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STABILITY OF DEGENERATE WAVE EQUATION WITH A SINGULAR POTENTIAL AND LOCAL DAMPING

MOHAMMAD AKIL¹, GENNI FRAGNELLI², IBTISSAM ISSA³

ABSTRACT. In this paper, we investigate the stability of a degenerate/singular wave equation featuring localized singular damping, along with a drift term and a leading operator in non-divergence form. We establish exponential stability results in this context under suitable conditions on the degeneracy and singularity coefficients.

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1. INTRODUCTION

In recent decades, the study of stability and controllability in nondegenerate hyperbolic/parabolic equations has garnered significant attention from multiple authors. Building upon pioneering works on controllability [18, 20], there has been substantial progress in unraveling the controllability properties of non-degenerate parabolic equations with variable coefficients. Notably, in [45], researchers employed local Carleman estimates for elliptic equations to delve into the null controllability for the heat equation on a manifold. Conversely, the study of degenerate equations historically lacked comprehensive results, despite their relevance to many real-world applications. However, in recent times, there has been a notable surge in interest in the study of degenerate parabolic/hyperbolic equations, marking a significant shift in focus.

In various real-world scenarios, such as camouflage (for rendering operators invisible to outside observation) [35], Lévy noise phenomena [10], meteorology [9] and biology [49], degenerate partial differential equations (PDEs) give rise to challenging control and inverse problems. The intricate mathematical challenges associated with degenerate PDEs have been illuminated by the myriad applications they underpin. For instance, in semiconductor physics and device engineering, degenerate PDEs are instrumental in understanding and optimizing

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the behavior of modern electronic devices.

These situations frequently involve operators whose diffusion coefficients vary spatially, resulting in a lack of uniform ellipticity. On the other hand, these operators become uniformly elliptic in localized portions of the spatial domain that are positive distances from the degenerate areas. Degeneracy can appear inside internal submanifolds or at the boundary. For degenerate parabolic equations, the authors explore well-posedness and global null controllability in publications like [15, 16, 17].

This paper is devoted to study a class of degenerate/singular wave equations. These equations are characterized by degeneracy in the non-divergence form, accompanied by a drift term, and also feature local singular damping. The system is defined as follows

$$(1.1) \quad \begin{cases} u_{tt} - au_{xx} - \frac{\lambda}{d}u - bu_x + \chi_{(x_1, x_2)}u_t = 0, & (x, t) \in \mathbb{R}_*^+ \times (0, 1), \\ u(t, 0) = u(t, 1) = 0, & t \in \mathbb{R}_*^+, \\ u(0, x) = u_0(x), u_t(0, x) = u_1(x), & x \in (0, 1), \end{cases}$$

where the damping coefficient is given by $\chi_{(x_1, x_2)}(x)$ with $0 \leq x_1 < x_2 \leq 1$ and

$$(1.2) \quad \begin{cases} a, b, d \in C^0[0, 1], \\ a, d > 0 \text{ on } (0, 1], a(0) = d(0) = 0, \\ \frac{b}{a} \in L^1(0, 1). \end{cases}$$

Hence, if $a(x) = x^K$, $K > 0$, we can consider $b(x) = x^m$, $m > 0$, for any $m > K - 1$. This condition is clearly satisfied if $K < 1$.

The degeneracy of a function θ at $x = 0$ is measured by the parameter K_θ defined by

$$(1.3) \quad K_\theta := \sup_{x \in (0, 1]} \frac{x|\theta'(x)|}{\theta(x)}.$$

In particular, we say that θ is

$$(1.4) \quad \begin{cases} \text{weakly degenerate (WD) if } \theta \in C^0[0, 1] \cap C^1(0, 1] \text{ and } K_\theta \in (0, 1), \\ \text{strongly degenerate (SD) if } \theta \in C^1[0, 1] \text{ and } K_\theta \in [1, 2). \end{cases}$$

Here we assume $K_a, K_d < 2$ because it is essential in the calculation that will be conducted below. Additionally, we will introduce the condition $K_a + 2K_d \leq 2$ later on.

Before delving into the system addressed in this paper, a literature review on the study of degenerate systems would be beneficial. It is widely recognized that the standard linear theory for transverse waves in a string of length L under tension \mathcal{T} leads to the classical wave equation:

$$\rho(x)u_{tt}(t, x) = \frac{\partial \mathcal{T}}{\partial x}u_x(t, x) + \mathcal{T}(x, t)u_{xx}(t, x),$$

where $u(t, x)$ denotes the vertical displacement of the string from the x axis at position $x \in (0, L)$ and time $t > 0$, $\rho(x)$ is the mass density of the string at position x , while $\mathcal{T}(t, x)$ denotes the tension in the string at position x and time t . Divide by $\rho(x)$, assume \mathcal{T} is independent of t , and set $a(x) = \mathcal{T}(x)\rho^{-1}(x)$, $b(x) = \mathcal{T}'(x)\rho^{-1}(x)$. In this way, we obtain

$$u_{tt}(t, x) = a(x)u_{xx}(t, x) + b(x)u_x(t, x).$$

Let's assume that the density is remarkably high at a particular point, for example, $x = 0$. In this case, the previous equation degenerates at $x = 0$, as we can treat $a(0) = 0$, and the remaining term becomes a drift term.

Little is known in the literature about cases where the coefficient $a(x)$ (in the equation $u_{tt} - a(x)u_{xx}$) exhibits degeneracy, despite the relevance of many applications described by hyperbolic equations that degenerate at the boundary of the spatial domain (refer to [5]). In [13], the authors consider the following degenerate wave equation with drift in the presence of a leading operator that is not in divergence form:

$$\begin{aligned} u_{tt} - a(x)u_{xx} - b(x)u_x &= 0, \\ u(t, 0) = 0, u(t, 1) &= f(t) \end{aligned}$$

and study boundary controllability. Regarding the same problem in divergence form, we refer to the pioneering paper [5], for a general function $a(x)$, and to [36], for the prototype case ($a(x) = x^\alpha$, $\alpha \in (0, 1)$). In a recent study in [31], the authors examine a one-dimensional degenerate wave equation with drift, featuring a leading degenerate operator in non-divergence form. They establish uniform exponential decay for the solutions of the system under certain conditions. A similar study is presented in [13]. Moreover, the authors in [3] explore a one-dimensional degenerate wave equation with degenerate damping, incorporating a drift term and a leading operator in non-divergence form. Additionally, they consider a system that couples degenerate and non-degenerate wave equations, interconnected through transmission, and subject to a single dissipation law at the boundary of the non-degenerate equation. In both scenarios, they derive exponential stability results. It's worth noting the work in [2], where the authors explore the stability of a transmission problem involving a degenerate wave equation and a heat equation under the Coleman-Gurtin heat conduction law or Gurtin-Pipkin law with memory effect.

In recent years, considerable attention has been directed towards controllability issues for parabolic problems, not only involving degenerate terms but also singular ones. Indeed, numerous challenges arising in Physics and Biology (as seen in [6, 43, 52]), Biology (refer to [11, 12, 19, 24, 25, 34]), or Mathematical Finance (as discussed in [37]) are described by degenerate parabolic equations featuring singular terms.

Demonstrating controllability often involves establishing global Carleman estimates for the adjoint operator, a common strategy employed in various contexts. Notably, in works such as [4, 16, 17, 27, 48], researchers successfully demonstrate such properties for systems with regular degenerate coefficients. Similarly, studies like [12] and [28] extend these results to systems with non-smooth degenerate coefficients. Moreover, investigations in works such as [22] and [30] delve into controllability issues in systems featuring degenerate and singular coefficients. In [46], the stability of an elastic string system with local Kelvin-Voigt damping is considered. Here the damping coefficient has a singularity at the interface of the damped and undamped regions and behaves like x^α near the interface. In particular, the authors prove that the semigroup corresponding to the system is polynomially or exponentially stable and the decay rate depends on the parameter $\alpha \in (0, 1]$ (see also [38] and [40] for more recent results). In a recent work [39], the stability of a multidimensional wave equation featuring localized Kelvin-Voigt damping within a cuboidal domain is considered. Notably, the damping region deviates from satisfying the geometric control condition (GCC), and the damping coefficient is considered degenerate near the interface. The authors establish polynomial stability of the system, with the decay rate contingent upon the degree of degeneration. Remarkably, this decay rate aligns with the optimal rate attained for the analogous system with a constant damping coefficient, as demonstrated in [53]. Moreover, in [41], the author studies Euler-Bernoulli beam equation with a local Kelvin-Voigt damping that degenerates near 0. He proves the stability of the system's semigroup, whose decay rate is influenced by the degeneracy speed α , and introduces a new method combining local analysis and classical iterative methods.

In [23], the author examines non-smooth general degenerate/singular parabolic equations in non-divergence form, with degeneracy and singularity occurring within the spatial domain, alongside Dirichlet or Neumann boundary conditions. She establishes both the well-posedness and Carleman estimates for the associated adjoint problem. Furthermore, in [29], the authors address well-posedness and null controllability for operators with Dirichlet boundary conditions in divergence form, featuring degeneracy and singularity (i.e., $\lambda \neq 0$), both within the domain (see also [26]).

A recent work in [7] tackles the null controllability of wave equations featuring degeneracy and singularity, with a particular focus on cases involving pure powers. In [32] and [33], the authors present the first results for a degenerate hyperbolic equation in non-divergence form with drift, where both the degeneracy and singularity are characterized by more general functions. In particular, in the first paper the authors give a controllability result considering a boundary control acting on the non degeneracy point; in the second one they study the stability substituting the boundary control with damping terms acting always at the non degeneracy point. The presence of both the drift term and singular term necessitates the utilization of different function spaces compared to prior works such as [7], [13] or [31], leading to the development of novel techniques. In the field of quantum physics, a single potential refers to a potential that either becomes infinitely large or displays unconventional behavior at specific points or inside specified regions of space. Potential, in this context, denotes the energy associated to a force field or a force exerted on a particle as a result of its location in space. The sign of

the singular potential term in the wave equation can make a significant difference in the physical interpretation and behavior of the system. In the event of a negative sign, the singular potential acts as a repulsive potential, causing the wave function to be pushed away from the region where the potential is concentrated. Conversely, in the case of a positive sign, it functions as an attractive potential.

As far as we know, in the literature there are no previous studies on the stability of the degenerate wave equation with a singular term and a *singular internal damping term*. We are interested in analyzing the stability of a degenerate/singular wave equation in non-divergence form with a drift and a singular damping.

The paper is organized as follows: in Section 2, we reformulate the system (1.1) into an evolution system and we prove the well-posedness of the system by semigroup approach. In Section 3 we give some technical lemmas that are crucial for the rest of the paper; in Section 4, we show the strong stability of the system under some conditions on the damped region. Finally, in Section 5, exponential stability is derived and in the Appendix we provide some examples that satisfy the conditions considered.

2. PRELIMINARIES, FUNCTIONAL SPACES AND WELL-POSEDNESS

This section is devoted to define the functional spaces that will be used throughout the entire paper and to establish a very modest assumption to prove the well-posedness of (1.1). We begin with the following hypothesis.

Hypothesis 2.1. The assumptions in (1.2) are satisfied. Moreover, a is (WD) or (SD), d is (WD) and such that $K_a + 2K_d \leq 2$.

