An estimate for an equation with holderian condition and weight and boundary singularity

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AN ESTIMATE FOR AN EQUATION WITH HOLDERIAN CONDITION AND WEIGHT AND BOUNDARY SINGULARITY.

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ABSTRACT. We consider the following problem on open set Ω of $\mathbb{R}^2$:

$$\begin{cases} -\Delta u_i = |x-x_0|^{2\beta} V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^2, \\ u_i = 0 & \text{in } \partial\Omega. \end{cases}$$

Here, $x_0 \in \partial\Omega$ and, $\beta \in (0,1)$.

We assume, for example that:

$$\int_\Omega |x-x_0|^{+2\beta} V_i e^{u_i} dy \leq 16\pi - \epsilon, \ \epsilon > 0$$

1) We give, a quantization analysis of the previous problem under the conditions:

$$\int_\Omega |x-x_0|^{+2\beta} e^{u_i} dy \leq C,$$

and,

$$0 \leq V_i \leq b < +\infty$$

2) In addition to the previous hypothesis we assume that V_i $s-$ holderian with $1/2 + \beta/2 < s \leq 1$, then we have a compactness result, namely:

$$\sup_\Omega u_i \leq c = c(b, C, A, s, \epsilon, \beta, x_0, \Omega).$$

where A is the holderian constant of V_i.

1. INTRODUCTION AND MAIN RESULTS

We set $\Delta = \partial_{11} + \partial_{22}$ on open set Ω of $\mathbb{R}^2$ with a smooth boundary. Here we assume that we are on the unit disk.

We consider the following problem on $\Omega \subset \mathbb{R}^2$:

$$(P) \begin{cases} -\Delta u_i = |x-x_0|^{2\beta} V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^2, \\ u_i = 0 & \text{in } \partial\Omega. \end{cases}$$

Here, $x_0 \in \partial\Omega$ and, $\beta \in (0,1)$.

We assume that,

$$0 \leq V_i \leq b < +\infty, \ \int_\Omega |x-x_0|^{2\beta} e^{u_i} dy \leq C, \ u_i \in W_0^{1,1}(\Omega)$$

The above equation is called, the Prescribed Scalar Curvature equation in relation with conformal change of metrics. The function V_i is the prescribed curvature.

Here, we try to find some a priori estimates for sequences of the previous problem.

Equations of this type (in dimension 2 and higher dimensions) were studied by many authors, see [1-24]. We can see in [8], different results for the solutions of those type of equations with or without boundaries conditions and, with minimal conditions on V, for example we suppose $V_i \geq 0$ and $V_i \in L^p(\Omega)$ or $V_i e^{u_i} \in L^p(\Omega)$ with $p \in [1, +\infty]$.
Among other results, we can see in [8], the following important Theorem,

Theorem A*(Brezis-Merle [8]).If $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the previous problem (P) with, $0 < a \leq V_i \leq b < +\infty$, then, for all compact set K of Ω,*

$$\sup_K u_i \leq c = c(a, b, K, \Omega).$$

A simple consequence of this theorem is that, if we assume $u_i = 0$ on $\partial\Omega$ then, the sequence $(u_i)_i$ is locally uniformly bounded. We can find in [8] an interior estimate if we assume $a = 0$, but we need an assumption on the integral of e^{u_i}. We have in [8]:

Theorem B *(Brezis-Merle [8]).If $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the previous problem (P) with, $0 \leq V_i \leq b < +\infty$, and,*

$$\int_\Omega e^{u_i} dy \leq C,$$

then, for all compact set K of Ω,

$$\sup_K u_i \leq c = c(b, C, K, \Omega).$$

If, we assume V with more regularity, we can have another type of estimates, $\sup + \inf$. It was proved, by Shafrir, see [23], that, if $(u_i)_i, (V_i)_i$ are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C\left(\frac{a}{b}\right) \sup_K u_i + \inf_\Omega u_i \leq c = c(a, b, K, \Omega).$$

We can see in [12], an explicit value of $C\left(\frac{a}{b}\right) = \sqrt{\frac{a}{b}}$. In his proof, Shafrir has used the Stokes formula and an isoperimetric inequality, see [6]. For Chen-Lin, they have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with A the Lipschitz constant, then, $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$, see Brezis-Li-Shafrir [7]. This result was extended for Hölderian sequences $(V_i)_i$ by Chen-Lin, see [12]. Also, we can see in [18], an extension of the Brezis-Li-Shafrir to compact Riemann surface without boundary. We can see in [19] explicit form, ($8\pi m, m \in \mathbb{N}^*$ exactly), for the numbers in front of the Dirac masses, when the solutions blow-up. Here, the notion of isolated blow-up point is used.

