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Generalized Trigonometric Functions and Their Properties Derived from Series Representations

David Pigeon

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Abstract

This paper introduces generalized trigonometric functions based on series representations of sine and cosine functions. We derive a set of properties that extend well-known formulas such as Euler's and Moivre's formulas. The approach yields insights into the algebraic properties of these functions and their applications in various contexts.

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1 Introduction

The **complex exponential** is defined on \mathbb{C} as:

$$\forall z \in \mathbb{C}, \exp(z) := e^z := \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

This series is absolutely convergent for any complex number and is uniformly convergent on any bounded subset of the complex plane (which justifies some infinite summation inversions in this exposition).

We will use the following properties of the exponential:

Proposition 1 (Usual properties of the exponential). (i) For all $z, z' \in \mathbb{C}$, we have $e^{z+z'} = e^z e^{z'}$.

(ii) The function \exp is holomorphic with derivative \exp , surjective from \mathbb{C} to \mathbb{C}^* .

We have four usual functions defined from the exponential that resemble each other. For all $z \in \mathbb{C}$, we have:

$$\begin{aligned}\cos(z) &= \frac{e^{iz} + e^{-iz}}{2} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} \\ \sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \\ \cosh(z) &= \frac{e^z + e^{-z}}{2} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} \\ \sinh(z) &= \frac{e^z - e^{-z}}{2} = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}\end{aligned}$$

The middle formulas are called **associated Euler formulas** and the ones on the right are called **associated power series**.

To encompass these four functions in the same notation, let's transform these Euler formulas and series. For all $z \in \mathbb{C}$, we have:

$$\begin{aligned}\cos(z) &= \frac{1}{2} \left(\frac{e^{iz}}{i^0} + \frac{e^{-iz}}{(-i)^0} \right) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+0}}{(2k+0)!} \\ \sin(z) &= \frac{1}{2} \left(\frac{e^{iz}}{i^1} + \frac{e^{-iz}}{(-i)^1} \right) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \\ \cosh(z) &= \frac{1}{2} \left(\frac{e^z}{1^0} + \frac{e^{-z}}{(-1)^0} \right) = \sum_{k=0}^{\infty} 1^k \frac{z^{2k+0}}{(2k+0)!} \\ \sinh(z) &= \frac{1}{2} \left(\frac{e^z}{1^1} + \frac{e^{-z}}{(-1)^1} \right) = \sum_{k=0}^{\infty} 1^k \frac{z^{2k+1}}{(2k+1)!}\end{aligned}$$

We also define $\pi/2$ as the first strictly positive zero of the cosine function restricted to \mathbb{R} . Thus, $e^{i\pi} = -1$, and the function \exp is $2i\pi$ -periodic.

2 Generalities

2.1 Notations

Notation 2. (1) We denote:

$$\mathbb{N}_n := \mathbb{N} \cap [n, +\infty[.$$

So $\mathbb{N} = \mathbb{N}_0$. We also denote $\mathbb{N}_* := \mathbb{N}_1$.

(2) We denote:

$$\Omega := \{(s, a, b) \in \mathbb{C}^* \times \mathbb{N}_* \times \mathbb{N}, 0 \leq b \leq a - 1\}.$$

For all $s \in \mathbb{C}^*$ and all $a \in \mathbb{N}_*$, we denote:

$$\Omega(s, a) := \{b \in \mathbb{N}, 0 \leq b \leq a - 1\}.$$

For all $a, b \in \mathbb{N}$ such that $0 \leq b < a$, we denote:

$$\Omega(a, b) := \{(s, a, b) \in \Gamma, s \in \mathbb{C}^*\}.$$

(3) For all $I \in \mathcal{P}(\mathbb{N})$ and all $a, b \in \mathbb{N}$, we denote:

$$\begin{aligned} I(a, b) &:= \{l \in \mathbb{N}, al + b \in I\} \\ aI + b &:= \{al + b, l \in I\} \end{aligned}$$

(4) Let x and y be two elements. We denote $\delta_{x,y}$ the **Kronecker symbol** defined by:

$$\delta_{x,y} := \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

By extension, for any set A :

$$\delta_{x,A} := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

which is nothing but the indicator function of A evaluated at x (sometimes denoted by $1_A(x)$).

Example 3. For all $I \in \mathcal{P}(\mathbb{N})$ and all $a, b \in \mathbb{N}$, we have:

$$\begin{aligned} aI(a, b) + b &:= a\{l \in \mathbb{N}, al + b \in I\} + b = I \cap (a\mathbb{N} + b) \\ (aI + b)(a, b) &:= \{l \in \mathbb{N}, al + b \in aI + b\} = I \end{aligned}$$

2.2 n -th Roots

Notation 4. Let $s := re^{i\theta} \in \mathbb{C}^*$ and $a, a', c \in \mathbb{N}_*$ such that $a = a'c$.

(1) We denote:

$$\mathbb{U}_a(s) := \{z \in \mathbb{C}, z^a = s\}.$$

We simply denote $\mathbb{U}_a := \mathbb{U}_a(1)$.

(2) We denote:

$$\xi_a := e^{2i\pi/a}.$$

For all $\tau \in \mathbb{U}_a(s)$, we have:

$$\begin{aligned} \mathbb{U}_a &= \{\xi^k, k \in \{0, \dots, a-1\}\} \\ \mathbb{U}_a(s) &= \left\{ \sqrt[a]{r} e^{i\theta/a} \xi^k, k \in \{0, \dots, a-1\} \right\} = \{\tau u, u \in \mathbb{U}_a\}. \\ &= \bigsqcup_{\tau \in \mathbb{U}_c(s)} \mathbb{U}_{a'}(\tau). \end{aligned}$$

(3) For all $\alpha := (s, a, b), \beta := (t, a, b) \in \Gamma(a, b)$, we denote:

$$\begin{aligned} \mathbb{U}(\alpha) &:= \mathbb{U}_a(s) \\ \mathbb{U}(\alpha\beta) &:= \mathbb{U}_a(st) \\ \mathbb{U}(\alpha/\beta) &:= \mathbb{U}_a(s/t) \end{aligned}$$

Example 5. Here we give the fundamental examples that will be used in the examples. Let $s \in \mathbb{C}^*$.

(1) We have $\mathbb{U}_1(s) = \{s\}$.

(2) Let $\tau \in \mathbb{C}$ such that $\tau^2 = s$. We have $\mathbb{U}_2(s) = \{\pm\tau\}$. Therefore, in particular, we have $\mathbb{U}_2 = \{\pm 1\}$ and $\mathbb{U}_2(-1) = \{\pm i\}$.

(3) As $j := \xi_3 := e^{2i\pi/3} = (1 + i\sqrt{3})/2$, we have $1 + j + j^2 = 0$, $j^3 = 1$, $\mathbb{U}_3 = \{1, j, j^2\}$ and $\mathbb{U}_3(-1) = \{-1, -j, -j^2\}$.

(4) As $\xi_4 = i$, we have:

$$\mathbb{U}_4 = \{\pm 1, \pm i\} = \{i^l, l \in \{0, 1, 2, 3\}\} = \mathbb{U}_2 \sqcup \mathbb{U}_2(-1).$$

and

$$\mathbb{U}_4(-1) = \{\pm \xi_8, \pm i\xi_8\} = \{i^l \xi_8, l \in \{0, 1, 2, 3\}\}$$

with $\xi_8 := e^{i\pi/4} = (1 + i)/\sqrt{2}$.

