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# Nonlinear dynamical eigenstructure for differential GES and contraction theory

Mario Spirito, Bernhard Maschke, Yann Le Gorrec

**Abstract**—In this paper, we describe, in the context of nonlinear autonomous systems, a sufficient condition to get differential Global Exponential Stability (dGES), i.e., the origin of the associated variational dynamics is Globally Exponentially Stable. This condition leverages the definition of nonlinear dynamical eigenvalues and the associated eigenvectors, i.e., the nonlinear dynamical eigenstructure. Moreover, by exploiting the system’s nonlinear eigenvectors, we show how to obtain a contractive metric for the autonomous system and thus the dGES condition via the contraction analysis. The paper is correlated with a numerical example, showing an explicit analytic contractive metric and the system dGES property.

## I. INTRODUCTION

In systems and control theory, stability is one of the fundamental concepts. One can determine the stability of an equilibrium point for a nonlinear system by exploiting a properly defined Lyapunov function, [15], and verifying that its time derivative is strictly negative except at the equilibrium point. For linear systems, the stability can be verified by assessing the real part of the state matrix eigenvalues. Along the same idea, for a nonlinear system we can define nonlinear (dynamical) eigenvalues that have been introduced in [12], although no stability result has been given until [13]. The interested reader can also consider the work [18] and the recent review article [2] for additional details. Moreover, the idea of dynamical eigenvectors has been used in [11] (see also the reference therein) to study the characteristics of linear time-varying systems analytic solutions.

On the other hand, it is well known, and accepted by the community, that the contraction approach plays a crucial role in studying the stability and/or the attractiveness of invariant manifolds for nonlinear autonomous systems, acting as a counterpart of Lyapunov stability theory, allowing to guarantee the attractiveness to an equilibrium point or convergence of system trajectories on each other. This approach has been introduced in works such as [7], [29], for the case of constant metric, and re-proposed recently in [20], [21], [22], while their extension to the more general case of nonlinear Riemannian metrics can be found in [17], [8] and [3]. See also [6] for the case of non-Euclidean  $L_1$  and  $L_\infty$  metrics.

In particular, by using this approach, one can determine the exponential convergence to equilibrium points by studying the properties of the variational dynamics. More specifically,

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this attractiveness property is equivalent to the existence of a positive definite quadratic form (called Lyapunov-Finsler metric in [8]) which is decreasing along the flow of the autonomous system under consideration. A weaker notion of a contractive system is the so-called partial-contractive dynamics given in [17], sometimes also called semi-contractive dynamics. Recently, a link between semi-contractive dynamics and dissipative Hamiltonian systems has been proposed in [26].

Although the useful properties we can study via a contraction approach, computing the contractive metric is in general a hard task. An approach is to use machine learning techniques and has been proposed and used in [30] and [9]. On the other hand, for the definition of a nonlinear control law, a link between the solution of a differential algebraic Riccati equation, whose solution is based on the nonlinear eigenvectors defined in [13], and the closed-loop contractive metric has been proposed in [14]. However, the analytic metric construction is an open topic and the interested reader is invited to delve into it via an overview of the argument such as [10].

The aim of this paper is twofold: describe sufficient conditions for GES of the variational dynamics and propose a method to construct a contractive metric.

The paper is organized as follows. We recall, in Section II, some preliminary concepts such as variational system, forward contraction, Lyapunov characteristic exponents, the definition of differential Global Exponential Stability, and the definition of the nonlinear eigenstructure associated with a nonlinear system. Then Section III is concerned with the analysis on the Global Exponential Stability of the variational dynamics by exploiting its nonlinear dynamical eigenstructure. Thereby we link such a nonlinear dynamical eigenstructure to the definition of a contractive metric for the autonomous system under consideration, through which we provide, in Section IV, a parallel differential GES result. In Section V, we propose a nonlinear example of a differential Global Exponential Stable system. We then give some conclusions in Section VI. The paper also contains an appendix in which we summarize some transversal notions needed in Proposition 2.