In [8], Brezis and Merle proposed the following Problem:

Problem *(Brezis-Merle [8]).If $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the previous problem (P) with,*

$$0 \leq V_i \to V \text{ in } C^0(\bar{\Omega}).$$

$$\int_\Omega e^{u_i} dy \leq C,$$

Is it possible to prove that:

$$\sup_\Omega u_i \leq c = c(C, V, \Omega) \ ?$$

Here, we assume more regularity on V_i, we suppose that $V_i \geq 0$ is C^s (s-holderian) $1/2 < s \leq 1$) and when we have a boundary singularity. We give the answer where $bC < 16\pi$ for an equation with boundary singularity.

The following first theorem can be proved with the assumption $\beta \in (0, +\infty)$ but we prove it in the case $\beta \in (0, 1)$.

Our main results are:

Theorem 1.1. *Assume $\Omega = B_1(0)$, $x_0 \in \partial\Omega$, $\beta \in (0, 1)$, and,*

$$\int_{B_1(0)} |x - x_0|^{2\beta} V_i e^{u_i} dy \leq 16\pi - \epsilon, \ \epsilon > 0,$$

$$u_i(x_i) = \sup_{B_1(0)} u_i \to +\infty.$$

There is a sequences $(x_i^0)_i, (\delta_i^0)$, such that:

$$(x_i^0)_i \equiv (x_i)_i,\ \delta_i^0 = \delta_i = d(x_i, \partial B_1(0)) \to 0,$$

and,

$$u_i(x_i) = \sup_{B_1(0)} u_i \to +\infty,$$

$$u_i(x_i) + 2\log\delta_i + 2\beta \log d(x_i, x_0) \to +\infty,$$

$$\forall\ \epsilon > 0,\ \limsup_{i\to+\infty} \int_{B(x_i,\delta_i\epsilon)} |x - x_0|^{+2\beta} V_i e^{u_i} dy \geq 4\pi > 0.$$

If we assume:

$$V_i \to V \text{ in } C^0(\bar{B}_1(0)),$$

then,

$$\forall\ \epsilon > 0,\ \sup_{B_1(0) - B(x_i,\delta_i\epsilon)} u_i \leq C_\epsilon$$

$$\forall\ \epsilon > 0,\ \limsup_{i\to+\infty} \int_{B(x_i,\delta_i\epsilon)} |x - x_0|^{+2\beta} V_i e^{u_i} dy = 8\pi.$$

And, thus, we have the following convergence in the sense of distributions:

$$\int_{B_1(0)} |x - x_0|^{2\beta} V_i e^{u_i} dy \to \int_{B_1(0)} |x - x_0|^{2\beta} V e^u dy + 8\pi\delta_{x_0^1}.$$

Theorem 1.2. *Assume that, V_i is uniformly $s-$holderian with $1/2 + \beta/2 < s \leq 1$, $x_0 \in \partial\Omega$, $\beta \in (0,1)$, and,*

$$\int_{B_1(0)} |x - x_0|^{2\beta} V_i e^{u_i} dy \leq 16\pi - \epsilon,\ \epsilon > 0,$$

then we have:

$$\sup_\Omega u_i \leq c = c(b, C, A, s, \beta, \epsilon, x_0, \Omega).$$

where A is the hölderian constant of V_i.

2. Proofs of the results

Proofs of the theorems:

Without loss of generality, we can assume that $\Omega = B_1(0)$ the unit ball centered on the origin.

Here, G is the Green function of the Laplacian with Dirichlet condition on $B_1(0)$. We have (in complex notation):

$$G(x,y) = \frac{1}{2\pi} \log \frac{|1 - \bar{x}y|}{|x - y|},$$

Since $u_i \in W_0^{1,1}(\Omega)$ and $\beta \in (0,1)$, we have by the Brezis-Merle result and the elliptic estimates, (see [1]):

$$u_i \in C^2(\Omega) \cap W^{2,p}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega})$$

for all $2 < p < +\infty$.

Set,

$$v_i(x) = \int_{B_1(0)} G(x,y)V_i(y)|x-x_0|^{2\beta}e^{u_i(y)}dy.$$

We decompose v_i in two terms (Newtionian potential):

$$v_i^1(x) = \int_{B_1(0)} -\frac{1}{2\pi}\log|x-y|V_i(y)|x-x_0|^{2\beta}e^{u_i(y)}dy,$$

and,

$$v_i^2(x) = \int_{B_1(0)} \frac{1}{2\pi}\log|1-\bar{x}y|V_i(y)|x-x_0|^{2\beta}e^{u_i(y)}dy,$$

According to the proof in the book of Gilbarg-Trudinger see [15], v_i^1, v_i^2 and thus v_i are $C^1(\bar{\Omega})$. Indeed, we use the same proof as in [15] (Chapter 4, Newtonian potential).