(5) As $\xi_6 = e^{i\pi/3} = (1 + i\sqrt{3})/2$, we have:

$$\begin{aligned}\mathbb{U}_6 &= \{\pm 1, \pm j, \pm j^2\} = \{(-1)^k j^l, k \in \{0, 1\}, l \in \{0, 1, 2\}\} \\ &= \mathbb{U}_3 \sqcup \mathbb{U}_3(-1) = \mathbb{U}_2 \sqcup \mathbb{U}_2(j) \sqcup \mathbb{U}_2(j^2)\end{aligned}$$

(6) We have:

$$\xi_8 := e^{i\pi/4} = (1 + i)/\sqrt{2}$$

and thus:

$$\begin{aligned}\mathbb{U}_8 &= \{\pm 1, \pm i, \pm \xi_8, \pm i\xi_8\} \\ &= \{i^k \xi_8^l, k \in \{0, 1, 2, 3\}, l \in \{0, 1\}\} \\ &= \{\xi_8^k, k \in \{0, \dots, 7\}\} \\ &= \mathbb{U}_2 \sqcup \mathbb{U}_2(-1) \sqcup \mathbb{U}_2(i) \sqcup \mathbb{U}_2(-i) = \mathbb{U}_4 \sqcup \mathbb{U}_4(-1)\end{aligned}$$

(7) We have

$$\xi_{12} := e^{i\pi/6} = i^3 j = (\sqrt{3} + i)/2$$

and thus:

$$\begin{aligned}\mathbb{U}_{12} &= \{\pm 1, \pm j, \pm j^2, \pm i, \pm ij, \pm ij^2\} \\ &= \{j^k i^l, k \in \{0, 1, 2\}, l \in \{0, 1, 2, 3\}\} \\ &= \{\xi_{12}^k, k \in \{0, \dots, 11\}\} \\ &= \mathbb{U}_6 \sqcup \mathbb{U}_6(-1) = \mathbb{U}_4 \sqcup \mathbb{U}_4(j) \sqcup \mathbb{U}_4(j^2) \\ &= \mathbb{U}_3 \sqcup \mathbb{U}_3(-1) \sqcup \mathbb{U}_3(i) \sqcup \mathbb{U}_3(-i) \\ &= \mathbb{U}_2 \sqcup \mathbb{U}_2(j) \sqcup \mathbb{U}_2(j^2) \sqcup \mathbb{U}_2(-1) \sqcup \mathbb{U}_2(j) \sqcup \mathbb{U}_2(j^2)\end{aligned}$$

2.3 Matrices

In this subsection, we fix $a \in \mathbb{N}_*$ and $w \in \mathbb{C}^*$.

Definition 6. *The w -identity matrix of size a is the matrix in $\mathcal{M}_a(\mathbb{C})$ defined by:*

$$I_a(w) := (w^{i-1} \delta_{i,j})_{i,j} = \begin{pmatrix} 1 & & & \\ w & (0) & & \\ (0) & \ddots & & \\ & & & w^{a-1} \end{pmatrix}_{[a]}.$$

Example 7. (a) We have $I_a(1) = I_a$.

(b) We have:

$$I_a(w)^{-1} = I_a\left(\frac{1}{w}\right).$$

Definition 8. The *unit matrix of size a* is the matrix in $\mathcal{M}_a(\mathbb{C})$ defined by:

$$U_a := (\xi_a^{(i-1)(j-1)})_{i,j} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \xi_a & \cdots & \xi_a^{a-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \xi_a^{a-1} & \cdots & (\xi_a^{a-1})^{a-1} \end{pmatrix}_{[a]}.$$

Remark 9. The matrix U_a is the Vandermonde matrix associated with the distinct elements $1, \xi_a, \dots, \xi_a^{a-1}$ and thus we have:

$$\det(U_a) = \prod_{0 \leq i < j \leq a-1} (\xi_a^i - \xi_a^j) \neq 0.$$

It is also called the *Vandermonde-Fourier matrix*.

2.4 Definitions

The four functions \cos , \sin , ch , and sh are then particular cases of the following function.

Definition 10. Let $\alpha \in \Omega(a, b)$ and $I \in \mathcal{P}(\mathbb{N})$.

(i) The (α, I) -*sine function* is the function defined on \mathbb{C} by:

$$\forall z \in \mathbb{C}, \mathbf{Sin}_{\alpha, I}(z) := \sum_{k \in I} s^k \frac{z^{ak+b}}{(ak+b)!}.$$

(ii) The I -*exponential function* is the function defined on \mathbb{C} by:

$$\forall z \in \mathbb{C}, \mathbf{Exp}_I(z) := \mathbf{Sin}_{(1,1,0), I}(z) = \sum_{k \in I} \frac{z^k}{k!}.$$

Notation 11. In the case $I := \mathbb{N}$, we simply denote:

$$\forall z \in \mathbb{C}, \mathbf{Sin}_{\alpha}(z) := \sum_{k=0}^{\infty} s^k \frac{z^{ak+b}}{(ak+b)!}.$$

called the α -*sine function*.

Example 12. (1) We have:

$$\exp = \mathbf{Exp}_{\mathbb{N}}$$

(2) We have:

$$\cos = \mathbf{Sin}_{(-1,2,0)}, \quad \sin = \mathbf{Sin}_{(-1,2,1)}, \quad \text{ch} = \mathbf{Sin}_{(1,2,0)}, \quad \text{sh} = \mathbf{Sin}_{(1,2,1)}.$$

3 Usual Properties

3.1 Symmetry Formula

The first proposition expresses any sine as an exponential.

Proposition 13 (Symmetry Formula). *Let $\alpha \in \Omega(a, b)$ and $I \in \mathcal{P}(\mathbb{N})$. For all $z \in \mathbb{C}$ and all $\tau \in \mathbb{U}(\alpha)$, we have:*

$$\mathbf{Sin}_{\alpha, I}(z) = \frac{1}{\tau^b} \mathbf{Exp}_{aI+b}(\tau z).$$

Proof. Let $z \in \mathbb{C}$ and $\tau \in \mathbb{U}(\alpha)$. We have:

$$\begin{aligned} \frac{1}{\tau^b} \mathbf{Exp}_{aI+b}(\tau z) &= \frac{1}{\tau^b} \sum_{k \in aI+b} \frac{(\tau z)^k}{k!} \\ &= \frac{1}{\tau^b} \sum_{l \in I} \frac{(\tau z)^{al+b}}{(al+b)!} \\ &= \sum_{l \in I} \tau^{al} \frac{z^{al+b}}{(al+b)!} \\ &= \sum_{l \in I} s^l \frac{z^{al+b}}{(al+b)!} \\ &= \mathbf{Sin}_{\alpha, I}(z) \end{aligned}$$