*Notation:* We denote with  $\mathbb{R}$  ( $\mathbb{C}$ ) the set of real (complex) numbers. Given a matrix  $P \in \mathbb{R}^{n \times n}$ , we denote by  $\sigma(P)$  its spectrum, while  $\lambda_i(P)$  is the  $i$ -th eigenvalue and  $\sigma_i(P)$  its  $i$ -th singular value, i.e.,  $\sigma_i(P) = \sqrt{\lambda_i(P^\top P)}$ , and the condition number of  $P$ , i.e.,  $\mu(P) = \sigma_{\max}(P)/\sigma_{\min}(P)$ . Given a matrix  $M \in \mathbb{R}^{n \times n}$ , we denote by  $\text{skew}(M)$  and  $\text{sym}(M)$  its antisymmetric and its symmetric parts, i.e.,  $\text{skew}(M) = (M - M^\top)/2$  and  $\text{sym}(M) = (M + M^\top)/2$ . Given any  $n \in \mathbb{N}$ , we define the

matrix  $\mathbb{J}_n := I_n + N_n$ , where  $I_n$  is the identity matrix of dimension  $n$  and  $N_n$  is a nilpotent matrix with ones on the upper diagonal and zeros elsewhere. Given a number  $\epsilon \in \mathbb{R}$  and a natural number  $n \in \mathbb{N}$ , we denote

$$D_n(\epsilon) := \begin{cases} \text{diag}(1, \epsilon, \dots, \epsilon^{n-1}), & \text{if } \epsilon \neq 0, \\ I_n, & \text{if } \epsilon = 0. \end{cases}$$

## II. PRELIMINARIES

### A. Variational and prolonged dynamics

A dynamical system with  $m$  inputs and  $p$  outputs, defined on the smooth manifold  $\mathcal{X}$ , can be prolonged (or ‘lifted’ as mentioned in [8]) to a system on the  $2n$ -dimensional tangent bundle  $\mathcal{T}\mathcal{X}$  of the manifold  $\mathcal{X}$ , with  $2m$  inputs and  $2p$  outputs, as shown in [5].

However, the same approach can be specialized for autonomous (closed) systems. We thus consider the autonomous dynamics

$$\Sigma : \begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases} \quad (1)$$

where  $x \in \mathcal{X} \subseteq \mathbb{R}^n$ , with  $f$  continuously differentiable for all  $x \in \mathcal{X}$ , where  $\mathcal{X}$  is classically considered as compact. Then, given an admissible state trajectory  $t \mapsto x(t)$  for  $\Sigma$ , we define the *variational system* along such a trajectory as the time-varying system

$$D\Sigma : \begin{cases} \dot{\xi} = \frac{\partial f}{\partial x}(x)\xi, \\ \xi(0) = \xi_0 \end{cases} \quad (2)$$

with state  $\xi \in \mathbb{R}^n$ .

The *prolonged system* of  $\Sigma$ , hence, corresponds to the augmented  $2n$ -dimensional system on the tangent bundle  $\mathcal{T}\mathcal{X}$  composed of the ‘parallel’ configuration of  $\Sigma$  and  $D\Sigma$ , i.e., the prolonged dynamics is given by the dynamics

$$\mathcal{T}\Sigma : \begin{cases} \dot{x} = f(x), & x(0) = x_0 \\ \dot{\xi} = \frac{\partial f}{\partial x}(x)\xi, & \xi(0) = \xi_0. \end{cases} \quad (3)$$

In the standard notation  $\xi = \delta x$ , as employed, e.g., in [28]. In the case of autonomous systems, as treated in [8], the variational state  $\xi = \delta x$  refers to the tangent vector to the parameterized curve connecting any two system trajectories. The infinitesimal variations  $\delta x(t)$  on the state  $x(t)$  can also be interpreted as being a generic ‘virtual displacement’ along any possible direction on the tangent space  $\mathcal{T}_x\mathcal{X}$ , see [5] and [28] for further details.

However, if we specialize such a direction to be the one of the ‘state velocity’, i.e., we impose  $\xi = \dot{x}$  (that still lays on the tangent bundle of  $\mathcal{X}$ ), we then describe the time evolution of the vectorfield  $f(x)$  and we can moreover determine the initial conditions of the variational system  $D\Sigma$ , i.e.,  $\xi_0 = f(x_0)$ . As highlighted in [27], this choice of the variational system direction recalls the definition of Forward Contraction introduced in [8, sec. III.C].

### B. Contractive dynamics

A weak notion of contractive dynamics is that of semi-contractive systems whose definition reads as follows.

*Definition 2.1 (Semi-contractive systems):* System (1) is said to be semi-contractive if there exists a  $C^1$  function  $P : \mathcal{X} \rightarrow \mathbb{R}^{n \times n}$ , two strictly positive real numbers  $\underline{p}$  and  $\bar{p}$  such that  $P$  has a time derivative computed along the flow of the vectorfield  $f$ , i.e., column-wise we have  $\dot{P}(x) = \frac{\partial P(x)}{\partial x} f(x)$ , and it satisfies

$$\underline{p}I \leq P(x) \leq \bar{p}I,$$

$$\dot{P}(x) + P(x) \frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}^\top(x) P(x) \leq 0, \quad \forall x \in \mathcal{X}. \quad (4)$$