Remark 2.1. We notice that, at this stage, b may not degenerate at $x = 0$. Indeed, if a is (WD) then $\frac{1}{a} \in L^1(0, 1)$ and the assumption $\frac{b}{a} \in L^1(0, 1)$ is always satisfied. If a is (SD) then $\frac{1}{a} \notin L^1(0, 1)$, hence, if we want $\frac{b}{a} \in L^1(0, 1)$ then b has to degenerate at 0. In this case b can be (WD) or (SD).

In order to study the well-posedness of (1.1), let us recall the well-known absolutely continuous weight function

$$\eta(x) := \exp \left\{ \int_{\frac{1}{2}}^x \frac{b(s)}{a(s)} ds \right\}, \quad x \in [0, 1]$$

introduced by Feller in a related context [21] and used later by several authors, see, for example, [14], [23] and the references therein. Under Hypothesis 2.1, it is clear that the function $\eta : [0, 1] \rightarrow \mathbb{R}$ introduced before is well defined and we immediately find that $\eta \in C^0[0, 1] \cap C^1(0, 1)$ is a strictly positive function, which is bounded above and below by a positive constant. Notice also that η can be extended to a function of class $C^1[0, 1]$ when b degenerates at 0 not slower than a , for instance if $a(x) = x^K$ and $b(x) = x^m$ with $K \leq m$.

Now we set the function σ as

$$(2.1) \quad \sigma(x) := \frac{a(x)}{\eta(x)},$$

which is a continuous function in $[0, 1]$, independent of the possible degeneracy of a . Moreover, observe that if u is a sufficiently smooth function, e.g. $u \in W_{loc}^{2,1}(0, 1)$, then we can write $Bu := au_{xx} + bu_x$ as

$$Bu = \sigma(\eta u_x)_x.$$

Using the definition of σ , the system (1.1) can be rewritten as

$$(2.2) \quad \begin{cases} u_{tt} - \sigma(\eta u_x)_x - \frac{\lambda}{d} u_x + \chi_{(x_1, x_2)} u_t = 0, & (t, x) \in \mathbb{R}_*^+ \times (0, 1), \\ u(t, 0) = u(t, 1) = 0, & t \in \mathbb{R}_*^+, \\ u(0, x) = u_0(x), u_t(0, x) = u_1(x), & x \in (0, 1). \end{cases}$$

We introduce the following Hilbert spaces

$$L_{\frac{1}{\sigma}}^2(0, 1) := \left\{ u \in L^2(0, 1); \|u\|_{\frac{1}{\sigma}} < \infty \right\}, \quad \langle u, v \rangle_{\frac{1}{\sigma}} := \int_0^1 \frac{1}{\sigma} u \bar{v} dx, \quad \text{for every } u, v \in L_{\frac{1}{\sigma}}^2(0, 1),$$

$$H_{\frac{1}{\sigma}}^1(0, 1) := L_{\frac{1}{\sigma}}^2(0, 1) \cap H^1(0, 1), \quad \langle u, v \rangle_1 := \langle u, v \rangle_{\frac{1}{\sigma}} + \int_0^1 \eta u_x \bar{v}_x dx, \quad \text{for every } u, v \in H_{\frac{1}{\sigma}}^1(0, 1),$$

and

$$H_{\frac{1}{\sigma}}^2(0, 1) := \left\{ u \in H_{\frac{1}{\sigma}}^1(0, 1); Bu \in L_{\frac{1}{\sigma}}^2(0, 1) \right\}, \quad \langle u, v \rangle_2 := \langle u, v \rangle_1 + \langle Bu, Bv \rangle_{\frac{1}{\sigma}}.$$

The previous inner products induce the related respective norms given by

$$\|u\|_{\frac{1}{\sigma}}^2 = \int_0^1 \frac{1}{\sigma} |u|^2 dx, \quad \|u\|_1^2 = \|u\|_{\frac{1}{\sigma}}^2 + \int_0^1 \eta |u_x|^2 dx \quad \text{and} \quad \|u\|_2^2 = \|u\|_1^2 + \int_0^1 \sigma |(\eta u_x)_x|^2 dx.$$

Also, we consider the following Hilbert spaces

$$H_{\frac{1}{\sigma}, 0}^1(0, 1) = L_{\frac{1}{\sigma}}^2(0, 1) \cap H_0^1(0, 1) \quad \text{and} \quad H_{\frac{1}{\sigma}, 0}^2(0, 1) := \left\{ u \in H_{\frac{1}{\sigma}, 0}^1(0, 1); Bu \in L_{\frac{1}{\sigma}}^2(0, 1) \right\}$$

endowed with the previous inner products and the previous norms. In the following, we will denote by $\|\cdot\|$ the usual norm in $L^2(0, 1)$, i.e. $\|\cdot\| := \|\cdot\|_{L^2(0,1)}$.

Proposition 2.2. (Hardy-Poincaré Inequality) Assume Hypothesis 2.1. Then, for all $u \in H_{\frac{1}{\sigma}, 0}^1(0, 1)$

$$(HP) \quad \int_0^1 |u|^2 \frac{1}{\sigma(x)d(x)} dx \leq C'_{HP} \int_0^1 \eta |u_x|^2 dx$$

$$\text{where } C'_{HP} = \max_{x \in [0,1]} \frac{1}{\eta} \left(\frac{4 \max_{x \in [0,1]} \eta(x)}{a(1)d(1)} \right).$$

Proof. Take $u \in H_{\frac{1}{\sigma}, 0}^1(0, 1)$ and using the definition of σ , we get

$$(2.3) \quad \int_0^1 \frac{|u|^2}{\sigma d} dx = \int_0^1 \eta \frac{|u|^2}{ad} dx \leq \max_{x \in [0,1]} \eta \int_0^1 \frac{|u|^2}{ad} dx.$$

So, it is enough to estimate $\int_0^1 \frac{|u|^2}{ad} dx$. To this aim, thanks to Hypothesis 2.1 and the fact that $K_a + K_d < K_a + 2K_d \leq 2$, we have

$$\int_0^1 \frac{|u|^2}{a(x)d(x)} dx \leq \frac{1}{a(1)d(1)} \int_0^1 \frac{|u|^2}{x^{K_a+K_d}} dx \leq \frac{1}{a(1)d(1)} \int_0^1 \frac{|u|^2}{x^2} dx.$$

By the Hardy inequality, we know that

$$(2.4) \quad \int_0^1 \frac{|u(x)|^2}{x^2} dx \leq 4 \int_0^1 |u_x|^2 dx,$$

for all $u \in H_0^1(0, 1)$, hence

$$\int_0^1 \frac{|u|^2}{a(x)d(x)} dx \leq \frac{4}{a(1)d(1)} \int_0^1 |u_x|^2 dx \leq \left(\frac{4}{a(1)d(1)} \right) \max_{x \in [0,1]} \frac{1}{\eta} \int_0^1 \eta |u_x|^2 dx,$$

combining the above inequality with (2.3), we get the desired result (HP). \square

We observe that the aforementioned proposition remains valid when $d(x) = 1$ (see also [3]). Consequently, we may infer that

$$(2.5) \quad \int_0^1 \frac{|u|^2}{\sigma} dx \leq \tilde{C}_{HP} \int_0^1 \eta |u_x|^2 dx \quad \forall u \in H_{\frac{1}{\sigma}, 0}^1(0, 1),$$

where $\tilde{C}_{HP} = \frac{4}{a(1)} \max_{x \in [0,1]} \frac{1}{\eta} \max_{x \in [0,1]} \eta(x)$. Let C_{HP} and \bar{C}_{HP} be the best constants of (HP) and (2.5), respectively, such that $C_{HP} \leq C'_{HP}$ and $\bar{C}_{HP} \leq \tilde{C}_{HP}$. Clearly, $\bar{C}_{HP} \leq \max_{x \in [0,1]} d(x) C_{HP}$.

Observe that $\|u\|_0 := \int_0^1 \eta |u_x|^2 dx$ and $\|u\|_1$ are equivalent in $H_{\frac{1}{\sigma}, 0}^1(0, 1)$. Indeed, from (2.5), we can deduce that

$$\|u\|_0 \leq \|u\|_1 \leq (1 + \bar{C}_{HP}) \|u\|_0,$$

for all $u \in H_{\frac{1}{\sigma}, 0}^1(0, 1)$.

In the rest of the paper we make the following assumption on the parameter λ .

Hypothesis 2.2. The constant $\lambda \in \mathbb{R}$ satisfies the condition $\lambda < \frac{1}{C_{HP}}$.

We define on $H^1_{\frac{1}{\sigma},0}(0,1)$ the following norm

$$\|u\|_{\lambda}^2 := \int_0^1 \eta |u_x|^2 dx - \lambda \int_0^1 \frac{|u|^2}{\sigma(x)d(x)} dx$$

and we have that the next equivalence holds.

Proposition 2.3. Under Hypotheses 2.1 and 2.2, the two norms $\|\cdot\|_0$ and $\|\cdot\|_{\lambda}$ are equivalent in $H^1_{\frac{1}{\sigma},0}(0,1)$.

Proof. Here we will distinguish two cases according to the value of λ . Take $u \in H^1_{\frac{1}{\sigma},0}(0,1)$

• If $\lambda > 0$: It is clear that $\|u\|_{\lambda}^2 \leq \|u\|_0^2$. And from Proposition 2.2, we get that

$$-\lambda \int_0^1 \frac{|u|^2}{\sigma(x)d(x)} dx \geq -\lambda C_{HP} \int_0^1 \eta |u_x|^2 dx.$$

Hypothesis 2.2 yields

$$\int_0^1 \eta |u_x|^2 dx \leq \frac{1}{1 - \lambda C_{HP}} \|u\|_{\lambda}^2.$$

Thus,

$$\|u\|_{\lambda}^2 \leq \|u\|_0^2 \leq \frac{1}{1 - \lambda C_{HP}} \|u\|_{\lambda}^2.$$

• If $\lambda < 0$: It is clear that $\|u\|_0^2 \leq \|u\|_{\lambda}^2$. Now, from Proposition 2.2 and Hypothesis 2.2, we get that $\|u\|_{\lambda}^2 \leq (1 - \lambda C_{HP}) \int_0^1 \eta |u_x|^2 dx$. Thus,

$$\frac{1}{1 - \lambda C_{HP}} \|u\|_{\lambda}^2 \leq \|u\|_0^2 \leq \|u\|_{\lambda}^2.$$

□

Let (u, u_t) be a regular solution of (2.2). The energy of the system is given by

$$(2.6) \quad \mathcal{E}(t) = \frac{1}{2} \int_0^1 \left(\frac{1}{\sigma} |u_t|^2 + \eta |u_x|^2 - \frac{\lambda}{\sigma d} |u|^2 \right) dx$$

and we obtain that

$$\frac{d}{dt} \mathcal{E}(t) = - \int_0^1 \frac{\chi_{(x_1, x_2)}}{\sigma} |u_t|^2 dx \leq 0.$$

Thus, the energy of the system is positive thanks to Hypothesis 2.2 and (2.3) and the system (2.2) is dissipative in the sense that its energy is a non increasing function with respect to the time variable t .

In the following we will prove that under suitable conditions, (2.2) has regular solutions, so the energy is well defined.