□

Example 14. *Let $z \in \mathbb{C}$. Thus in the case $a := 2$, we simply have:*

$$\begin{aligned} \cos(z) &= \mathbf{Sin}_{(-1, 2, 0)}(z) = \mathbf{Exp}_{2\mathbb{N}}(iz) \\ \sin(z) &= \mathbf{Sin}_{(-1, 2, 1)}(z) = -i \mathbf{Exp}_{2\mathbb{N}+1}(iz) \\ \operatorname{ch}(z) &= \mathbf{Sin}_{(1, 2, 0)}(z) = \mathbf{Exp}_{2\mathbb{N}}(z) \\ \operatorname{sh}(z) &= \mathbf{Sin}_{(1, 2, 1)}(z) = \mathbf{Exp}_{2\mathbb{N}+1}(z) \end{aligned}$$

Corollary 15. *Let $\alpha, \beta \in \Omega(a, b)$. For all $z \in \mathbb{C}$ and all $\tau \in \mathbb{U}(\beta/\alpha)$, we have:*

$$\mathbf{Sin}_{\alpha, I}(\tau z) = \tau^b \mathbf{Sin}_{\beta, I}(z).$$

Proof. Let $z \in \mathbb{C}$ and $\tau \in \mathbb{U}(\beta/\alpha)$. There exist $\mu \in \mathbb{U}(\beta)$ and $\nu \in \mathbb{U}(\alpha)$ such that $\tau = \mu/\nu$. Then, by Proposition 13, we have:

$$\begin{aligned} \mathbf{Sin}_{\alpha, I}(\tau z) &= \frac{1}{\nu^b} \mathbf{Exp}_{aI+b}(\nu \tau z) \\ &= \tau^b \frac{1}{\mu^b} \mathbf{Exp}_{aI+b}(\mu z) \\ &= \tau^b \mathbf{Sin}_{\beta, I}(z) \end{aligned}$$

□

Example 16. Let $z \in \mathbb{C}$. By applying Corollary 15 in the case $\beta := (1, a, b)$ and $I := \mathbb{N}$, we have for all $\nu \in \mathbb{U}(\alpha)$:

$$\mathbf{Sin}_\alpha(z) = \frac{1}{\nu^b} \mathbf{Sin}_{(1,a,b)}(\nu z).$$

Thus in the case $(s, a) := (-1, 2)$ and $\nu := i$, we retrieve the usual formulas:

$$\begin{aligned} \cos(z) &= \mathbf{Sin}_{(-1,2,0)}(z) = \frac{1}{i^0} \mathbf{Sin}_{(1,2,0)}(iz) = \text{ch}(iz) \\ \sin(z) &= \mathbf{Sin}_{(-1,2,1)}(z) = \frac{1}{i^1} \mathbf{Sin}_{(1,2,1)}(iz) = -\text{ish}(iz) \end{aligned}$$

3.2 Euler Formula

Example 17. Let $\alpha \in \Omega(a, b)$ and $I \in \mathcal{P}(\mathbb{N})$. A first simple relation is obtained by averaging over $\mathbb{U}(\alpha)$ the relation of Proposition 18. We have for all $z \in \mathbb{C}$:

$$\mathbf{Sin}_{\alpha, I}(z) = \frac{1}{a} \sum_{\tau \in \mathbb{U}(\alpha)} \frac{\mathbf{Exp}_{aI+b}(\tau z)}{\tau^b}.$$

This relation, which resembles a first Euler relation, is not useful. For example, for \cos we have:

$$\cos(z) = \mathbf{Sin}_{(1,2,0)}(z) = \frac{1}{2} (\mathbf{Exp}_{2\mathbb{N}}(iz) + \mathbf{Exp}_{2\mathbb{N}}(-iz)).$$

this is not the known Euler formula for \cos .

Proposition 18 (Euler Formula - Version 1). Let $\alpha \in \Omega(a, b)$ and $I \in \mathcal{P}(\mathbb{N})$. We have for all $z \in \mathbb{C}$:

$$\mathbf{Sin}_{\alpha, I(a,b)}(z) = \frac{1}{a} \sum_{\tau \in \mathbb{U}(\alpha)} \frac{\exp_I(\tau z)}{\tau^b}.$$

Proof. Let $z \in \mathbb{C}$. For all $d \in \mathbb{N}$ we have:

$$\sum_{\tau \in \mathbb{U}(\alpha)} \tau^d = a s^{d/a} \delta_{d, a\mathbb{N}}.$$

thus we have:

$$\begin{aligned} \frac{1}{a} \sum_{\tau \in \mathbb{U}(\alpha)} \frac{\exp_I(\tau z)}{\tau^b} &= \frac{1}{a} \sum_{\tau \in \mathbb{U}(\alpha)} \frac{1}{\tau^b} \sum_{k \in I} \frac{(\tau z)^k}{k!} \\ &= \sum_{k \in I} \frac{z^k}{k!} \frac{1}{a} \sum_{\tau \in \mathbb{U}(\alpha)} \tau^{k-b} \\ &= \sum_{k \in I \cap (a\mathbb{N}+b)} \frac{z^k}{a k!} a s(\alpha)^{(k-b)/a} \delta_{k-b, a\mathbb{N}} \end{aligned}$$

$$\begin{aligned}
&= \sum_{l \in I(a,b)} s(\alpha)^l \frac{z^{al+b}}{(al+b)!} \\
&= \mathbf{Sin}_{\alpha, I(a,b)}(z)
\end{aligned}$$

□

Example 19. Let $\alpha \in \Omega(a, b)$. For any $n \in \mathbb{N}$, we have:

$$\mathbb{N}_{an+b}(a, b) = \{l \in \mathbb{N}, al + b \in \mathbb{N}_{an+b}\} = \mathbb{N}_n.$$

Thus for any $z \in \mathbb{C}$, we have:

$$\mathbf{Sin}_{\alpha, \mathbb{N}_n}(z) = \frac{1}{a} \sum_{\tau \in \mathbb{U}(\alpha)} \frac{\exp_{\mathbb{N}_{an+b}}(\tau z)}{\tau^b}.$$

Corollary 20. Let $\alpha \in \Omega(a, b)$. For any $z \in \mathbb{C}$, we have:

$$\mathbf{Sin}_{\alpha}(z) = \frac{1}{a} \sum_{\tau \in \mathbb{U}(\alpha)} \frac{e^{\tau z}}{\tau^b}.$$

Proof. We apply Proposition 18 in the case $I := \mathbb{N}$. This directly follows from the fact that $\mathbb{N}(a, b) = \mathbb{N}$. □

Example 21. (1) Case where $a := 2$. Let $\nu \in \mathbb{U}_2(s)$. Since $\mathbb{U}_2(s) = \{\pm\nu\}$, we have:

$$\mathbf{Sin}_{(s,2,b)}(z) = \frac{1}{2} \left(\frac{e^{\nu z}}{\nu^b} + \frac{e^{-\nu z}}{(-\nu)^b} \right).$$

We retrieve the four Euler formulas stated in the introduction.

