.$$

*Proof.* We use proposition 4.2.3.

(i) Since:

$$\mathcal{R}(e_k, e_l)e_j + \mathcal{R}(e_l, e_j)e_k + \mathcal{R}(e_j, e_k)e_l$$

we have:

$$\begin{aligned} \mathbf{R}^i{}_{jkl} + \mathbf{R}^i{}_{klj} + \mathbf{R}^i{}_{ljk} &= e^i (\mathcal{R}(e_k, e_l)e_j) + e^i (\mathcal{R}(e_l, e_j)e_k) + e^i (\mathcal{R}(e_j, e_k)e_l) \\ &= e^i (\mathcal{R}(e_k, e_l)e_j + \mathcal{R}(e_l, e_j)e_k + \mathcal{R}(e_j, e_k)e_l) \\ &= e^i (0) \\ &= 0 \\ \mathbf{R}_{ijkl} + \mathbf{R}_{iklj} + \mathbf{R}_{iljk} &= \mathcal{R}m(e_i, e_j, e_k, e_l) + \mathcal{R}m(e_i, e_k, e_l, e_j) + \mathcal{R}m(e_i, e_l, e_j, e_k) \\ &= 0 \end{aligned}$$

(ii) We have:

$$\begin{aligned} \mathbf{R}^i{}_{jkl} &= e^i (\mathcal{R}(e_k, e_l)e_j) \\ &= e^i (-\mathcal{R}(e_l, e_k)e_j) \\ &= -e^i (\mathcal{R}(e_l, e_k)e_j) \\ &= -\mathbf{R}^i{}_{kjl} \\ \mathbf{R}_{ijkl} &= \mathcal{R}m(e_i, e_j, e_k, e_l) \\ &= -\mathcal{R}m(e_i, e_j, e_l, e_k) \\ &= -\mathbf{R}_{ijlk} \end{aligned}$$

(iii) We have:

$$\begin{aligned} \mathbf{R}_{ijkl} &= \mathcal{R}m(e_i, e_j, e_k, e_l) \\ &= -\mathcal{R}m(e_j, e_i, e_k, e_l) \\ &= -\mathbf{R}_{jikl} \end{aligned}$$

(iv) We have:

$$\begin{aligned} \mathbf{R}_{ijkl} &= \mathcal{R}m(e_i, e_j, e_k, e_l) \\ &= \mathcal{R}m(e_k, e_l, e_i, e_j) \\ &= \mathbf{R}_{klij} \end{aligned}$$

(v) We have:

$$\begin{aligned} \nabla_n \mathbf{R}_{ijkl} + \nabla_k \mathbf{R}_{ijln} + \nabla_l \mathbf{R}_{ijnk} &= \nabla_{e_n} \mathcal{R}m(e_i, e_j, e_k, e_l) + \nabla_{e_k} \mathcal{R}m(e_i, e_j, e_l, e_n) + \nabla_{e_l} \mathcal{R}m(e_i, e_j, e_n, e_k) \\ &= 0 \end{aligned}$$

□

This implies the nullity of a class of components.

**Corollary 4.2.4: Nullities and symmetries of certain components**

Let  $i, j, k, l \in \{0, 1, 2, 3\}$ .

(i) We have:

$$\mathbf{R}_{jkii} = \mathbf{R}_{iijk} = 0.$$

(ii) We have:

$$\mathbf{R}_{ijk i} = \mathbf{R}_{k i i j} = -\mathbf{R}_{k i j i} = -\mathbf{R}_{i k i j} = -\mathbf{R}_{i j i k} = -\mathbf{R}_{j i k i} = \mathbf{R}_{j i i k} = \mathbf{R}_{i k j i}.$$

*Proof.* (i) We have from point (ii) of proposition 4.2.3:

$$\mathbf{R}_{ijll} = -\mathbf{R}_{ijll}$$

*i.e.*, we have:

$$\mathbf{R}_{ijll} = 0.$$

We have from point (iii) of proposition 4.2.3:

$$\mathbf{R}_{iikl} = -\mathbf{R}_{iikl}$$

*i.e.*, we have:

$$\mathbf{R}_{iikl} = 0.$$

(ii) From (i) and point (i) of proposition 4.2.3, we have:

$$\begin{aligned} 0 &= \mathbf{R}_{ijll} + \mathbf{R}_{illj} + \mathbf{R}_{iljl} \\ &= 0 + \mathbf{R}_{illj} + \mathbf{R}_{iljl} \end{aligned}$$

*i.e.*, we have:

$$\mathbf{R}_{illj} = -\mathbf{R}_{iljl}.$$

And we have:

$$\begin{aligned} 0 &= \mathbf{R}_{ikl} + \mathbf{R}_{ikli} + \mathbf{R}_{ilik} \\ &= 0 + \mathbf{R}_{ikli} + \mathbf{R}_{ilik} \end{aligned}$$

i.e., we have:

$$\mathbf{R}_{ikli} = -\mathbf{R}_{ilik}.$$

By point (iv) of proposition 4.2.3, we have directly:

$$\begin{aligned} \mathbf{R}_{ijil} &= \mathbf{R}_{ilij} \\ \mathbf{R}_{ijk i} &= \mathbf{R}_{k iij} \end{aligned}$$

□

### Example 4.2.5: Examples of components

(1) (a) The **components of the curvature tensor in the bases  $\mathcal{C}$  and  $\mathcal{C}^*$**  are:

$$\mathbf{R}^i{}_{jkl} := dx^i (\mathcal{R} (\partial_k, \partial_l) \partial_j).$$

(b) The **components of the Riemann tensor in the bases  $\mathcal{C}$  and  $\mathcal{C}^*$**  are:

$$\mathcal{R}m_{ijkl} := \mathcal{R}m (\partial_i, \partial_j, \partial_k, \partial_l).$$

Therefore, we have the decompositions:

$$\begin{aligned} \mathcal{R} &= \mathbf{R}^i{}_{jkl} \partial_i \otimes dx^j \otimes dx^k \otimes dx^l \\ \mathcal{R}m &= \mathbf{R}_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l \end{aligned}$$

By lemma 4.2.2, we have the relations:

$$\begin{aligned} \mathbf{R}_{ijkl} &= g_{in} \mathbf{R}^n{}_{jkl} \\ &= g_{in} \mathbf{R}^i{}_{jkl} \\ &= \eta_i e^{u_i} \mathbf{R}^i{}_{jkl} \end{aligned}$$

(2) (a) The **components of the curvature tensor in the bases  $\mathcal{C}_\perp$  and  $\mathcal{C}_\perp^*$**  are:

$$\mathbb{R}^i{}_{jkl} := \theta^i (\mathcal{R} (\theta_k, \theta_l) \theta_j).$$

(b) The **components of the Riemann tensor in the bases  $\mathcal{C}_\perp$  and  $\mathcal{C}_\perp^*$**  are:

$$\mathbb{R}_{ijkl} := \mathcal{R}m (\theta_i, \theta_j, \theta_k, \theta_l).$$

Therefore, we have the decompositions:

$$\begin{aligned} \mathcal{R} &= \mathbb{R}^i{}_{jkl} \theta_i \otimes \theta^j \otimes \theta^k \otimes \theta^l \\ \mathcal{R}m &= \mathbb{R}_{ijkl} \theta^i \otimes \theta^j \otimes \theta^k \otimes \theta^l \end{aligned}$$

By lemma 4.2.2, we have the relations:

$$\begin{aligned} \mathbb{R}_{ijkl} &= \eta_{ii} \mathbb{R}^i{}_{jkl} \\ &= \eta_i \mathbb{R}^i{}_{jkl} \end{aligned}$$

### 4.3 Practical calculations in $\mathcal{C}$ and $\mathcal{C}^*$

We begin with a proposition that establishes a link between the components of the curvature tensor and the Christoffel symbols.

#### Proposition 4.3.1: Usual properties

Let  $i, j, k, l \in \{0, 1, 2, 3\}$ . We have:

$$R^i{}_{jkl} = \partial_k \Gamma^i{}_{jl} - \partial_l \Gamma^i{}_{kj} + \Gamma^m{}_{jl} \Gamma^i{}_{km} - \Gamma^m{}_{kj} \Gamma^i{}_{lm}.$$

*Proof.* By definition of  $\mathcal{R}$ :

$$\begin{aligned} \mathcal{R}(\partial_k, \partial_l) \partial_j &= \nabla_{\partial_k} \nabla_{\partial_l} \partial_j - \nabla_{\partial_l} \nabla_{\partial_k} \partial_j - \nabla_{[\partial_k, \partial_l]} \partial_j \\ &= \nabla_{\partial_k} (\Gamma^m{}_{jl} \partial_m) - \nabla_{\partial_l} (\Gamma^m{}_{kj} \partial_m) \\ &= d\Gamma^m{}_{jl}(\partial_k) \partial_m + \Gamma^m{}_{jl} \nabla_{\partial_k} \partial_m - d\Gamma^m{}_{kj}(\partial_l) \partial_m + \Gamma^m{}_{kj} \nabla_{\partial_l} \partial_m \\ &= \partial_k \Gamma^m{}_{jl} \partial_m + \Gamma^m{}_{jl} \Gamma^n{}_{km} \partial_n - \partial_l \Gamma^m{}_{kj} \partial_m - \Gamma^m{}_{kj} \Gamma^n{}_{lm} \partial_n \end{aligned}$$

Therefore, we have:

$$\begin{aligned} R^i{}_{jkl} &= dx^i(\mathcal{R}(\partial_k, \partial_l) \partial_j) \\ &= dx^i(\partial_k \Gamma^m{}_{jl} \partial_m + \Gamma^m{}_{jl} \Gamma^n{}_{km} \partial_n - \partial_l \Gamma^m{}_{kj} \partial_m - \Gamma^m{}_{kj} \Gamma^n{}_{lm} \partial_n) \\ &= \partial_k \Gamma^m{}_{jl} \delta_m^i + \Gamma^m{}_{jl} \Gamma^n{}_{km} \delta_n^i - \partial_l \Gamma^m{}_{kj} \delta_m^i - \Gamma^m{}_{kj} \Gamma^n{}_{lm} \delta_n^i \\ &= \partial_k \Gamma^i{}_{jl} - \partial_l \Gamma^i{}_{kj} + \Gamma^m{}_{jl} \Gamma^i{}_{km} - \Gamma^m{}_{kj} \Gamma^i{}_{lm} \end{aligned}$$

□

According to Corollary 4.2.4, there are only two types of components to calculate, as the others can be obtained through symmetry:

$$R^i{}_{jil} \quad , \quad R^i{}_{jij}.$$

#### Theorem 4.3.2: Values of the curvature and Riemann tensors

Let  $i, j, k, l \in \{0, 1, 2, 3\}$  **distinct**, i.e.,  $\{0, 1, 2, 3\} = \{i, j, k, l\}$ .

(i) We have:

$$\begin{aligned} R^i{}_{jil} &= -u_{i,jl} + u_{i,j}(u_{j,l} - u_{i,l}) + u_{l,j}u_{i,l} \\ R^j{}_{iil} &= -\eta_i \eta_j e^{2u_i - 2u_j} R^i{}_{jil} \\ R^j{}_{ilj} &= -R^j{}_{iil} \\ R^i{}_{jli} &= -R^i{}_{jil} \end{aligned}$$

(ii) We have:

$$\begin{aligned} R^i{}_{jij} &= -(u_{i,jj} + u_{i,j}(u_{i,j} - u_{j,j})) - \eta_i \eta_j (u_{j,ii} + u_{j,i}(u_{j,i} - u_{i,i})) e^{2u_j - 2u_i} \\ &\quad - \eta_k \eta_j u_{i,k} u_{j,k} e^{2u_j - 2u_k} - \eta_l \eta_j u_{i,l} u_{j,l} e^{2u_j - 2u_l} \end{aligned}$$

and:

$$R^j{}_{iij} = -\eta_i \eta_j R^i{}_{jij}.$$

(iii) All other components of the Riemann tensor are zero.

*Proof.* (i) We have:

$$\begin{aligned} R^i{}_{jil} &= \partial_i \Gamma_{jl}^i - \partial_l \Gamma_{ij}^i + \Gamma_{jl}^m \Gamma_{im}^i - \Gamma_{ij}^m \Gamma_{lm}^i \\ &= -\partial_l \Gamma_{ij}^i + \Gamma_{jl}^j \Gamma_{ij}^i + \Gamma_{jl}^l \Gamma_{il}^i - \Gamma_{ij}^i \Gamma_{li}^i \\ &= -\partial_l (u_{i,j}) + u_{j,l} u_{i,j} + u_{l,j} u_{i,l} - u_{i,j} u_{i,l} \\ &= -u_{i,jl} + u_{i,j} (u_{j,l} - u_{i,l}) + u_{l,j} u_{i,l} \\ &= -u_{i,jl} + u_{i,j} (u_{j,l} - u_{i,l}) + u_{l,j} u_{i,l} \end{aligned}$$

(ii) We have:

$$\begin{aligned} R^i{}_{jij} &= \partial_i \Gamma_{jj}^i - \partial_j \Gamma_{ij}^i + \Gamma_{jj}^m \Gamma_{im}^i - \Gamma_{ij}^m \Gamma_{jm}^i \\ &= \partial_i \Gamma_{jj}^i - \partial_j \Gamma_{ij}^i + \Gamma_{jj}^m \Gamma_{im}^i - \Gamma_{ij}^i \Gamma_{ji}^i - \Gamma_{ij}^j \Gamma_{jj}^i \\ &= \partial_i (-\eta_i \eta_j u_{j,i} e^{2u_j - 2u_i}) - \partial_j (u_{i,j}) + (-\eta_m \eta_j u_{j,m} e^{2u_j - 2u_m}) u_{i,m} - u_{i,j}^2 - u_{j,i} (-\eta_i \eta_j u_{j,i} e^{2u_j - 2u_i}) \\ &= -(u_{i,jj} + u_{i,j} (u_{i,j} - u_{j,j})) - \eta_i \eta_j (u_{j,ii} + u_{j,i} (u_{j,i} - u_{i,i})) e^{2u_j - 2u_i} \\ &\quad - \eta_k \eta_j u_{i,k} u_{j,k} e^{2u_j - 2u_k} - \eta_i \eta_j u_{i,l} u_{j,l} e^{2u_j - 2u_l} \end{aligned}$$

□

#### 4.4 Practical computations of the curvature 2-form in $\mathcal{C}_\perp$ and $\mathcal{C}_\perp^*$

Let  $X$  and  $Y$  be two vector fields. For any vector field  $\partial_j$ , the element  $\mathcal{R}(X, Y)\partial_j$  is a vector field, so it can be decomposed in the  $\mathcal{C}_\perp$  basis. This leads to the definition of the curvature 2-forms.

##### Definition 4.4.1: Curvature applications

Let  $X$  and  $Y$  be two vector fields, and  $i, j \in \{0, 1, 2, 3\}$ . We define the scalar field  $\Omega_j^i(X, Y)$  by:

$$\Omega_j^i(X, Y) := \theta^i (\mathcal{R}(X, Y)\theta_j).$$

Thus, we have a 2-form  $\Omega_j^i \in \Lambda^2(U)$  that associates the scalar field  $\Omega_j^i(X, Y)$  to every pair of vector fields  $X$  and  $Y$ . We have the decomposition:

$$\mathcal{R}(X, Y)\theta_j = \Omega_j^i(X, Y)\theta_i.$$

##### Lemma 4.4.2: Usual properties

Let  $i, j, k \in \{0, 1, 2, 3\}$ .