Now, we define the energy Hilbert space \mathcal{H} by

$$\mathcal{H} = H^1_{\frac{1}{\sigma},0}(0,1) \times L^2_{\frac{1}{\sigma}}(0,1),$$

equipped with the following inner product

$$\langle U, \tilde{U} \rangle_{\mathcal{H}} = \int_0^1 \left(\eta u_x \tilde{u}_x - \lambda \frac{u \tilde{u}}{\sigma d} \right) dx + \int_0^1 \frac{1}{\sigma} v \tilde{v} dx$$

and endowed with the associated norm $\|U\|_{\mathcal{H}}^2 = \int_0^1 \left(\eta |u_x|^2 + \frac{1}{\sigma} |v|^2 - \frac{\lambda}{\sigma d} |u|^2 \right) dx$, for all $U = (u, v)^{\top}$ and $\tilde{U} = (\tilde{u}, \tilde{v})^{\top}$ in \mathcal{H} . Moreover, consider the unbounded linear operator \mathcal{A} defined by

$$\mathcal{A}(u, v)^{\top} = \left(v, \sigma(\eta u_x)_x + \frac{\lambda}{d(x)} u - \chi_{(x_1, x_2)} v \right)^{\top}$$

for all $U = (u, v) \in D(\mathcal{A})$, where

$$D(\mathcal{A}) := H^2_{\frac{1}{\sigma},0}(0,1) \times H^1_{\frac{1}{\sigma},0}(0,1) \subset \mathcal{H}.$$

Indeed, the inclusion of $u \in H_{\frac{1}{\sigma},0}^2(0,1)$ in the definition of $D(\mathcal{A})$ can be rationalized as follows: starting with $u \in H_{\frac{1}{\sigma},0}^1$ we have $\frac{u}{d} \in L_{\frac{1}{\sigma}}^2(0,1)$. Indeed, using the definition of σ , Remark 2.1 and the fact that $K_a + 2K_d \leq 2$ (see Hypothesis 2.1), one has

$$(2.7) \quad \begin{aligned} \int_0^1 \frac{|u|^2}{\sigma d^2} dx &= \int_0^1 \eta \frac{|u|^2}{ad^2} dx \leq \max_{x \in [0,1]} \eta(x) \int_0^1 \frac{|u|^2}{ad^2} dx \leq \frac{\max_{x \in [0,1]} \eta(x)}{a(1)d^2(1)} \int_0^1 \frac{1}{x^{K_a+2K_d}} |u|^2 dx \\ &\leq \frac{\max_{x \in [0,1]} \eta(x)}{a(1)d^2(1)} \int_0^1 \frac{1}{x^2} |u|^2 dx \leq \frac{4 \max_{x \in [0,1]} \eta(x)}{a(1)d^2(1)} \int_0^1 |u_x|^2 dx. \end{aligned}$$

Thus, we can rewrite (2.2) as the following evolution equation

$$(2.8) \quad U_t = \mathcal{A}U, \quad U(0) = U_0, \quad \text{where } U_0 = (u_0, u_1)^\top.$$

Proposition 2.4. The unbounded linear operator \mathcal{A} is m-dissipative in the energy space \mathcal{H} .

Proof. For all $U = (u, v)^\top \in D(\mathcal{A})$, we have

$$(2.9) \quad \Re(\langle \mathcal{A}U, U \rangle_{\mathcal{H}}) = - \int_0^1 \frac{\chi(x_1, x_2)}{\sigma(x)} |v|^2 dx \leq 0,$$

which implies that \mathcal{A} is dissipative. Now, let $F = (f_1, f_2)^\top \in \mathcal{H}$, we need to prove the existence of $U = (u, v)^\top \in D(\mathcal{A})$ unique solution of the equation

$$(2.10) \quad -\mathcal{A}U = F.$$

Equivalently, we have the following system

$$-v = f_1 \quad \text{and} \quad -\sigma(\eta u_x)_x - \frac{\lambda}{d(x)} u + \chi(x_1, x_2) v = f_2.$$

Combining the above two equations, we get

$$(2.11) \quad -\sigma(\eta u_x)_x - \frac{\lambda}{d(x)} u = f_2 + \chi(x_1, x_2) f_1.$$

Let $\varphi \in H_{\frac{1}{\sigma},0}^1(0,1)$. Multiplying (2.11) by $\frac{1}{\sigma} \bar{\varphi}$ and integrating over $(0,1)$, we obtain

$$(2.12) \quad \Lambda(u, \varphi) = L(\varphi), \quad \forall \varphi \in H_{\frac{1}{\sigma},0}^1(0,1),$$

where

$$\Lambda(u, \varphi) = \int_0^1 \eta u_x \bar{\varphi}_x dx - \int_0^1 \frac{\lambda}{d(x)\sigma(x)} u \bar{\varphi} dx \quad \text{and} \quad L(\varphi) = \int_0^1 \frac{1}{\sigma} f_2 \bar{\varphi} dx + \int_0^1 \frac{1}{\sigma} \chi(x_1, x_2) f_1 \bar{\varphi} dx.$$

We have that Λ is a sesquilinear, continuous and coercive form on $H_{\frac{1}{\sigma},0}^1(0,1)$, and L is a continuous form on $H_{\frac{1}{\sigma},0}^1(0,1)$. Then, using the Lax-Milgram Theorem, we deduce that there exists $u \in H_{\frac{1}{\sigma},0}^1(0,1)$ unique solution of the variational problem (2.12). Now, taking $v := -f_1$, we have $v \in H_{\frac{1}{\sigma},0}^1(0,1)$. It remains to prove that $U \in D(\mathcal{A})$ and solves (2.10). To this aim observe that equation (2.12) holds for every $\varphi \in C_c^\infty(0,1)$, thus we have $-(\eta u_x)_x - \frac{\lambda}{\sigma} u = \frac{1}{\sigma} (f_2 + \chi(x_1, x_2) f_1)$ a.e. in $(0,1)$. This implies that $-\sigma(\eta u_x)_x = \frac{\lambda}{d} u + f_2 + \chi(x_1, x_2) f_1 \in L_{\frac{1}{\sigma}}^2(0,1)$, i.e. $Bu \in L_{\frac{1}{\sigma}}^2(0,1)$. Thus, $U \in D(\mathcal{A})$; therefore, $(u, v) \in D(\mathcal{A})$ is the unique solution of (2.10). Then, \mathcal{A} is an isomorphism and since $\rho(\mathcal{A})$ is an open set of \mathbb{C} (see [44, Theorem 6.7 (Chapter III)]), we easily get $R(\lambda I - \mathcal{A}) = \mathcal{H}$ for a sufficiently small $\lambda > 0$. This, together with the dissipativeness of \mathcal{A} , implies that $D(\mathcal{A})$ is dense in \mathcal{H} and \mathcal{A} is m-dissipative in \mathcal{H} (see [50, Theorems 4.5 and 4.6]). The proof is thus complete. \square

According to the Lumer-Phillips Theorem (see [50]), Proposition 2.4 implies that the operator \mathcal{A} generates a C_0 -semigroup of contractions $(\mathcal{T}(t))_{t \geq 0} = (e^{t\mathcal{A}})_{t \geq 0}$ in \mathcal{H} which gives the well-posedness of (2.8). Then, we have the following result

Theorem 1. For any $U_0 \in \mathcal{H}$, problem (2.8) admits a unique weak solution satisfying

$$U(t) \in C^0(\mathbb{R}^+; \mathcal{H}).$$

Moreover, if $U_0 \in D(\mathcal{A})$, (2.8) admits a unique strong solution U satisfying

$$U(t) \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A})).$$

Remark 2.5.

- (1) We note that (HP) is valid when $K_a + K_d < 2$. However, (2.7) holds if $K_a + 2K_d \leq 2$. Moreover, the latter condition imposes a constraint on d to be (WD).
- (2) We note that the validity of the Hardy Poincaré inequality in (HP), as well as the well-posedness and Lemma 3.1, persists even when considering the function a that is not necessarily (WD) or (SD), provided the following assumption holds: a satisfies $a(0) = 0$, $a > 0$ on $(0, 1]$, and there exists $\gamma > 0$ such that the function $x \rightarrow \frac{x^\gamma}{a(x)}$ is non-decreasing in a right neighborhood of $x = 0$.

3. TECHNICAL LEMMAS

This section is dedicated to introduce technical lemmas that will be utilized throughout the paper. It aims to prevent some repeated calculations that may arise in the sections on strong and exponential stability. Below are the essential lemmas presented:

Lemma 3.1. (See [31, Lemma 2.2], [33, Lemma 2.1] and [3, Lemma 2.5]) Assume $\frac{b}{a} \in L^1(0, 1)$.

- (1) If a is (WD) or (SD), then $\lim_{x \rightarrow 0} v(x)u_x(x) = 0$, for all $u \in H_{\frac{1}{\sigma}}^2(0, 1)$ and for all $v \in H_{\frac{1}{\sigma}, 0}^1(0, 1)$.
- (2) If a is (WD) or (SD), then $xu_x(\eta u_x)_x \in L^1(0, 1)$, for all $u \in H_{\frac{1}{\sigma}, 0}^2(0, 1)$.
- (3) If $K_a \leq 1$, then $\lim_{x \rightarrow 0} x|u_x|^2 = 0$, for all $u \in H_{\frac{1}{\sigma}, 0}^2(0, 1)$.
- (4) If $K_a > 1$ and $\frac{xb}{a} \in L^\infty(0, 1)$, then $\lim_{x \rightarrow 0} x|u_x|^2 = 0$, for all $u \in H_{\frac{1}{\sigma}, 0}^2(0, 1)$.
- (5) If a is (WD) or (SD), then $\lim_{x \rightarrow 0} \frac{x}{a}|u(x)|^2 = 0$, for all $u \in H_{\frac{1}{\sigma}, 0}^1(0, 1)$.
- (6) Assume Hypothesis 2.1. If $u \in H_{\frac{1}{\sigma}}^1(0, 1)$, then $\lim_{x \rightarrow 0} \frac{x}{\sigma d}|u(x)|^2 = 0$.

Proof. For the reader's convenience, we will prove only the last point; we refer to [31, Lemma 2.2], [33, Lemma 2.1] and [3, Lemma 2.5] for the other points.

Take $u \in H_{\frac{1}{\sigma}, 0}^1(0, 1)$. Then, $u(x) = \int_0^x u_z dz$ and applying the Cauchy-Schwarz inequality, we have

$$|u(x)|^2 \leq x \|u_x\|^2.$$

Thus, from the above equation we get

$$(3.1) \quad \frac{x}{ad} |u|^2 \leq \frac{x^2}{ad} \|u_x\|^2.$$

Combining this inequality with $\frac{1}{a(x)d(x)} \leq \frac{1}{x^{K_a+K_d}a(1)d(1)}$, we obtain

$$\frac{x}{ad} |u|^2 \leq \frac{x^2}{x^{K_a+K_d}a(1)d(1)} \|u_x\|^2 \rightarrow 0, \text{ as } x \rightarrow 0,$$

since $K_a + K_d < K_a + 2K_d \leq 2$ and the proof is complete. \square

Now, to ensure clarity and avoid repetition, we adhere to the following approach. Let $\{\beta_n, U^n = (u^n, v^n)^\top\}_{n \geq 1} \subset \mathbb{R}^* \times D(\mathcal{A})$ such that

$$(3.2) \quad (i\beta_n I - \mathcal{A})U^n = F_n := (f_n^1, f_n^2) \rightarrow 0 \quad \text{in } \mathcal{H},$$

as $n \rightarrow +\infty$. Detailing (3.2), we get

$$(3.3) \quad i\beta_n u^n - v^n = f_n^1 \quad \text{in } H_{\frac{1}{\sigma}, L}^1(0, 1)$$

and

$$(3.4) \quad i\beta_n v^n - \sigma(\eta u_x^n)_x - \frac{\lambda}{\sigma d} u^n + \chi_{(x_1, x_2)} i\beta_n u^n = f_n^2 \quad \text{in } L_{\frac{1}{\sigma}}^2(0, 1).$$

Inserting (3.4) into (3.3), we get

$$(3.5) \quad \beta_n^2 u^n + \sigma(x)(\eta u_x^n)_x + \frac{\lambda}{d(x)} u^n - \chi_{(x_1, x_2)} i \beta_n u^n = - (f_n^2 + i \beta_n f_n^1 + \chi_{(x_1, x_2)} f_n^1).$$

By equation (3.3) and (2.5), we get

$$(3.6) \quad \int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx \leq 2 \int_0^1 \frac{1}{\sigma} |v^n|^2 dx + 2\bar{C}_{HP} \int_0^1 \eta |(f_x^1)_n|^2 dx \leq c_1 [\|U^n\|_{\mathcal{H}} + \|F_n\|_{\mathcal{H}}],$$

where $c_1 = 2 \max\{1, \bar{C}_{HP}\}$. Thus, $\frac{1}{\sqrt{\sigma}} \beta_n u^n$ is uniformly bounded in $L^2(0, 1)$.