(2) Case where $a := 3$. Let $\nu \in \mathbb{U}_3(s)$. Since $\mathbb{U}_3(s) = \{\nu, \nu j, \nu j^2\}$, we have:

$$\mathbf{Sin}_{(s,2,b)}(z) = \frac{1}{3} \left(\frac{e^{\nu z}}{\nu^b} + \frac{e^{j\nu z}}{(\nu j)^b} + \frac{e^{\nu j^2 z}}{(\nu j^2)^b} \right).$$

Proposition 22 (Euler Formula - Version 2). Let $(s, a, b) \in \Omega$, $I \in \mathcal{P}(\mathbb{N})$, and $a', c \in \mathbb{N}_*$ such that $a = a'c$. We have:

$$\mathbf{Sin}_{(s,a,b), I(a,b)} = \frac{1}{c} \sum_{\nu \in \mathbb{U}_c(s)} \frac{1}{\nu^q} \mathbf{Sin}_{(\nu, a', b'), I(a', b')}$$

with q and b' respectively the quotient and remainder of the Euclidean division of b by a' , i.e., $b = qa' + b'$ and $0 \leq b' < a'$.

Proof. We have:

$$\mathbb{U}_a(s) = \bigsqcup_{\nu \in \mathbb{U}_c(s)} \mathbb{U}_{a'}(\nu)$$

Thus we have for all $z \in \mathbb{C}$:

$$\begin{aligned} \mathbf{Sin}_{(s,a,b),I(a,b)}(z) &= \frac{1}{a} \sum_{\tau \in \mathbb{U}_a(s)} \frac{\mathbf{Exp}_I(\tau z)}{\tau^b} \\ &= \frac{1}{c} \sum_{\nu \in \mathbb{U}_c(s)} \frac{1}{a'} \sum_{\mu \in \mathbb{U}_{a'}(\nu)} \frac{\mathbf{Exp}_I(\mu z)}{\mu^b} \\ &= \frac{1}{c} \sum_{\nu \in \mathbb{U}_c(s)} \frac{1}{a' \mu^{qa'}} \sum_{\mu \in \mathbb{U}_{a'}(\nu)} \frac{\mathbf{Exp}_I(\mu z)}{\mu^{b'}} \\ &= \frac{1}{c} \sum_{\nu \in \mathbb{U}_c(s)} \frac{1}{\mu^{qa'}} \left(\frac{1}{a'} \sum_{\mu \in \mathbb{U}_{a'}(\nu)} \frac{\mathbf{Exp}_I(\mu z)}{\mu^{b'}} \right) \\ &= \frac{1}{c} \sum_{\nu \in \mathbb{U}_c(s)} \frac{1}{\nu^q} \mathbf{Sin}_{(\nu,a',b'),I(a',b')}(z) \end{aligned}$$

□

Example 23. In the case where $I := \mathbb{N}$, we have:

$$\mathbf{Sin}_{(s,a,b)} = \frac{1}{c} \sum_{\nu \in \mathbb{U}_c(s)} \frac{\mathbf{Sin}_{(\nu,a',b')}}{\nu^q}.$$

For example:

$$\begin{aligned} &\mathbf{Sin}_{(1,12,0)}(z) \\ &= \frac{1}{12} \left(e^z + e^{-z} + e^{jz} + e^{-jz} + e^{j^2z} + e^{-j^2z} + e^{iz} + e^{-iz} + e^{jiz} + e^{-jiz} + e^{j^2iz} + e^{-j^2iz} \right) \end{aligned}$$

Therefore, we have:

$$\begin{aligned} &\mathbf{Sin}_{(1,12,0)} \\ &= \frac{1}{2} \left(\mathbf{Sin}_{(1,6,0)} + \mathbf{Sin}_{(-1,6,0)} \right) \\ &= \frac{1}{3} \left(\mathbf{Sin}_{(1,4,0)} + \mathbf{Sin}_{(j,4,0)} + \mathbf{Sin}_{(j^2,4,0)} \right) \\ &= \frac{1}{4} \left(\mathbf{Sin}_{(1,3,0)} + \mathbf{Sin}_{(-1,3,0)} + \mathbf{Sin}_{(i,3,0)} + \mathbf{Sin}_{(-i,3,0)} \right) \\ &= \frac{1}{6} \left(\mathbf{Sin}_{(1,2,0)} + \mathbf{Sin}_{(-1,2,0)} + \mathbf{Sin}_{(j,2,0)} + \mathbf{Sin}_{(-j,2,0)} + \mathbf{Sin}_{(j^2,2,0)} + \mathbf{Sin}_{(-j^2,2,0)} \right) \end{aligned}$$

3.3 Moivre's Formula

Proposition 24 (Exponential Decomposition). *Let $I \in \mathcal{P}(\mathbb{N})$, $s \in \mathbb{C}^*$, and $a \in \mathbb{N}_*$. For all $z \in \mathbb{C}$ and all $\tau \in \mathbb{U}_a(s)$, we have:*

$$\sum_{b=0}^{a-1} \tau^b \mathbf{Sin}_{(s,a,b),I(a,b)}(z) = \exp_I(\tau z).$$

Proof. By Proposition 18, we have:

$$\begin{aligned} \sum_{b=0}^{a-1} \tau^b \mathbf{Sin}_{(s,a,b),I(a,b)}(z) &= \sum_{b=0}^{a-1} \tau^b \frac{1}{a} \sum_{\nu \in \mathbb{U}(\alpha)} \frac{\exp_I(\nu z)}{\nu^b} \\ &= \sum_{\nu \in \mathbb{U}(\alpha)} \frac{\exp_I(\nu z)}{a} \sum_{b=0}^{a-1} (\tau/\nu)^b \\ &= \sum_{\nu \in \mathbb{U}(\alpha)} \frac{\exp_I(\nu z)}{a} a \delta_{\tau,\nu} \\ &= \frac{\exp_I(\tau z)}{a} \\ &= \exp_I(\tau z) \end{aligned}$$

□

Example 25. *Let $s \in \mathbb{C}^*$. Since $\mathbb{N}(a,b) = \mathbb{N}$, for all $z \in \mathbb{C}$ and all $\tau \in \mathbb{U}_a(s)$, we have:*

$$\sum_{b=0}^{a-1} \tau^b \mathbf{Sin}_{(s,a,b)}(z) = e^{\tau z}.$$

In the case $a := 2$, we recover the usual relations ($\tau := 1, -1, i, -i$ respectively). For all $z \in \mathbb{C}$:

$$\begin{aligned} \operatorname{ch}(z) + \operatorname{sh}(z) &= \mathbf{Sin}_{(1,2,0)}(z) + \mathbf{Sin}_{(1,2,1)}(z) = e^z \\ \operatorname{ch}(z) - \operatorname{sh}(z) &= \mathbf{Sin}_{(1,2,0)}(z) - \mathbf{Sin}_{(1,2,1)}(z) = e^{-z} \\ \cos(z) + i \sin(z) &= \mathbf{Sin}_{(1,2,0)}(z) + i \mathbf{Sin}_{(1,2,1)}(z) = e^{iz} \\ \cos(z) - i \sin(z) &= \mathbf{Sin}_{(1,2,0)}(z) - i \mathbf{Sin}_{(1,2,1)}(z) = e^{-iz} \end{aligned}$$

Corollary 26 (Generalized de Moivre's Formula). *Let $m \in \mathbb{Z}$. For all $z \in \mathbb{C}$ and all $\tau \in \mathbb{U}_a(s)$, we have:*

$$\left(\sum_{b=0}^{a-1} \tau^b \mathbf{Sin}_{(s,a,b)}(z) \right)^m = \sum_{b=0}^{a-1} \tau^b \mathbf{Sin}_{(s,a,b)}(mz).$$