We use this fact and the maximum principle to have $v_i = u_i$.

Also, we can use integration by part (the Green representation formula, see its proof in the first chapter of [15]) to have in Ω (and not $\bar{\Omega}$):

$$u_i(x) = -\int_{B_1(0)} G(x,y)\Delta u_i(y)dy = \int_{B_1(0)} G(x,y)V_i(y)|x-x_0|^{2\beta}e^{u_i(y)}dy.$$

We write,

$$u_i(x_i) = \int_\Omega G(x_i,y)|x-x_0|^{2\beta}V_i(y)e^{u_i(y)}dx = \int_{\Omega - B(x_i,\delta_i/2)} G(x_i,y)|x-x_0|^{2\beta}V_ie^{u_i(y)}dy +$$

$$+\int_{B(x_i,\delta_i/2)} G(x_i,y)|x-x_0|^{2\beta}V_ie^{u_i(y)}dy$$

According to the maximum principle, the harmonic function $G(x_i,.)$ on $\Omega - B(x_i,\delta_i/2)$ take its maximum on the boundary of $B(x_i,\delta_i/2)$, we can compute this maximum:

$$G(x_i,y_i) = \frac{1}{2\pi}\log\frac{|1-\bar{x}_iy_i|}{|x_i-y_i|} = \frac{1}{2\pi}\log\frac{|1-\bar{x}_i(x_i+\delta_i\theta_i)|}{|\delta_i/2|} = \frac{1}{2\pi}\log(2|(1+|x_i|)+\theta_i|) < +\infty$$

with $|\theta_i| = 1/2$.

Thus,

$$u_i(x_i) \leq C + \int_{B(x_i,\delta_i/2)} G(x_i,y)|x-x_0|^{2\beta}V_ie^{u_i(y)}dy \leq C + e^{u_i(x_i)+2\beta\log d(x_i,x_0)}\int_{B(x_i,\delta_i/2)} G(x_i,y)dy$$

Now, we compute $\int_{B(x_i,\delta_i/2)} G(x_i,y)dy$

we set in polar coordinates,

$$y = x_i + \delta_i t\theta$$

we find:

$$\int_{B(x_i,\delta_i/2)} G(x_i,y)dy = \int_{B(x_i,\delta_i/2)} \frac{1}{2\pi}\log\frac{|1-\bar{x}_iy|}{|x_i-y|} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{1/2}\delta_i^2\log\frac{|1-\bar{x}_i(x_i+\delta_it\theta)|}{\delta_it}tdtd\theta =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{1/2}\delta_i^2(\log(|1+|x_i|+t\theta|) - \log t)tdtd\theta \leq C\delta_i^2.$$

Thus,

$$u_i(x_i) \leq C + C\delta_i^2 e^{u_i(x_i)+2\beta\log d(x_i,x_0)},$$

which we can write, because $u_i(x_i) \to +\infty$,

$$u_i(x_i) \leq C'\delta_i^2 e^{u_i(x_i)+2\beta\log d(x_i,x_0)},$$

We can conclude that:

$$u_i(x_i) + 2\log\delta_i + 2\beta\log d(x_i, x_0) \to +\infty.$$

Since in $B(x_i, \delta_i\epsilon)$, $d(x, x_0)$ is equivalent to $d(x_i, x_0)$ we can consider the following functions:

$$v_i(y) = u_i(x_i + \delta_i y) + 2\log\delta_i + 2\beta\log d(x_i, x_0), \quad y \in B(0, 1/2)$$

The function satisfies all conditions of the Brezis-Merle hypothesis, we can conclude that, on each compact set:

$$v_i \to -\infty$$

we can assume, without loss of generality that for $1/2 > \epsilon > 0$, we have:

$$v_i \to -\infty, \quad y \in B(0, 2\epsilon) - B(0, \epsilon),$$

Lemma 2.1. *For all $1/4 > \epsilon > 0$, we have:*

$$\sup_{B(x_i,(3/2)\delta_i\epsilon)-B(x_i,\delta_i\epsilon)} u_i \leq C_\epsilon.$$

Proof of the lemma

Let t_i' and t_i the points of $B(x_i, 2\delta_i\epsilon) - B(x_i, (1/2)\delta_i\epsilon)$ and $B(x_i, (3/2)\delta_i\epsilon) - B(x_i, \delta_i\epsilon)$ respectively where u_i takes its maximum.