Let us define the following function

$$(3.7) \quad \varphi = \sum_{j=1}^2 (x - j + 1) \varphi_j, \quad \varphi_j := \begin{cases} 2 - j & \text{on } [0, x_1], \\ j - 1 & \text{on } [x_2, 1], \end{cases} \quad \text{where } \varphi_j \in C^1([0, 1]) \text{ such that } 0 \leq \varphi_j \leq 1.$$

Clearly, $\varphi' = \varphi_1' + \varphi_2' + \hat{\varphi}$, where $\hat{\varphi} = x\varphi_1' + (x-1)\varphi_2'$ with $\text{supp } \hat{\varphi} = [x_1, x_2]$.

Lemma 3.2. Let $\varphi \in C^1([0, 1])$ such that $\varphi(1) = 0$. Then, the solution $(u^n, v^n)^\top \in D(\mathcal{A})$ of (3.3)-(3.4) satisfies

$$(3.8) \quad \begin{aligned} & \int_0^1 \frac{\varphi'}{\sigma} |\beta_n u^n|^2 dx + \int_0^1 \eta \varphi' |u_x^n|^2 dx + \int_0^1 \frac{\lambda \varphi'}{\sigma d} |u^n|^2 dx = \int_0^1 \frac{\varphi}{\sigma} \left(\frac{a' - b}{a} \right) |\beta_n u^n|^2 dx + \int_0^1 \varphi \frac{b}{a} \eta |u_x^n|^2 dx \\ & - 2\Re \left(i \int_0^1 \frac{\chi_{(x_1, x_2)}}{\sigma} \beta_n u^n \varphi \overline{u_x^n} dx \right) + \int_0^1 \frac{\lambda \varphi}{\sigma d} \left(\frac{a' - b}{a} \right) |u^n|^2 dx + \int_0^1 \frac{\lambda \varphi d'}{\sigma d} |u^n|^2 dx \\ & + 2\Re \left(\int_0^1 (f_n^2 + i \beta_n f_n^1 + \chi_{(x_1, x_2)} f_n^1) \frac{\varphi}{\sigma} \overline{u_x^n} dx \right). \end{aligned}$$

Proof. First, multiplying (3.5) by $-\frac{2\varphi}{\sigma} \overline{u_x^n}$, integrating over $(0, 1)$ and taking the real part, we get

$$(3.9) \quad \begin{aligned} & \int_0^1 \left(\frac{\varphi}{\sigma} \right)' |\beta_n u^n|^2 dx + \lim_{x \rightarrow 0} \frac{\varphi}{\sigma} |\beta_n u^n|^2 - 2\Re \left(\int_0^1 (\eta u_x^n)_x \varphi \overline{u_x^n} dx \right) + \int_0^1 \left(\frac{\lambda \varphi}{\sigma d} \right)' |u^n|^2 dx \\ & + \lim_{x \rightarrow 0} \frac{\lambda \varphi}{\sigma d} |u^n|^2 + 2\Re \left(i \int_0^1 \frac{\chi_{(x_1, x_2)}}{\sigma} \beta_n u^n \varphi \overline{u_x^n} dx \right) = 2\Re \left(\int_0^1 (f_n^2 + i \beta_n f_n^1 + \chi_{(x_1, x_2)} f_n^1) \frac{\varphi}{\sigma} \overline{u_x^n} dx \right). \end{aligned}$$

For the first term in the above equation, we have that $\left(\frac{\varphi}{\sigma} \right)' = \frac{\varphi'}{\sigma} - \frac{\varphi}{\sigma} \left(\frac{a' - b}{a} \right)$, $\eta' = \frac{b}{a} \eta$ and $\left(\frac{\varphi}{\sigma d} \right)' = \frac{\varphi'}{\sigma d} - \frac{\varphi}{\sigma d} \left(\frac{a' - b}{a} \right) - \frac{\varphi d'}{\sigma d^2}$; thus we obtain

$$\begin{aligned} & \int_0^1 \left(\frac{\varphi}{\sigma} \right)' |\beta_n u^n|^2 dx = \int_0^1 \frac{\varphi'}{\sigma} |\beta_n u^n|^2 dx - \int_0^1 \frac{\varphi}{\sigma} \left(\frac{a' - b}{a} \right) |\beta_n u^n|^2 dx, \\ & - 2\Re \left(\int_0^1 (\eta u_x^n)_x \varphi \overline{u_x^n} dx \right) = 2\Re \left(\int_0^1 \eta u_x^n (\varphi \overline{u_x^n})_x dx \right) + 2 \lim_{x \rightarrow 0} \eta \varphi |u_x^n|^2 = 2 \int_0^1 \eta \varphi' |u_x^n|^2 dx, \\ & - \int_0^1 (\eta \varphi)' |u_x^n|^2 dx + \lim_{x \rightarrow 0} \eta \varphi |u_x^n|^2 = \int_0^1 \eta \varphi' |u_x^n|^2 dx - \int_0^1 \varphi \frac{b}{a} \eta |u_x^n|^2 dx + \lim_{x \rightarrow 0} \eta \varphi |u_x^n|^2 \end{aligned}$$

and

$$\int_0^1 \left(\frac{\lambda \varphi}{\sigma d} \right)' |u^n|^2 dx = \int_0^1 \frac{\lambda \varphi'}{\sigma d} |u^n|^2 dx - \int_0^1 \frac{\lambda \varphi}{\sigma d} \left(\frac{a' - b}{a} \right) |u^n|^2 dx - \int_0^1 \frac{\lambda \varphi d'}{\sigma d} |u^n|^2 dx.$$

By Lemma 3.1 and the definition of φ , we obtain

$$(3.10) \quad \lim_{x \rightarrow 0} \eta \varphi |u_x^n|^2 = 0, \quad \beta^2 \lim_{x \rightarrow 0} \frac{\varphi(x)}{\sigma(x)} |u^n(x)| = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{\lambda \varphi}{\sigma d} |u^n(x)|^2 = 0.$$

Substituting the above three equations into (3.9) and using the limits above, we get (3.8). \square

Lemma 3.3. The solution $(u^n, v^n)^\top \in D(\mathcal{A})$ of (3.3)-(3.4) satisfies

$$(3.11) \quad \begin{aligned} & \left(1 + \frac{K_a}{2}\right) \int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx + \left(1 - \frac{K_a}{2}\right) \int_0^1 \eta |u_x^n|^2 dx + \left(1 + \frac{K_a}{2}\right) \int_0^1 \frac{\lambda}{\sigma d} |u^n|^2 dx \\ & \leq (\mathcal{M}_0 + \mathcal{M}_1) \int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx + (\mathcal{N}_0 + \mathcal{N}_1) \int_0^1 \frac{b}{a} \eta |u_x^n|^2 dx \\ & \quad + (\mathcal{M}_0 + \mathcal{M}_1 + \mathcal{D}_0 + \mathcal{D}_1) \int_0^1 \frac{|\lambda|}{\sigma d} |u^n|^2 dx + |\mathbf{C}(n)| + |\mathbf{D}(n)|, \end{aligned}$$

where

$$(3.12) \quad \begin{aligned} \mathbf{C}(n) &= 2\Re \left(i \int_0^1 \frac{\chi_{(x_1, x_2)}}{\sigma} \beta_n u^n \overline{\varphi u_x^n} dx \right) + 2\Re \left(\int_0^1 (f_n^2 + i\beta_n f_n^1 + \chi_{(x_1, x_2)} f_n^1) \frac{\varphi}{\sigma} \overline{u_x^n} dx \right) \\ &+ \int_{x_1}^{x_2} \frac{\varphi_1 + \varphi_2}{\sigma} |\beta_n u^n|^2 dx + \int_{x_1}^{x_2} (\varphi_1 + \varphi_2) \eta |u_x^n|^2 dx + \int_{x_1}^{x_2} \frac{\lambda(\varphi_1 + \varphi_2)}{\sigma d} |u^n|^2 dx \\ &+ \int_0^{x_1} \frac{\hat{\varphi}}{\sigma} |\beta_n u^n|^2 dx + \int_0^{x_1} \hat{\varphi} \eta |u_x^n|^2 dx + \int_0^{x_1} \frac{\lambda \hat{\varphi}}{\sigma d} |u^n|^2 dx + \int_{x_1}^{x_2} \frac{1}{\sigma} |\beta_n u^n|^2 dx \\ &+ \int_{x_1}^{x_2} \eta |u_x^n|^2 dx + \int_{x_1}^{x_2} \frac{\lambda}{\sigma d} |u^n|^2 dx + \int_{x_1}^{x_2} \frac{\varphi}{\sigma} \left(\frac{a' - b}{a} \right) |\beta_n u^n|^2 dx + \int_{x_1}^{x_2} \varphi \frac{b}{a} \eta |u_x^n|^2 dx \\ &+ \int_{x_1}^{x_2} \frac{\lambda \varphi}{\sigma d} \left(\frac{a' - b}{a} \right) |u^n|^2 dx + \int_{x_1}^{x_2} \frac{\lambda \varphi d'}{\sigma d^2} |u^n|^2 dx \end{aligned}$$

and

$$(3.13) \quad \begin{aligned} \mathbf{D}(n) &= \frac{K_a}{2} \Re \left(\int_0^1 \frac{\chi_{(x_1, x_2)}}{\sigma} i \beta_n |u^n|^2 dx \right) - \frac{K_a}{2} \Re \left(\int_0^1 \frac{1}{\sigma} f_n^2 \overline{u^n} dx \right) - \frac{K_a}{2} \Re \left(i \int_0^1 \frac{1}{\sigma} f_n^1 \beta_n \overline{u^n} dx \right) \\ &- \frac{K_a}{2} \Re \left(\int_0^1 \frac{1}{\sigma} \chi_{(x_1, x_2)} f_n^1 \overline{u^n} dx \right). \end{aligned}$$

Proof. The proof will be divided into two steps.