(i) We have:

$$\Omega_j^i = d\varpi_j^i + \varpi_k^i \wedge \varpi_k^j$$

(ii) We have:

$$\Omega_j^k = -\eta_k \eta_j \Omega_j^k.$$

(iii) We have:

$$\Omega_k^k = 0.$$

*Proof.* (i) For all vector fields  $X$  and  $Y$ , we have:

$$\begin{aligned}
 \Omega_j^i(X, Y)\theta_i &= R(X, Y)\theta_j = \nabla_X \nabla_Y \theta_j - \nabla_Y \nabla_X \theta_j - \nabla_{[X, Y]}\theta_j \\
 &= \nabla_X (\varpi_j^i(Y)\theta_i) - \nabla_Y (\varpi_j^i(X)\theta_i) - \varpi_j^i([X, Y])\theta_i \\
 &= X(\varpi_j^k(Y))\theta_k + \varpi_j^k(Y)\nabla_X \theta_k - Y(\varpi_j^k(X))\theta_k - \varpi_j^k(X)\nabla_Y \theta_k - \varpi_j^i([X, Y])\theta_i \\
 &= X(\varpi_j^k(Y))\theta_k + \varpi_j^k(Y)\varpi_k^i(X)\theta_i - Y(\varpi_j^k(X))\theta_k - \varpi_j^k(X)\varpi_k^i(Y)\theta_i - \varpi_j^i([X, Y])\theta_i \\
 &= X(\varpi_j^i(Y))\theta_i + \varpi_j^k(Y)\varpi_k^i(X)\theta_i - Y(\varpi_j^i(X))\theta_i - \varpi_j^k(X)\varpi_k^i(Y)\theta_i - \varpi_j^i([X, Y])\theta_i \\
 &= (X(\varpi_j^i(Y)) + \varpi_j^k(Y)\varpi_k^i(X) - Y(\varpi_j^i(X)) - \varpi_j^k(X)\varpi_k^i(Y) - \varpi_j^i([X, Y]))\theta_i \\
 &= (X(\varpi_j^i(Y)) - Y(\varpi_j^i(X)) - \varpi_j^i([X, Y]) + \varpi_j^k(Y)\varpi_k^i(X) - \varpi_j^k(X)\varpi_k^i(Y))\theta_i \\
 &= (d\varpi_j^i(X, Y) + (\varpi_k^i \otimes \varpi_j^k - \varpi_j^k \otimes \varpi_k^i)(X, Y))\theta_i \\
 &= (d\varpi_j^i + \varpi_k^i \wedge \varpi_j^k)(X, Y)\theta_i
 \end{aligned}$$

Since this holds for all vector fields  $X$ ,  $Y$ , and  $\theta_i$ , we have the result.

(ii) We have:

$$\begin{aligned}
 -\eta_k \eta_j \Omega_k^j &= -\eta_k \eta_j (d\varpi_k^j + \varpi_i^j \wedge \varpi_k^i) \\
 &= -\eta_k \eta_j d\varpi_k^j - \eta_k \eta_j \varpi_i^j \wedge \varpi_k^i \\
 &= d(-\eta_k \eta_j \varpi_k^j) - (-\eta_i \eta_j \varpi_i^j) \wedge (-\eta_k \eta_i \varpi_k^i) \\
 &= d\varpi_k^j - \varpi_j^i \wedge \varpi_k^i \\
 &= d\varpi_k^j + \varpi_k^i \wedge \varpi_j^i \\
 &= \Omega_k^j
 \end{aligned}$$

(iii) We have by (ii):

$$\Omega_k^k = -\eta_k \eta_k \Omega_k^k = -\Omega_k^k$$

In other words, we have  $\Omega_k^k = 0$ . □

**Lemma 4.4.3:** Lemma for the computation of the two forms  $\Omega_j^i$

Let  $i, j, k, l \in \{0, 1, 2, 3\}$  **distinct** such that  $\{0, 1, 2, 3\} = \{i, j, k, l\}$ , and  $p, q \neq i, j$ .

(i) We have:

$$\varpi_p^i \wedge \varpi_j^p = u_{i,p} u_{p,j} e^{-u_p - u_j} \theta^{i,p} - \eta_p \eta_j u_{i,p} u_{j,p} e^{-2u_p} \theta^{i,j} + \eta_i \eta_j u_{p,i} u_{j,p} e^{-u_i - u_p} \theta^{p,j}.$$

(ii) We have:

$$d\varpi_j^i = (u_{i,j,q} + u_{i,j}(u_{i,q} - u_{j,q})) e^{-u_j - u_q} \theta^{q,i} - \eta_i \eta_j (u_{j,i,q} + u_{j,i}(u_{j,q} - u_{i,q})) e^{-u_i - u_q} \theta^{q,j}.$$

*Proof.* (i) We have:

$$\begin{aligned}
 \varpi_p^i \wedge \varpi_j^p &= (u_{i,p} e^{-u_p} \theta^i - \eta_i \eta_p u_{p,i} e^{-u_i} \theta^p) \wedge (u_{p,j} e^{-u_j} \theta^p - \eta_p \eta_j u_{j,p} e^{-u_p} \theta^j) \\
 &= u_{i,p} u_{p,j} e^{-u_p - u_j} \theta^{i,p} - \eta_p \eta_j u_{i,p} u_{j,p} e^{-2u_p} \theta^{i,j} + \eta_i \eta_j u_{p,i} u_{j,p} e^{-u_i - u_p} \theta^{p,j}
 \end{aligned}$$

(ii) We have:

$$\begin{aligned}
 d\varpi^i_j &= d(u_{i,j}e^{-u_j}\theta^i - \eta_i\eta_j u_{j,i}e^{-u_i}\theta^j) \\
 &= d(u_{i,j}e^{-u_j}\theta^i) - \eta_i\eta_j d(u_{j,i}e^{-u_i}\theta^j) \\
 &= d(u_{i,j}e^{-u_j}) \wedge \theta^i + u_{i,j}e^{-u_j} d(\theta^i) - \eta_i\eta_j d(u_{j,i}e^{-u_i}) \wedge \theta^j - \eta_i\eta_j u_{j,i}e^{-u_i} d(\theta^j) \\
 &= (u_{i,jq} - u_{i,j}u_{j,q}) e^{-u_j} dx^q \wedge \theta^i + u_{i,j}e^{-u_j} u_{i,q} e^{-u_q} \theta^{q,i} \\
 &\quad - \eta_i\eta_j (u_{j,iq} - u_{j,i}u_{i,q}) e^{-u_i} dx^q \wedge \theta^j - \eta_i\eta_j u_{j,i} e^{-u_i} u_{j,q} e^{-u_q} \theta^{q,j} \\
 &= (u_{i,jq} + u_{i,j}(u_{i,q} - u_{j,q})) e^{-u_j - u_q} \theta^{q,i} - \eta_i\eta_j (u_{j,iq} + u_{j,i}(u_{j,q} - u_{i,q})) e^{-u_i - u_q} \theta^{q,j}
 \end{aligned}$$

□

**Proposition 4.4.4:** Exact values of the two forms  $\Omega_j^i$

We have:

$$\begin{aligned}
 \Omega_j^i &= \sum_{p \neq i,j} \left[ (u_{i,jp} + u_{i,j}(u_{i,p} - u_{j,p}) - u_{i,p}u_{p,j}) e^{-u_p - u_j} \theta^{p,i} + \eta_i\eta_j (u_{p,i}u_{j,p} - u_{j,ip} + u_{j,i}(u_{i,p} - u_{j,p})) e^{-u_i - u_p} \theta^{p,j} \right] \\
 &\quad + \left[ (u_{i,jj} + u_{i,j}(u_{i,j} - u_{j,j})) e^{-2u_j} + \eta_i\eta_j (u_{j,ii} + u_{j,i}(u_{j,i} - u_{i,i})) e^{-2u_i} + \eta_j \sum_{p \neq i,j} \eta_p u_{i,p} u_{j,p} e^{-2u_p} \right] \theta^{j,i}
 \end{aligned}$$

*Proof.* We have:

$$\begin{aligned}
 \Omega_j^i &= d\varpi^i_j + \sum_{p \neq i,j} \varpi^i_p \wedge \varpi^p_j \\
 &= (u_{i,jq} + u_{i,j}(u_{i,q} - u_{j,q})) e^{-u_j - u_q} \theta^{q,i} - \eta_i\eta_j (u_{j,iq} + u_{j,i}(u_{j,q} - u_{i,q})) e^{-u_i - u_q} \theta^{q,j} \\
 &\quad + \sum_{p \neq i,j} u_{i,p}u_{p,j} e^{-u_p - u_j} \theta^{i,p} - \eta_p\eta_j u_{i,p}u_{j,p} e^{-2u_p} \theta^{i,j} + \eta_i\eta_j u_{p,i}u_{j,p} e^{-u_i - u_p} \theta^{p,j} \\
 &= (u_{i,jj} + u_{i,j}(u_{i,j} - u_{j,j})) e^{-u_j - u_j} \theta^{j,i} + (u_{i,jk} + u_{i,j}(u_{i,k} - u_{j,k})) e^{-u_j - u_k} \theta^{k,i} \\
 &\quad + (u_{i,jl} + u_{i,j}(u_{i,l} - u_{j,l})) e^{-u_j - u_l} \theta^{l,i} - \eta_i\eta_j (u_{j,ii} + u_{j,i}(u_{j,i} - u_{i,i})) e^{-u_i - u_i} \theta^{i,j} \\
 &\quad - \eta_i\eta_j (u_{j,ik} + u_{j,i}(u_{j,k} - u_{i,k})) e^{-u_i - u_k} \theta^{k,j} - \eta_i\eta_j (u_{j,il} + u_{j,i}(u_{j,l} - u_{i,l})) e^{-u_i - u_l} \theta^{l,j} \\
 &\quad + u_{i,k}u_{k,j} e^{-u_k - u_j} \theta^{i,k} - \eta_k\eta_j u_{i,k}u_{j,k} e^{-2u_k} \theta^{i,j} + \eta_i\eta_j u_{k,i}u_{j,k} e^{-u_i - u_k} \theta^{k,j} \\
 &\quad + u_{i,l}u_{l,j} e^{-u_l - u_j} \theta^{i,l} - \eta_l\eta_j u_{i,l}u_{j,l} e^{-2u_l} \theta^{i,j} + \eta_i\eta_j u_{l,i}u_{j,l} e^{-u_i - u_l} \theta^{l,j} \\
 &= \sum_{p \neq i,j} \left[ (u_{i,jp} + u_{i,j}(u_{i,p} - u_{j,p}) - u_{i,p}u_{p,j}) e^{-u_j - u_p} \theta^{p,i} - \eta_i\eta_j (u_{j,ip} + u_{j,i}(u_{j,p} - u_{i,p}) - u_{p,i}u_{j,p}) e^{-u_i - u_p} \theta^{p,j} \right] \\
 &\quad + \left[ (u_{i,jj} + u_{i,j}(u_{i,j} - u_{j,j})) e^{-2u_j} + \eta_i\eta_j (u_{j,ii} + u_{j,i}(u_{j,i} - u_{i,i})) e^{-2u_i} + \eta_j \sum_{p \neq i,j} \eta_p u_{i,p} u_{j,p} e^{-2u_p} \right] \theta^{j,i}
 \end{aligned}$$

□

**Example 4.4.5:** Sequence 5 of metric  $h$

We continue the example with the metric  $h$  (see 2.4.1, 2.5.1, 3.5.5, 3.6.3, and 3.6.5) with :

$$\begin{aligned}
 u_0(t, r) &:= u(t, r) & u_1(t, r) &:= v(t, r) \\
 u_2(t, r) &:= b(t) + \ln(r) & u_3(t, r, \vartheta) &:= b(t) + \ln(r) + \ln \sin(\vartheta)
 \end{aligned}$$

• We have :

$$\Omega_0^1 = (u_{1,02} + u_{1,0}(u_{1,2} - u_{0,2}) - u_{1,2}u_{2,0}) e^{-u_2 - u_0} \theta^{2,1}$$

$$\begin{aligned}
 & + \eta_1 \eta_0 (u_{2,1} u_{0,2} - u_{0,12} + u_{0,1} (u_{1,2} - u_{0,2})) e^{-u_1 - u_2} \theta^{2,0} \\
 & + (u_{1,03} + u_{1,0} (u_{1,3} - u_{0,3}) - u_{1,3} u_{3,0}) e^{-u_3 - u_0} \theta^{3,1} \\
 & + \eta_1 \eta_0 (u_{3,1} u_{0,3} - u_{0,13} + u_{0,1} (u_{1,3} - u_{0,3})) e^{-u_1 - u_3} \theta^{3,0} \\
 & + [(u_{1,00} + u_{1,0} (u_{1,0} - u_{0,0})) e^{-2u_0} + \eta_1 \eta_0 (u_{0,11} + u_{0,1} (u_{0,1} - u_{1,1})) e^{-2u_1}] \theta^{0,1} \\
 & + [\eta_0 \eta_2 u_{1,2} u_{0,2} e^{-2u_2} + \eta_0 \eta_3 u_{1,3} u_{0,3} e^{-2u_3}] \theta^{0,1} \\
 = & [(u_{1,00} + u_{1,0} (u_{1,0} - u_{0,0})) e^{-2u_0} + \eta_1 \eta_0 (u_{0,11} + u_{0,1} (u_{0,1} - u_{1,1})) e^{-2u_1}] \theta^{0,1} \\
 = & [(\ddot{v} + \dot{v} (\dot{v} - \dot{u})) e^{-2u} - (u'' + u' (u' - v')) e^{-2v}] \theta^{0,1}
 \end{aligned}$$

• We have :

$$\begin{aligned}
 \Omega_0^2 = & (u_{2,01} + u_{2,0} (u_{2,1} - u_{0,1}) - u_{2,1} u_{1,0}) e^{-u_1 - u_0} \theta^{1,2} \\
 & + \eta_2 \eta_0 (u_{1,2} u_{0,1} - u_{0,21} + u_{0,2} (u_{2,1} - u_{0,1})) e^{-u_2 - u_1} \theta^{1,0} \\
 & + (u_{2,03} + u_{2,0} (u_{2,3} - u_{0,3}) - u_{2,3} u_{3,0}) e^{-u_3 - u_0} \theta^{3,2} \\
 & + \eta_2 \eta_0 (u_{3,2} u_{0,3} - u_{0,23} + u_{0,2} (u_{2,3} - u_{0,3})) e^{-u_2 - u_3} \theta^{3,0} \\
 & + [(u_{2,00} + u_{2,0} (u_{2,0} - u_{0,0})) e^{-2u_0} + \eta_2 \eta_0 (u_{0,22} + u_{0,2} (u_{0,2} - u_{2,2})) e^{-2u_2}] \theta^{0,2} \\
 & + [\eta_0 \eta_1 u_{2,1} u_{0,1} e^{-2u_1} + \eta_0 \eta_3 u_{2,3} u_{0,3} e^{-2u_3}] \theta^{0,2} \\
 = & (u_{2,0} (u_{2,1} - u_{0,1}) - u_{2,1} u_{1,0}) e^{-u_1 - u_0} \theta^{1,2} + [(u_{2,00} + u_{2,0} (u_{2,0} - u_{0,0})) e^{-2u_0}] \theta^{0,2} \\
 & + \eta_0 \eta_1 u_{2,1} u_{0,1} e^{-2u_1} \theta^{0,2} \\
 = & (\dot{b} (r^{-1} - u') - r^{-1} \dot{v}) e^{-u-v} \theta^{1,2} + [(\ddot{b} + \dot{b} (\dot{b} - \dot{u})) e^{-2u} - r^{-1} u' e^{-2v}] \theta^{0,2}
 \end{aligned}$$