According to the Brezis-Merle work, we have:

$$u_i(t_i') + 2\log\delta_i + 2\beta\log d(x_i, x_0) \to -\infty$$

We write,

$$u_i(t_i) = \int_\Omega G(t_i, y)|x - x_0|^{2\beta}V_i(y)e^{u_i(y)}dx = \int_{\Omega - B(x_i,2\delta_i\epsilon)} G(t_i, y)|x - x_0|^{2\beta}V_i e^{u_i(y)}dy +$$
$$+ \int_{B(x_i,2\delta_i\epsilon)-B(x_i,(1/2)\delta_i\epsilon)} G(t_i, y)|x - x_0|^{2\beta}V_i e^{u_i(y)}dy +$$
$$+ \int_{B(x_i,(1/2)\delta_i\epsilon)} G(t_i, y)|x - x_0|^{2\beta}V_i e^{u_i(y)}dy$$

But, in the first and the third integrale, the point t_i is far from the singularity x_i and we know that the Green function is bounded. For the second integrale, after a change of variable, we can see that this integale is bounded by (we take the supremum in the annulus and use Brezis-Merle theorem)

$$\delta_i^2 e^{u_i(t_i')+2\beta\log d(x_i,x_0)} \times I_j$$

where I_j is a Jensen integrale (of the form $\int_0^1\int_0^{2\pi}(\log(|1 + |x_i| + t\theta) - \log|\theta_i - t\theta|)tdtd\theta$ which is bounded).

we conclude the lemma.

From the lemma, we see that far from the singularity the sequence is bounded, thus if we take the supremum on the set $B_1(0) - B(x_i, \delta_i\epsilon)$ we can see that this supremum is bounded and thus the sequence of functions is uniformly bounded or tends to infinity and we use the same arguments as for x_i to conclude that around this point and far from the singularity, the seqence is bounded.

The process will be finished , because, according to Brezis-Merle estimate, around each supremum constructed and tending to infinity, we have:

$$\forall\ \epsilon > 0,\ \limsup_{i\to+\infty}\int_{B(x_i,\delta_i\epsilon)} |x - x_0|^{2\beta}V_i e^{u_i}dy \geq 4\pi > 0.$$

Finaly, with this construction, we have a finite number of "exterior "blow-up points and outside the singularities the sequence is bounded uniformly, for example, in the case of one "exterior" blow-up point, we have:

$$u_i(x_i) \to +\infty$$

$$\forall\ \epsilon > 0,\ \sup_{B_1(0)-B(x_i,\delta_i\epsilon)} u_i \leq C_\epsilon$$

$$\forall\ \epsilon > 0,\ \limsup_{i\to+\infty} \int_{B(x_i,\delta_i\epsilon)} |x-x_0|^{2\beta} V_i e^{u_i} dy \geq 4\pi > 0.$$

$$x_i \to x_0 \in \partial B_1(0).$$

Remark: For the general case, the process of quantization can be extended to more than one blow-up points.

We have the following lemma:

Lemma 2.2. *Each δ_i^k is of order $d(x_i^k, \partial B_1(0))$. Namely: there is a positive constant $C > 0$ such that for $\epsilon > 0$ small enough:*

$$\delta_i^k \leq d(x_i^k, \partial B_1(0)) \leq (2 + \frac{C}{\epsilon})\delta_i^k.$$

Proof of the lemma

Now, if we suppose that there is another "exterior" blow-up $(t_i)_i$, we have, because $(u_i)_i$ is uniformly bounded in a neighborhood of $\partial B(x_i, \delta_i\epsilon)$, we have :

$$d(t_i, \partial B(x_i, \delta_i\epsilon)) \geq \delta_i\epsilon$$

If we set,

$$\delta_i' = d(t_i, \partial(B_1(0) - B(x_i, \delta_i\epsilon))) = \inf\{d(t_i, \partial B(x_i, \delta_i\epsilon)), d(t_i, \partial(B_1(0)))\}$$

then, δ_i' is of order $d(t_i, \partial B_1(0))$. To see this, we write:

$$d(t_i, \partial B_1(0)) \leq d(t_i, \partial B(x_i, \delta_i\epsilon)) + d(\partial B(x_i, \delta_i\epsilon), x_i) + d(x_i, \partial B_1(0)),$$

Thus,

$$\frac{d(t_i, \partial B_1(0))}{d(t_i, \partial B(x_i, \delta_i\epsilon))} \leq 2 + \frac{1}{\epsilon},$$

Thus,

$$\delta_i' \leq d(t_i, \partial B_1(0)) \leq \delta_i'(2 + \frac{1}{\epsilon}).$$

Now, the general case follow by induction. We use the same argument for three, four,..., n blow-up points.