Step 1: We have that (3.8) holds true. Thus, the terms in the right hand side of (3.8) become

$$\begin{aligned} & \int_0^1 \frac{\varphi'}{\sigma} |\beta_n u^n|^2 dx + \int_0^1 \eta \varphi' |u_x^n|^2 dx + \int_0^1 \frac{\lambda \varphi'}{\sigma d} |u^n|^2 dx = \int_0^{x_1} \frac{1}{\sigma} |\beta_n u^n|^2 dx + \int_{x_2}^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx + \int_{x_2}^1 \eta |u_x^n|^2 dx \\ &+ \int_0^{x_1} \eta |u_x^n|^2 dx + \int_0^{x_1} \frac{\lambda}{\sigma d} |u^n|^2 dx + \int_{x_2}^1 \frac{\lambda}{\sigma d} |u^n|^2 dx + \int_{x_1}^{x_2} \frac{\varphi_1 + \varphi_2}{\sigma} |\beta_n u^n|^2 dx + \int_{x_1}^{x_2} (\varphi_1 + \varphi_2) \eta |u_x^n|^2 dx \\ &+ \int_{x_1}^{x_2} \frac{\lambda(\varphi_1 + \varphi_2)}{\sigma d} |u^n|^2 dx + \int_0^{x_1} \frac{\hat{\varphi}}{\sigma} |\beta_n u^n|^2 dx + \int_0^{x_1} \hat{\varphi} \eta |u_x^n|^2 dx + \int_0^{x_1} \frac{\lambda \hat{\varphi}}{\sigma d} |u^n|^2 dx. \end{aligned}$$

Substituting the above equation into (3.8) and using the definition of φ , we get

$$(3.14) \quad \begin{aligned} & \int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx + \int_0^1 \eta |u_x^n|^2 dx + \int_0^1 \frac{\lambda}{\sigma d} |u^n|^2 dx \leq (\mathcal{M}_0 + \mathcal{M}_1) \int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx \\ & \quad + (\mathcal{N}_0 + \mathcal{N}_1) \int_0^1 \frac{b}{a} \eta |u_x^n|^2 dx + (\mathcal{M}_0 + \mathcal{M}_1 + \mathcal{D}_0 + \mathcal{D}_1) \int_0^1 \frac{|\lambda|}{\sigma d} |u^n|^2 dx + |\mathbf{C}(n)|. \end{aligned}$$

where $C(n)$ is defined as in (3.12). □

Step 2: Now, multiplying (3.5) by $\frac{K_a}{2\sigma} \overline{u^n}$, integrating over $(0, 1)$, taking the real part and using Lemma 3.1, we get

$$(3.15) \quad \frac{K_a}{2} \int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx - \frac{K_a}{2} \int_0^1 \eta |u_x^n|^2 dx + \frac{K_a}{2} \int_0^1 \frac{\lambda}{\sigma d} |u^n|^2 dx \leq |\mathbf{D}(n)|,$$

where $D(n)$ is defined as in (3.13). Adding (3.14) and (3.15), we obtain (3.11).

4. STRONG STABILITY

In this section we prove the strong stability of the system (1.1) when a is (WD) or (SD). First, for convenience, we introduce the following constants

$$(4.1) \quad \begin{cases} \mathcal{M}_0 := \left\| x \frac{a' - b}{a} \right\|_{L^\infty(0, x_1)}, & \mathcal{M}_1 := \left\| (x - 1) \frac{a' - b}{a} \right\|_{L^\infty(x_2, 1)}, \\ \mathcal{N}_0 := \left\| x \frac{b}{a} \right\|_{L^\infty(0, x_1)}, & \mathcal{N}_1 := \left\| (x - 1) \frac{b}{a} \right\|_{L^\infty(x_2, 1)}, \\ \mathcal{D}_0 := \left\| x \frac{d'}{d} \right\|_{L^\infty(0, x_1)} \quad \text{and} \quad \mathcal{D}_1 := \left\| (x - 1) \frac{d'}{d} \right\|_{L^\infty(x_2, 1)}. \end{cases}$$

Hypothesis 4.1. Assume Hypotheses 2.1 and 2.2. Moreover, suppose that

$$(4.2) \quad \begin{cases} \text{If } \lambda > 0 : & \begin{cases} N_0 + N_1 < 1 - \frac{K_a}{2} \\ \mathcal{M}_0 + \mathcal{M}_1 + \left\| \frac{d'}{d} \right\|_{L^\infty(x_2, 1)} + 2K_d < 1 + \frac{K_a}{2}. \end{cases} \\ \text{If } \lambda \leq 0 : & \begin{cases} \mathcal{M}_0 + \mathcal{M}_1 < 1 + \frac{K_a}{2} \\ N_0 + N_1 < 1 - \frac{K_a}{2} + \left(4 + \left\| \frac{d'}{d} \right\|_{L^\infty(x_2, 1)} \right) \lambda C_{HP}. \end{cases} \end{cases}$$

Remark 4.1. We remark that the choice of the functions a and b gives reliance on the length of the damped interval; i.e. the choice of x_1 and x_2 depends on the choice of the functions a and b . Also, we provide some examples that satisfy the conditions in Hypothesis 4.1, see Appendix A.

Theorem 4.2. Assume Hypothesis 4.1. Then, the C_0 -semigroup of contractions $(e^{t\mathcal{A}})_{t \geq 0}$ is strongly stable in \mathcal{H} , i.e., for all $U_0 \in \mathcal{H}$, the solution of (2.8) satisfies $\mathcal{E}(t) \xrightarrow[t \rightarrow \infty]{} 0$.

According to Theorem of Arendt-Batty [8], to prove Theorem 4.2, we need to prove that the operator \mathcal{A} has no pure imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable. The proof of Theorem 4.2 is based on the following proposition.

Proposition 4.3. Assume Hypothesis 4.1. Then,

$$(4.3) \quad i\mathbb{R} \subseteq \rho(\mathcal{A}).$$

We will prove Proposition 4.3 by a contradiction argument. Remark that, it has been proved in Proposition 2.4 that $0 \in \rho(\mathcal{A})$. Now, suppose that (4.3) is false, then there exists $\omega \in \mathbb{R}^*$ such that $i\omega \notin \rho(\mathcal{A})$. According to Remark A.3 in [1] and page 25 in [47], there exists $\{\beta_n, U^n = (u^n, v^n, y^n, \gamma^n)^\top\}_{n \geq 1} \subset \mathbb{R}^* \times D(\mathcal{A})$, such that

$$(4.4) \quad \beta_n \rightarrow \omega \quad \text{as} \quad n \rightarrow \infty \quad \text{and} \quad |\beta_n| < |\omega|,$$

$$(4.5) \quad \|U^n\|_{\mathcal{H}} = \|(u^n, v^n)^\top\|_{\mathcal{H}} = 1$$

$$(4.6) \quad \text{and} \quad (i\beta_n I - \mathcal{A})U^n = F_n := (f_n^1, f_n^2) \rightarrow 0 \quad \text{in} \quad \mathcal{H}, \quad \text{as} \quad n \rightarrow \infty.$$

Detailing as in (3.3)-(3.5), we have that (3.5) and (3.6) hold here. We will prove condition (4.3) finding a contradiction with (4.5) such as $\|U^n\|_{\mathcal{H}} \rightarrow 0$. The proof of this fact will rely on the following Lemmas.

Lemma 4.4. Assume Hypothesis 4.1. Then, the solution $(u^n, v^n)^\top \in D(\mathcal{A})$ of (3.3)-(3.4) satisfies

$$(4.7) \quad \int_{x_1}^{x_2} \frac{1}{\sigma} |v^n|^2 dx \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{and} \quad \int_{x_1}^{x_2} \frac{1}{\sigma} |\beta_n u^n|^2 dx \xrightarrow[n \rightarrow \infty]{} 0.$$

Proof. Taking the inner product of (4.6) with U^n in \mathcal{H} and using the fact that $\|F_n\|_{\mathcal{H}} \rightarrow 0$ and $\|U^n\|_{\mathcal{H}} = 1$, we obtain

$$\int_0^1 \frac{\chi_{(x_1, x_2)}}{\sigma} |v^n|^2 dx = -\Re(\langle \mathcal{A}U^n, U^n \rangle_{\mathcal{H}}) \leq \|F_n\|_{\mathcal{H}} \|U^n\|_{\mathcal{H}} \xrightarrow[n \rightarrow \infty]{} 0.$$

By (3.3), the above equation and by (2.5) we have $\int_{x_1}^{x_2} \frac{1}{\sigma} |f_n^1|^2 dx \leq \bar{C}_{HP} \int_0^1 \eta |(f_n^1)_x|^2 dx \leq \|F_n\|_{\mathcal{H}} \rightarrow 0$; this implies

$$\int_{x_1}^{x_2} \frac{1}{\sigma} |\beta_n u^n|^2 dx \leq 2 \int_{x_1}^{x_2} \frac{1}{\sigma} |v^n|^2 dx + 2 \int_{x_1}^{x_2} \frac{1}{\sigma} |f_n^1|^2 dx \xrightarrow{n \rightarrow \infty} 0$$

and the thesis follows. \square

Lemma 4.5. Assume Hypothesis 4.1. Then, the solution $(u^n, v^n)^\top \in D(\mathcal{A})$ of (3.3)-(3.4) satisfies

$$(4.8) \quad \int_{x_1}^{x_2} \eta |u_x^n|^2 dx \xrightarrow{n \rightarrow \infty} 0 \quad \text{and} \quad \int_{x_1}^{x_2} \eta |u_x^n|^2 dx - \int_{x_1}^{x_2} \frac{\lambda}{\sigma d} |u_x^n|^2 dx \xrightarrow{n \rightarrow \infty} 0.$$

Proof. We will divide the proof of this Lemma into three steps.

Step 1. Multiplying (3.5) by $\frac{1}{\sigma} \overline{u^n}$ and integrating over (x_1, x_2) , we obtain

$$(4.9) \quad \int_{x_1}^{x_2} \eta |u_x^n|^2 dx = \Re([\eta u_x^n \overline{u^n}]_{x_1}^{x_2}) + \mathbf{A}(n),$$

where

$$(4.10) \quad \mathbf{A}(n) = \int_{x_1}^{x_2} \frac{1}{\sigma} |\beta_n u^n|^2 dx + \int_{x_1}^{x_2} \frac{\lambda}{\sigma d} |u^n|^2 dx + \Re\left(\int_{x_1}^{x_2} (f_n^2 + i\beta_n f_n^1 + \chi_{(x_1, x_2)} f_n^1) \frac{1}{\sigma} \overline{u^n} dx\right).$$

Step 2. In order to get estimation on the interval (x_1, x_2) , we define a function g such that $g(x) = \frac{1}{x_2 - x_1} [2x - (x_1 + x_2)]$. Now, multiplying (3.5) by $\frac{2g}{\sigma} \overline{u_x^n}$, integrating over (x_1, x_2) and taking the real part, one has

$$(4.11) \quad \begin{aligned} & - \int_{x_1}^{x_2} \left(\frac{g}{\sigma}\right)' |\beta_n u^n|^2 dx + \left[\frac{g}{\sigma} |\beta_n u^n|^2\right]_{x_1}^{x_2} - 2\Re\left(\int_{x_1}^{x_2} \eta u_x^n (g \overline{u_x^n})_x dx\right) + 2\Re[\eta g |u_x^n|^2]_{x_1}^{x_2} \\ & + 2\lambda \Re\left(\int_{x_1}^{x_2} \frac{g}{\sigma d} u^n \overline{u_x^n} dx\right) - 2\Re\left(i \int_{x_1}^{x_2} \frac{\beta_n g}{\sigma} u^n \overline{u_x^n} dx\right) \\ & = -2\Re\left(\int_{x_1}^{x_2} \frac{1}{\sigma} (f_n^2 + i\beta_n f_n^1 + \chi_{(x_1, x_2)} f_n^1) g \overline{u_x^n} dx\right). \end{aligned}$$