A system is said to be *contractive* if (4) holds with a strict sign, i.e.,

$$\dot{P}(x) + P(x) \frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}^\top(x) P(x) < 0, \quad \forall x \in \mathcal{X}, \quad (5)$$

where  $\mathcal{X} \subseteq \mathbb{R}^n$  is referred to as the contractive region. A strictly related, although independent, concept is that of incremental stability, that is the exponential attractiveness between any two trajectories of a system dynamics. In particular, it has been shown in [3, Prop. 1] that if the vectorfield  $f(x)$  is globally Lipschitz with bounded second derivative, the two concepts of contractive and incrementally stable systems are *equivalent*. Additionally, they are both equivalent, see [3, Prop. 1], to the global exponential attractiveness of the manifold  $\mathcal{E} = \{(x, \xi) : \xi = 0\}$  for the prolonged systems  $\mathcal{T}\Sigma$ . Furthermore, thanks to the direction choice  $\xi = \dot{x}$ , we can conclude that the manifold  $\mathcal{E}$  is the set of equilibrium points of the dynamics contained in  $\mathcal{X}$ , which consequently implies that the system has a single asymptotically stable equilibrium in the contraction region.

### C. Lyapunov characteristic exponents

Lyapunov characteristic exponent has been introduced for the analysis of system stability, see [4], [1, Ch.2], and [23] for some recent results. In simple words, the idea behind the characteristic exponent, [1], is to compare the asymptotic evolution of the function  $f(t)$  with an exponential function.

*Definition 2.2: (Characteristic exponent)* Let  $f(t)$  be a complex-valued function defined on the interval  $[t_0, \infty)$ . The value defined as

$$\chi[f(t)] = \limsup_{t \rightarrow \infty} \frac{1}{t} \ln |f(t)| \quad (6)$$

is called the characteristic exponent of the function  $f(t)$ . When  $f(t)$  is a finite-dimensional matrix, then its characteristic exponent coincides with the one of its norm, i.e.  $\chi[f(t)] = \chi[\|f(t)\|]$ , see Lemma 2.2.1 in [1].

### D. Global Exponential Stability

The standard definition of global exponentially stable equilibrium point can be found in classical books on nonlinear systems such as [15] and it refers to the norm evolution  $|x(t)|$  of the system (1) state trajectory  $x(t)$ .

*Definition 2.3 (Global Exponential Stability):* The equilibrium point  $x^* \in \mathcal{X}$  of (1) is GES if there exists real  $\kappa > 0$  and  $\alpha < 0$  such that for any initial conditions  $x_0 \in \mathbb{R}^n$

$$|x(t)| \leq \kappa \exp(\alpha t) |x_0|.$$

With the contraction theory, as a consequence of the forward contraction approach, we can characterize the global attractiveness of the system's equilibrium point by analyzing the stability properties of the variational dynamics (2). For this reason, we introduce the concept of differential Global Exponential Stability (dGES) as follows.

*Definition 2.4 (differential Global Exponential Stability):* The single equilibrium point  $x^* \in \mathcal{X}$  of (1) is differential Global Exponentially Stable if there exists real  $\kappa > 0$  and  $\alpha < 0$  such that for any initial conditions  $\xi_0 \in \mathbb{R}^n$

$$|\xi(t)| \leq \kappa \exp(\alpha t) |\xi_0|.$$

According to this definition, we can directly conclude the following.

*Lemma 2.1:* A contractive dynamics (1) whose associated metric  $P(x)$  satisfies a stronger inequality than (5), i.e.,

$$\dot{P}(x) + P(x) \frac{\partial f(x)}{\partial x} + \frac{\partial f(x)}{\partial x}^\top P(x) \leq 2\alpha P(x)$$

with  $\alpha < 0$ , is differentially Globally Exponentially Stable with  $\kappa = \sqrt{\bar{p}/p}$ .

The proof of the lemma is an application of the definition of contractive system with the usage of the standard Lyapunov function upper and lower bounds and it is thus omitted.

In some sense, the standard GES definition is an integral version of the differential one. In particular, we can still conclude that the system equilibrium point  $x = x^*$  is reached by any system initial condition in an exponential manner but when seen from the perspective of the differential dynamics.