We have, by induction and, here we use the fact that u_i is uniformly bounded outside a small ball centered at $x_i^j, j = 0, \ldots, k-1$:

$$\delta_i^j \leq d(x_i^j, \partial B_1(0)) \leq C_1\delta_i^j,\ j = 0, \ldots, k-1,$$

.

$$d(x_i^k, \partial B(x_i^j, \delta_i^j\epsilon/2)) \geq \epsilon\delta_i^j, \epsilon > 0,\ j = 0, \ldots, k-1,$$

.

and let's consider x_i^k such that:

$$u_i(x_i^k) = \sup_{B_1(0)-\cup_{j=0}^{k-1} B(x_i^j, \delta_i^j\epsilon)} u_i \to +\infty,$$

take,

$$\delta_i^k = \inf\{d(x_i^k, \partial B_1(0)), d(x_i^k, \partial(B_1(0) - \cup_{j=0}^{k-1} B(x_i^j, \delta_i^j\epsilon/2))\},$$

if, we have,

$$\delta_i^k = d(x_i^k, \partial B(x_i^j, \delta_i^j \epsilon/2)),\ j \in \{0, \dots, k-1\}.$$

Then,

$$\delta_i^k \leq d(x_i^k, \partial B_1(0)) \leq$$

$$\leq d(x_i^k, \partial B(x_i^j, \delta_i^j \epsilon/2)) + d(\partial B(x_i^j, \delta_i^j \epsilon/2), x_i^j) + d(x_i^j, \partial B_1(0))$$

$$\leq (2 + \frac{C_1}{\epsilon})\delta_i^k.$$

To apply lemma 2.1 for m blow-up points, we use an induction:

We do directly the same approch for t_i as x_i by using directly the Green function of the unit ball.

If we look to the blow-up points, we can see, with this work that, after finite steps, the sequence will be bounded outside a finite number of balls , because of Brezis-Merle estimate:

$$\forall\ \epsilon > 0,\ \limsup_{i \to +\infty} \int_{B(x_i^k, \delta_i^k \epsilon)} |x - x_0|^{2\beta} V_i e^{u_i} dy \geq 4\pi > 0.$$

Here, we can take the functions:

$$u_i^k(y) = u_i(x_i^k + \delta_i^k y) + 2 \log \delta_i^k + 2\beta \log d(x_i, x_0).$$

Indeed, by corollary 4 of the paper of Brezis-Merle, if we have:

$$\limsup_{i \to +\infty} \int_{B(x_i^k, \delta_i^k \epsilon)} |x - x_0|^{2\beta} V_i e^{u_i} dy \leq 4\pi - \epsilon_0 < 4\pi,$$

then, $(u_i^k)^+$ would be locally bounded and this contradict the fact that $u_i^k(0) \to +\infty$.

Finaly, we can say that, there is a finite number of sequences $(x_i^k)_i, (\delta_i^k), 0 \leq k \leq m$, such that:

$$(x_i^0)_i \equiv (x_i)_i,\ \delta_i^0 = \delta_i = d(x_i, \partial B_1(0)),$$

$$(x_i^1)_i \equiv (t_i)_i,\ \delta_i^1 = \delta_i' = d(t_i, \partial(B_1(0) - B(x_i, \delta_i \epsilon)),$$

and each δ_i^k is of order $d(x_i^k, \partial B_1(0))$.
and,

$$u_i(x_i^k) = \sup_{B_1(0) - \cup_{j=0}^{k-1} B(x_i^j, \delta_i^j \epsilon)} u_i \to +\infty,$$

$$u_i(x_i^k) + 2 \log \delta_i^k + 2\beta \log d(x_i^k, x_0) \to +\infty,$$

$$\forall\ \epsilon > 0, \sup_{B_1(0) - \cup_{j=0}^{m} B(x_i^j, \delta_i^j \epsilon)} u_i \leq C_\epsilon$$

$$\forall\ \epsilon > 0,\ \limsup_{i \to +\infty} \int_{B(x_i^k, \delta_i^k \epsilon)} |x - x_0|^{2\beta} V_i e^{u_i} dy \geq 4\pi > 0.$$