Integrating by parts the third term in the above equation and using that fact that $\eta' = \frac{b}{a}\eta$, we get

$$\sum_{i=1}^2 \left(\frac{1}{\sigma(x_i)} |\beta_n u^n(x_i)|^2 + \eta(x_i) |u_x^n(x_i)|^2\right) = \int_{x_1}^{x_2} \eta \left(g' - \frac{b}{a} g\right) |u_x^n|^2 dx + \mathbf{B}(n),$$

where

$$\mathbf{B}(n) = \int_{x_1}^{x_2} \left(\frac{g}{\sigma}\right)' |\beta_n u^n|^2 dx + 2\Re\left(\int_{x_1}^{x_2} \frac{g}{\sigma} (i\beta_n - \frac{\lambda}{d}) u^n \overline{u_x^n} dx\right) - 2\Re\left(\int_{x_1}^{x_2} \frac{1}{\sigma} (f_n^2 + i\beta_n f_n^1 + \chi_{(x_1, x_2)} f_n^1) g \overline{u_x^n} dx\right).$$

Thus, we have

$$(4.12) \quad \sum_{i=1}^2 \left(\frac{1}{\sigma(x_i)} |\beta_n u^n(x_i)|^2 + \eta(x_i) |u_x^n(x_i)|^2\right) \leq \mathbf{T} \int_{x_1}^{x_2} \eta |u_x^n|^2 dx + |\mathbf{B}(n)|,$$

where $\mathbf{T} = \frac{2}{x_2 - x_1} + \left\|\frac{b}{a}\right\|_{L^\infty(x_1, x_2)}$.

Step 3. The goal of this step is to show that $|\mathbf{A}(n)| \xrightarrow{n \rightarrow \infty} 0$ and $|\mathbf{B}(n)| \xrightarrow{n \rightarrow \infty} 0$. Starting with $\mathbf{A}(n)$, using the second limit in (4.7) and the fact that $\beta_n \rightarrow \omega$, we get

$$\left|\int_{x_1}^{x_2} \frac{\lambda}{\sigma d} |u^n|^2 dx\right| \leq |\lambda| \max_{x \in [x_1, x_2]} \left(\frac{1}{d(x)}\right) \int_{x_1}^{x_2} \frac{1}{\sigma} |u^n|^2 dx \xrightarrow{n \rightarrow \infty} 0.$$

Using again the second limit in (4.7), (2.5) and the fact that $\|F_n\|_{\mathcal{H}} \rightarrow 0$, $n \rightarrow \infty$, we obtain

$$\begin{aligned} \left|\int_{x_1}^{x_2} \frac{1}{\sigma} f_n^2 \overline{u^n} dx\right| & \leq \left(\int_{x_1}^{x_2} \frac{1}{\sigma} |f_n^2|^2 dx\right)^{\frac{1}{2}} \left(\int_{x_1}^{x_2} \frac{1}{\sigma} |u^n|^2 dx\right)^{\frac{1}{2}} \xrightarrow{n \rightarrow \infty} 0, \\ \left|\int_{x_1}^{x_2} \frac{\chi_{(x_1, x_2)}}{\sigma} f_n^1 \overline{u^n} dx\right| & \leq \sqrt{\bar{C}_{HP}} \left(\int_0^1 \eta |(f_n^1)_x|^2 dx\right)^{\frac{1}{2}} \left(\int_{x_1}^{x_2} \frac{1}{\sigma} |u^n|^2 dx\right)^{\frac{1}{2}} \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

and

$$\left| \Re \left(i\beta_n \int_{x_1}^{x_2} \frac{1}{\sigma} f_n^1 \overline{u^n} dx \right) \right| \leq |\omega| \sqrt{\overline{C}_{HP}} \left(\int_0^1 \eta |(f_n^1)_x|^2 dx \right)^{\frac{1}{2}} \left(\int_{x_1}^{x_2} \frac{1}{\sigma} |u^n|^2 dx \right)^{\frac{1}{2}} \xrightarrow{n \rightarrow \infty} 0.$$

Thus, using the above limits and the second limit in (4.7), we obtain $|\mathbf{A}(n)| \xrightarrow{n \rightarrow \infty} 0$.

Now, for $\mathbf{B}(n)$, we will estimate each term in $\mathbf{B}(n)$ separately. Recalling that $\sigma = \frac{a}{\eta}$ and that $\eta' = \frac{b}{a}\eta$, we have

$$\sigma' = \frac{a'\eta - a\eta'}{\eta^2} = \frac{a' - b}{\eta} \quad \text{and} \quad \left(\frac{g}{\sigma} \right)' = \frac{g'}{\sigma} - g \frac{\sigma'}{\sigma^2} = \frac{g'}{\sigma} - \frac{g}{\sigma} \left(\frac{a' - b}{a} \right), \quad \text{where} \quad g' = \frac{2}{x_2 - x_1}.$$

Using the above equation, the second limit in (4.7) and the fact that $\beta_n \rightarrow \omega$, we get

$$\int_{x_1}^{x_2} \left(\frac{g}{\sigma} \right)' |\beta_n u^n|^2 dx \leq \left(\frac{2}{x_2 - x_1} + \left\| \frac{a' - b}{a} \right\|_{L^\infty(x_1, x_2)} \right) \int_{x_1}^{x_2} \frac{1}{\sigma} |\beta_n u^n|^2 dx \xrightarrow{n \rightarrow \infty} 0.$$

Using again the second limit in (4.7), the Cauchy-Schwarz inequality and the fact that $\|U^n\| = 1$, $\|F_n\|_{\mathcal{H}} \rightarrow 0$, as $n \rightarrow 0$, we obtain

$$\begin{aligned} \left| \Re \left(\int_{x_1}^{x_2} \frac{g}{\sigma} (i\beta_n - \frac{\lambda}{d}) u^n \overline{u_x^n} dx \right) \right| &\leq \left(|\omega| + |\lambda| \max_{[x_1, x_2]} \frac{1}{d} \right) \max_{[x_1, x_2]} \frac{1}{\sqrt{a(x)}} \left(\int_{x_1}^{x_2} \frac{1}{\sigma} |u^n|^2 dx \right)^{\frac{1}{2}} \left(\int_{x_1}^{x_2} \eta |u_x^n|^2 dx \right)^{\frac{1}{2}} \rightarrow 0, \\ \left| \Re \left(\int_{x_1}^{x_2} \frac{1}{\sigma} f_n^2 g \overline{u_x^n} dx \right) \right| &\leq \max_{x \in [x_1, x_2]} \left(\frac{1}{\sqrt{a(x)}} \right) \left(\int_{x_1}^{x_2} \frac{1}{\sigma} |f_n^2|^2 dx \right)^{\frac{1}{2}} \left(\int_{x_1}^{x_2} \eta |u_x^n|^2 dx \right)^{\frac{1}{2}} \rightarrow 0 \end{aligned}$$

and

$$\left| \Re \left(\int_{x_1}^{x_2} \frac{1}{\sigma} \beta_n f_n^1 g \overline{u_x^n} dx \right) \right| \leq |\omega| \sqrt{\tilde{C}_{HP}} \max_{x \in [x_1, x_2]} \left(\frac{1}{\sqrt{a(x)}} \right) \left(\int_0^1 \eta |(f_n^1)_x|^2 dx \right)^{\frac{1}{2}} \left(\int_{x_1}^{x_2} \eta |u_x^n|^2 dx \right)^{\frac{1}{2}} \rightarrow 0,$$

as $n \rightarrow \infty$ and, similarly, we can get

$$\left| \Re \left(\int_{x_1}^{x_2} \chi_{(x_1, x_2)} \frac{1}{\sigma} f_n^1 g \overline{u_x^n} dx \right) \right| \xrightarrow{n \rightarrow \infty} 0,$$

as $n \rightarrow \infty$. Thus, using the above limits we conclude that $\mathbf{B}(n) \xrightarrow{n \rightarrow \infty} 0$.

Step 4. The aim of this step is to prove (4.8). From Step 1 and applying Young's inequality, we get

$$(4.13) \quad \int_{x_1}^{x_2} \eta |u_x^n|^2 dx \leq \sum_{i=1}^2 \left(\frac{\eta(x_i)}{4\mathbf{T}} |u_x^n(x_i)|^2 + \mathbf{T}\eta(x_i) |u^n(x_i)|^2 \right) + |\mathbf{A}(n)|.$$

From (4.12), we have that

$$\eta(x_1) |u_x^n(x_1)|^2 \leq \mathbf{T} \int_{x_1}^{x_2} \eta |u_x^n|^2 dx + |\mathbf{B}(n)| \quad \text{and} \quad \eta(x_2) |u_x^n(x_2)|^2 \leq \mathbf{T} \int_{x_1}^{x_2} \eta |u_x^n|^2 dx + |\mathbf{B}(n)|.$$

Inserting the above inequalities into (4.13), we obtain

$$(4.14) \quad \frac{1}{2} \int_{x_1}^{x_2} \eta |u_x^n|^2 dx \leq \mathbf{T}\eta(x_2) |u^n(x_2)|^2 + \mathbf{T}\eta(x_1) |u^n(x_1)|^2 + \frac{1}{2\mathbf{T}} |\mathbf{B}(n)| + |\mathbf{A}(n)|.$$

Applying the Gagliardo-Nirenberg inequality, one has

$$\begin{aligned} |u^n(\xi)| &\leq \|u^n\|_{L^2(x_1, x_2)}^{\frac{1}{2}} \|u_x^n\|_{L^2(x_1, x_2)}^{\frac{1}{2}} + \|u^n\|_{L^2(x_1, x_2)} \\ &\leq \left(\max_{x \in [x_1, x_2]} \sigma \int_{x_1}^{x_2} \frac{1}{\sigma} |u^n|^2 dx \right)^{\frac{1}{2}} \left(\max_{x \in [x_1, x_2]} \frac{1}{\eta} \int_{x_1}^{x_2} \eta |u_x^n|^2 dx \right)^{\frac{1}{2}} + \max_{x \in [x_1, x_2]} \sigma \int_{x_1}^{x_2} \frac{1}{\sigma} |u^n|^2 dx, \end{aligned}$$

where $\xi = x_1, x_2$. From the above inequality, the second limit in (4.7) and using the fact that $\|U^n\|_{\mathcal{H}} = 1$, we deduce that $|u^n(\xi)| \xrightarrow{n \rightarrow \infty} 0$, where $\xi = x_1, x_2$. Therefore, using this limit and passing to the limit in (4.14), we obtain the first limit in (4.8). Using the equivalence between the norms given in Proposition 2.3, we deduce that the second limit in (4.8) holds. \square

Lemma 4.6. Assume Hypothesis 4.1. Then, the solution $(u^n, v^n)^\top \in D(\mathcal{A})$ of (3.3)-(3.4) satisfies

$$(4.15) \quad \int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx \xrightarrow{n \rightarrow \infty} 0 \quad \text{and} \quad \int_0^1 \eta |u_x^n|^2 dx - \int_0^1 \frac{\lambda}{\sigma d} |u^n|^2 dx \xrightarrow{n \rightarrow \infty} 0.$$