### E. Nonlinear eigenvectors and eigenvalues

Let us recall the theory developed in [13] regarding the nonlinear system eigenstructure. In particular, the authors in [13] analyze the existence of a diffeomorphism that brings the dynamics (1) into a diagonal form, i.e., a system (1) is said to be diagonalizable on a compact set  $\mathcal{X}_d$  if there exists, on such a set, a diffeomorphic change of coordinate  $\zeta = \phi(x)$ , where  $\phi : \mathcal{X}_d \rightarrow \mathbb{C}^n$  where  $\zeta_i = g_i(\zeta_i)$ ,  $i = 1, \dots, n$ , where  $\zeta_i(t) \in \mathbb{C}$  and the elements of the vectorfield  $g_i : \mathbb{C} \rightarrow \mathbb{C}$  are analytic. Moreover, they define the nonlinear eigenvalue and eigenvector of system (1) as follows.

*Definition 2.5 (Def. 2 in [13]):* The complex analytic functions  $\lambda : \mathcal{X}_d \rightarrow \mathbb{C}$  and  $v : \mathcal{X}_d \rightarrow \mathbb{C}^n$ , where at equilibrium points  $x = x^*$  we have  $v(x^*) \neq 0$ , are respectively called an eigenvalue and left eigenvector of system (1) if for all  $x \in \mathcal{X}_d$  the following holds

$$v^\top(x) \frac{\partial f(x)}{\partial x} + \left( \frac{\partial v(x)}{\partial x} f(x) \right)^\top = \lambda(x) v^\top(x). \quad (7)$$

At an equilibrium point  $x = x^*$ ,  $\lambda(x^*)$  and  $v(x^*)$  are nothing but an eigenvalue and a left eigenvector of the system Jacobian evaluated at the equilibrium, i.e.,

$$v^\top(x^*) \frac{\partial f(x^*)}{\partial x} = \lambda(x^*) v^\top(x^*). \quad (8)$$

In their work, the authors also show conditions to describe the diagonalizability property of system (1), as described in the following theorem

*Theorem 2.1 (Th.1 in [13]):* Let the complex analytic functions  $\lambda_i : \mathcal{X}_d \rightarrow \mathbb{C}$  and  $v_i : \mathcal{X}_d \rightarrow \mathbb{C}^n$  be respectively the eigenvalues and the associated left eigenvectors of system (1). The system (1) is diagonalizable on  $\mathcal{X}_d \subset \mathbb{R}^n$  if and only if the following holds

- 1) the distribution of  $(v_1(x), \dots, v_2(x))$  spans  $\mathbb{R}^n$  at each  $x \in \mathcal{X}_d$ ;
- 2) there exists complex analytic functions  $\psi_i : \mathcal{X}_d \rightarrow \mathbb{C}^n$  such that  $v_i^\top(x) dx = d\psi$  on  $\mathcal{X}_d$ ,  $i = 1, \dots, n$ .

The first condition reported in the theorem above is related to the linear independence among the eigenvectors at each point of the domain  $\mathcal{X}_d$ . The second one instead is referred to as an integrability condition on the eigenvectors.

### III. DIFFERENTIAL GLOBAL STABILITY

Before providing the dGES result, we define a more general (and more compact version) of the nonlinear eigenvalue and eigenvector  $(\lambda, v)$  solution of (7) as exploited in [13]. We thus introduce the solution pair  $(\Pi(x), \mathbb{A}(x))$ , with  $\Pi, \mathbb{A} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ , to

$$\dot{\Pi}(x) + \Pi(x) \frac{\partial f(x)}{\partial x} - \mathbb{A}(x) \Pi(x) = 0 \quad (9)$$

with  $\Pi(x)$  invertible for all  $x \in \mathcal{X}$ . In particular, the real matrix  $\Pi(x)$  defines a 'dynamical' change of coordinates for the variational system (2), so that the state  $z = \Pi(x)\xi$  has  $\mathbb{A}(x)$  as state matrix, i.e., its dynamics reads as

$$\dot{z} = \left( \dot{\Pi} + \Pi \frac{\partial f(x)}{\partial x} \right) \xi = \mathbb{A}z, \quad z(0) = \Pi(x_0)\xi(0). \quad (10)$$

We claim that (9) is more general than (7) for two reasons, i.e.,

- the matrix  $\mathbb{A}(x)$  can be taken as non-diagonal, thus allowing, for example, to get a real representation of the nonlinear eigenvalues and eigenvectors defined in [13];
- whenever  $\mathbb{A}(x)$  is diagonal<sup>1</sup>, the  $i$ -th row of  $\Pi(x)$ , i.e.,  $\Pi_i(x)$ , and  $\mathbb{A}_{ii}(x)$ , for  $i = 1, \dots, n$ , satisfy a weaker definition of the nonlinear left eigenvectors and eigenvalues than in (7), we because we do not require  $\Pi_i(x)$  to be differentiable<sup>2</sup> in  $x$ .