Proof. From Corollary 24, we directly have:

$$\left(\sum_{b=0}^{a-1} \tau^b \mathbf{Sin}_{(s,a,b)}(z) \right)^m = (e^{\tau z})^m = e^{\tau m z} = \sum_{b=0}^{a-1} \tau^b \mathbf{Sin}_{(s,a,b)}(mz).$$

□

3.4 Algebraic Equations

Proposition 27. *Let $s \in \mathbb{C}^*$ and $a \in \mathbb{N}_*$. We have:*

$$\prod_{\tau \in \mathbb{U}_a(s)} \sum_{b=0}^{a-1} \tau^b \mathbf{Sin}_{(s,a,b)} = 1.$$

Proof. From Corollary 24, for all $z \in \mathbb{C}$:

$$\prod_{\tau \in \mathbb{U}_a(s)} \sum_{b=0}^{a-1} \tau^b \mathbf{Sin}_{(s,a,b)}(z) = \prod_{\tau \in \mathbb{U}_a(s)} e^{\tau z} = \exp\left(z \sum_{\tau \in \mathbb{U}_a(s)} \tau\right) = 1$$

□

Example 28 (Case $a := 2$). *Let $a := 2$ and $s \in \mathbb{C}^*$, we have $\mathbb{U}_2(s) = \{\pm\nu\}$ with $\nu^2 = s$. Let's define:*

$$X := \mathbf{Sin}_{(s,2,0)}, \quad Y := \mathbf{Sin}_{(s,2,1)}$$

By Proposition 27, we have:

$$\begin{aligned} 1 &= \prod_{\tau \in \{\pm\nu\}} (X + \tau Y) \\ &= (X + \nu Y)(X - \nu Y) = X^2 - sY^2 \end{aligned}$$

(1) *In the case where $s := 1$, we retrieve the usual parametrization of the hyperbola $X^2 - Y^2 = 1$ using the functions ch and sh .*

(2) *In the case where $s := -1$, we retrieve the usual parametrization of the circle $X^2 + Y^2 = 1$ using the functions \cos and \sin .*

Example 29. *Let $a := 3$ and $s := re^{i\theta} \in \mathbb{C}^*$. We have $\mathbb{U}_3(s) = \{\nu, j\nu, j^2\nu\}$ with $j := e^{2i\pi/3}$ and $\nu := \sqrt[3]{r}e^{i\theta/3}$. Let's define:*

$$X := \mathbf{Sin}_{(s,3,0)}, \quad Y := \mathbf{Sin}_{(s,3,1)}, \quad Z := \mathbf{Sin}_{(s,3,2)}$$

By Proposition 27, for all $z \in \mathbb{C}$:

$$\begin{aligned} 1 &= \prod_{\tau \in \{\nu, j\nu, j^2\nu\}} (X + \tau Y + \tau^2 Z) \\ &= (X + \nu Y + \nu^2 Z)(X + j\nu Y + j^2\nu^2 Z)(X + j^2\nu Y + j\nu^2 Z) \\ &= X^3 + sY^3 + s^2Z^3 - 3sXYZ. \end{aligned}$$

Example 30. *Let $a := 4$ and $s := re^{i\theta} \in \mathbb{C}^*$. We have $\mathbb{U}_4(s) = \{\pm\nu, \pm i\nu\}$ with $\nu := \sqrt[4]{r}e^{i\theta/4}$. Let's define:*

$$X := \mathbf{Sin}_{(s,4,0)}, \quad Y := \mathbf{Sin}_{(s,4,1)}, \quad Z := \mathbf{Sin}_{(s,4,2)}, \quad T := \mathbf{Sin}_{(s,4,3)}$$

By Proposition 27, for all $z \in \mathbb{C}$:

$$\begin{aligned}
1 &= \prod_{\tau \in \{\pm\nu, \pm i\nu\}} (X + \tau Y + \tau^2 Z + \tau^3 T) \\
&= (X + \nu Y + \nu^2 Z + \nu^3 T) (X - \nu Y + \nu^2 Z - \nu^3 T) \\
&\quad (X + i\nu Y - \nu^2 Z - i\nu^3 T) (X - i\nu Y - \nu^2 Z + i\nu^3 T) \\
&= X^4 - sY^4 + s^2Z^4 - s^3T^4 + 4s(XZ(Y^2 + sT^2) - YT(X^2 + sZ^2)) + 2s(sY^2T^2 - X^2Z^2)
\end{aligned}$$

3.5 Differential Equations

Proposition 31. Let $\alpha := (s, a, b) \in \Omega(a, b)$. We have:

$$\mathbf{Sin}'_{\alpha} = s^{\delta_{0,b}} \mathbf{Sin}_{\alpha-1}$$

with

$$\forall k \in \{0, \dots, a\}, \quad \alpha - k := \begin{cases} (s, a, b - k) & \text{if } b \geq k \\ (s, a, a + b - k) & \text{if } b < k. \end{cases}$$

Proof. We consider two cases.

- Case $b = 0$. Directly from the Euler's formula 18 for all $z \in \mathbb{C}$:

$$\mathbf{Sin}'_{\alpha}(z) = \mathbf{Sin}'_{(s,a,0)}(z) = \frac{1}{a} \sum_{\tau \in \mathbb{U}_a(s)} \tau e^{\tau z} = \frac{s}{a} \sum_{\tau \in \mathbb{U}_a(s)} \frac{e^{\tau z}}{\tau^{a-1}} = s \mathbf{Sin}_{\alpha-1}$$

- Case $b \neq 0$. Directly from the Euler's formula 18 for all $z \in \mathbb{C}$:

$$\mathbf{Sin}'_{\alpha}(z) = \mathbf{Sin}'_{(s,a,0)}(z) = \frac{1}{a} \sum_{\tau \in \mathbb{U}_a(s)} \frac{e^{\tau z}}{\tau^{b-1}} = \mathbf{Sin}_{\alpha-1}$$

Hence the result. □

Corollary 32. Let $\alpha := (s, a, b) \in \Omega(a, b)$. We have:

$$\mathbf{Sin}_{\alpha}^{(a)} = s \mathbf{Sin}_{\alpha}$$

Proof. We have, by simple recurrence using proposition 31:

$$\mathbf{Sin}_{\alpha}^{(a)} = \mathbf{Sin}_{\alpha-b}^{(a-b)} = s \mathbf{Sin}_{\alpha-b-1}^{(a-b-1)} = s \mathbf{Sin}_{\alpha-a}^{(0)} = s \mathbf{Sin}_{\alpha}.$$

□

Corollary 33. Let $\alpha := (s, a, b) \in \Omega(a, b)$. Let's define:

$$(E_{(s,a)}) : y^{(a)} = sy, \quad \mathbb{C}.$$

Then we have:

$$\text{Sol}(E_{(s,a)}) = \text{Vect}_{\mathbb{C}}(\mathbf{Sin}_{(s,a,b)}, b \in \{0, \dots, a-1\}).$$

Proof. By Cauchy-Lin's theorem, we know that:

$$\dim_{\mathbb{C}} \text{Sol}(E_{(s,a)}) = a.$$

By corollary 32, we have for all $b \in \{0, \dots, a-1\}$, $\mathbf{Sin}_{(s,a,b)}^{(a)} = s\mathbf{Sin}_{(s,a,b)}$ i.e. $\mathbf{Sin}_{(s,a,b)}^{(a)} \in \text{Sol}(E_{(s,a)})$. Hence we have:

$$\text{Vect}_{\mathbb{C}}(\mathbf{Sin}_{(s,a,b)}, b \in \{0, \dots, a-1\}) \subset \text{Sol}(E_{(s,a)})$$

We just need to show that the family $\{\mathbf{Sin}_{(s,a,b)}, b \in \{0, \dots, a-1\}\}$ is free.