Proof. From Lemma 3.3, we have that (3.11) holds. Firstly, we need to prove that

$$(4.16) \quad |\mathbf{C}(n)| \xrightarrow{n \rightarrow \infty} 0 \quad \text{and} \quad |\mathbf{D}(n)| \xrightarrow{n \rightarrow \infty} 0.$$

To this aim, firstly we will estimate each term in $\mathbf{C}(n)$ and $\mathbf{D}(n)$, where $\mathbf{C}(n)$ and $\mathbf{D}(n)$ are defined in (3.12) and (3.13), respectively. Starting with $\mathbf{C}(n)$, using the Cauchy Schwarz inequality and the Hardy Poincaré inequality in (2.5), the fact that $\beta_n \rightarrow \omega$, $\|U^n\|_{\mathcal{H}} = 1$ and the second limit in (4.7), we obtain

$$(4.17) \quad \left| \Re \left(i \int_0^1 \frac{\chi_{(x_1, x_2)}}{\sigma} \beta_n u^n \overline{\varphi u_x^n} dx \right) \right| \leq |\omega| \max_{x \in [x_1, x_2]} \frac{1}{\sqrt{a}} \left(\int_{x_1}^{x_2} \frac{1}{\sigma} |u^n|^2 dx \right)^{\frac{1}{2}} \left(\int_{x_1}^{x_2} \eta |u_x^n|^2 dx \right)^{\frac{1}{2}} \xrightarrow{n \rightarrow \infty} 0$$

and

$$(4.18) \quad \left| \Re \left(\int_0^1 \chi_{(x_1, x_2)} f_n^1 \frac{\varphi}{\sigma} \overline{u_x^n} dx \right) \right| \leq \max_{x \in [x_1, x_2]} \frac{1}{\sqrt{a}} \sqrt{\tilde{C}_{HP}} \left(\int_0^1 \eta |(f_n^1)_x|^2 dx \right)^{\frac{1}{2}} \left(\int_{x_1}^{x_2} \eta |u_x^n|^2 dx \right)^{\frac{1}{2}} \xrightarrow{n \rightarrow \infty} 0.$$

Now, consider the term $2\Re \left(\int_0^1 f_n^2 \frac{\varphi}{\sigma} \overline{u_x^n} dx \right)$. From the definition of φ , we have

$$(4.19) \quad 2\Re \left(\int_0^1 f_n^2 \frac{\varphi}{\sigma} \overline{u_x^n} dx \right) = 2\Re \left(\int_0^1 \frac{x\varphi_1}{\sigma} f_n^2 \overline{u_x^n} dx \right) + 2\Re \left(\int_0^1 \frac{(x-1)\varphi_2}{\sigma} f_n^2 \overline{u_x^n} dx \right).$$

For the first term in the right hand side of the above equation, using the Cauchy-Schwarz inequality, the fact that a doesn't vanish on the interval (x_1, x_2) , $\frac{x}{\sqrt{a}}$ is non-decreasing in the neighborhood of 0 if a is (WD) or (SD) and the fact that $K_a < 2$, we obtain

$$\left| \Re \left(\int_0^1 \frac{x\varphi_1}{\sigma} f_n^2 \overline{u_x^n} dx \right) \right| = \left| \Re \left(\int_0^{x_2} \frac{x\varphi_1}{\sigma} f_n^2 \overline{u_x^n} dx \right) \right| \leq \frac{x_2}{\sqrt{a(x_2)}} \left(\int_0^{x_2} \frac{1}{\sigma} |f_n^2|^2 dx \right)^{\frac{1}{2}} \left(\int_0^{x_2} \eta |u_x^n|^2 dx \right)^{\frac{1}{2}} \xrightarrow{n \rightarrow \infty} 0.$$

Now, for the second term in (4.19) we can use a similar argument as above, obtaining

$$\left| 2\Re \left(\int_0^1 \frac{(x-1)\varphi_2}{\sigma} f_n^2 \overline{u_x^n} dx \right) \right| \leq 2(1-x_1) \left(\int_{x_1}^1 \frac{1}{\sigma} |f_n^2|^2 dx \right)^{\frac{1}{2}} \left(\int_{x_1}^1 \eta |u_x^n|^2 dx \right)^{\frac{1}{2}} \xrightarrow{n \rightarrow \infty} 0.$$

Thus, using the above two limits in (4.19), we get

$$(4.20) \quad \left| 2\Re \left(\int_0^1 f_n^2 \frac{\varphi}{\sigma} \overline{u_x^n} dx \right) \right| \xrightarrow{n \rightarrow \infty} 0.$$

We can deduce in a similar way and using the fact that $\beta_n \xrightarrow{n \rightarrow \infty} \omega$ that

$$(4.21) \quad \left| 2\Re \left(\int_0^1 i \beta_n f_n^1 \frac{\varphi}{\sigma} \overline{u_x^n} dx \right) \right| \xrightarrow{n \rightarrow \infty} 0.$$

Using the second limit in (4.7) and (4.8), the fact that $\beta_n \rightarrow \omega$, $\text{supp } \hat{\varphi} = [x_1, x_2]$, we get

$$(4.22) \quad \begin{aligned} & \int_{x_1}^{x_2} \frac{\varphi_1 + \varphi_2}{\sigma} |\beta_n u^n|^2 dx + \int_{x_1}^{x_2} (\varphi_1 + \varphi_2) \eta |u_x^n|^2 dx + \int_{x_1}^{x_2} \frac{\lambda(\varphi_1 + \varphi_2)}{\sigma d} |u^n|^2 dx \\ & + \int_0^1 \frac{\hat{\varphi}}{\sigma} |\beta_n u^n|^2 dx + \int_0^1 \hat{\varphi} \eta |u_x^n|^2 dx + \int_0^1 \frac{\lambda \hat{\varphi}}{\sigma d} |u^n|^2 dx + \int_{x_1}^{x_2} \frac{1}{\sigma} |\beta_n u^n|^2 dx \\ & + \int_{x_1}^{x_2} \eta |u_x^n|^2 dx + \int_{x_1}^{x_2} \frac{\lambda}{\sigma d} |u^n|^2 dx \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

For the last four terms of $\mathbf{C}(n)$, using again the second limit in (4.7), (4.8) and the fact that $\beta_n \xrightarrow{n \rightarrow \infty} \omega$, we have

$$(4.23) \quad \int_{x_1}^{x_2} \frac{\varphi}{\sigma} \left(\frac{a' - b}{a} \right) |\beta_n u^n|^2 dx \leq \left\| \left(\frac{a' - b}{a} \right) \right\|_{L^\infty(x_1, x_2)} \int_{x_1}^{x_2} \frac{1}{\sigma} |\beta_n u^n|^2 dx \xrightarrow{n \rightarrow \infty} 0$$

and similarly

$$(4.24) \quad \int_{x_1}^{x_2} \varphi \frac{b}{a} \eta |u_x^n|^2 dx + \int_{x_1}^{x_2} \frac{\lambda \varphi}{\sigma d} \left(\frac{a' - b}{a} \right) |u^n|^2 dx + \int_{x_1}^{x_2} \frac{\lambda \varphi d'}{\sigma d^2} |u^n|^2 dx \xrightarrow{n \rightarrow \infty} 0.$$

Thus, from (4.17), (4.18), (4.20)-(4.24), we obtain that $|\mathbf{C}(n)| \rightarrow 0$, as $n \rightarrow \infty$. Now, for $\mathbf{D}(n)$, using the second limit in (4.7) and the Hardy Poincaré inequality in (2.5), we can deduce that

$$(4.25) \quad \frac{K_a}{2} \mathfrak{R} \left(\int_0^1 \frac{\chi_{(x_1, x_2)}}{\sigma} i \beta_n |u^n|^2 dx \right) \xrightarrow{n \rightarrow \infty} 0 \quad \text{and} \quad \frac{K_a}{2} \mathfrak{R} \left(\int_0^1 \frac{1}{\sigma} \chi_{(x_1, x_2)} f_n^1 \overline{u^n} dx \right) \xrightarrow{n \rightarrow \infty} 0.$$

Using (3.6), we obtain

$$(4.26) \quad \left| \frac{K_a}{2} \mathfrak{R} \left(\int_0^1 \frac{1}{\sigma} f_n^2 \overline{u^n} dx \right) \right| \leq \frac{K_a}{2} \left(\int_0^1 \frac{1}{\sigma} |f_n^2|^2 dx \right)^{\frac{1}{2}} \left(\int_0^1 \frac{1}{\sigma} |u^n|^2 dx \right)^{\frac{1}{2}} \xrightarrow{n \rightarrow \infty} 0$$

and

$$(4.27) \quad \left| \frac{K_a}{2} \mathfrak{R} \left(i \int_0^1 \frac{1}{\sigma} f_n^1 \beta_n \overline{u^n} dx \right) \right| \leq \frac{K_a \sqrt{\tilde{C}_{HP}}}{2} \left(\int_0^1 \eta |(f_n^1)_x|^2 dx \right)^{\frac{1}{2}} \left(\int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx \right)^{\frac{1}{2}} \xrightarrow{n \rightarrow \infty} 0.$$

Thus, from (4.25)-(4.27) we get that $|\mathbf{D}(n)| \xrightarrow{n \rightarrow \infty} 0$, as $n \rightarrow \infty$.

Secondly, in order to prove (4.15), we will distinguish two cases according to the values of λ .

Case 1. If $\lambda > 0$:

Inequality (3.11) yields that

$$\begin{aligned} & \left(1 + \frac{K_a}{2} - \mathcal{M}_0 - \mathcal{M}_1 \right) \int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx + \left(1 - \frac{K_a}{2} - \mathcal{N}_0 - \mathcal{N}_1 \right) \int_0^1 \eta |u_x^n|^2 dx + \\ & \left(1 + \frac{K_a}{2} - \mathcal{M}_0 - \mathcal{M}_1 - \mathcal{D}_0 - \mathcal{D}_1 \right) \int_0^1 \frac{\lambda}{\sigma d} |u^n|^2 dx \leq |\mathbf{C}(n)| + |\mathbf{D}(n)|. \end{aligned}$$

By Hypothesis 4.1, we obtain

$$1 + \frac{K_a}{2} - \mathcal{M}_0 - \mathcal{M}_1 - \mathcal{D}_0 - \mathcal{D}_1 \geq 1 + \frac{K_a}{2} - \mathcal{M}_0 - \mathcal{M}_1 - 2K_d - \left\| \frac{d'}{d} \right\|_{L^\infty(x_2, 1)} > 0.$$

Thus, from the above inequality, (4.16) and by Hypothesis 4.1, we conclude that

$$\int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx \xrightarrow{n \rightarrow \infty} 0, \quad \int_0^1 \eta |u_x^n|^2 dx \xrightarrow{n \rightarrow \infty} 0 \quad \text{and} \quad \int_0^1 \frac{\lambda}{\sigma d} |u^n|^2 dx \xrightarrow{n \rightarrow \infty} 0.$$

Case 2. If $\lambda \leq 0$:

From (3.11) we have that

$$(4.28) \quad \begin{aligned} & \left(1 + \frac{K_a}{2} - \mathcal{M}_0 - \mathcal{M}_1 \right) \int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx + \left(1 - \frac{K_a}{2} - \mathcal{N}_0 - \mathcal{N}_1 \right) \int_0^1 \eta |u_x^n|^2 dx + \\ & \left(1 + \frac{K_a}{2} + \mathcal{M}_0 + \mathcal{M}_1 + \mathcal{D}_0 + \mathcal{D}_1 \right) \int_0^1 \frac{\lambda}{\sigma d} |u^n|^2 dx \leq |\mathbf{C}(n)| + |\mathbf{D}(n)|. \end{aligned}$$