• We have :

$$\begin{aligned}
 \Omega_0^3 = & (u_{3,01} + u_{3,0} (u_{3,1} - u_{0,1}) - u_{3,1} u_{1,0}) e^{-u_1 - u_0} \theta^{1,3} \\
 & + \eta_3 \eta_0 (u_{1,3} u_{0,1} - u_{0,31} + u_{0,3} (u_{3,1} - u_{0,1})) e^{-u_3 - u_1} \theta^{1,0} \\
 & + (u_{3,02} + u_{3,0} (u_{3,2} - u_{0,2}) - u_{3,2} u_{2,0}) e^{-u_2 - u_0} \theta^{2,3} \\
 & + \eta_3 \eta_0 (u_{2,3} u_{0,2} - u_{0,32} + u_{0,3} (u_{3,2} - u_{0,2})) e^{-u_3 - u_2} \theta^{2,0} \\
 & + [(u_{3,00} + u_{3,0} (u_{3,0} - u_{0,0})) e^{-2u_0} + \eta_3 \eta_0 (u_{0,33} + u_{0,3} (u_{0,3} - u_{3,3})) e^{-2u_3}] \theta^{0,3} \\
 & + [\eta_0 \eta_1 u_{3,1} u_{0,1} e^{-2u_1} + \eta_0 \eta_2 u_{3,2} u_{0,2} e^{-2u_2}] \theta^{0,3} \\
 = & (u_{3,0} (u_{3,1} - u_{0,1}) - u_{3,1} u_{1,0}) e^{-u_1 - u_0} \theta^{1,3} + (u_{3,0} u_{3,2} - u_{3,2} u_{2,0}) e^{-u_2 - u_0} \theta^{2,3} \\
 & + [(u_{3,00} + u_{3,0} (u_{3,0} - u_{0,0})) e^{-2u_0} + \eta_0 \eta_1 u_{3,1} u_{0,1} e^{-2u_1}] \theta^{0,3} \\
 = & (\dot{b} (r^{-1} - u') - r^{-1} \dot{v}) e^{-u-v} \theta^{1,3} + [(\ddot{b} + \dot{b} (\dot{b} - \dot{u})) e^{-2u} - r^{-1} u' e^{-2v}] \theta^{0,3}
 \end{aligned}$$

• We have :

$$\begin{aligned}
 \Omega_1^2 = & (u_{2,10} + u_{2,1} (u_{2,0} - u_{1,0}) - u_{2,0} u_{0,1}) e^{-u_0 - u_1} \theta^{0,2} \\
 & + \eta_2 \eta_1 (u_{0,2} u_{1,0} - u_{1,20} + u_{1,2} (u_{2,0} - u_{1,0})) e^{-u_2 - u_0} \theta^{0,1} \\
 & + (u_{2,13} + u_{2,1} (u_{2,3} - u_{1,3}) - u_{2,3} u_{3,1}) e^{-u_3 - u_1} \theta^{3,2} \\
 & + \eta_2 \eta_1 (u_{3,2} u_{1,3} - u_{1,23} + u_{1,2} (u_{2,3} - u_{1,3})) e^{-u_2 - u_3} \theta^{3,1} \\
 & + [(u_{2,11} + u_{2,1} (u_{2,1} - u_{1,1})) e^{-2u_1} + \eta_2 \eta_1 (u_{1,22} + u_{1,2} (u_{1,2} - u_{2,2})) e^{-2u_2}] \theta^{1,2} \\
 & + [\eta_1 \eta_0 u_{2,0} u_{1,0} e^{-2u_0} + \eta_1 \eta_3 u_{2,3} u_{1,3} e^{-2u_3}] \theta^{1,2} \\
 = & (u_{2,1} (u_{2,0} - u_{1,0}) - u_{2,0} u_{0,1}) e^{-u_0 - u_1} \theta^{0,2} \\
 & + [(u_{2,11} + u_{2,1} (u_{2,1} - u_{1,1})) e^{-2u_1} + \eta_1 \eta_0 u_{2,0} u_{1,0} e^{-2u_0}] \theta^{1,2} \\
 = & (r^{-1} (\dot{b} - \dot{v}) - \dot{b} u') e^{-u-v} \theta^{0,2} + [-r^{-1} v' e^{-2v} - \dot{v} \dot{b} e^{-2u}] \theta^{1,2}
 \end{aligned}$$

- We have :

$$\begin{aligned}
\Omega_1^3 &= (u_{3,10} + u_{3,1}(u_{3,0} - u_{1,0}) - u_{3,0}u_{0,1}) e^{-u_0 - u_1} \theta^{0,3} \\
&\quad + \eta_3 \eta_1 (u_{0,3}u_{1,0} - u_{1,30} + u_{1,3}(u_{3,0} - u_{1,0})) e^{-u_3 - u_0} \theta^{0,1} \\
&\quad + (u_{3,12} + u_{3,1}(u_{3,2} - u_{1,2}) - u_{3,2}u_{2,1}) e^{-u_2 - u_1} \theta^{2,3} \\
&\quad + \eta_3 \eta_1 (u_{2,3}u_{1,2} - u_{1,32} + u_{1,3}(u_{3,2} - u_{1,2})) e^{-u_3 - u_2} \theta^{2,1} \\
&\quad + [(u_{3,11} + u_{3,1}(u_{3,1} - u_{1,1})) e^{-2u_1} + \eta_3 \eta_1 (u_{1,33} + u_{1,3}(u_{1,3} - u_{3,3})) e^{-2u_3}] \theta^{1,3} \\
&\quad + [\eta_1 \eta_0 u_{3,0} u_{1,0} e^{-2u_0} + \eta_1 \eta_2 u_{3,2} u_{1,2} e^{-2u_2}] \theta^{1,3} \\
&= (u_{3,1}(u_{3,0} - u_{1,0}) - u_{3,0}u_{0,1}) e^{-u_0 - u_1} \theta^{0,3} + (u_{3,1}u_{3,2} - u_{3,2}u_{2,1}) e^{-u_2 - u_1} \theta^{2,3} \\
&\quad + [(u_{3,11} + u_{3,1}(u_{3,1} - u_{1,1})) e^{-2u_1} + \eta_1 \eta_0 u_{3,0} u_{1,0} e^{-2u_0}] \theta^{1,3} \\
&= (r^{-1}(\dot{b} - \dot{v}) - \dot{b}u') e^{-u-v} \theta^{0,3} + [-r^{-1}v' e^{-2v} - \dot{v}b e^{-2u}] \theta^{1,3}
\end{aligned}$$

- We have :

$$\begin{aligned}
\Omega_2^3 &= (u_{3,20} + u_{3,2}(u_{3,0} - u_{2,0}) - u_{3,0}u_{0,2}) e^{-u_0 - u_2} \theta^{0,3} \\
&\quad + \eta_3 \eta_2 (u_{0,3}u_{2,0} - u_{2,30} + u_{2,3}(u_{3,0} - u_{2,0})) e^{-u_3 - u_0} \theta^{0,2} \\
&\quad + (u_{3,21} + u_{3,2}(u_{3,1} - u_{2,1}) - u_{3,1}u_{1,2}) e^{-u_1 - u_2} \theta^{1,3} \\
&\quad + \eta_3 \eta_2 (u_{1,3}u_{2,1} - u_{2,31} + u_{2,3}(u_{3,1} - u_{2,1})) e^{-u_3 - u_1} \theta^{1,2} \\
&\quad + [(u_{3,22} + u_{3,2}(u_{3,2} - u_{2,2})) e^{-2u_2} + \eta_3 \eta_2 (u_{2,33} + u_{2,3}(u_{2,3} - u_{3,3})) e^{-2u_3}] \theta^{2,3} \\
&\quad + [\eta_2 \eta_0 u_{3,0} u_{2,0} e^{-2u_0} + \eta_2 \eta_1 u_{3,1} u_{2,1} e^{-2u_1}] \theta^{2,3} \\
&= u_{3,2}(u_{3,0} - u_{2,0}) e^{-u_0 - u_2} \theta^{0,3} + u_{3,2}(u_{3,1} - u_{2,1}) e^{-u_1 - u_2} \theta^{1,3} \\
&\quad + [(u_{3,22} + (u_{3,2})^2) e^{-2u_2} + \eta_2 \eta_0 u_{3,0} u_{2,0} e^{-2u_0} + \eta_2 \eta_1 u_{3,1} u_{2,1} e^{-2u_1}] \theta^{2,3} \\
&= [-r^{-2}e^{-2b} - (\dot{b})^2 e^{-2u} + r^{-2}e^{-2v}] \theta^{2,3}
\end{aligned}$$

#### 4.5 Practical computations in $\mathcal{C}_\perp$ and $\mathcal{C}_\perp^*$

We start with a link between  $\mathbb{R}^i_{jkl}$  and  $\Omega_j^i$ .

Lemma 4.5.1: Link with the 2-forms  $\Omega_j^i$

We have:

$$\mathbb{R}^i_{jkl} = \Omega_j^i(\theta_k, \theta_l).$$

*Proof.* By definition of  $\mathbb{R}^i_{jkl}$ , we have:

$$\begin{aligned}
\mathbb{R}^i_{jkl} &= \theta^i(R(\theta_k, \theta_l)\theta_j) \\
&= \theta^i(\Omega_j^l(\theta_k, \theta_l)\theta_l) \\
&= \Omega_j^l(\theta_k, \theta_l)\delta_l^i \\
&= \Omega_j^i(\theta_k, \theta_l)
\end{aligned}$$

□

Therefore, we have:

$$\begin{aligned}
\mathbb{R}^i_{jil} &= \Omega_j^i(\theta_i, \theta_l) \\
&= d\varpi^i_j(\theta_i, \theta_l) + \varpi^i_p \wedge \varpi^p_j(\theta_i, \theta_l)
\end{aligned}$$

The following lemma gives the value of each term on the right-hand side.

**Lemma 4.5.2: Preliminary lemma for the calculation of Riemann tensors**

Let  $i, j, k, l$  such that  $\{0, 1, 2, 3\} = \{i, j, k, l\}$  and  $p \neq i, j$ .

(i) (a) We have:

$$\varpi_p^i \wedge \varpi_j^p(\theta_i, \theta_l) = \delta_l^p u_{i,l} u_{l,j} e^{-u_l - u_j}.$$

(b) We have:

$$\varpi_p^i \wedge \varpi_j^p(\theta_i, \theta_j) = -\eta_p \eta_j u_{i,p} u_{j,p} e^{-2u_p}.$$

(ii) (a) We have:

$$d\varpi_j^i(\theta_i, \theta_l) = -(u_{i,jl} + u_{i,j}(u_{i,l} - u_{j,l})) e^{-u_j - u_l}.$$

(b) We have:

$$d\varpi_j^i(\theta_i, \theta_j) = -(u_{i,jj} + u_{i,j}(u_{i,j} - u_{j,j})) e^{-2u_j} - \eta_i \eta_j (u_{j,ii} + u_{j,i}(u_{j,i} - u_{i,i})) e^{-2u_i}.$$

*Proof.* (i) (a) We have:

$$\begin{aligned} \varpi_p^i \wedge \varpi_j^p(\theta_i, \theta_l) &= u_{i,p} u_{p,j} e^{-u_p - u_j} \theta^{i,p}(\theta_i, \theta_l) - \eta_p \eta_j u_{i,p} u_{j,p} e^{-2u_p} \theta^{i,j}(\theta_i, \theta_l) \\ &\quad + \eta_i \eta_j u_{p,i} u_{j,p} e^{-u_i - u_p} \theta^{p,j}(\theta_i, \theta_l) \\ &= u_{i,p} u_{p,j} e^{-u_p - u_j} \theta^{i,p}(\theta_i, \theta_l) \\ &= \delta_l^p u_{i,l} u_{l,j} e^{-u_l - u_j} \end{aligned}$$

(b) We have:

$$\begin{aligned} \varpi_p^i \wedge \varpi_j^p(\theta_i, \theta_j) &= u_{i,p} u_{p,j} e^{-u_p - u_j} \theta^{i,p}(\theta_i, \theta_j) - \eta_p \eta_j u_{i,p} u_{j,p} e^{-2u_p} \theta^{i,j}(\theta_i, \theta_j) \\ &\quad + \eta_i \eta_j u_{p,i} u_{j,p} e^{-u_i - u_p} \theta^{p,j}(\theta_i, \theta_j) \\ &= -\eta_p \eta_j u_{i,p} u_{j,p} e^{-2u_p} \theta^{i,j}(\theta_i, \theta_j) \\ &= -\eta_p \eta_j u_{i,p} u_{j,p} e^{-2u_p} \end{aligned}$$

(ii) (a) We have:

$$\begin{aligned} d\varpi_j^i(\theta_i, \theta_l) &= (u_{i,jq} + u_{i,j}(u_{i,q} - u_{j,q})) e^{-u_j - u_q} \theta^{q,i}(\theta_i, \theta_l) \\ &\quad - \eta_i \eta_j (u_{j,iq} + u_{j,i}(u_{j,q} - u_{i,q})) e^{-u_i - u_q} \theta^{q,j}(\theta_i, \theta_l) \\ &= (u_{i,jq} + u_{i,j}(u_{i,q} - u_{j,q})) e^{-u_j - u_q} \theta^{q,i}(\theta_i, \theta_l) \\ &= -(u_{i,jl} + u_{i,j}(u_{i,l} - u_{j,l})) e^{-u_j - u_l} \end{aligned}$$

(b) We have:

$$\begin{aligned} d\varpi_j^i(\theta_i, \theta_j) &= (u_{i,jq} + u_{i,j}(u_{i,q} - u_{j,q})) e^{-u_j - u_q} \theta^{q,i}(\theta_i, \theta_j) \\ &\quad - \eta_i \eta_j (u_{j,iq} + u_{j,i}(u_{j,q} - u_{i,q})) e^{-u_i - u_q} \theta^{q,j}(\theta_i, \theta_j) \\ &= -\delta_j^q (u_{i,jq} + u_{i,j}(u_{i,q} - u_{j,q})) e^{-u_j - u_q} - \delta_i^q \eta_i \eta_j (u_{j,iq} + u_{j,i}(u_{j,q} - u_{i,q})) e^{-u_i - u_q} \\ &= -(u_{i,jj} + u_{i,j}(u_{i,j} - u_{j,j})) e^{-2u_j} - \eta_i \eta_j (u_{j,ii} + u_{j,i}(u_{j,i} - u_{i,i})) e^{-2u_i} \end{aligned}$$

□

We deduce the exact values of the components of the Riemann tensor  $\mathbb{R}_{ijkl}^i$ .

**Theorem 4.5.3: Values of the Riemann tensor**

Let  $i, j, k, l$  be such that  $\{0, 1, 2, 3\} = \{i, j, k, l\}$ .

(i) We have:

$$\begin{aligned}\mathbb{R}^i_{jil} &= -(u_{i,jl} + u_{i,j}(u_{i,l} - u_{j,l}) - u_{i,l}u_{l,j})e^{-u_j - u_i} \\ \mathbb{R}^j_{iil} &= -\eta_i \eta_j \mathbb{R}^i_{jil} \\ \mathbb{R}^j_{ili} &= -\mathbb{R}^j_{iil} \\ \mathbb{R}^i_{jli} &= -\mathbb{R}^i_{jil}\end{aligned}$$

(ii) We have:

$$\begin{aligned}\mathbb{R}^i_{jij} &= -(u_{i,jj} + u_{i,j}(u_{i,j} - u_{j,j}))e^{-2u_j} - \eta_i \eta_j (u_{j,ii} + u_{j,i}(u_{j,i} - u_{i,i}))e^{-2u_i} \\ &\quad - \eta_k \eta_j u_{i,k} u_{j,k} e^{-2u_k} - \eta_l \eta_j u_{i,l} u_{j,l} e^{-2u_l}\end{aligned}$$

and:

$$\mathbb{R}^j_{iij} = -\eta_i \eta_j \mathbb{R}^i_{jij}.$$

(iii) All other values of the Riemann tensor are zero.