Let $\tau \in \mathbb{U}_a(s)$. We have:

$$\mathbb{U}_a(s) = \{\tau \xi_a^k, k \in \{0, \dots, a-1\}\}.$$

Let $\alpha_1, \dots, \alpha_a \in \mathbb{C}$. Since the family $\{z \mapsto e^{\nu z}, \nu \in \mathbb{U}_a(s)\}$ is linearly independent over \mathbb{C} , we have:

$$\begin{aligned} \sum_{b=0}^{a-1} \alpha_b \mathbf{Sin}_{(s,a,b)} = 0 &\iff \forall z \in \mathbb{C}, \sum_{b=0}^{a-1} \alpha_b \mathbf{Sin}_{(s,a,b)}(z) = 0 \\ &\iff \forall z \in \mathbb{C}, \sum_{b=0}^{a-1} \alpha_b \frac{1}{a} \sum_{\nu \in \mathbb{U}_a(s)} \frac{e^{\nu z}}{\nu^b} = 0 \\ &\iff \forall z \in \mathbb{C}, \sum_{\nu \in \mathbb{U}_a(s)} e^{\nu z} \sum_{b=0}^{a-1} \frac{\alpha_b}{\nu^b} = 0 \\ &\iff \forall \nu \in \mathbb{U}_a(s), \sum_{b=0}^{a-1} \frac{\alpha_b}{\nu^b} = 0 \\ &\iff \forall \nu \in \mathbb{U}_a(s), \sum_{b=0}^{a-1} \alpha_b \nu^{a-1-b} = 0 \\ &\iff \forall k \in \{0, \dots, a-1\}, \sum_{b=0}^{a-1} \alpha_b (\tau \xi_a^k)^{a-1-b} = 0 \\ &\iff U_a I(\tau) \begin{pmatrix} \alpha_{a-1} \\ \alpha_{a-2} \\ \vdots \\ \alpha_0 \end{pmatrix} = 0 \\ &\iff \begin{pmatrix} \alpha_{a-1} \\ \alpha_{a-2} \\ \vdots \\ \alpha_0 \end{pmatrix} = 0 \text{ since } U_a I(\tau) \text{ is invertible} \\ &\iff \forall b \in \{0, \dots, a-1\}, \alpha_b = 0 \end{aligned}$$

Thus the family is linearly independent. Hence the result. \square

Example 34. (1) Case $a := 1$. Then we find that $\exp = \mathbf{Sin}_{(1,1,0)}$ satisfies:

$$(E_{(1,1)}) : y' = y, \mathbb{C}.$$

and that:

$$\text{Sol}(E_{(1,1)}) = \text{Vect}_{\mathbb{C}}(\exp).$$

(2) Case $a := 2$.

(a) Case $s := 1$. Then we find that $\text{ch} = \mathbf{Sin}_{(1,2,0)}$ and $\text{sh} = \mathbf{Sin}_{(1,2,1)}$ satisfy:

$$(E_{(1,2)}) : y'' = y, \mathbb{C}.$$

and that:

$$\text{Sol}(E_{(1,2)}) = \text{Vect}_{\mathbb{C}}(\text{ch}, \text{sh}).$$

(b) Case $s := -1$. Then we find that $\text{cos} = \mathbf{Sin}_{(-1,2,0)}$ and $\text{sin} = \mathbf{Sin}_{(-1,2,1)}$ satisfy:

$$(E_{(-1,2)}) : y'' = -y, \mathbb{C}.$$

and that:

$$\text{Sol}(E_{(-1,2)}) = \text{Vect}_{\mathbb{C}}(\text{cos}, \text{sin}).$$

3.6 Sine of a Sum

Proposition 35. Let $\alpha := (s, a, b) \in \Omega(a, b)$. For all $z, z' \in \mathbb{C}$, we have:

$$\mathbf{Sin}_{\alpha}(z+z') = \sum_{b'=0}^b \mathbf{Sin}_{(s,a,b')}(z) \mathbf{Sin}_{(s,a,b-b')}(z') + \sum_{b'=b+1}^{a-1} s \mathbf{Sin}_{(s,a,b')}(z) \mathbf{Sin}_{(s,a,a+b-b')}(z')$$

with the convention that $\sum_{\emptyset} := 0$.

Proof. Let $\alpha := (s, a, b) \in \Omega(a, b)$. For all $z, z' \in \mathbb{C}$, we have:

$$\begin{aligned} \mathbf{Sin}_{\alpha}(z+z') &= \frac{1}{a} \sum_{\tau \in \mathbb{U}_a(s)} \frac{e^{\tau(z+z')}}{\tau^b} \\ &= \frac{1}{a} \sum_{\tau \in \mathbb{U}_a(s)} \frac{1}{\tau^b} e^{\tau z} e^{\tau z'} \\ &= \frac{1}{a} \sum_{\tau \in \mathbb{U}_a(s)} \frac{1}{\tau^b} \left(\sum_{b'=0}^{a-1} \tau^{b'} \mathbf{Sin}_{(s,a,b')}(z) \right) \left(\sum_{b''=0}^{a-1} \tau^{b''} \mathbf{Sin}_{(s,a,b'')}(z') \right) \\ &= \frac{1}{a} \sum_{0 \leq b', b'' \leq a-1} \mathbf{Sin}_{(s,a,b')}(z) \mathbf{Sin}_{(s,a,b'')}(z') \sum_{\tau \in \mathbb{U}_a(s)} \tau^{b'+b''-b} \\ &= \frac{1}{a} \sum_{0 \leq b', b'' \leq a-1} \mathbf{Sin}_{(s,a,b')}(z) \mathbf{Sin}_{(s,a,b'')}(z') a s^{(b'+b''-b)/a} \delta_{b'+b''-b, a\mathbb{N}} \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{0 \leq b', b'' \leq a-1 \\ b'+b''=b[a]}} s^{(b'+b''-b)/a} \mathbf{Sin}_{(s,a,b')}(z') \mathbf{Sin}_{(s,a,b'')}(z) \\
&= \sum_{b'=0}^b \mathbf{Sin}_{(s,a,b')}(z) \mathbf{Sin}_{(s,a,b-b')}(z') + \sum_{b'=b+1}^{a-1} s \mathbf{Sin}_{(s,a,b')}(z) \mathbf{Sin}_{(s,a,a+b-b')}(z').
\end{aligned}$$

□

Corollary 36. Let $\alpha := (s, a, b) \in \Omega(a, b)$. For all $z, z' \in \mathbb{C}$ and any $u \in \mathbb{U}_a$:

$$\begin{aligned}
\mathbf{Sin}_\alpha(z + uz') &= \sum_{b'=0}^b u^{b-b'} \mathbf{Sin}_{(s,a,b')}(z) \mathbf{Sin}_{(s,a,b-b')}(z') \\
&\quad + \sum_{b'=b+1}^{a-1} s u^{b-b'} \mathbf{Sin}_{(s,a,b')}(z) \mathbf{Sin}_{(s,a,a+b-b')}(z').
\end{aligned}$$