In the rest of the paper, we assume that the solution pair  $(\Pi(x), \mathbb{A}(x))$  to (9) is available<sup>3</sup> for system (2). A major difference with respect to the change of coordinates introduced in [13], i.e.,  $\zeta = \phi(x)$  such that  $\dot{\zeta}_i = g_i(\zeta_i)$ ,  $i = 1, \dots, n$ , is

<sup>1</sup>By relaxing the condition on the real solution pair  $(\Pi, \mathbb{A})$ , i.e., considering  $\Pi, \mathbb{A} : \mathbb{R}^n \rightarrow \mathbb{C}^{n \times n}$ .

<sup>2</sup>See for example also the discussion on the solution of the regulator equations in [19].

<sup>3</sup>Note that in principle one can set  $\mathbb{A}$  to be a desired matrix and then seek a solution  $\Pi$  satisfying (9). In principle, we may thus have different solution pairs satisfying the same matrix ordinary differential equation (9).

that in case when  $\mathbb{A}(x)$  diagonal we can explicitly write the solution of (10), in terms of  $x(t)$ , i.e.,

$$z(t) = \exp\left(\int_0^t \mathbb{A}(x(s))ds\right) z(0), \quad (11)$$

since in (10), we have  $n$  linear time-varying scalar ordinary differential equations, i.e., for  $i = 1, \dots, n$ ,  $\dot{z}_i = \mathbb{A}_{ii}(x(t))z_i$  and thus

$$z_i(t) = \exp\left(\int_0^t \mathbb{A}_{ii}(x(s))ds\right) z_i(0).$$

While, on the other hand, the analytic solution of a general nonlinear scalar equation, such as  $\dot{\zeta}_i = g_i(\zeta_i)$ ,  $i = 1, \dots, n$ , might be too hard or even impossible to obtain. Instead, by focusing on the variational dynamics, we can characterize the local behavior of the system according to the actual state variable  $x$ . The above discussion can then be summarized in the following Lemma.

*Lemma 3.1:* Consider the dynamical system (1) with initial condition  $x(0) = x_0 \in \mathbb{R}^n$ , the related variational dynamics (2) with initial condition  $\xi(0) = f(x_0)$ , and assume to have a solution pair  $(\Pi(x), \mathbb{A}(x))$  of (9), with  $\Pi(x)$  invertible and  $\mathbb{A}$  diagonal. Then an analytic solution of the variational system  $\xi(t)$ , whenever it exists, is given by

$$\xi(t) = \Pi^{-1}(x(t)) \exp\left(\int_0^t \mathbb{A}(x(s))ds\right) \Pi(x_0)\xi(0). \quad (12)$$

*Proof:* The proof simply comes from the analytic solution of  $z(t)$  in (11), from which we obtain  $\xi(t)$  by considering the reversed change of coordinates  $\xi = \Pi^{-1}z$  and substituting  $z(0) = \Pi(x_0)\xi(0)$ . ■

A possible interpretation of equation (12) is that the time evolution of  $\xi(t)$  can be ‘decomposed’ into two contributions, i.e., one provided by  $\Pi(x)$  and one by  $\mathbb{A}(x)$ , and they are *dynamically* interconnected, via equation (9), to the Jacobian of  $f(x)$ . This gives the reason to call the solution pair  $(\Pi, \mathbb{A})$ , with  $\mathbb{A}$  diagonal (or block diagonal in case of complex conjugate eigenvalues), the *nonlinear dynamical eigenstructure* of  $\partial f(x)/\partial x$ .

*Remark 3.1:* Note that in this scenario it is fundamental to define the initial condition of the variational dynamics, i.e.,  $\xi(0) = f(x_0)$ , in order to write its explicit solution (12). This is possible only by selecting the vectorfield direction on the tangent space at each  $x$ . Otherwise, taking as variational dynamics state the virtual displacement  $\xi(t) = \delta x(t)$  does not allow us to define any initial condition.

A direct consequence of Lemma 3.1 is the following proposition, which describes a global asymptotic stability property of the variational dynamics whenever its solution exists for all positive time.