*Proof.* (i) We have:

$$\begin{aligned}\mathbb{R}^i_{jil} &= \Omega_j^i(\theta_i, \theta_l) \\ &= d\varpi^i_j(\theta_i, \theta_l) + \varpi^i_p \wedge \varpi^p_j(\theta_i, \theta_l) \\ &= d\varpi^i_j(\theta_i, \theta_l) + \varpi^i_k \wedge \varpi^k_j(\theta_i, \theta_l) + \varpi^i_l \wedge \varpi^l_j(\theta_i, \theta_l) \\ &= -(u_{i,jl} + u_{i,j}(u_{i,l} - u_{j,l}))e^{-u_j - u_i} + u_{i,l}u_{l,j}e^{-u_i - u_j} \\ &= -(u_{i,jl} + u_{i,j}(u_{i,l} - u_{j,l}) - u_{i,l}u_{l,j})e^{-u_j - u_i}\end{aligned}$$

(ii) We have:

$$\begin{aligned}\mathbb{R}^i_{jij} &= \Omega_j^i(\theta_i, \theta_j) \\ &= d\varpi^i_j(\theta_i, \theta_j) + \varpi^i_p \wedge \varpi^p_j(\theta_i, \theta_j) \\ &= d\varpi^i_j(\theta_i, \theta_j) + \varpi^i_k \wedge \varpi^k_j(\theta_i, \theta_j) + \varpi^i_l \wedge \varpi^l_j(\theta_i, \theta_j) \\ &= -(u_{i,jj} + u_{i,j}(u_{i,j} - u_{j,j}))e^{-2u_j} - \eta_i \eta_j (u_{j,ii} + u_{j,i}(u_{j,i} - u_{i,i}))e^{-2u_i} \\ &\quad - \eta_k \eta_j u_{i,k} u_{j,k} e^{-2u_k} - \eta_l \eta_j u_{i,l} u_{j,l} e^{-2u_l}\end{aligned}$$

(iii) Since  $\Omega_j^i = 0$ , we have for all  $p, q$ :

$$\mathbb{R}^i_{ipq} = 0.$$

Since  $\Omega_j^i$  is an alternating form, we have:

$$\mathbb{R}^p_{qii} = 0.$$

Using the general expression found for  $\Omega_j^i$  in Proposition 4.4.4, we have  $\mathbb{R}^i_{jkl} = 0$ .

□

The values can be deduced in the case of the metric  $h$ .

**Example 4.5.4: Sequence 6 of the metric  $h$** 

We resume the example with the metric  $h$  (see 2.4.1, 2.5.1, 3.5.5, 3.6.3, 3.6.5, and 4.4.5) with:

$$\begin{aligned} u_0(t, r) &:= u(t, r) & u_1(t, r) &:= v(t, r) \\ u_2(t, r) &:= b(t) + \ln(r) & u_3(t, r, \vartheta) &:= b(t) + \ln(r) + \ln \sin(\vartheta) \end{aligned}$$

We consider two cases:

$$(1) \mathbb{R}^i_{jij} \quad , \quad (2) \mathbb{R}^i_{jil}.$$

**(1) Calculations of  $\mathbb{R}^i_{jij}$ .**

- Case where  $i := 0$ . We have:

$$\begin{aligned} \mathbb{R}^0_{101} &= -(u_{0,11} + u_{0,1}(u_{0,1} - u_{1,1})) e^{-2u_1} \\ &\quad - \eta_0 \eta_1 (u_{1,00} + u_{1,0}(u_{1,0} - u_{0,0})) e^{-2u_0} \\ &\quad - \eta_2 \eta_1 u_{0,2} u_{1,2} e^{-2u_2} - \eta_3 \eta_1 u_{0,3} u_{1,3} e^{-2u_3} \\ &= -(u_{0,11} + u_{0,1}(u_{0,1} - u_{1,1})) e^{-2u_1} - \eta_0 \eta_1 (u_{1,00} + u_{1,0}(u_{1,0} - u_{0,0})) e^{-2u_0} \\ &= -(u'' + u'(u' - v')) e^{-2v-2b} + (\ddot{v} + \dot{v}(\dot{v} - \dot{u})) e^{-2b-2u} \end{aligned}$$

- Case where  $j := 2, 3$  ( $k := 3, 2$ ). We have:

$$\begin{aligned} \mathbb{R}^0_{j0j} &= -(u_{0,jj} + u_{0,j}(u_{0,j} - u_{j,j})) e^{-2u_j} \\ &\quad - \eta_0 \eta_j (u_{j,00} + u_{j,0}(u_{j,0} - u_{0,0})) e^{-2u_0} \\ &\quad - \eta_1 \eta_j u_{0,1} u_{j,1} e^{-2u_1} - \eta_k \eta_j u_{0,k} u_{j,k} e^{-2u_k} \\ &= -\eta_0 \eta_j (u_{j,00} + u_{j,0}(u_{j,0} - u_{0,0})) e^{-2u_0} - \eta_1 \eta_j u_{0,1} u_{j,1} e^{-2u_1} \\ &= (\ddot{b} + \dot{b}(\dot{b} - \dot{u})) e^{-2u} - r^{-1} u' e^{-2v} \\ \mathbb{R}^1_{j1j} &= -(u_{1,jj} + u_{1,j}(u_{1,j} - u_{j,j})) e^{-2u_j} \\ &\quad - \eta_1 \eta_j (u_{j,11} + u_{j,1}(u_{j,1} - u_{1,1})) e^{-2u_1} \\ &\quad - \eta_0 \eta_j u_{1,0} u_{j,0} e^{-2u_0} - \eta_k \eta_j u_{1,k} u_{j,k} e^{-2u_k} \\ &= -\eta_1 \eta_j (u_{j,11} + u_{j,1}(u_{j,1} - u_{1,1})) e^{-2u_1} - \eta_0 \eta_j u_{1,0} u_{j,0} e^{-2u_0} \\ &= -r^{-1} v' e^{-2v} + \dot{v} \dot{b} e^{-2u} \end{aligned}$$

- Case where  $i := 2$ . We have:

$$\begin{aligned} \mathbb{R}^2_{323} &= -(u_{2,33} + u_{2,3}(u_{2,3} - u_{3,3})) e^{-2u_3} \\ &\quad - \eta_2 \eta_3 (u_{3,22} + u_{3,2}(u_{3,2} - u_{2,2})) e^{-2u_2} \\ &\quad - \eta_0 \eta_3 u_{2,0} u_{3,0} e^{-2u_0} - \eta_1 \eta_3 u_{2,1} u_{3,1} e^{-2u_1} \\ &= -\eta_2 \eta_3 (u_{3,22} + (u_{3,2})^2) e^{-2u_2} \\ &\quad - \eta_0 \eta_3 u_{2,0} u_{3,0} e^{-2u_0} - \eta_1 \eta_3 u_{2,1} u_{3,1} e^{-2u_1} \\ &= r^{-2} e^{-2b} + (\dot{b})^2 e^{-2u} - r^{-2} e^{-2v} \end{aligned}$$

The other terms are obtained by symmetry.

**(2) Calculations of  $\mathbb{R}^i_{jil}$ .** We have:

$$\begin{aligned} \mathbb{R}^i_{jil} &= -(u_{i,jl} + u_{i,j}(u_{i,l} - u_{j,l}) + u_{i,l} u_{l,j}) e^{-u_j - u_l} \\ &= -(u_{i,j}(u_{i,l} - u_{j,l}) + u_{i,l} u_{l,j}) e^{-u_j - u_l} \end{aligned}$$

- Case where  $j, l = 3$ . We have:

$$\begin{aligned}\mathbb{R}^i_{3il} &= -\left(u_{i,3}(u_{i,l} - u_{3,l}) + u_{i,l}u_{l,3}\right)e^{-u_3 - u_l} = 0 \\ \mathbb{R}^i_{ji3} &= -\left(u_{i,j}(u_{i,3} - u_{j,3}) + u_{i,3}u_{l,j}\right)e^{-u_j - u_3} = 0\end{aligned}$$

- Case where  $i := 0, 1$  (thus  $j, l = 1, 0 = 1 - i$ ). We have:

$$\begin{aligned}\mathbb{R}^i_{2il} &= -\left(u_{i,2}(u_{i,l} - u_{2,l}) + u_{i,l}u_{l,2}\right)e^{-u_2 - u_l} = 0 \\ \mathbb{R}^i_{ji2} &= -\left(u_{i,j}(u_{i,2} - u_{j,2}) + u_{i,2}u_{2,j}\right)e^{-u_j - u_2} = 0\end{aligned}$$

- Case where  $i := 3$ . For  $j, l = 0, 1$  we have:

$$\begin{aligned}\mathbb{R}^3_{230} &= -\left(u_{3,2}(u_{3,l} - u_{2,l}) - u_{3,l}u_{l,2}\right)e^{-u_2 - u_l} \\ &= -\left(u_{3,2}(u_{2,l} - u_{2,l})\right)e^{-u_2 - u_l} \\ &= 0 \\ \mathbb{R}^3_{j32} &= -\left(u_{3,j}(u_{3,2} - u_{j,2}) - u_{3,2}u_{2,j}\right)e^{-u_j - u_2} \\ &= -\left(u_{3,j}u_{3,2} - u_{3,2}u_{2,j}\right)e^{-u_j - u_2} \\ &= -\left(u_{2,j}u_{3,2} - u_{3,2}u_{2,j}\right)e^{-u_j - u_2} \\ &= 0\end{aligned}$$

- Case where  $i := 2, 3$ . We have:

$$\begin{aligned}\mathbb{R}^i_{0i1} &= -\left(u_{i,0}(u_{i,1} - u_{0,1}) - u_{i,1}u_{1,0}\right)e^{-u_0 - u_1} \\ &= -\left(\dot{b}(r^{-1} - u') - r^{-1}\dot{v}\right)e^{-u - v} \\ \mathbb{R}^i_{1i0} &= -\left(u_{i,1}(u_{i,0} - u_{1,0}) - u_{i,0}u_{0,1}\right)e^{-u_1 - u_0} \\ &= -\left(r^{-1}(\dot{b} - \dot{v}) - u'\dot{b}\right)e^{-u - v}\end{aligned}$$

These are the only non-zero terms of the form  $\mathbb{R}^i_{jil}$ .

## 5 Tensor of Ricci

### 5.1 Generalities

The curvature tensor  $\mathcal{R}$  is of type  $(1, 3)$ , so we can contract the upper index with the second index.

**Definition 5.1.1:** The Ricci tensor

The **Ricci tensor** is defined by:

$$\mathcal{R}ic = [\mathcal{R}]_2^1$$

That is, for any vector fields  $Y$  and  $Z$ :

$$\mathcal{R}ic(Y, Z) = [\mathcal{R}]_2^1(Y, Z).$$

## 5.2 Computation of the components in $\mathcal{E}$ and $\mathcal{E}^*$

**Definition 5.2.1:** The Ricci tensor

The components of the Ricci tensor in  $\mathcal{E}$  and  $\mathcal{E}^*$  are:

$$\mathbf{R}_{jl} := \mathcal{R}ic(e_j, e_l).$$

Therefore, we have the decomposition:

$$\mathcal{R}ic = \mathbf{R}_{jl} e_j \otimes e_l.$$

Next, we demonstrate the usual properties of the Ricci tensor. For part (ii), note that for any vector fields  $Y$  and  $Z$ , we have a linear mapping from  $TM$  (at each point  $p$  of  $M$ , we have a linear mapping from  $T_pM$ ):

$$X \mapsto \mathcal{R}(X, Y)Z.$$

Thus, we can consider the trace of this linear mapping.

**Proposition 5.2.2:** Usual properties

Let  $j, k, l, n \in \{0, 1, 2, 3\}$  and two vector fields  $Y$  and  $Z$ .

(i) **(Symmetry)** We have:

$$\mathbf{R}_{jl} = \mathbf{R}_{lj}.$$

(ii) (a) We have:

$$\mathcal{R}ic(Y, Z) = \text{tr}(X \mapsto \mathcal{R}(X, Y)Z).$$

(b) We have:

$$\mathbf{R}_{jl} = \mathbf{R}^i{}_{jil}.$$

(iii) **(Contracted Bianchi identity)** We have:

$$\nabla_n \mathbf{R}_{jl} - \nabla_l \mathbf{R}_{jn} = -\nabla_k \mathbf{R}^k{}_{jln}.$$

*Proof.* (i) We use the first three points of Proposition 4.2.3.

• By point (i), we have:

$$\mathbf{R}_{ijkl} + \mathbf{R}_{iklj} + \mathbf{R}_{iljk} = 0$$

• By point (iii), we have:

$$\begin{aligned} g^{ki} \mathbf{R}_{iklj} &= g^{ik} \mathbf{R}_{iklj} \\ &= -g^{ik} \mathbf{R}_{kilj} \\ &= -g^{ki} \mathbf{R}_{ikjl} \end{aligned}$$

Thus, we have:

$$g^{ki} \mathbf{R}_{iklj} = 0.$$

• By point (ii), we have:

$$\begin{aligned} 0 &= g^{ki} (\mathbf{R}_{ijkl} + \mathbf{R}_{iklj} + \mathbf{R}_{iljk}) \\ &= \mathbf{R}^k{}_{jkl} + 0 + \mathbf{R}^k{}_{lj k} \\ &= \mathbf{R}_{jl} - \mathbf{R}^k{}_{lkj} \\ &= \mathbf{R}_{jl} - \mathbf{R}_{lj} \end{aligned}$$

Thus, we have:

$$\mathbf{R}_{jl} = \mathbf{R}_{lj}.$$

- (ii) We prove both points simultaneously. Recall that for any endomorphism  $\phi$  of  $TM$ , the **trace of  $\phi$**  is defined by:

$$\text{tr}(\phi) := e^i(\phi(e_i)).$$

The trace is independent of the chosen basis.

Since:

$$\mathcal{R} = \mathbf{R}^i_{mkn} e_i \otimes e^m \otimes e^k \otimes e^n$$

we have:

$$\begin{aligned} [\mathcal{R}]_2^1 &= \mathbf{R}^i_{mkn} e^k(e_i) e^m \otimes e^n \\ &= \mathbf{R}^i_{mkn} \delta_i^k e^m \otimes e^n \\ &= \mathbf{R}^i_{min} e^m \otimes e^n \end{aligned}$$

Thus, we have:

$$\begin{aligned} \mathcal{R}\text{ic}(e_j, e_l) &= [\mathcal{R}]_2^1(e_j, e_l) \\ &= (\mathbf{R}^i_{min} e^m \otimes e^n)(e_j, e_l) \\ &= \mathbf{R}^i_{mkn} e^m(e_j) e^n(e_l) \\ &= \mathbf{R}^i_{min} \delta_j^m \delta_l^n \\ &= \mathbf{R}^i_{jil} \end{aligned}$$

On the other hand, we have:

$$\begin{aligned} \text{tr}(X \mapsto \mathcal{R}(X, e_j)e_l) &= e^p(\mathcal{R}(e_p, e_p)e_l) \\ &= e^p(\mathbf{R}^i_{mkn} e^m(e_l) e^k(e_p) e^n(e_j) e_i) \\ &= \mathbf{R}^i_{mkn} \delta_l^m \delta_p^k \delta_j^n e^p(e_i) \\ &= \mathbf{R}^i_{lpj} \delta_i^p \\ &= \mathbf{R}^i_{lij} \\ &= \mathbf{R}^i_{jil} \end{aligned}$$

Hence, the result follows from the bilinearity of  $\mathcal{R}\text{ic}$  and  $(Y, Z) \mapsto \text{tr}(X \mapsto \mathcal{R}(X, Y)Z)$ .

- (iii) We use points (ii) and (iv) of Proposition 4.2.3.