Proof. Let $\alpha := (s, a, b) \in \Omega(a, b)$, $z, z' \in \mathbb{C}$, and $u \in \mathbb{U}_a$. By Proposition ??, we have:

$$\begin{aligned}
&\mathbf{Sin}_\alpha(z + uz') \\
&= \sum_{b'=0}^b \mathbf{Sin}_{(s,a,b')}(z) \mathbf{Sin}_{(s,a,b-b')}(uz') + \sum_{b'=b+1}^{a-1} s \mathbf{Sin}_{(s,a,b')}(z) \mathbf{Sin}_{(s,a,a+b-b')}(uz') \\
&= \sum_{b'=0}^b u^{b-b'} \mathbf{Sin}_{(s,a,b')}(z) \mathbf{Sin}_{(s,a,b-b')}(z') + \sum_{b'=b+1}^{a-1} s u^{a+b-b'} \mathbf{Sin}_{(s,a,b')}(z) \mathbf{Sin}_{(s,a,a+b-b')}(z') \\
&= \sum_{b'=0}^b u^{b-b'} \mathbf{Sin}_{(s,a,b')}(z) \mathbf{Sin}_{(s,a,b-b')}(z') + \sum_{b'=b+1}^{a-1} s u^{b-b'} \mathbf{Sin}_{(s,a,b')}(z) \mathbf{Sin}_{(s,a,a+b-b')}(z')
\end{aligned}$$

□

3.7 Rotation Matrix

We will retrieve the results from the previous subsection using rotation matrices.

Definition 37. Let $\alpha := (s, a, b) \in \Omega(a, b)$. The *rotation matrix associated with α* is the matrix in $\mathcal{M}_a(\mathcal{F}(\mathbb{C}))$ defined by:

$$\forall i, j \in \{1, \dots, a\}, \quad [\text{Rot}_\alpha]_{i,j} := \begin{cases} \mathbf{Sin}_{(s,a,j-i)} & \text{if } j \geq i \\ s \mathbf{Sin}_{(s,a,a+j-i)} & \text{if } j < i. \end{cases}$$

Example 38. Let $\alpha \in \Omega(a, b)$. Since $\mathbf{Sin}_\alpha(0) = \delta_{0,b}$, we have:

$$\text{Rot}_\alpha(0) = (\delta_{i,j})_{i,j} = I_a.$$

Proposition 39. Let $\alpha := (s, a, b) \in \Omega(a, b)$.

(i) For all $z, z' \in \mathbb{C}$:

$$\text{Rot}_\alpha(z + z') = \text{Rot}_\alpha(z)\text{Rot}_\alpha(z').$$

(ii) For all $z \in \mathbb{C}$:

$$\text{Rot}_\alpha(z)^{-1} = \text{Rot}_\alpha(-z).$$

(iii) For all $z \in \mathbb{C}$ and all $u \in \mathbb{U}_a$:

$$\text{Rot}_\alpha(uz) = I_a(u)^{-1}\text{Rot}_\alpha(z)I_a(u).$$

(iv) We have:

$$\det(\text{Rot}_\alpha) = 1.$$

Proof. Let $S_b := \mathbf{Sin}_{(s,a,b)}$ for all $b \in \{0, \dots, a-1\}$ and $R := \text{Rot}_\alpha$.

(i) Let $z, z' \in \mathbb{C}$ and $i, j \in \{1, \dots, a\}$.

We have two cases.

(1) Case $i \leq j$. By Proposition 35, we have:

$$\begin{aligned} & [R(z)R(z')]_{i,j} \\ &= \sum_{k=1}^a [R(z)]_{i,k} \cdot [R(z')]_{k,j} \\ &= \sum_{k=1}^{i-1} [R(z)]_{i,k} \cdot [R(z')]_{k,j} + \sum_{k=i}^j [R(z)]_{i,k} \cdot [R(z')]_{k,j} + \sum_{k=j+1}^a [R(z)]_{i,k} \cdot [R(z')]_{k,j} \\ &= \sum_{k=1}^{i-1} sS_{a+k-i}(z) \cdot S_{j-k}(z') + \sum_{k=i}^j S_{k-i}(z) \cdot S_{j-k}(z') + \sum_{k=j+1}^a S_{k-i}(z) \cdot sS_{a+j-k}(z') \\ &= \sum_{b'=a-i+1}^{a-1} sS_{b'}(z) \cdot S_{a+j-i-b'}(z') + \sum_{b'=0}^{j-i} S_{b'}(z) \cdot S_{j-i-b'}(z') + \sum_{b'=j-i+1}^{a-i} S_{b'}(z) \cdot sS_{a+j-i-b'}(z') \\ &= \sum_{b'=0}^{j-i} S_{b'}(z) \cdot S_{j-i-b'}(z') + \sum_{b'=j-i+1}^{a-1} S_{b'}(z) \cdot sS_{a+j-i-b'}(z') \\ &= S_{j-i}(z + z') \\ &= [R(z + z')]_{i,j} \end{aligned}$$

(2) Case $i > j \Rightarrow i \geq j + 1$. By Proposition 35:

$$\begin{aligned} & [R(z)R(z')]_{i,j} \\ &= \sum_{k=1}^a [R(z)]_{i,k} \cdot [R(z')]_{k,j} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^j [R(z)]_{i,k} \cdot [R(z')]_{k,j} + \sum_{k=j+1}^{i-1} [R(z)]_{i,k} \cdot [R(z')]_{k,j} + \sum_{k=i}^a [R(z)]_{i,k} \cdot [R(z')]_{k,j} \\
&= \sum_{k=1}^j sS_{a+k-i} \cdot S_{j-k} + \sum_{k=j+1}^{i-1} sS_{a+k-i} \cdot sS_{a+j-k} + \sum_{k=i}^a S_{k-i} \cdot sS_{a+j-k} \\
&= s \left(\sum_{b'=a-i+1}^{a-i+j} S_{b'} \cdot S_{a+j-i-b'} + \sum_{b'=a+j-i+1}^{a-1} sS_{b'} \cdot S_{a+(a+j-i)-b'} + \sum_{b'=0}^{a-i} S_{b'} \cdot S_{a+(a+j-i)-b'} \right) \\
&= s \left(\sum_{b'=0}^{a-i+j} S_{b'} \cdot S_{(a+j-i)-b'} + \sum_{b'=a+j-i+1}^{a-1} sS_{b'} \cdot S_{a+(a+j-i)-b'} \right) \\
&= sS_{a+j-i}(z+z') \\
&= [R(z+z')]_{i,j}
\end{aligned}$$

Hence the result.