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Since in $B(x_i, \delta_i \epsilon)$, $d(x, x_0)$ is equivalent to $d(x_i, x_0)$ we can consider the following functions:

$$v_i(y) = u_i(x_i + \delta_i y) + 2 \log \delta_i + 2\beta \log d(x_i, x_0).$$

With the previous method, we have a finite number of "exterior" blow-up points (perhaps the same) and the sequences tend to the boundary. With the aid of proposition 1 of the paper of Li-Shafrir, we see that around each exterior blow-up, we have a finite number of "interior"

blow-ups. Around, each exterior blow-up, we have after rescaling with δ_i^k, the same situation as around a fixed ball with positive radius. If we assume:

$$V_i \to V \text{ in } C^0(\bar{B}_1(0)),$$

then,

$$\forall\ \epsilon > 0,\ \limsup_{i\to+\infty} \int_{B(x_i^k,\delta_i^k\epsilon)} |x-x_0|^{2\beta} V_i e^{u_i} dy = 8\pi m_k,\ m_k \in \mathbb{N}^*.$$

And, thus, we have the following convergence in the sense of distributions:

$$\int_{B_1(0)} |x-x_0|^{2\beta} V_i e^{u_i} dy \to \int_{B_1(0)} |x-x_0|^{2\beta} V e^u dy + \sum_{k=0}^{m} 8\pi m'_k \delta_{x_0^k},\ m'_k \in \mathbb{N}^*,\ x_0^k \in \partial B_1(0).$$

Consequence: using a Pohozaev-type identity, proof of theorem 2

By a conformal transformation, we can assume that our domain $\Omega = B^+$ is a half ball centered at the origin, $B^+ = \{x, |x| \leq 1, x_1 \geq 0\}$, and, $x_0 = 0$. In this case the normal at the boundary is $\nu = (-1, 0)$ and $u_i(0, x_2) \equiv 0$. Also, we set x_i the blow-up point and $x_i^2 = (0, x_i^2)$ and $x_i^1 = (x_i^1, 0)$ respectevely the second and the first part of x_i. Let ∂B^+ the part of the boundary for which u_i and its derivatives are uniformly bounded and thus converge to the corresponding function.

The case of one blow-up point:

Theorem 2.3. *If V_i is s-Holderian with $1/2 + \beta/2 < s \leq 1$ and,*

$$\int_\Omega |x|^{2\beta} V_i e^{u_i} dy \leq 16\pi - \epsilon,\ \epsilon > 0,$$

we have :

$$2V_i(x_i) \int_{B(x_i,\delta_i\epsilon)} |x|^{2\beta} e^{u_i} dy = o(1),$$

which means that there are no blow-up points.

Proof of the theorem

In order to use the Pohozaev identity we need to have a good function u_i, since $\beta \in (0, 1)$, we have a function u_i such that:

$$u_i \in C^{2,\epsilon}(\bar{\Omega})$$

Thus, we can use integration by parts. The Pohozaev identity gives us the following formula:

$$\int_\Omega < (x - x_2^i)|\nabla u_i > (-\Delta u_i) dy = \int_\Omega < (x - x_2^i)|\nabla u_i > |x|^{2\beta} V_i e^{u_i} dy = A_i$$

$$A_i = \int_{\partial B^+} < (x - x_2^i)|\nabla u_i >< \nu|\nabla u_i > d\sigma + \int_{\partial B^+} < (x - x_2^i)|\nu > |\nabla u_i|^2 d\sigma$$

We can write it as:

$$\int_\Omega < (x-x_2^i)|\nabla u_i > (V_i - V_i(x_i))|x|^{2\beta} e^{u_i} dy = A_i + V_i(x_i) \int_\Omega < (x-x_2^i)|\nabla u_i > |x|^{2\beta} e^{u_i} dy =$$

$$= A_i + V_i(x_i) \int_\Omega < (x - x_2^i)|x|^{2\beta} |\nabla(e^{u_i}) > dy$$

And, if we integrate by part the second term, we have (because $x_1 = 0$ on the boundary and $\nu_2 = 0$):

$$\int_\Omega < (x-x_2^i)|\nabla u_i > (V_i - V_i(x_i))|x|^{2\beta} e^{u_i} dy = -2(1+\beta)V_i(x_i)\int_\Omega |x|^{2\beta} e^{u_i} dy +$$

$$+2\beta V_i(x_i)\int_{B(x_i,\delta_i\epsilon)} x_2 x_2^i |x|^{2\beta-2} e^{u_i} dy - 2\beta V_i(x_i)\int_{\Omega - B(x_i,\delta_i\epsilon)} x_2 x_2^i |x|^{2\beta-2} e^{u_i} dy + B_i$$