- By point (iv), we have:

$$\nabla_n \mathbf{R}_{ijkl} + \nabla_k \mathbf{R}_{ijln} + \nabla_l \mathbf{R}_{ijnk} = 0.$$

$$\nabla_n \mathbf{R}_{ijkl} + \nabla_j \mathbf{R}_{iknl} + \nabla_k \mathbf{R}_{injl} = 0.$$

- Thus, with point (ii), we have:

$$\begin{aligned} 0 &= g^{ki} (\nabla_n \mathbf{R}_{ijkl} + \nabla_k \mathbf{R}_{ijln} + \nabla_l \mathbf{R}_{ijnk}) \\ &= \nabla_n \mathbf{R}^k_{jkl} + \nabla_k \mathbf{R}^k_{jln} + \nabla_l \mathbf{R}^k_{jnk} \\ &= \nabla_n \mathbf{R}_{jl} + \nabla_k \mathbf{R}^k_{jln} - \nabla_l \mathbf{R}^k_{jkn} \end{aligned}$$

Thus, we have:

$$\nabla_n \mathbf{R}_{jl} - \nabla_l \mathbf{R}_{jn} = -\nabla_k \mathbf{R}^k_{jln}.$$

□

**Example 5.2.3: Components of the Ricci tensor in bases  $\mathcal{C}$  and  $\mathcal{C}_\perp$** 

(i) The **components of the Ricci tensor in  $\mathcal{C}$  and  $\mathcal{C}^*$**  are given by:

$$R_{ij} := R^k{}_{ikj}.$$

Thus, we have the decomposition:

$$\mathcal{R}ic = R_{jl} dx^j \otimes dx^l.$$

(ii) The **components of the Ricci tensor in  $\mathcal{C}_\perp$  and  $\mathcal{C}_\perp^*$**  are given by:

$$\mathbb{R}_{ij} := \mathbb{R}^k{}_{ikj}.$$

Thus, we have the decomposition:

$$\mathcal{R}ic = \mathbb{R}_{jl} \theta^j \otimes \theta^l.$$

We also have:

$$\mathbb{R}_{ij} = e^{-u_i - u_j} R_{ij}.$$

**5.3 Calculation of components in  $\mathcal{C}$  and  $\mathcal{C}^*$** **Lemma 5.3.1: Usual properties**

We have:

$$R_{jl} = \partial_i \Gamma_{jl}^i - \partial_l \Gamma_{ij}^i + \Gamma_{jl}^p \Gamma_{ip}^i - \Gamma_{ij}^p \Gamma_{lp}^i.$$

*Proof.* By Proposition 4.3.1, we have:

$$\begin{aligned} R_{jl} &= R^i{}_{jil} \\ &= \partial_i \Gamma_{jl}^i - \partial_l \Gamma_{ij}^i + \Gamma_{jl}^p \Gamma_{ip}^i - \Gamma_{ij}^p \Gamma_{lp}^i \end{aligned}$$

□

**Notation 5.3.2: Hat notation**

We denote:

$$\hat{u}_{i,i} := \sum_{k \neq i} u_{k,i} \quad \hat{u}_{i,jj} := \sum_{k \neq i} u_{k,jj} \quad \hat{u}_{i,i}^2 := \sum_{k \neq i} u_{k,i}^2 \quad \hat{u}_{ij,j} := \sum_{k \neq i,j} u_{k,j} \quad \hat{u}_{ij,jj} := \sum_{k \neq i,j} u_{k,jj}$$

**Theorem 5.3.3: Exact values of  $R_{jl}$** 

Let  $j, l \in \{0, 1, 2, 3\}$  be distinct. We have:

$$\begin{aligned} R_{jl} &= - \left[ \sum_{p \neq j,l} u_{p,jl} - u_{j,l} u_{p,j} - u_{l,j} u_{p,l} + u_{p,j} u_{p,l} \right] \\ R_{jj} &= - \left( \hat{u}_{j,jj} + \hat{u}_{j,j}^2 - u_{j,j} \hat{u}_{j,j} \right) - \eta_j \sum_{p \neq j} \eta_p [u_{j,pp} + u_{j,p} (\hat{u}_{p,p} - u_{p,p})] e^{2u_j - 2u_p} \end{aligned}$$

*Proof.* We have two cases.

- We have:

$$\begin{aligned} \mathbb{R}_{jl} &= \partial_i \Gamma_{jl}^i - \partial_j \Gamma_{li}^i + \Gamma_{ip}^i \Gamma_{jl}^p - \Gamma_{jp}^i \Gamma_{il}^p \\ &= \partial_j \Gamma_{jl}^j + \partial_l \Gamma_{jl}^l - \partial_j \Gamma_{li}^i + \Gamma_{ij}^i \Gamma_{jl}^j + \Gamma_{il}^i \Gamma_{jl}^l - \Gamma_{jp}^i \Gamma_{il}^p \\ &= - \left[ \sum_{p \neq j, l} u_{p,jl} - u_{j,l} u_{p,j} - u_{l,j} u_{p,l} + u_{p,j} u_{p,l} \right] \end{aligned}$$

- We have:

$$\begin{aligned} \mathbb{R}_{jj} &= \partial_i \Gamma_{jj}^i - \partial_j \Gamma_{ji}^i + \Gamma_{ip}^i \Gamma_{jj}^p - \Gamma_{ji}^i \Gamma_{ij}^i - \Gamma_{jj}^i \Gamma_{ij}^j \\ &= \partial_i \left( -\eta_j \eta_i u_{j,i} e^{2u_j - 2u_i} \right) - \partial_j u_{i,j} + u_{i,p} \left( -\eta_j \eta_p u_{j,p} e^{2u_j - 2u_p} \right) - u_{i,j}^2 - \left( -\eta_j \eta_i u_{j,i} e^{2u_j - 2u_i} \right) u_{j,i} \\ &= -\eta_j \eta_i \left( u_{j,ii} + 2u_{j,i} (u_{j,i} - u_{i,i}) \right) e^{2u_j - 2u_i} - u_{i,jj} - \eta_j \eta_p u_{i,p} u_{j,p} e^{2u_j - 2u_p} - u_{i,j}^2 + \eta_j \eta_i u_{j,i}^2 e^{2u_j - 2u_i} \\ &= - \left( \hat{u}_{j,jj} + \hat{u}_{j,j}^2 - u_{j,j} \hat{u}_{j,j} \right) - \eta_j \sum_{p \neq j} \eta_p \left[ u_{j,pp} + u_{j,p} (\hat{u}_{p,p} - u_{p,p}) \right] e^{-2u_p + 2u_j} \end{aligned}$$

□

We will calculate the values of the components of the Ricci tensor in the following subsection.

#### 5.4 Calculation of components in $\mathcal{C}_\perp$ and $\mathcal{C}_\perp^*$

Lemma 5.4.1: Usual properties

We have:

$$\mathbb{R}_{jl} = d\omega^i_j (\theta_i, \theta_l) + \omega^i_k \wedge \omega^k_j (\theta_i, \theta_l).$$

*Proof.* We have directly:

$$\begin{aligned} \mathbb{R}_{jl} &= \Omega_j^i (\theta_i, \theta_l) \\ &= d\omega^i_j (\theta_i, \theta_l) + \omega^i_k \wedge \omega^k_j (\theta_i, \theta_l) \end{aligned}$$

□

We deduce the values of the components of the Ricci tensor.

Theorem 5.4.2: Exact values of  $\mathbb{R}_{jl}$

Let  $i, j, k, l$  such that  $\{0, 1, 2, 3\} = \{i, j, k, l\}$  and  $p \neq j, l$ . We have:

$$\begin{aligned} \mathbb{R}_{jl} &= - \left[ \sum_{p \neq j, l} u_{p,jl} - u_{j,l} u_{p,j} - u_{l,j} u_{p,l} + u_{p,j} u_{p,l} \right] e^{-u_j - u_l} \\ \mathbb{R}_{jj} &= - \left( \hat{u}_{j,jj} + \hat{u}_{j,j}^2 - u_{j,j} \hat{u}_{j,j} \right) e^{-2u_j} - \eta_j \sum_{p \neq j} \eta_p \left[ u_{j,pp} + u_{j,p} (\hat{u}_{p,p} - u_{p,p}) \right] e^{-2u_p} \end{aligned}$$

*Proof.* We have two cases.

- Since:

$$\mathbb{R}^j_{jll} = 0 \quad , \quad \mathbb{R}^l_{jll}$$

we have:

$$\begin{aligned}\mathbb{R}_{jl} &= \mathbb{R}^p_{jpl} \\ &= \sum_{p \neq j, l} \mathbb{R}^p_{jpl} \\ &= - \left[ \sum_{p \neq j, l} u_{p,jl} - u_{j,l}u_{p,j} - u_{l,j}u_{p,l} + u_{p,j}u_{p,l} \right] e^{-u_j - u_l}\end{aligned}$$

• Since:

$$\mathbb{R}^j_{jjj} = 0$$

we have:

$$\begin{aligned}\mathbb{R}_{jj} &= \mathbb{R}^p_{jpp} \\ &= \mathbb{R}^i_{jij} + \mathbb{R}^k_{jkk} + \mathbb{R}^l_{jll} \\ &= - (u_{i,jj} + u_{i,j}(u_{i,j} - u_{j,j})) e^{-2u_j} - \eta_i \eta_j (u_{j,ii} + u_{j,i}(u_{j,i} - u_{i,i})) e^{-2u_i} \\ &\quad - \eta_k \eta_j u_{i,k} u_{j,k} e^{-2u_k} - \eta_l \eta_j u_{i,l} u_{j,l} e^{-2u_l} \\ &\quad - (u_{k,jj} + u_{k,j}(u_{k,j} - u_{j,j})) e^{-2u_j} - \eta_k \eta_j (u_{j,kk} + u_{j,k}(u_{j,k} - u_{k,k})) e^{-2u_k} \\ &\quad - \eta_i \eta_j u_{k,i} u_{j,i} e^{-2u_i} - \eta_l \eta_j u_{k,l} u_{j,l} e^{-2u_l} \\ &\quad - (u_{l,jj} + u_{l,j}(u_{l,j} - u_{j,j})) e^{-2u_j} - \eta_l \eta_j (u_{j,ll} + u_{j,l}(u_{j,l} - u_{l,l})) e^{-2u_l} \\ &\quad - \eta_k \eta_j u_{l,k} u_{j,k} e^{-2u_k} - \eta_i \eta_j u_{l,i} u_{j,i} e^{-2u_i} \\ &= - \sum_{p \neq j} (u_{p,jj} + u_{p,j}(u_{p,j} - u_{j,j})) e^{-2u_j} \\ &\quad - \eta_l \eta_j [(u_{j,ll} + u_{j,l}(u_{j,l} + u_{k,l} + u_{i,l} - u_{l,l}))] e^{-2u_l} \\ &= - (\hat{u}_{j,jj} + \hat{u}_{j,j}^2 - u_{j,j} \hat{u}_{j,j}) e^{-2u_j} - \eta_j \sum_{p \neq j} \eta_p [u_{j,pp} + u_{j,p}(\hat{u}_{p,p} - u_{p,p})] e^{-2u_p}\end{aligned}$$

□

We deduce the values of the components for the metric  $h$ .

### Example 5.4.3: Example 7 for the metric $h$

We revisit the example with the metric  $h$  (see 2.4.1, 2.5.1, 3.5.5, 3.6.3, 3.6.5, 4.4.5, and 4.5.4) with:

$$\begin{aligned}u_0(t, r) &:= u(t, r) & u_1(t, r) &:= v(t, r) \\ u_2(t, r) &:= b(t) + \ln(r) & u_3(t, r, \vartheta) &:= b(t) + \ln(r) + \ln \sin(\vartheta)\end{aligned}$$

(1) **First method.** We use Theorem 5.3.3. We deal with the off-diagonal elements first, and then the diagonal elements.

• **Off-diagonal zero terms.** We have:

$$\begin{aligned}\mathbb{R}_{3l} &= - \left[ \sum_{p \neq 3, l} -u_{3,l}u_{p,3} - u_{l,3}u_{p,l} + u_{p,3}u_{p,l} \right] e^{-u_3 - u_l} = 0 \\ \mathbb{R}_{j3} &= - \left[ \sum_{p \neq j, 3} -u_{j,3}u_{p,j} - u_{l,j}u_{p,3} + u_{p,j}u_{p,3} \right] e^{-u_j - u_3} = 0. \\ \mathbb{R}_{12} &= - \left[ \sum_{p \neq 1, 2} -u_{1,2}u_{p,1} - u_{2,1}u_{p,2} + u_{p,1}u_{p,2} \right] e^{-u_1 - u_2} = - [-u_{2,1}u_{3,2} + u_{3,1}u_{3,2}] e^{-u_1 - u_2} = 0\end{aligned}$$

$$\mathbb{R}_{2l} = - \left[ \sum_{p \neq 2, l} -u_{2,l} u_{p,2} - u_{l,2} u_{p,l} + u_{p,2} u_{p,l} \right] e^{-u_2 - u_l} = - \left[ -u_{2,l} u_{3,2} + u_{3,2} u_{3,l} \right] e^{-u_2 - u_l} = 0$$

• **Non-zero terms.** We have :

$$\begin{aligned} \mathbb{R}_{10} &= - \left[ \sum_{p \neq 1, 0} u_{p,10} - u_{1,0} u_{p,1} - u_{0,1} u_{p,0} + u_{p,1} u_{p,0} \right] e^{-u_1 - u_0} \\ &= - \left[ -u_{1,0} u_{2,1} - u_{0,1} u_{2,0} + u_{2,1} u_{2,0} - u_{1,0} u_{3,1} - u_{0,1} u_{3,0} + u_{3,1} u_{3,0} \right] e^{-u_1 - u_0} \\ &= -2 \left[ -u_{1,0} u_{2,1} - u_{0,1} u_{2,0} + u_{2,1} u_{2,0} \right] e^{-u_1 - u_0} \\ &= -2 \left[ -\dot{v} r^{-1} - \dot{b} u' + \dot{b} r^{-1} \right] e^{-u-v} \\ &= 2 \left[ \dot{v} r^{-1} + \dot{b} (u' - r^{-1}) \right] e^{-u-v} \\ \mathbb{R}_{00} &= - \left[ \hat{u}_{0,00} + \hat{u}_{0,0}^2 - u_{0,0} \hat{u}_{0,0} \right] e^{-2u_0} - \eta_0 \eta_1 \left[ u_{0,11} + u_{0,1} (\hat{u}_{1,1} - u_{1,1}) \right] e^{-2u_1} \\ &\quad - \eta_0 \eta_2 \left[ u_{0,22} + u_{0,2} (\hat{u}_{2,2} - u_{2,2}) \right] e^{-2u_2} - \eta_0 \eta_3 \left[ u_{0,33} + u_{0,3} (\hat{u}_{3,3} - u_{3,3}) \right] e^{-2u_3} \\ &= - \left[ \hat{u}_{0,00} + \hat{u}_{0,0}^2 - u_{0,0} \hat{u}_{0,0} \right] e^{-2u_0} - \eta_0 \eta_1 \left[ u_{0,11} + u_{0,1} (\hat{u}_{1,1} - u_{1,1}) \right] e^{-2u_1} \\ &= - \left[ \ddot{v} + 2\ddot{b} + (\dot{v})^2 + 2(\dot{b})^2 - \dot{u}(\dot{v} + 2\dot{b}) \right] e^{-2u} + \left[ u'' + u'(u' + 2r^{-1} - v') \right] e^{-2v} \\ \mathbb{R}_{11} &= - \left[ \hat{u}_{1,11} + \hat{u}_{1,1}^2 - u_{1,1} \hat{u}_{1,1} \right] e^{-2u_1} - \eta_1 \eta_0 \left[ u_{1,00} + u_{1,0} (\hat{u}_{0,0} - u_{0,0}) \right] e^{-2u_0} \\ &\quad - \eta_1 \eta_2 \left[ u_{1,22} + u_{1,2} (\hat{u}_{2,2} - u_{2,2}) \right] e^{-2u_2} - \eta_1 \eta_3 \left[ u_{1,33} + u_{1,3} (\hat{u}_{3,3} - u_{3,3}) \right] e^{-2u_3} \\ &= - \left[ \hat{u}_{1,11} + \hat{u}_{1,1}^2 - u_{1,1} \hat{u}_{1,1} \right] e^{-2u_1} - \eta_1 \eta_0 \left[ u_{1,00} + u_{1,0} (\hat{u}_{0,0} - u_{0,0}) \right] e^{-2u_0} \\ &= - \left[ u'' + (u')^2 - v'(u' + 2r^{-1}) \right] e^{-2v} + \left[ \ddot{v} + \dot{v}(\dot{v} + 2\dot{b} - \dot{u}) \right] e^{-2u} \\ \mathbb{R}_{22} &= - \left( \hat{u}_{2,22} + \hat{u}_{2,2}^2 - u_{2,2} \hat{u}_{2,2} \right) e^{-2u_2} - \eta_2 \eta_0 \left[ u_{2,00} + u_{2,0} (\hat{u}_{0,0} - u_{0,0}) \right] e^{-2u_0} \\ &\quad - \eta_2 \eta_1 \left[ u_{2,11} + u_{2,1} (\hat{u}_{1,1} - u_{1,1}) \right] e^{-2u_1} - \eta_2 \eta_3 \left[ u_{2,33} + u_{2,3} (\hat{u}_{3,3} - u_{3,3}) \right] e^{-2u_3} \\ &= - \left( \hat{u}_{2,22} + \hat{u}_{2,2}^2 \right) e^{-2u_2} - \eta_2 \eta_0 \left[ u_{2,00} + u_{2,0} (\hat{u}_{0,0} - u_{0,0}) \right] e^{-2u_0} \\ &\quad - \eta_2 \eta_1 \left[ u_{2,11} + u_{2,1} (\hat{u}_{1,1} - u_{1,1}) \right] e^{-2u_1} \\ &= r^{-2} e^{-2b} + \left[ \ddot{b} + \dot{b}(\dot{v} + 2\dot{b} - \dot{u}) \right] e^{-2u} - r^{-1} (u' - v' + r^{-1}) e^{-2v} \\ \mathbb{R}_{33} &= - \left( \hat{u}_{3,33} + \hat{u}_{3,3}^2 - u_{3,3} \hat{u}_{3,3} \right) e^{-2u_3} - \eta_3 \eta_0 \left[ u_{3,00} + u_{3,0} (\hat{u}_{0,0} - u_{0,0}) \right] e^{-2u_0} \\ &\quad - \eta_3 \eta_1 \left[ u_{3,11} + u_{3,1} (\hat{u}_{1,1} - u_{1,1}) \right] e^{-2u_1} - \eta_3 \eta_2 \left[ u_{3,22} + u_{3,2} (\hat{u}_{2,2} - u_{2,2}) \right] e^{-2u_2} \\ &= - \eta_3 \eta_0 \left[ u_{3,00} + u_{3,0} (\hat{u}_{0,0} - u_{0,0}) \right] e^{-2u_0} - \eta_3 \eta_1 \left[ u_{3,11} + u_{3,1} (\hat{u}_{1,1} - u_{1,1}) \right] e^{-2u_1} \\ &\quad - \eta_3 \eta_2 \left[ u_{3,22} + u_{3,2} (\hat{u}_{2,2} - u_{2,2}) \right] e^{-2u_2} \\ &= r^{-2} e^{-2b} + \left[ \ddot{b} + \dot{b}(\dot{v} + 2\dot{b} - \dot{u}) \right] e^{-2u} - r^{-1} (u' - v' + r^{-1}) e^{-2v} \end{aligned}$$

(2) **Second method.** We use definition 5.1.1.