*Proposition 1:* Consider the dynamical system (1) with initial condition  $x(0) = x_0 \in \mathbb{R}^n$ , the related variational dynamics (2) with initial condition  $\xi(0) = f(x_0)$  and assume  $\|\partial f(x)/\partial x\| < M_f$  for all  $x \in \mathbb{R}^n$  for some positive real  $M_f$ . Furthermore, assume that the solution pair  $(\Pi(x), \mathbb{A}(x))$  of (9), is such that  $\Pi(x)$  is invertible and  $\mathbb{A}$  diagonal and

bounded, i.e.,  $|\mathbb{A}(x)| < M_{\mathbb{A}}$  for all  $x \in \mathbb{R}^n$  for some positive real  $M_{\mathbb{A}}$ . Then the origin of (2) is globally asymptotically stable if

$$\chi\left(\Pi^{-1}(x(t)) \exp\left(\int_0^t \mathbb{A}(x(s))ds\right)\right) < 0. \quad (13)$$

*Proof:* By the assumption on the boundedness of the system’s Jacobian, i.e.,  $\|\partial f(x)/\partial x\| < M_f$ , we guarantee the existence of solution for system (1) for all  $t \geq 0$ , see [15, Th. 2.3]. Then the assumption on the boundedness of  $\mathbb{A}$ , i.e.,  $\|\mathbb{A}(x)\| < M_{\mathbb{A}}$ , provide the global existence of the solution  $z(t)$  in (11) for all positive time. Then the invertibility of  $\Pi(x)$  allows us to determine the global existence of a solution  $\xi(t)$  as in (11). Finally, the condition on the Lyapunov characteristic exponent guarantees that, asymptotically, all the elements of  $\xi(t)$  have an exponential-like convergence independently from the initial condition. ■ A second direct consequence of the Lemma above is the following Proposition describing the global exponential stability of the variational dynamics.

*Proposition 2 (differential Global Exponential Stability):* Assume there exists (at least) a solution pair  $(\Pi, \mathbb{A})$  of (9) such that  $\mathbb{A}$  is constant and Hurwitz and there exists a positive real scalar  $M_{\Pi}$  such that  $\mu(\Pi(x)) < M_{\Pi}$ , then

$$|\xi(t)| \leq M_{\Pi} \mu(\mathbb{D}) \exp(\alpha t) |\xi(0)| \quad (14)$$

with

$$\alpha = \max_{i=1, \dots, m} \Re\{\bar{\lambda}_i\} \left(1 + \cos\left(\pi \frac{g_i}{g_i + 1}\right)\right) \quad (15)$$

and  $\mathbb{D}$  defined in (19), with  $\bar{\lambda}_i \in \sigma(\mathbb{A})$ .

*Proof:* The proof is an application of the above theorem in which we consider, without loss of generality,  $\mathbb{A}$  to be in Jordan form. It has been proven in [24] that for any Jordan form, with simple eigenvalues in the origin, we can write

$$\|\exp(\mathbb{A}t)\| \leq \mu(\mathbb{D}) \exp(\alpha t)$$

with  $\alpha$  defined in (15). Then the rest of the proof comes from the definition of the solution  $\xi(t)$  in (12). ■

## IV. ON CONTRACTIVE DYNAMICS

### A. Constructing a contractive metric

Equation (9) is more general than the nonlinear left eigenvectors defined in (7) also for the fact that, in principle, one can choose a generic matrix  $\mathbb{A}(x)$  and look for (the possibly not so easy to find) matrix  $\Pi(x)$ . In this particular, if we choose a  $\mathbb{A}$  with a positive definite symmetric part and we assume that there exists a solution  $\Pi(x)$  whose singular values are everywhere bounded. This particular pair  $(\Pi(x), \mathbb{A}(x))$  allows us to define a contractive metric  $P(x)$  for system (1).

*Lemma 4.1:* Assume that for system (1) we obtain a solution pair  $(\Pi(x), \mathbb{A}(x))$  to (9) such that for all  $x \in \mathcal{X}$

- 1) there exists positive real  $\underline{p}$  and  $\bar{p}$  such that  $\underline{p} \leq \sigma_{\min}^2(\Pi)$  and  $\sigma_{\max}^2(\Pi) \leq \bar{p}$ ,
- 2)  $\text{sym}(\mathbb{A}(x)) < 0$ .

Then  $P(x) = \Pi^\top(x)\Pi(x)$ , for  $x \in \mathcal{X}$ , defines a contractive metric for (1).

*Proof:* In order to be a contractive metric,  $P(x)$  should satisfy (5) with  $P(x)$  being everywhere bounded from below and from above, i.e.,  $\underline{p}I \leq P(x) \leq \bar{p}I$  for some  $\underline{p}$  and  $\bar{p}$ . This boundedness property is satisfied by assumption on the singular values of  $\Pi(x)$ , i.e.,  $\underline{p}I \leq \Pi^\top(x)\Pi(x) \leq \bar{p}I$ . Then, to show that  $P(x)$  satisfies the (5) we consider the Lyapunov function  $V = \xi^\top \Pi^\top(x)\Pi(x)\xi$ , whose dynamics reads as

$$\dot{V} = 2\xi^\top \Pi^\top \left( \dot{\Pi} + \Pi \frac{\partial f}{\partial x} \right) \xi = 2\xi^\top \Pi^\top \mathbb{A} \Pi \xi < 0$$

for  $\xi \neq 0$ , thus satisfying (5). ■

### B. Differential GES in the contractive framework

Whenever  $\mathbb{A}$  has a symmetric part that is upper-bounded by a negative definite constant matrix we can obtain differential GES. For the sake of exposition we consider  $\mathbb{A}$  to be constant with a negative definite symmetric part. Then we have the following parallel result of Proposition 2.