where B_i is,

$$B_i = V_i(x_i)\int_{\partial B^+} < (x-x_2^i)|\nu > |x|^{2\beta} e^{u_i} dy$$

applying the same procedure to u, we can write:

$$\int_\Omega < (x-x_2^i)|\nabla u > (V - V(0))|x|^{2\beta} e^{u} dy = -2(1+\beta)V(0)\int_\Omega |x|^{2\beta} e^{u} dy +$$

$$-2\beta V(0)\int_{B(x_i,\delta_i\epsilon)} x_2 x_2^i |x|^{2\beta-2} e^{u} dy - 2\beta V(0)\int_{\Omega - B(x_i,\delta_i\epsilon)} x_2 x_2^i |x|^{2\beta-2} e^{u} dy + B,$$

with,

$$B = V(0)\int_{\partial B^+} < (x-x_2^i)|\nu > |x|^{2\beta} e^{u} dy$$

we use the fact that, u_i is bounded outside $B(x_i,\delta_i\epsilon)$ and the convergence of u_i to u on compact set of $\bar{\Omega} - \{0\}$, and the fact that $\beta \in (0,+\infty)$, to write the following:

$$2V_i(x_i)\int_{B(x_i,\delta_i\epsilon)} |x|^{2\beta} e^{u_i} dy + o(1) =$$

$$= \int_\Omega < (x-x_2^i)|\nabla u_i > (V_i - V_i(x_i))|x|^{2\beta} e^{u_i} dy - \int_\Omega < (x-x_2^i)|\nabla u > (V - V(0))|x|^{2\beta} e^{u} dy +$$

$$+(A_i - A) + (B_i - B),$$

where A and B are,

$$A = \int_{\partial B^+} < (x-x_2^i)|\nabla u >< \nu|\nabla u > d\sigma + \int_{\partial B^+} < (x-x_2^i)|\nu > |\nabla u|^2 d\sigma$$

$$B = V(0)\int_{\partial B^+} < (x-x_2^i)|\nu > |x|^{2\beta} e^{u} dy$$

and, because of the uniform convergence of u_i and its derivatives on ∂B^+, we have:

$$A_i - A = o(1) \text{ and } B_i - B = o(1)$$

which we can write as:

$$2V_i(x_i)\int_{B(x_i,\delta_i\epsilon)} |x|^{2\beta} e^{u_i} dy + o(1) =$$

$$= \int_\Omega < (x-x_2^i)|\nabla(u_i - u) > (V_i - V_i(x_i))|x|^{2\beta} e^{u_i} dy +$$

$$+ \int_\Omega < (x-x_2^i)|\nabla u > (V_i - V_i(x_i))|x|^{2\beta} (e^{u_i} - e^{u}) dy +$$

$$+ \int_\Omega < (x-x_2^i)|\nabla u > (V_i - V_i(x_i) - (V - V(0)))|x|^{2\beta} e^{u} dy + o(1)$$

We can write the second term as:

$$\int_\Omega < (x-x_2^i)|\nabla u > (V_i - V_i(x_i))|x|^{2\beta} (e^{u_i} - e^{u}) dy = \int_{\Omega - B(0,\epsilon)} < (x-x_2^i)|\nabla u > (V_i - V_i(x_i))(e^{u_i} - e^{u})|x|^{2\beta} dy +$$

$$+ \int_{B(0,\epsilon)} < (x-x_2^i)|\nabla u > (V_i - V_i(x_i))(e^{u_i} - e^{u})|x|^{2\beta} dy = o(1),$$

because of the uniform convergence of u_i to u outside a region which contain the blow-up and the uniform convergence of V_i. For the third integral we have the same result:

$$\int_\Omega < (x - x_2^i)|\nabla u > (V_i - V_i(x_i) - (V - V(0)))|x|^{2\beta} e^u dy = o(1),$$

because of the uniform convergence of V_i to V.