• Case  $j := 0$ . We have:

$$\begin{aligned} \mathbb{R}_{00} &= \mathbb{R}_{000}^0 + \mathbb{R}_{010}^1 + \mathbb{R}_{020}^2 + \mathbb{R}_{030}^3 \\ &= \mathbb{R}_{010}^1 + 2\mathbb{R}_{020}^2 \\ &= (u'' + u'(u' - v')) e^{-2v} - (\ddot{v} + \dot{v}(\dot{v} - \dot{u})) e^{-2u} - 2(\ddot{b} + \dot{b}(\dot{b} - \dot{u})) e^{-2u} + 2r^{-1} u' e^{-2v} \\ &= - \left[ \ddot{v} + 2\ddot{b} + (\dot{v})^2 + 2(\dot{b})^2 - \dot{u}(\dot{v} + 2\dot{b}) \right] e^{-2u} + \left[ u'' + u'(u' + 2r^{-1} - v') \right] e^{-2v} \\ \mathbb{R}_{01} &= \mathbb{R}_{001}^0 + \mathbb{R}_{011}^1 + \mathbb{R}_{021}^2 + \mathbb{R}_{031}^3 \\ &= 2\mathbb{R}_{021}^2 \\ &= 2(\dot{b}(u' - r^{-1}) + r^{-1} \dot{v}) e^{-u-v} \\ \mathbb{R}_{02} &= \mathbb{R}_{002}^0 + \mathbb{R}_{012}^1 + \mathbb{R}_{022}^2 + \mathbb{R}_{032}^3 = 0 \end{aligned}$$

$$\mathbb{R}_{03} = \mathbb{R}_{003}^0 + \mathbb{R}_{013}^1 + \mathbb{R}_{023}^2 + \mathbb{R}_{033}^3 = 0$$

- Case  $j := 1$ . We have:

$$\begin{aligned} \mathbb{R}_{10} &= \mathbb{R}_{100}^0 + \mathbb{R}_{110}^1 + \mathbb{R}_{120}^2 + \mathbb{R}_{130}^3 \\ &= 2\mathbb{R}_{120}^2 \\ &= 2\mathbb{R}_{021}^2 \\ &= \mathbb{R}_{01} \\ \mathbb{R}_{11} &= \mathbb{R}_{101}^0 + \mathbb{R}_{111}^1 + \mathbb{R}_{121}^2 + \mathbb{R}_{131}^3 \\ &= \mathbb{R}_{101}^0 + 2\mathbb{R}_{121}^2 \\ &= -\left(u'' + u'(u' - v')\right)e^{-2v} + (\ddot{v} + \dot{v}(\dot{v} - \dot{u}))e^{-2u} \\ &\quad + 2r^{-1}v'e^{-2v} + 2\dot{v}\dot{b}e^{-2u} \\ &= -\left[u'' + (u')^2 - v'(u' + 2r^{-1})\right]e^{-2v} + [\ddot{v} + \dot{v}(\dot{v} + 2\dot{b} - \dot{u})]e^{-2u} \\ \mathbb{R}_{12} &= \mathbb{R}_{102}^0 + \mathbb{R}_{112}^1 + \mathbb{R}_{122}^2 + \mathbb{R}_{132}^3 = 0 \\ \mathbb{R}_{13} &= \mathbb{R}_{103}^0 + \mathbb{R}_{113}^1 + \mathbb{R}_{123}^2 + \mathbb{R}_{133}^3 = 0 \end{aligned}$$

- Case  $j := 2$ . We have:

$$\begin{aligned} \mathbb{R}_{20} &= \mathbb{R}_{200}^0 + \mathbb{R}_{210}^1 + \mathbb{R}_{220}^2 + \mathbb{R}_{230}^3 = 0 \\ \mathbb{R}_{21} &= \mathbb{R}_{201}^0 + \mathbb{R}_{211}^1 + \mathbb{R}_{221}^2 + \mathbb{R}_{231}^3 = 0 \\ \mathbb{R}_{22} &= \mathbb{R}_{202}^0 + \mathbb{R}_{212}^1 + \mathbb{R}_{222}^2 + \mathbb{R}_{232}^3 \\ &= \mathbb{R}_{202}^0 + \mathbb{R}_{212}^1 + \mathbb{R}_{232}^3 \\ &= (\ddot{b} + \dot{b}(\dot{b} - \dot{u}))e^{-2u} - u'r^{-1}e^{-2v} \\ &\quad + r^{-1}v'e^{-2v} + \dot{v}\dot{b}e^{-2u} \\ &\quad + r^{-2} + (\dot{b})^2e^{-2u} - r^{-2}e^{-2v} \\ &= r^{-2} + [\ddot{b} + \dot{b}(\dot{v} + 2\dot{b} - \dot{u})]e^{-2u} - r^{-1}(u' - v' + r^{-1})e^{-2v} \\ \mathbb{R}_{23} &= \mathbb{R}_{203}^0 + \mathbb{R}_{213}^1 + \mathbb{R}_{223}^2 + \mathbb{R}_{233}^3 = 0 \end{aligned}$$

- Case  $j := 3$ . We have:

$$\begin{aligned} \mathbb{R}_{30} &= \mathbb{R}_{300}^0 + \mathbb{R}_{310}^1 + \mathbb{R}_{320}^2 + \mathbb{R}_{330}^3 = 0 \\ \mathbb{R}_{31} &= \mathbb{R}_{301}^0 + \mathbb{R}_{311}^1 + \mathbb{R}_{321}^2 + \mathbb{R}_{331}^3 = 0 \\ \mathbb{R}_{32} &= \mathbb{R}_{302}^0 + \mathbb{R}_{312}^1 + \mathbb{R}_{322}^2 + \mathbb{R}_{332}^3 = 0 \\ \mathbb{R}_{33} &= \mathbb{R}_{303}^0 + \mathbb{R}_{313}^1 + \mathbb{R}_{323}^2 + \mathbb{R}_{333}^3 \\ &= \mathbb{R}_{303}^0 + \mathbb{R}_{313}^1 + \mathbb{R}_{323}^2 \\ &= \mathbb{R}_{202}^0 + \mathbb{R}_{212}^1 + \mathbb{R}_{232}^3 \\ &= \mathbb{R}_{22} \end{aligned}$$

Therefore, the matrices  $\mathbb{R}_{jl}$  and  $\mathbb{R}_{jl}$  have the form :

$$\mathbb{R}_{jl} = \begin{pmatrix} \mathbb{R}_{00} & \mathbb{R}_{01} & 0 & 0 \\ \mathbb{R}_{10} & \mathbb{R}_{11} & 0 & 0 \\ 0 & 0 & \mathbb{R}_{22} & 0 \\ 0 & 0 & 0 & \mathbb{R}_{33} \end{pmatrix}, \quad \mathbb{R}_{jl} = \begin{pmatrix} \mathbb{R}_{00} & \mathbb{R}_{01} & 0 & 0 \\ \mathbb{R}_{10} & \mathbb{R}_{11} & 0 & 0 \\ 0 & 0 & \mathbb{R}_{22} & 0 \\ 0 & 0 & 0 & \mathbb{R}_{33} \end{pmatrix}$$

with:

$$\begin{aligned}\mathbb{R}_{00} &= -[\ddot{v} + 2\ddot{b} + (\dot{v})^2 + 2(\dot{b})^2 - \dot{v}(\dot{v} + 2\dot{b})]e^{-2u} + [u'' + u'(u' + 2r^{-1} - v')]e^{-2v} \\ \mathbf{R}_{00} &= -[\ddot{v} + 2\ddot{b} + (\dot{v})^2 + 2(\dot{b})^2 - \dot{v}(\dot{v} + 2\dot{b})] + [u'' + u'(u' + 2r^{-1} - v')]e^{2u-2v} \\ \mathbb{R}_{01} &= 2(\dot{b}(u' - r^{-1}) + r^{-1}\dot{v})e^{-u-v} \\ \mathbf{R}_{01} &= 2(\dot{b}(u' - r^{-1}) + r^{-1}\dot{v}) \\ \mathbb{R}_{11} &= -[u'' + (u')^2 - v'(u' + 2r^{-1})]e^{-2v} + [\ddot{v} + \dot{v}(\dot{v} + 2\dot{b} - \dot{u})]e^{-2u} \\ \mathbf{R}_{11} &= -[u'' + (u')^2 - v'(u' + 2r^{-1})] + [\ddot{v} + \dot{v}(\dot{v} + 2\dot{b} - \dot{u})]e^{-2u+2v} \\ \mathbb{R}_{22} = \mathbb{R}_{33} &= r^{-2} + [\ddot{b} + \dot{b}(\dot{v} + 2\dot{b} - \dot{u})]e^{-2u} - r^{-1}(u' - v' + r^{-1})e^{-2v} \\ \mathbf{R}_{22} &= 1 + r^2[\ddot{b} + \dot{b}(\dot{v} + 2\dot{b} - \dot{u})]e^{-2u} - r(u' - v' + r^{-1})e^{-2v} \\ \mathbf{R}_{33} &= \sin^2 \vartheta \mathbf{R}_{22}\end{aligned}$$

## 6 Scalar Curvature

### Definition 6.0.1: Scalar Curvature

The **scalar curvature** is defined by:

$$\mathcal{S} := [[Ric]^1]_1^1.$$

Since the operations  $[\bullet]_1^1$  and  $[\bullet]^1$  are independent of the chosen basis,  $\mathcal{S}$  is also independent of the chosen basis.

### Proposition 6.0.2: Usual Properties

(i) (a) We have:

$$\mathcal{S} = \text{tr}_g Ric.$$

(b) We have:

$$\mathcal{S} = \mathbf{g}^{ik} \mathbf{R}_{ik}.$$

(ii) (**Twice Contracted Bianchi Identity**)

(a) We have:

$$\begin{aligned}\nabla^l \mathbf{R}_{lj} &= \frac{1}{2} \nabla_j \mathcal{S} \\ \nabla_k \mathbf{R}^k_j &= \frac{1}{2} \nabla_j \mathcal{S} \\ \mathbf{g}^{kl} \nabla_k \mathbf{R}_{lj} &= \frac{1}{2} \nabla_j \mathcal{S}\end{aligned}$$

(b) We have:

$$\begin{aligned}[[Ric]^1]_2^1 &= \frac{1}{2} d\mathcal{S} \\ \text{div}[Ric]^1 &= \frac{1}{2} d\mathcal{S} \\ \text{div}_g Ric &= \frac{1}{2} d\mathcal{S}\end{aligned}$$

*Proof.* (i) Since:

$$[\mathcal{R}ic]^1 : (\alpha, X) \mapsto g^*(\alpha, \mathcal{R}ic(\bullet, X))$$

$$\mathcal{R}ic = \mathbf{R}_{kl} e^k \otimes e^l$$

we have:

$$\begin{aligned} [\mathcal{R}ic]^1(e^i, e_j) &= g^*(e^i, \mathcal{R}ic(\bullet, e_j)) \\ &= g^*(e^i, \mathbf{R}_{kl} e^l(e_j)e^k) \\ &= g^*(e^i, \mathbf{R}_{kl} \delta_j^l e^k) \\ &= g^*(e^i, \mathbf{R}_{kj} e^k) \\ &= \mathbf{R}_{kj} g^*(e^i, e^k) \\ &= \mathbf{R}_{kj} \mathbf{g}^{ik} \end{aligned}$$

and we have:

$$\begin{aligned} \mathcal{S} &= \text{tr}([\mathcal{R}ic]^1) \\ &= e^i([\mathcal{R}ic]^1(\bullet, e_i)) \\ &= e^i(\mathbf{R}^j_l e^l(e_i)e_j) \\ &= \mathbf{R}^j_l \delta_i^l e^i(e_j) \\ &= \mathbf{R}^j_i \delta_j^i \\ &= \mathbf{R}^i_i \\ &= \mathbf{g}^{ik} \mathbf{R}_{ki} \end{aligned}$$

(ii) (a) By the twice contracted Bianchi identity (see proposition 5.2.2), we have:

$$\nabla_n \mathbf{R}_{jl} - \nabla_l \mathbf{R}_{jn} + \nabla_k \mathbf{R}^k_{jln} = 0.$$

Since:

$$\begin{aligned} \mathbf{g}^{nl} \nabla_k \mathbf{R}^k_{njl} &= \mathbf{g}^{nl} \mathbf{g}^{kp} \nabla_k \mathbf{R}_{pnjl} \\ &= \mathbf{g}^{nl} \nabla^p \mathbf{R}_{pnjl} \\ &= \mathbf{g}^{nl} \nabla^p \mathbf{R}_{nplj} \\ &= \nabla^p \mathbf{R}^l_{plj} \\ &= \nabla^p \mathbf{R}_{pj} \\ &= \nabla^p \mathbf{R}_{j p} \end{aligned}$$

we then contract:

$$\begin{aligned} 0 &= \mathbf{g}^{nl} (\nabla_n \mathbf{R}_{jl} - \nabla_j \mathbf{R}_{nl} + \nabla_k \mathbf{R}^k_{njl}) \\ &= \mathbf{g}^{nl} \nabla_n \mathbf{R}_{jl} - \mathbf{g}^{nl} \nabla_j \mathbf{R}_{nl} + \mathbf{g}^{nl} \nabla_k \mathbf{R}^k_{njl} \\ &= \nabla^l \mathbf{R}_{jl} - \nabla_j \mathcal{S} + \nabla^p \mathbf{R}_{j p} \end{aligned}$$

*i.e.*, we have:

$$\nabla^l \mathbf{R}_{jl} = \frac{1}{2} \nabla_j \mathcal{S}.$$

For the other two equalities, we just need to notice that:

$$\begin{aligned} \nabla^l \mathbf{R}_{jl} &= \mathbf{g}^{kl} \nabla_k \mathbf{R}_{lj} \\ &= \frac{1}{2} \nabla_j \mathcal{S} \end{aligned}$$