*Corollary 4.1:* Assume there exist (at least) a solution pair  $(\Pi, \mathbb{A})$  of (9) such that  $\mathbb{A}$  is constant with negative definite symmetric part, and there exists positive real  $\underline{p}$  and  $\bar{p}$  such that  $\underline{p} \leq \sigma_{\min}^2(\Pi)$  and  $\sigma_{\max}^2(\Pi) \leq \bar{p}$  for all  $x \in \mathbb{R}^n$ , then

$$|\xi(t)| \leq \sqrt{\frac{\bar{p}}{\underline{p}}} \exp(\alpha t) |\xi(0)| \quad (16)$$

with  $\alpha = \lambda_{\min}(\text{sym } \mathbb{A})$ .

The proof of the corollary simply follows as an application of the previous lemma, thus it is omitted.

1) *The particular case of  $\Pi$  is constant:* whenever  $\Pi$  (and thus  $P = \Pi^\top \Pi$ ) is constant we can always determine the differential exponential stability of the system equilibrium point, with  $\text{sym}(\mathbb{A})$  being a strictly negative definite matrix.

*Corollary 4.2:* Assume there exists (at least) a solution pair  $(\Pi, \mathbb{A})$  of (9) such that  $\mathbb{A}$  has a negative definite symmetric part, and  $\Pi$  being full rank and constant, which implies that  $0 < \sigma_{\min}^2(\Pi)I \leq \Pi^\top \Pi \leq \sigma_{\max}^2(\Pi)I$ , then

$$|x(t) - x^*| \leq \mu(\Pi) \exp(\alpha t) |x(0) - x^*| \quad (17)$$

with  $\alpha$  being such that  $\Pi^\top \mathbb{A}(x)\Pi \leq \alpha \Pi^\top \Pi < 0$ , for all  $x \in \mathbb{R}^n$ .

The proof comes from standard arguments in the convergence dynamics formalism as described in [20], by considering the constant metric  $P = \Pi^\top \Pi$ .

## V. A NONLINEAR EXAMPLE

Consider the nonlinear autonomous system in [13, ex.8], with dynamics

$$\begin{aligned} \dot{x}_1 &= -x_2(-1 + 4x_1 + 4x_2 - 4x_2^2) \\ \dot{x}_2 &= -2(x_1 + x_2 - x_2^2) \end{aligned} \quad (18)$$

and Jacobian matrix given by

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} -4x_2 & 1 - 8x_2 + 12x_2^2 - 4x_1 \\ -2 & -2 + 4x_2 \end{bmatrix}.$$

For this system, it has been shown that the origin is globally asymptotically stable, by solving for the nonlinear eigenvectors equations (7). In particular, the solutions to (7) obtained in [13, ex.8] are complex pairs of nonlinear eigenvalues and eigenvectors. While, by applying the proposed method we obtain a real-valued solution pair  $(\Pi, \mathbb{A})$  to (9) given by

$$\Pi(x) = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} - 2x_2 \end{bmatrix}, \quad \mathbb{A} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}.$$

It is worth noticing that, in this case,  $\mathbb{A}$  is not diagonal but constant, thus we can apply the analysis described in Section IV. In particular, by defining the metric

$$P(x) = \Pi^\top(x)\Pi(x) = \begin{bmatrix} 1 & \frac{1}{2} - 2x_2 \\ \frac{1}{2} - 2x_2 & \frac{1}{4} + (2x_2 - \frac{1}{2})^2 \end{bmatrix}$$

whose principle minors are  $1 > 0$  and its determinant  $\det(P) = 1/4 > 0$ , thus showing it is everywhere positive definite and, for the sake of completeness, its inverse is given by

$$P^{-1}(x) = \begin{bmatrix} 16x_2^2 - 8x_2 + 2 & 8x_2 - 2 \\ 8x_2 - 2 & 4 \end{bmatrix}.$$