Now, we look to the first integral:

$$\int_\Omega < (x - x_2^i)|\nabla(u_i - u) > (V_i - V_i(x_i))|x|^{2\beta} e^{u_i} dy,$$

we can write it as:

$$\int_\Omega < (x-x_2^i)|\nabla(u_i-u) > (V_i-V_i(x_i))|x|^{2\beta} e^{u_i} dy = \int_\Omega < (x-x_i)|\nabla(u_i-u) > (V_i-V_i(x_i))|x|^{2\beta} e^{u_i} dy +$$

$$+ \int_\Omega < x_1^i|\nabla(u_i - u) > (V_i - V_i(x_i))|x|^{2\beta} e^{u_i} dy,$$

Thus, we have proved by using the Pohozaev identity the following equality:

$$\int_\Omega < (x - x_i)|\nabla(u_i - u) > (V_i - V_i(x_i))|x|^{2\beta} e^{u_i} dy +$$

$$+ \int_\Omega < x_1^i|\nabla(u_i - u) > (V_i - V_i(x_i))|x|^{2\beta} e^{u_i} dy =$$

$$= 2V_i(x_i) \int_{B(x_i, \delta_i\epsilon)} |x|^{2\beta} e^{u_i} dy + o(1)$$

We can see, because of the uniform boundedness of u_i outside $B(x_i, \delta_i\epsilon)$ and the fact that :

$$||\nabla(u_i - u)||_1 = o(1),$$

it is sufficient to look to the integral on $B(x_i, \delta_i\epsilon)$.

Assume that we are in the case of one blow-up, it must be (x_i) and isolated, we can write the following inequality as a consequence of YY.Li-I.Shafrir result:

$$u_i(x) + 2\log|x - x_i| + 2\beta \log d(x, 0) \leq C,$$

We use this fact and the fact that V_i is s-holderian to have that, on $B(x_i, \delta_i\epsilon)$,

$$|(x - x_i)(V_i - V_i(x_i))|x|^{2\beta} e^{u_i}| \leq \frac{C}{|x - x_i|^{1-s}} \in L^{(2-\epsilon')/(1-s)}, \ \forall\ \epsilon' > 0,$$

and, we use the fact that:

$$||\nabla(u_i - u)||_q = o(1), \ \forall\ 1 \leq q < 2$$

to conclude by the Hölder inequality that:

$$\int_{B(x_i, \delta_i\epsilon)} < (x - x_i)|\nabla(u_i - u) > (V_i - V_i(x_i))|x|^{2\beta} e^{u_i} dy = o(1),$$

For the other integral, namely:

$$\int_{B(x_i, \delta_i\epsilon)} < x_1^i|\nabla(u_i - u) > (V_i - V_i(x_i))|x|^{2\beta} e^{u_i} dy,$$

We use the fact that, because our domain is a half ball, and the $\sup + \inf$ inequality to have:

$$x_1^i = \delta_i,$$

$$u_i(x) + 4\log\delta_i + 4\beta \log d(x, 0) \leq C$$

and,

$$|x|^{s\beta}e^{(s/2)u_i(x)} \leq |x-x_i|^{-s},$$

$$|V_i - V_i(x_i)| \leq |x-x_i|^s,$$

Finaly, we have:

$$|\int_{B(x_i,\delta_i\epsilon)} < x_1^i|\nabla(u_i-u) > (V_i - V_i(x_i))|x|^{2\beta}e^{u_i}dy| \leq$$

$$\leq C\int_{B(x_i,\delta_i\epsilon)} |\nabla(u_i-u)||x|^{2\beta(1/2-s/2)}(e^{u_i})^{(3/4-s/2)},$$

But in the second member, for $1/2 < s \leq 1$, we have $q_s = 1/(3/4 - s/2) > 2$ and thus $q_s' < 2$ and,

$$(|x|^{2\beta}e^{u_i})^{3/4-s/2} \in L^{q_s}$$

$$|x|^{2\beta(-1/4)} \in L^r,\ r = \frac{4}{\beta+\epsilon},\ \epsilon > 0,$$

with,

$$\frac{1}{r} + \frac{1}{q_s} = \frac{1}{t} < 1/2,$$

and,

$$||\nabla(u_i-u)||_{t'} = o(1),$$

This is possible if,

$$\beta < 2(s-1/2).$$

or,

$$s > \frac{\beta}{2} + \frac{1}{2},$$

This is possisble for $\beta \in (0,1)$.
one conclude that:

$$\int_{B(x_i,\delta_i\epsilon)} < x_1^i|\nabla(u_i-u) > (V_i - V_i(x_i))|x|^{2\beta}e^{u_i}dy = o(1)$$

Finaly, with this method, we conclude that, in the case of one blow-up point and V_i is s-Holderian with $1/2 < s \leq 1$:

$$2V_i(x_i)\int_{B(x_i,\delta_i\epsilon)} |x|^{2\beta}e^{u_i}dy = o(1)$$

which means that there is no blow-up, which is a contradiction.

Finaly, for one blow-up point and V_i is s-Hölderian with $1/2 < s \leq 1$, the sequence (u_i) is uniformly bounded on Ω.

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