(ii) For all $z \in \mathbb{C}$:

$$I_a = R(z-z) = R(z)R(-z)$$

Which implies:

$$R(z)^{-1} = R(-z).$$

(iii) Let $z \in \mathbb{C}$, $u \in \mathbb{U}_a$, and $i, j \in \{1, \dots, a\}$. We have:

$$\begin{aligned}
[I_a(u)^{-1}R(z)I_a(u)]_{i,j} &= \sum_{k=1}^a [I_a(u)^{-1}]_{i,k} \cdot [R(z)I_a(u)]_{k,j} \\
&= \sum_{k=1}^a u^{a-i+1} \delta_{i,k} \cdot [R(z)I_a(u)]_{k,j} \\
&= u^{a-i+1} [R(z)I_a(u)]_{i,j} \\
&= u^{a-i+1} \sum_{k=1}^a [R(z)]_{i,k} \cdot [I_a(u)]_{k,j} \\
&= u^{a-i+1} \sum_{k=1}^a [R(z)]_{i,k} \cdot u^{j-1} \delta_{k,j} \\
&= u^{a-i+1} [R(z)]_{i,j} \cdot u^{j-1} \\
&= [u^{j-i}R(z)]_{i,j}
\end{aligned}$$

We have two cases.

(1) Case $i \leq j$. We have:

$$\begin{aligned}
[I_a(u)^{-1}R(z)I_a(u)]_{i,j} &= [u^{j-i}R(z)]_{i,j} \\
&= [u^{j-i}S_{j-i}(z)]_{i,j} \\
&= [S_{j-i}(uz)]_{i,j} \\
&= [R(uz)]_{i,j}
\end{aligned}$$

(2) Case $i > j$. We have:

$$\begin{aligned}
[I_a(u)^{-1}R(z)I_a(u)]_{i,j} &= [u^{j-i}R(z)]_{i,j} \\
&= [u^{-a}u^{a+j-i}sS_{a+j-i}(z)]_{i,j} \\
&= [sS_{a+j-i}(uz)]_{i,j} \\
&= [R(uz)]_{i,j}
\end{aligned}$$

Hence the result.

(iv) Let $z \in \mathbb{C}$. For all $u \in \mathbb{U}_a$:

$$\det(R(uz)) = \det(I_a(u))^{-1} \det(R(z)) \det(I_a(u)) = \det(R(z)).$$

So we have:

$$\begin{aligned}
1 &= \det(R(0)) \\
&= \det\left(R\left(\sum_{u \in \mathbb{U}_a} uz\right)\right) = \det\left(\prod_{u \in \mathbb{U}_a} R(uz)\right) \\
&= \prod_{u \in \mathbb{U}_a} \det(R(uz)) = \prod_{u \in \mathbb{U}_a} \det(R(z)) \\
&= \det(R(z))^a
\end{aligned}$$

So we have:

$$\det(R(z)) \in \mathbb{U}_a.$$

Now, the function $z \mapsto \det(R(z))$ is continuous and:

$$\det(R(0)) = \det(I_a) = 1.$$

Hence we have:

$$\det(R(z)) = 1.$$

□

Example 40. Let $s \in \mathbb{C}$.

(1) Case $a := 1$. Then we have:

$$\text{Rot}_{(s,1)} = (\mathbf{Sin}_{(s,1,0)}) = (\exp).$$

The corollary simply states that for all $z, z' \in \mathbb{C}$:

$$\exp(z + z') = \exp(z) \exp(z').$$

(2) Case $a := 2$. Let's define:

$$S_0 := \mathbf{Sin}_{(s,2,0)} \quad , \quad S_1 := \mathbf{Sin}_{(s,2,1)}.$$

Then we have:

$$\text{Rot}_{(s,2)} = \begin{pmatrix} \mathbf{Sin}_{(s,2,0)} & \mathbf{Sin}_{(s,2,1)} \\ s\mathbf{Sin}_{(s,2,1)} & \mathbf{Sin}_{(s,2,0)} \end{pmatrix} = \begin{pmatrix} S_0 & S_1 \\ sS_1 & S_0 \end{pmatrix}.$$

The corollary simply states that for all $z, z' \in \mathbb{C}$ and all $u \in \{\pm 1\}$:

$$\begin{aligned} S_0(z + uz') &= S_0(z)S_0(z') + suS_1(z)S_1(z') \\ S_1(z + uz') &= uS_0(z)S_1(z') + S_1(z)S_0(z') \end{aligned}$$

(a) Case $s := 1$. Hence we have the usual formulas:

$$\begin{aligned} \text{ch}(z \pm z') &= \text{ch}(z)\text{ch}(z') \pm \text{sh}(z)\text{sh}(z') \\ \text{sh}(z \pm z') &= \pm \text{ch}(z)\text{sh}(z') + \text{sh}(z)\text{ch}(z') \end{aligned}$$

(b) Case $s := -1$. Hence we have the usual formulas:

$$\begin{aligned} \cos(z \pm z') &= \cos(z)\cos(z') \mp \sin(z)\sin(z') \\ \sin(z \pm z') &= \pm \cos(z)\sin(z') + \sin(z)\cos(z') \end{aligned}$$

(3) Case $a := 3$. Let's define:

$$S_0 := \mathbf{Sin}_{(s,3,0)} \quad , \quad S_1 := \mathbf{Sin}_{(s,3,1)} \quad , \quad S_2 := \mathbf{Sin}_{(s,3,2)}.$$

Then we have:

$$\text{Rot}_{(s,3)} = \begin{pmatrix} \mathbf{Sin}_{(s,3,0)} & \mathbf{Sin}_{(s,3,1)} & \mathbf{Sin}_{(s,3,2)} \\ s\mathbf{Sin}_{(s,3,2)} & \mathbf{Sin}_{(s,3,0)} & \mathbf{Sin}_{(s,3,1)} \\ s\mathbf{Sin}_{(s,3,1)} & s\mathbf{Sin}_{(s,3,2)} & \mathbf{Sin}_{(s,3,0)} \end{pmatrix} = \begin{pmatrix} S_0 & S_1 & S_2 \\ sS_2 & S_0 & S_1 \\ sS_1 & sS_2 & S_0 \end{pmatrix}.$$

The corollary simply states that for all $z, z' \in \mathbb{C}$ and all $u \in \mathbb{U}_a$:

$$\begin{aligned} S_0(z + uz') &= S_0(z)S_0(z') + su^2S_1(z)S_2(z') + suS_2(z)S_1(z') \\ S_1(z + uz') &= uS_0(z)S_1(z') + S_1(z)S_0(z') + su^2S_2(z)S_2(z') \\ S_2(z + uz') &= u^2S_0(z)S_2(z') + S_2(z)S_0(z') + uS_1(z)S_1(z') \end{aligned}$$

Corollary 41. Let $\alpha \in \Omega(a, b)$. Then for all $z \in \mathbb{C}$:

$$\text{Sp}(\text{Rot}_\alpha(z)) = \left\{ \sum_{b=0}^{a-1} \tau^b \mathbf{Sin}_{(s,a,b)}(z), \tau \in \mathbb{U}_a(s) \right\}.$$

Proof. We have:

$$\det(\text{Rot}_\alpha) = 1 = \prod_{\tau \in \mathbb{U}_a(s)} \sum_{b=0}^{a-1} \tau^b \mathbf{Sin}_{(s,a,b)}.$$

So for all $z \in \mathbb{C}$:

$$\det(XI_a - \text{Rot}_\alpha(z)) = \prod_{\tau \in \mathbb{U}_a(s)} \left(X - \sum_{b=0}^{a-1} \tau^b \mathbf{Sin}_{(s,a,b)}(z) \right).$$

Hence the result. □