With the constructed metric we can write the time derivative of  $V = \xi^\top \Pi^\top(x)\Pi(x)\xi$  as

$$\begin{aligned} \dot{V} &= 2\xi^\top \Pi^\top \left( \dot{\Pi} + \Pi \frac{\partial f(x)}{\partial x} \right) \xi = 2\xi^\top \Pi^\top \mathbb{A} \Pi \xi \\ &= -2\xi^\top \Pi^\top \Pi \xi = -2V, \end{aligned}$$

Thus  $V(t) = \exp(-2t)V(0)$ , from which we have

$$\begin{aligned} \lambda_{\min}(P)|\xi(t)|^2 &\leq \exp(-2t)V(0) \\ &\leq \exp(-2t)\lambda_{\max}(P)|\xi(0)|^2 \end{aligned}$$

and thus

$$|\xi(t)|^2 \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \exp(-2t) |\xi(0)|^2.$$

The eigenvalues of  $P(x)$  are two equal and simple eigenvalues for every  $x \in \mathbb{R}^n$ , i.e.,

$$\lambda_{1/2} = 2x_2^2 - x_2 + \frac{3 - \sqrt{(8x_2^2 - 4x_2 + 1)(8x_2^2 - 4x_2 + 5)}}{4}$$

and thus globally we have

$$\sqrt{\mu(P(x))} = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} = \mu(\Pi) = 1.$$

Which implies that

$$|\xi(t)| \leq \exp(-t) |\xi(0)|,$$

which proves the differential GES, rather than the global asymptotic stability as shown in [13].

*Remark 5.1:* It is worth noticing that to show analytically the differential GES property, it is enough to have a globally upper-bounded conditioning number of  $P(x)$  rather than considering  $P(x)$  globally upper-bounded.

## VI. CONCLUSIONS

In this paper, we introduce the concept of dynamical change of coordinates, whose particular solution provides the nonlinear dynamical eigenstructure of the variational dynamics. Through the obtained eigenstructure, we can determine an analytic expression for the solution of the variational system and consequently a sufficient condition to the introduced notion of differential GES. Moreover, for contractive systems, when available the transformation matrix of the dynamical change of coordinates allows us to determine an analytical expression of the contractive metric associated with the system dynamics. We also provide the conditions of differential GES in the contraction formalism. We show the validity of the approach via a nonlinear example.

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## APPENDIX

Given a matrix  $A$  of dimensions  $n \times n$ , let  $m \leq n$  be the total number of linearly independent (non-generalized) eigenvectors  $T_i^1 \neq 0$  associated with an eigenvalue  $\bar{\lambda}_i \in \sigma(A)$ ,  $i = 1, \dots, m$ , such that

$$AT_i^1 = \bar{\lambda}_i T_i^1 \quad \forall i = 1, \dots, m.$$

*Definition 1.1 (Jordan blocks dimension):* We define the values  $g_i \geq 1$  satisfying  $\sum_{i=1}^m g_i = n$ , for  $i \in \{1, \dots, m\}$ , such that there exist  $g_i - 1$  linearly independent generalized eigenvectors  $T_i^k \neq 0$ , for  $k = 2, \dots, g_i$ , associated to the corresponding eigenvalue  $\bar{\lambda}_i$  and satisfying

$$(A - \bar{\lambda}_i I)T_i^k = T_i^{k-1} \quad \forall k = 2, \dots, g_i.$$

To introduce the Jordan form modification by defining the matrix

$$\mathbb{D} := \text{blkdiag} \left( D_{g_1}(\bar{\lambda}_1), \dots, D_{g_m}(\bar{\lambda}_m) \right) \quad (19)$$

and we determine the matrix  $\mathbb{J}$  as  $\mathbb{J} := \mathbb{T}^{-1}A\mathbb{T}$  with  $\mathbb{T} := T\mathbb{D}$ , in which  $T$  is the matrix of standard right eigenvalues of  $A$  defining the matrix transformation that put  $A$  into its Jordan normal form  $J$ , see, e.g. [16]. The modified Jordan form  $\mathbb{J}$  as the following particular structure

$$\mathbb{J} = \text{blkdiag} \left( \bar{\lambda}_1 \mathbb{J}_{g_1}, \dots, \bar{\lambda}_m \mathbb{J}_{g_m} \right), \quad (20)$$

for which each block  $\bar{\lambda}_k \mathbb{J}_{g_k}$ , for  $k \in \{1, \dots, m\}$ , has symmetric part eigenvalues that can be written, for  $i \in \{1, \dots, g_k\}$ , as

$$\lambda_i(\text{sym}(\bar{\lambda}_k \mathbb{J}_{g_k})) = \bar{\lambda}_k \cdot \left( 1 + \cos \left( \pi \frac{i}{g_k + 1} \right) \right). \quad (21)$$

The interested reader can find further details in [24] and [25].

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