(b) We consider the base  $\mathcal{C}$ . Since:

$$\operatorname{div}_g \mathcal{R}\text{ic} = \left[ \nabla [\mathcal{R}\text{ic}]^1 \right]_2^1,$$

we have:

$$\begin{aligned} \operatorname{div}_g \mathcal{R}\text{ic}(\partial_j) &= \left[ \nabla [\mathcal{R}\text{ic}]^1 \right]_2^1(\partial_j) \\ &= \nabla_k \mathbf{R}^k_j \\ &= \frac{1}{2} \nabla_j \mathcal{S} \\ &= \frac{1}{2} d\mathcal{S}(\partial_j) \end{aligned}$$

Since this is true for all  $j$ , we have:

$$\operatorname{div}_g \mathcal{R}\text{ic} = \frac{1}{2} d\mathcal{S}.$$

□

### Example 6.0.3: Examples of calculations in bases $\mathcal{C}$ and $\mathcal{C}_\perp$

Therefore, we have:

$$\mathcal{S} = g^{jj} \mathbf{R}_{jj} = \eta^{jj} \mathbb{R}_{jj}.$$

We can directly show the last equality without using the invariance under change of basis. Since  $g_{jl}$  and  $\eta_{jl}$  are diagonal, we have:

$$\begin{aligned} \mathcal{S} &= g^{jl} \mathbf{R}_{jl} \\ &= g^{jj} \mathbf{R}_{jj} \\ &= g^{jj} e^{2u_j} \mathbb{R}_{jj} \\ &= g^{jj} \eta^j g_{jj} \mathbb{R}_{jj} \\ &= \eta^j \mathbb{R}_{jj} \end{aligned}$$

## 6.1 Practical calculations in $\mathcal{C}$ and $\mathcal{C}^*$

### Proposition 6.1.1: Value in terms of Christoffel symbols

We have:

$$\mathcal{S} = g^{kl} \left( \partial_i \Gamma_{kl}^i - \partial_k \Gamma_{li}^i + \Gamma_{ip}^i \Gamma_{kl}^p - \Gamma_{kp}^i \Gamma_{il}^p \right).$$

*Proof.* Using lemma 6.0.2, we have:

$$\mathcal{S} = g^{kl} \mathbf{R}_{kl}.$$

Since:

$$\mathbf{R}_{kl} = \partial_i \Gamma_{kl}^i - \partial_k \Gamma_{li}^i + \Gamma_{ip}^i \Gamma_{kl}^p - \Gamma_{kp}^i \Gamma_{il}^p,$$

we can conclude that:

$$\mathcal{S} = g^{kl} \left( \partial_i \Gamma_{kl}^i - \partial_k \Gamma_{li}^i + \Gamma_{ip}^i \Gamma_{kl}^p - \Gamma_{kp}^i \Gamma_{il}^p \right).$$

□

### Corollary 6.1.2: Exact value

We have:

$$\mathcal{S} = - \sum_k \eta_k \left( 2\hat{u}_{k,kk} + \hat{u}_{k,k}^2 - 2u_{k,k} \hat{u}_{k,k} + (\hat{u}_{k,k})^2 \right) e^{-2u_k}.$$

*Proof.* We have:

$$\begin{aligned}
\mathcal{S} &= g^{kl} (\partial_i \Gamma_{kl}^i - \partial_k \Gamma_{li}^i + \Gamma_{ip}^i \Gamma_{kl}^p - \Gamma_{kp}^i \Gamma_{il}^p) \\
&= g^{kk} (\partial_i \Gamma_{kk}^i - \partial_k \Gamma_{ki}^i + \Gamma_{ip}^i \Gamma_{kk}^p - \Gamma_{ki}^i \Gamma_{ik}^i - \Gamma_{kk}^i \Gamma_{ik}^k) \\
&= \eta_k (\partial_i (-\eta_k \eta_i u_{k,i} e^{2u_k - 2u_i}) - \partial_k u_{i,k} + u_{i,p} (-\eta_k \eta_p u_{k,p} e^{2u_k - 2u_p}) - u_{i,k}^2 - (-\eta_k \eta_i u_{k,i} e^{2u_k - 2u_i}) u_{k,i}) e^{-2u_k} \\
&= \eta_k (-\eta_k \eta_i (u_{k,ii} + 2u_{k,i} (u_{k,i} - u_{i,i})) e^{2u_k - 2u_i} - u_{i,kk} - \eta_k \eta_p u_{i,p} u_{k,p} e^{2u_k - 2u_p} - u_{i,k}^2 + \eta_k \eta_i u_{k,i}^2 e^{2u_k - 2u_i}) e^{-2u_k} \\
&= -\eta_k (u_{k,ii} + 2u_{k,i} (u_{k,i} - u_{i,i})) e^{-2u_i} - \eta_k u_{i,kk} e^{-2u_k} - \eta_p u_{i,p} u_{k,p} e^{-2u_p} - \eta_k u_{i,k}^2 e^{-2u_k} + \eta_i u_{k,i}^2 e^{-2u_i} \\
&= -\eta_k (u_{i,kk} + 2u_{i,k} (u_{i,k} - u_{k,k})) e^{-2u_k} - \eta_k u_{i,kk} e^{-2u_k} - \eta_k u_{i,k} u_{p,k} e^{-2u_k} - \eta_k u_{i,k}^2 e^{-2u_k} + \eta_k u_{i,k}^2 e^{-2u_k} \\
&= -\eta_k (u_{i,kk} + 2u_{i,k} (u_{i,k} - u_{k,k}) - u_{i,kk} - u_{i,k} u_{p,k} + u_{i,k}^2 + u_{i,k}^2) e^{-2u_k} \\
&= -\eta_k (2u_{i,k} (u_{i,k} - u_{k,k}) - u_{i,k} u_{p,k} + u_{i,k}^2) e^{-2u_k} \\
&= -\eta_k (2(u_{i,k}^2 - u_{i,k} u_{k,k}) - u_{i,k} u_{p,k}) e^{-2u_k} \\
&= -\sum_k \eta_k (2\hat{u}_{k,kk} + \hat{u}_{k,k}^2 - 2u_{k,k} \hat{u}_{k,k} + (\hat{u}_{k,k})^2) e^{-2u_k}
\end{aligned}$$

□

We will calculate the value of  $\mathcal{S}$  using the metric  $h$  in the next subsection.

## 6.2 Practical calculations in $\mathcal{C}_\perp$ and $\mathcal{C}_\perp^*$

Using the formula:

$$\mathcal{S} = \eta_j \mathbb{R}_{jj},$$

we can determine the value of the scalar curvature.

### Theorem 6.2.1: Value of the scalar tensor

We have:

$$\mathcal{S} = -\sum_k \eta_k (2\hat{u}_{k,kk} + \hat{u}_{k,k}^2 - 2u_{k,k} \hat{u}_{k,k} + (\hat{u}_{k,k})^2) e^{-2u_k}.$$

*Proof.* We have:

$$\begin{aligned}
\mathcal{S} &= \eta_j \mathbb{R}_{jj} \\
&= -\sum_j \eta_j e^{-2u_j} \sum_{k \neq j} (u_{k,jj} + u_{k,j} (u_k - u_j)_{,j}) - \sum_j \sum_{k \neq j} \eta_k \left( u_{j,kk} + u_{j,k} \left( -u_k + \sum_{i \neq k} u_i \right)_{,k} \right) e^{-2u_k} \\
&= -\sum_k \sum_{j \neq k} \eta_k \left( u_{j,kk} + u_{j,k} \left( -u_k + \sum_{i \neq k} u_i \right)_{,k} \right) e^{-2u_k} - \sum_k \sum_{j \neq k} \eta_k e^{-2u_k} (u_{j,kk} + u_{j,k} (u_j - u_k)_{,k}) \\
&= -\sum_k \eta_k e^{-2u_k} \sum_{j \neq k} \left( 2u_{j,kk} + u_{j,k} \left( 2u_j - 2u_k + \sum_{i \neq j,k} u_i \right)_{,k} \right) \\
&= -\sum_k \eta_k (2\hat{u}_{k,kk} + \hat{u}_{k,k}^2 - 2u_{k,k} \hat{u}_{k,k} + (\hat{u}_{k,k})^2) e^{-2u_k}
\end{aligned}$$

□

**Example 6.2.2: Example 8 of the metric  $h$** 

We revisit the example with the metric  $h$  (see 2.4.1, 2.5.1, 3.5.5, 3.6.3, 3.6.5, 4.4.5, 4.5.4, and 5.4.3) with:

$$\begin{aligned} u_0(t, r) &:= u(t, r) & u_1(t, r) &:= v(t, r) \\ u_2(t, r) &:= b(t) + \ln(r) & u_3(t, r, \vartheta) &:= b(t) + \ln(r) + \ln \sin(\vartheta) \end{aligned}$$

We calculate Scal in two different ways.

(1) Using Theorem 6.2.1, we have:

$$\begin{aligned} \mathcal{S} &= -\eta_0 (2\hat{u}_{0,00} + \hat{u}_{0,0}^2 - 2u_{0,0}\hat{u}_{0,0} + (\hat{u}_{0,0})^2) e^{-2u_0} \\ &\quad - \eta_1 (2\hat{u}_{1,11} + \hat{u}_{1,1}^2 - 2u_{1,1}\hat{u}_{1,1} + (\hat{u}_{1,1})^2) e^{-2u_1} \\ &\quad - \eta_2 (2\hat{u}_{2,22} + \hat{u}_{2,2}^2 - 2u_{2,2}\hat{u}_{2,2} + (\hat{u}_{2,2})^2) e^{-2u_2} \\ &= - \left( 2(\ddot{v} + 2\ddot{b}) + (\dot{v})^2 + 2(\dot{b})^2 - 2\dot{v}(\dot{v} + 2\dot{b}) + (\dot{v} + 2\dot{b})^2 \right) e^{-2u} \\ &\quad + \left( 2(u'' - 2r^{-2}) + (u')^2 + 2r^{-2} - 2v'(u' + 2r^{-1}) + (u' + 2r^{-1})^2 \right) e^{-2v} \\ &\quad - 2r^{-2}e^{-2b} \\ &= -2 \left[ \ddot{v} + 2\ddot{b} + (\dot{v})^2 + 3(\dot{b})^2 - \dot{v}\dot{v} - 2\dot{v}\dot{b} + 2\dot{v}\dot{b} \right] e^{-2u} \\ &\quad + 2 \left[ u'' - 2r^{-2} + (u')^2 + 3r^{-2} - u'v' + 2r^{-1}(u' - v')u' \right] e^{-2v} - 2r^{-2}e^{-2b} \end{aligned}$$

(2) Using Definition 6.0.1, we have:

$$\begin{aligned} \mathcal{S} &= \eta_j \mathbb{R}_{jj} \\ &= \eta_0 \mathbb{R}_{00} + \eta_1 \mathbb{R}_{11} + \eta_2 \mathbb{R}_{22} + \eta_3 \mathbb{R}_{33} \\ &= \mathbb{R}_{00} - \mathbb{R}_{11} - 2\mathbb{R}_{22} \\ &= - \left[ \ddot{v} + 2\ddot{b} + (\dot{v})^2 + 2(\dot{b})^2 - \dot{v}(\dot{v} + 2\dot{b}) \right] e^{-2u} + \left[ u'' + u'(u' + 2r^{-1} - v') \right] e^{-2v} \\ &\quad + \left[ u'' + (u')^2 - v'(u' + 2r^{-1}) \right] e^{-2v} - \left[ \ddot{v} + \dot{v}(\dot{v} + 2\dot{b} - \dot{v}) \right] e^{-2u} \\ &\quad - 2r^{-2} - 2 \left[ \ddot{b} + \dot{b}(\dot{v} + 2\dot{b} - \dot{v}) \right] e^{-2u} + 2r^{-1}(u' - v' + r^{-1}) e^{-2v} \\ &= -2 \left[ \ddot{v} + 2\ddot{b} + (\dot{v})^2 + 3(\dot{b})^2 - \dot{v}\dot{v} - 2\dot{v}\dot{b} + 2\dot{v}\dot{b} \right] e^{-2u} \\ &\quad + 2 \left[ u'' - 2r^{-2} + (u')^2 + 3r^{-2} - u'v' + 2r^{-1}(u' - v')u' \right] e^{-2v} - 2r^{-2}e^{-2b} \end{aligned}$$

We see the efficiency of the first method, which directly gives a very good factorization.

## 7 Tensor of Einstein

**Definition 7.0.1: The tensor of Einstein**

The **tensor of Einstein** is defined by:

$$\mathcal{G} := \mathcal{R} - \frac{1}{2}\mathcal{S}g.$$

That is, for all vector fields  $X$  and  $Y$ , we have:

$$\mathcal{G}(X, Y) = \mathcal{R}(X, Y) - \frac{1}{2}\mathcal{S}g(X, Y).$$

Next, we define the components of  $\mathcal{G}$  in the  $\mathcal{E}$  basis.

**Definition 7.0.2: The components of the tensor of Einstein in  $\mathcal{E}$  and  $\mathcal{E}^*$** 

The **components of the Ricci tensor in  $\mathcal{E}$  and  $\mathcal{E}^*$**  are:

$$\mathbf{G}_{jl} := \mathcal{G}(e_j, e_l).$$

**Proposition 7.0.3: Simple properties of the tensor of Einstein in  $\mathcal{E}$  and  $\mathcal{E}^*$** 

(i) **(Symmetry)**

(a) We have:

$$\mathcal{G}(X, Y) = \mathcal{G}(Y, X).$$

(b) We have:

$$\mathbf{G}_{jl} = \mathbf{G}_{lj}.$$

(ii) **(Twice contracted Bianchi identity)**

(a) We have:

$$\begin{aligned}\nabla^l \mathbf{G}_{lj} &= 0 \\ \nabla_k \mathbf{G}^k_j &= 0 \\ \mathbf{g}^{kl} \nabla_k \mathbf{G}_{lj} &= 0\end{aligned}$$

(b) We have:

$$\begin{aligned}[[\mathcal{G}]^1]_2^1 &= 0 \\ \operatorname{div}[\mathcal{G}]^1 &= 0 \\ \operatorname{div}_g \mathcal{G} &= 0\end{aligned}$$

*Proof.* (i) (Symmetry)

(a) Since  $\mathcal{R}$  and  $g$  are symmetric, we have:

$$\begin{aligned}\mathcal{G}(X, Y) &= \mathcal{R}(X, Y) - \frac{1}{2}\mathcal{S}g(X, Y) \\ &= \mathcal{R}(Y, X) - \frac{1}{2}\mathcal{S}g(Y, X) \\ &= \mathcal{G}(Y, X)\end{aligned}$$

(b) We have:

$$\begin{aligned}\mathbf{G}_{jl} &= \mathcal{G}(e_j, e_l) \\ &= \mathcal{G}(e_l, e_j) \\ &= \mathbf{G}_{lj}\end{aligned}$$

(ii) By proposition 6.0.2, we have:

$$\nabla^n \mathbf{R}_{nl} = \frac{1}{2} \nabla_l \mathcal{S}$$

Hence, we have:

$$\nabla^n \mathbf{R}_{nl} = \frac{1}{2} \mathbf{g}_{nl} \nabla^n \mathcal{S}.$$

Therefore, by linearity of  $\nabla$ , we obtain:

$$\begin{aligned}\nabla^n \mathbf{G}_{nl} &= \nabla^n \left( \mathbf{R}_{nl} - \frac{1}{2} \mathbf{g}_{nl} \mathcal{S} \right) \\ &= 0\end{aligned}$$

□

#### Example 7.0.4: Components of the Einstein tensor in the bases $\mathcal{C}$ and $\mathcal{C}_\perp$

(i) The components of the Einstein tensor in  $\mathcal{C}$  and  $\mathcal{C}^*$  are:

$$\mathbf{G}_{jl} := \mathbf{R}_{jl} - \frac{1}{2} \mathbf{g}_{jl} \mathcal{S}.$$

We have the decomposition:

$$\mathcal{G} = \mathbf{G}_{jl} dx^j \otimes dx^l.$$

(ii) The components of the Einstein tensor in  $\mathcal{C}_\perp$  and  $\mathcal{C}_\perp^*$  are:

$$\mathbb{G}_{jl} := \mathbb{R}_{jl} - \frac{1}{2} \eta_{jl} \mathcal{S}.$$

We have the decomposition:

$$\mathcal{G} = \mathbb{G}_{jl} \theta^j \otimes \theta^l.$$

We have:

$$\mathbb{G}_{ij} = e^{-u_i - u_j} \mathbf{G}_{ij}.$$

#### Theorem 7.0.5: Value of the Einstein tensor

Let  $j, l \in \{0, 1, 2, 3\}$  **distinct**.

(i) We have:

$$\begin{aligned}\mathbb{G}_{jl} &= - \left[ \sum_{p \neq j, l} u_{p, jl} - u_{j, l} u_{p, j} - u_{l, j} u_{p, l} + u_{p, j} u_{p, l} \right] e^{-u_j - u_l} \\ \mathbf{G}_{jl} &= - \left[ \sum_{p \neq j, l} u_{p, jl} - u_{j, l} u_{p, j} - u_{l, j} u_{p, l} + u_{p, j} u_{p, l} \right]\end{aligned}$$

(ii) We have:

$$\mathbb{G}_{jj} = \frac{1}{2} \left[ (\hat{u}_{j, j})^2 - \hat{u}_{j, j}^2 \right] e^{-2u_j} + \eta_j \sum_{k \neq j} \eta_k \left[ \hat{u}_{jk, k} + \frac{1}{2} \hat{u}_{k, k}^2 + \frac{1}{2} \hat{u}_{k, k} (\hat{u}_{jk, k} - 2u_{k, k} - u_{j, k}) + u_{j, k} u_{k, k} \right] e^{-2u_k}$$

$$\mathbb{G}_{jj} = \frac{1}{2} [(\hat{u}_{j,j})^2 - \hat{u}_{j,j}^2] + \eta_j \sum_{k \neq j} \eta_k \left[ \hat{u}_{jk,kk} + \frac{1}{2} \hat{u}_{k,k}^2 + \frac{1}{2} \hat{u}_{k,k} (\hat{u}_{jk,k} - 2u_{k,k} - u_{j,k}) + u_{j,k} u_{k,k} \right] e^{2u_j - 2u_k}$$

*Proof.* (i) We have:

$$\mathbb{G}_{jl} = \mathbb{R}_{jl} = - \left[ \sum_{p \neq j,l} u_{p,jl} - u_{j,l} u_{p,j} - u_{l,j} u_{p,l} + u_{p,j} u_{p,l} \right] e^{-u_j - u_l}.$$

(ii) We have:

$$\begin{aligned} \mathbb{G}_{jj} &= \mathbb{R}_{jj} - \frac{1}{2} \eta_j \mathbb{R} \\ &= - (\hat{u}_{j,jj} + \hat{u}_{j,j}^2 - \hat{u}_{j,j} u_{j,j}) e^{-2u_j} - \eta_j \sum_{k \neq j} \eta_k (u_{j,kk} + u_{j,k} (\hat{u}_{k,k} - u_{k,k})) e^{-2u_k} \\ &\quad + \frac{1}{2} \eta_j \sum_k \eta_k (2\hat{u}_{k,kk} + \hat{u}_{k,k}^2 - 2u_{k,k} \hat{u}_{k,k} + (\hat{u}_{k,k})^2) e^{-2u_k} \\ &= - (\hat{u}_{j,jj} + \hat{u}_{j,j}^2 - \hat{u}_{j,j} u_{j,j}) e^{-2u_j} - \eta_j \sum_{k \neq j} \eta_k (u_{j,kk} + u_{j,k} (\hat{u}_{k,k} - u_{k,k})) e^{-2u_k} \\ &\quad + \frac{1}{2} \eta_j \sum_{k \neq j} \eta_k (2\hat{u}_{k,kk} + \hat{u}_{k,k}^2 - 2u_{k,k} \hat{u}_{k,k} + (\hat{u}_{k,k})^2) e^{-2u_k} \\ &\quad + \frac{1}{2} (2\hat{u}_{j,jj} + \hat{u}_{j,j}^2 - 2u_{j,j} \hat{u}_{j,j} + (\hat{u}_{j,j})^2) e^{-2u_j} \\ &= \frac{1}{2} [(\hat{u}_{j,j})^2 - \hat{u}_{j,j}^2] e^{-2u_j} + \frac{1}{2} \eta_j \sum_{k \neq j} \eta_k (2\hat{u}_{k,kk} + \hat{u}_{k,k}^2 - 2u_{k,k} \hat{u}_{k,k} + (\hat{u}_{k,k})^2 - 2u_{j,kk} - 2u_{j,k} (\hat{u}_{k,k} - u_{k,k})) e^{-2u_k} \\ &= \frac{1}{2} [(\hat{u}_{j,j})^2 - \hat{u}_{j,j}^2] e^{-2u_j} + \eta_j \sum_{k \neq j} \eta_k \left[ \hat{u}_{k,kk} - u_{j,kk} + \frac{1}{2} \hat{u}_{k,k}^2 + \frac{1}{2} \hat{u}_{k,k} (\hat{u}_{k,k} - 2u_{k,k} - u_{j,k}) + u_{j,k} u_{k,k} \right] e^{-2u_k} \\ &= \frac{1}{2} [(\hat{u}_{j,j})^2 - \hat{u}_{j,j}^2] e^{-2u_j} + \eta_j \sum_{k \neq j} \eta_k \left[ \hat{u}_{jk,kk} + \frac{1}{2} \hat{u}_{k,k}^2 + \frac{1}{2} \hat{u}_{k,k} (\hat{u}_{jk,k} - 2u_{k,k} - u_{j,k}) + u_{j,k} u_{k,k} \right] e^{-2u_k} \end{aligned}$$

□

### Example 7.0.6: Sequence 9 of the metric $h$

We revisit the example with the metric  $h$  (see 2.4.1, 2.5.1, 3.5.5, 3.6.3, 3.6.5, 4.4.5, 4.5.4, 5.4.3, and 6.2.2) with :

$$\begin{aligned} u_0(t, r) &:= u(t, r) & u_1(t, r) &:= v(t, r) \\ u_2(t, r) &:= b(t) + \ln(r) & u_3(t, r, \vartheta) &:= b(t) + \ln(r) + \ln(\sin(\vartheta)) \end{aligned}$$

We have:

$$\mathbb{G}_{jj} = \frac{1}{2} [(\hat{u}_{j,j})^2 - \hat{u}_{j,j}^2] e^{-2u_j} + \eta_j \sum_{k \neq j} \eta_k \left[ \hat{u}_{jk,kk} + \frac{1}{2} \hat{u}_{k,k}^2 + \frac{1}{2} \hat{u}_{k,k} (\hat{u}_{jk,k} - 2u_{k,k} - u_{j,k}) + u_{j,k} u_{k,k} \right] e^{-2u_k}$$

• **Case  $j := 0$ .** We have :

$$\begin{aligned} \mathbb{G}_{00} &= \frac{1}{2} [(\hat{u}_{0,0})^2 - \hat{u}_{0,0}^2] e^{-2u_0} \\ &\quad + \eta_0 \eta_1 \left[ \hat{u}_{01,11} + \frac{1}{2} \hat{u}_{1,1}^2 + \frac{1}{2} \hat{u}_{1,1} (\hat{u}_{01,1} - 2u_{1,1} - u_{0,1}) + u_{0,1} u_{1,1} \right] e^{-2u_1} \\ &\quad + \eta_0 \eta_2 \left[ \hat{u}_{02,22} + \frac{1}{2} \hat{u}_{2,2}^2 + \frac{1}{2} \hat{u}_{2,2} (\hat{u}_{02,2} - 2u_{2,2} - u_{0,2}) + u_{0,2} u_{2,2} \right] e^{-2u_2} \\ &\quad + \eta_0 \eta_3 \left[ \hat{u}_{03,33} + \frac{1}{2} \hat{u}_{3,3}^2 + \frac{1}{2} \hat{u}_{3,3} (\hat{u}_{03,3} - 2u_{3,3} - u_{0,3}) + u_{0,3} u_{3,3} \right] e^{-2u_3} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left[ (\dot{v} + 2\dot{b})^2 - (\dot{v})^2 - 2(\dot{b})^2 \right] e^{-2u} \\
&\quad - \left[ -2r^{-2} + \frac{1}{2} \left( (u')^2 + 2r^{-2} \right) + \frac{1}{2} (u' + 2r^{-1}) (2r^{-1} - 2v' - u') + u'v' \right] e^{-2v} \\
&\quad + r^{-2} e^{-2b} \\
&= \dot{b} (2\dot{v} + \dot{b}) e^{-2u} - r^{-1} (r^{-1} - 2v') e^{-2v} + r^{-2} e^{-2b}
\end{aligned}$$

- **Case  $j := 1$ .** We have :

$$\begin{aligned}
\mathbb{G}_{11} &= \frac{1}{2} \left[ (\hat{u}_{1,1})^2 - \hat{u}_{1,1}^2 \right] e^{-2u_1} \\
&\quad + \eta_1 \eta_0 \left[ \hat{u}_{10,00} + \frac{1}{2} \hat{u}_{0,0}^2 + \frac{1}{2} \hat{u}_{0,0} (\hat{u}_{10,0} - 2u_{0,0} - u_{1,0}) + u_{1,0} u_{0,0} \right] e^{-2u_0} \\
&\quad + \eta_1 \eta_2 \left[ \hat{u}_{12,22} + \frac{1}{2} \hat{u}_{2,2}^2 + \frac{1}{2} \hat{u}_{2,2} (\hat{u}_{12,2} - 2u_{2,2} - u_{1,2}) + u_{1,2} u_{2,2} \right] e^{-2u_2} \\
&\quad + \eta_1 \eta_3 \left[ \hat{u}_{13,33} + \frac{1}{2} \hat{u}_{3,3}^2 + \frac{1}{2} \hat{u}_{3,3} (\hat{u}_{13,3} - 2u_{3,3} - u_{1,3}) + u_{1,3} u_{3,3} \right] e^{-2u_3} \\
&= \frac{1}{2} \left[ (u' + 2r^{-1})^2 - (u')^2 - 2(u')^2 \right] e^{-2v} \\
&\quad - \left[ 2\ddot{b} + \frac{1}{2} \left( (\dot{v})^2 + 2(\dot{b})^2 \right) + \frac{1}{2} (\dot{v} + 2\dot{b}) (2\dot{b} - 2\dot{u} - \dot{v}) + \dot{u}\dot{v} \right] e^{-2u} \\
&\quad - r^{-2} e^{-2b} \\
&= [\dot{b} (2\dot{u} - 3\dot{b}) - 2\ddot{b}] e^{-2u} + r^{-1} (2u' + r^{-1}) e^{-2v} - r^{-2} e^{-2b}
\end{aligned}$$

- **Case  $j := 2$ .** We have :

$$\begin{aligned}
\mathbb{G}_{22} &= \frac{1}{2} \left[ (\hat{u}_{2,2})^2 - \hat{u}_{2,2}^2 \right] e^{-2u_2} \\
&\quad + \eta_2 \eta_0 \left[ \hat{u}_{20,00} + \frac{1}{2} \hat{u}_{0,0}^2 + \frac{1}{2} \hat{u}_{0,0} (\hat{u}_{20,0} - 2u_{0,0} - u_{2,0}) + u_{2,0} u_{0,0} \right] e^{-2u_0} \\
&\quad + \eta_2 \eta_1 \left[ \hat{u}_{21,11} + \frac{1}{2} \hat{u}_{1,1}^2 + \frac{1}{2} \hat{u}_{1,1} (\hat{u}_{21,1} - 2u_{1,1} - u_{2,1}) + u_{2,1} u_{1,1} \right] e^{-2u_1} \\
&\quad + \eta_2 \eta_3 \left[ \hat{u}_{23,33} + \frac{1}{2} \hat{u}_{3,3}^2 + \frac{1}{2} \hat{u}_{3,3} (\hat{u}_{23,3} - 2u_{3,3} - u_{2,3}) + u_{2,3} u_{3,3} \right] e^{-2u_3} \\
&= - \left[ \ddot{v} + \ddot{b} + \frac{1}{2} \left( (\dot{v})^2 + 2(\dot{b})^2 \right) + \frac{1}{2} (\dot{v} + 2\dot{b}) (\dot{v} + \dot{b} - 2\dot{u} - \dot{b}) + \dot{u}\dot{b} \right] e^{-2u} \\
&\quad + \left[ u'' - r^{-2} + \frac{1}{2} \left( (u')^2 + 2r^{-2} \right) + \frac{1}{2} (u' + 2r^{-1}) (u' - 2v') + r^{-1} v' \right] e^{-2v} \\
&= [-\ddot{v} - \ddot{b} + (\dot{u} - \dot{v}) (\dot{v} + \dot{b}) - (\dot{b})^2] e^{-2u} + [u'' + u' (u' - v' + r^{-1}) - r^{-1} v'] e^{-2v}
\end{aligned}$$

- **Case  $j := 3$ .** Since  $\mathbb{R}_{33} = \mathbb{R}_{22}$ , we have:

$$\mathbb{G}_{33} = \mathbb{G}_{22}.$$

- We have:

$$\mathbb{G}_{01} = \mathbb{G}_{10} = \mathbb{R}_{01} = 2(\dot{b}(u' - r^{-1}) + r^{-1}\dot{v}) e^{-u-v}.$$

Therefore, the matrices  $\mathbb{G}_{jl}$  and  $G_{jl}$  are of the form:

$$\mathbb{G}_{jl} = \begin{pmatrix} \mathbb{G}_{00} & \mathbb{G}_{01} & 0 & 0 \\ \mathbb{G}_{10} & \mathbb{G}_{11} & 0 & 0 \\ 0 & 0 & \mathbb{G}_{22} & 0 \\ 0 & 0 & 0 & \mathbb{G}_{33} \end{pmatrix}, \quad G_{jl} = \begin{pmatrix} G_{00} & G_{01} & 0 & 0 \\ G_{10} & G_{11} & 0 & 0 \\ 0 & 0 & G_{22} & 0 \\ 0 & 0 & 0 & G_{33} \end{pmatrix}$$

with:

$$\begin{aligned} \mathbb{G}_{00} &= \dot{b}(2\dot{v} + \dot{b})e^{-2u} - r^{-1}(r^{-1} - 2v')e^{-2v} + r^{-2}e^{-2b} \\ G_{00} &= \dot{b}(2\dot{v} + \dot{b}) - r^{-1}(r^{-1} - 2v')e^{2u-2v} + r^{-2}e^{-2b+2u} \\ \mathbb{G}_{01} &= 2(\dot{b}(u' - r^{-1}) + r^{-1}\dot{v})e^{-u-v} \\ G_{01} &= 2(\dot{b}(u' - r^{-1}) + r^{-1}\dot{v}) \\ \mathbb{G}_{11} &= [\dot{b}(2\dot{u} - 3\dot{b}) - 2\ddot{b}]e^{-2u} + r^{-1}(2u' + r^{-1})e^{-2v} - r^{-2}e^{-2b} \\ G_{11} &= [\dot{b}(2\dot{u} - 3\dot{b}) - 2\ddot{b}]e^{-2u+2v} + r^{-1}(2u' + r^{-1}) - r^{-2}e^{-2b+2v} \\ \mathbb{G}_{22} = \mathbb{G}_{33} &= [-\ddot{v} - \ddot{b} + (\dot{u} - \dot{v})(\dot{v} + \dot{b}) - (\dot{b})^2]e^{-2u} + [u'' + u'(u' - v' + r^{-1}) - r^{-1}v']e^{-2v} \\ G_{22} &= r^2[-\ddot{v} - \ddot{b} + (\dot{u} - \dot{v})(\dot{v} + \dot{b}) - (\dot{b})^2]e^{-2u} + r^2[u'' + u'(u' - v' + r^{-1}) - r^{-1}v']e^{-2v} \\ G_{33} &= \sin^2 \vartheta G_{22} \end{aligned}$$

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