By the Hardy-Poincaré inequality in Proposition 2.2 and the fact that $\lambda < 0$, we get

$$\int_0^1 \frac{\lambda}{\sigma d} |u^n|^2 dx \geq \lambda C_{HP} \int_0^1 \eta |u_x^n|^2 dx.$$

Using the above inequality, the fact that $K_a + 2K_d \leq 2$ and Hypothesis 4.1, we get

$$\begin{aligned} & \left(1 + \frac{K_a}{2} + \sum_{i=0}^1 (\mathcal{M}_i + \mathcal{D}_i) \right) \int_0^1 \frac{\lambda}{\sigma d} |u^n|^2 dx \geq \left(1 + \frac{K_a}{2} + \mathcal{M}_0 + \mathcal{M}_1 + 2K_d + \left\| \frac{d'}{d} \right\|_{L^\infty(x_2, 1)} \right) \int_0^1 \frac{\lambda}{\sigma d} |u^n|^2 dx \\ & \geq \left(4 + \left\| \frac{d'}{d} \right\|_{L^\infty(x_2, 1)} \right) \lambda C_{HP} \int_0^1 \eta |u_x^n|^2 dx. \end{aligned}$$

Thus, combining the above inequality with (4.28), we have

$$\begin{aligned} & \left(1 + \frac{K_a}{2} - \mathcal{M}_0 - \mathcal{M}_1\right) \int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx + \left(1 - \frac{K_a}{2} - \mathcal{N}_0 - \mathcal{N}_1 + \left(4 + \left\| \frac{d'}{d} \right\|_{L^\infty(x_2,1)}\right) \lambda C_{HP}\right) \int_0^1 \eta |u_x^n|^2 dx \\ & \leq |\mathbf{C}(n)| + |\mathbf{D}(n)|. \end{aligned}$$

Therefore, by Hypothesis 4.1 in the case $\lambda \leq 0$ and the limits in (4.16), we can deduce that

$$\int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx \xrightarrow{n \rightarrow \infty} 0 \quad \text{and} \quad \int_0^1 \eta |u_x^n|^2 dx \xrightarrow{n \rightarrow \infty} 0.$$

From the equivalence between the norms in Proposition 4.3, we obtain that the second limit in (4.15) holds. \square

Thus, we obtain that $\|U^n\|_{\mathcal{H}} \rightarrow 0$ as $n \rightarrow \infty$, which contradicts that $\|U^n\|_{\mathcal{H}} = 1$ in (4.5). Thus, (4.3) holds true and the proof of Proposition 4.3 is complete.

Proof.[Proof of Theorem 4.2] By Proposition 4.3, we have $i\mathbb{R} \subseteq \rho(\mathcal{A})$ and consequently $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. Therefore, according to Arendt-Batty's Theorem, we get that the C_0 -semigroup $(e^{t\mathcal{A}})_{t \geq 0}$ of contractions is strongly stable and the proof is complete. \square

5. EXPONENTIAL STABILITY

In this section we will study the exponential stability for (1.1). The following theorem gives the main result of this section.

Theorem 5.1. Assume Hypothesis 4.1. Then, the C_0 -semigroup of contractions $(e^{t\mathcal{A}})_{t \geq 0}$ is exponentially stable, i.e. there exist constants $C \geq 1$ and $\tau > 0$ independent of U_0 such that

$$\|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} \leq Ce^{-\tau t}\|U_0\|_{\mathcal{H}}, \quad t \geq 0.$$

According to Huang [42] and Prüss [51], we have to check the following conditions:

$$(E) \quad i\mathbb{R} \subseteq \rho(\mathcal{A}) \quad \text{and} \quad \sup_{\lambda \in \mathbb{R}} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} = O(1).$$

The first condition in (E) is already proved in Proposition 2.2. We will prove the second condition in (E) using a contradiction argument. To this aim, suppose it is false, then there exists $\{\beta_n, U^n = (u^n, v^n)^\top\}_{n \geq 1} \subset \mathbb{R}^* \times D(\mathcal{A})$, such that $\beta_n \rightarrow +\infty$, as $n \rightarrow \infty$, $\|U^n\|_{\mathcal{H}} = \|(u^n, v^n)^\top\|_{\mathcal{H}} = 1$ and

$$(5.1) \quad (i\beta_n I - \mathcal{A})U^n = F_n := (f_n^1, f_n^2) \rightarrow 0 \quad \text{in } \mathcal{H}, \quad \text{as } n \rightarrow +\infty.$$

Detailing the above equation as in (3.3)-(3.5), we observe that (3.5) and (3.6) still hold here. Now, we proceed to verify the second condition in (E) by seeking a contradiction with $\|U^n\| = 1$ such as $\|U^n\| = o(1)$. To ensure clarity, we divide the proof into several lemmas.

Lemma 5.2. Assume Hypothesis 4.1. Then, the solution $(u^n, v^n)^\top \in D(\mathcal{A})$ of (3.3)-(3.4) satisfies

$$(5.2) \quad \int_{x_1}^{x_2} \frac{1}{\sigma} |v^n|^2 dx = o(1) \quad \text{and} \quad \int_{x_1}^{x_2} \frac{1}{\sigma} |\beta_n u^n|^2 dx = o(1).$$

Proof. Taking the inner product of (5.1) with U^n in \mathcal{H} , taking the real part and using the fact that $\|F_n\|_{\mathcal{H}} = o(1)$ and $\|U^n\|_{\mathcal{H}} = 1$, we obtain

$$\int_0^1 \frac{\chi(x_1, x_2)}{\sigma} |v^n|^2 dx = -\Re(\langle \mathcal{A}U^n, U^n \rangle_{\mathcal{H}}) \leq \|F_n\|_{\mathcal{H}} \|U^n\|_{\mathcal{H}} = o(1).$$

By (3.3), using the above equation and the fact that $\int_{x_1}^{x_2} \frac{1}{\sigma} |f_n^1|^2 dx \leq \bar{C}_{HP} \int_0^1 \eta |(f_n^1)_x|^2 dx \leq \|F_n\|_{\mathcal{H}} = o(1)$, we get

$$\int_{x_1}^{x_2} \frac{1}{\sigma} |\beta_n u^n|^2 dx \leq 2 \int_{x_1}^{x_2} \frac{1}{\sigma} |v^n|^2 dx + 2 \int_{x_1}^{x_2} \frac{1}{\sigma} |f_n^1|^2 dx = o(1).$$

\square

Lemma 5.3. Assume Hypothesis 4.1. Then, the solution $(u^n, v^n)^\top \in D(\mathcal{A})$ of (3.3)-(3.4) satisfies

$$(5.3) \quad \int_{x_1}^{x_2} \eta |u_x^n|^2 dx = o(1) \quad \text{and} \quad \int_{x_1}^{x_2} \eta |u_x^n|^2 dx - \int_{x_1}^{x_2} \frac{\lambda}{\sigma d} |u_x^n|^2 dx = o(1).$$

Proof. For the proof of this Lemma we will use similar techniques used in Lemma 4.5. To this aim, we will divide the proof into several steps.

Step 1. Proceeding as in Step 1 in Lemma 4.5, we can get that

$$(5.4) \quad \int_{x_1}^{x_2} \eta |u_x^n|^2 dx = \Re([\eta u_x^n \overline{u_x^n}]_{x_1}^{x_2}) + \mathbf{A}(n),$$

where $\mathbf{A}(n)$ is defined in (4.10).

Step 2. The aim of this step is to show that the solution $(u^n, v^n)^\top \in D(\mathcal{A})$ of (3.3)-(3.4) satisfies

$$(5.5) \quad \sum_{i=1}^2 \left(\frac{1}{2\sigma(x_i)} |\beta_n u^n(x_i)|^2 + \eta(x_i) |u_x(x_i)|^2 \right) \leq \mathbf{T} \int_{x_1}^{x_2} \eta |u_x^n|^2 dx + \sum_{i=1}^2 \frac{2}{\sigma(x_i)} |f_n^1(x_i)|^2 + |\mathbf{G}(n)|,$$

where \mathbf{T} is the same as in (4.12) and $\mathbf{G}(n)$ is to be determined such that $|\mathbf{G}(n)| = o(1)$.

Multiplying (3.5) by $\frac{2g}{\sigma} \overline{u_x^n}$, where g is the same function defined in Lemma 4.5, integrating over (x_1, x_2) , and taking the real part we obtain

$$(5.6) \quad \begin{aligned} & \frac{1}{\sigma(x_2)} |\beta_n u^n(x_2)|^2 + \frac{1}{\sigma(x_1)} |\beta_n u^n(x_1)|^2 + \eta(x_2) |u_x^n(x_2)|^2 + \eta(x_1) |u_x^n(x_1)|^2 = \int_{x_1}^{x_2} \eta \left(g' - \frac{b}{a} g \right) |u_x^n|^2 dx \\ & + \int_{x_1}^{x_2} \left(\frac{g}{\sigma} \right)' |\beta_n u^n|^2 dx - 2\lambda \Re \left(\int_{x_1}^{x_2} \frac{g}{\sigma d} u^n \overline{u_x^n} dx \right) + 2\Re \left(i \int_{x_1}^{x_2} \frac{\beta_n g}{\sigma} u^n \overline{u_x^n} dx \right) - 2\Re \left(\int_{x_1}^{x_2} \frac{1}{\sigma} f_n^2 g \overline{u_x^n} dx \right) \\ & + 2\Re \left(i \int_{x_1}^{x_2} \left(\frac{g}{\sigma} \right)' f_n^1 \beta_n \overline{u_x^n} dx \right) + 2\Re \left(i \int_{x_1}^{x_2} \frac{g}{\sigma} (f_n^1)_x \beta_n \overline{u_x^n} dx \right) - 2\Re \left(\int_{x_1}^{x_2} \frac{1}{\sigma} \chi_{(x_1, x_2)} f_n^1 g \overline{u_x^n} dx \right) \\ & - 2\Re \left(i f_n^1(x_2) \frac{1}{\sigma(x_2)} \beta_n u^n(x_2) \right) + 2\Re \left(i f_n^1(x_1) \frac{1}{\sigma(x_1)} \beta_n u^n(x_1) \right). \end{aligned}$$

For the first and the last two terms on the right hand side of the above equation, we have

$$\left| 2\Re \left(i f_n^1(x_i) \frac{1}{\sigma(x_i)} \beta_n u^n(x_i) \right) \right| \leq \frac{1}{2\sigma(x_i)} |\beta_n u^n(x_i)|^2 + \frac{2}{\sigma(x_i)} |f_n^1(x_i)|^2, \quad i = 1, 2,$$

and

$$\left| \int_{x_1}^{x_2} \eta \left(g' - \frac{b}{a} g \right) |u_x^n|^2 dx \right| \leq \mathbf{T} \int_{x_1}^{x_2} \eta |u_x^n|^2 dx.$$

Using the above estimations in (5.6), we get (5.5) with

$$\begin{aligned} \mathbf{G}(n) &= \int_{x_1}^{x_2} \left(\frac{g}{\sigma} \right)' |\beta_n u^n|^2 dx - 2\lambda \Re \left(\int_{x_1}^{x_2} \frac{g}{\sigma d} u^n \overline{u_x^n} dx \right) + 2\Re \left(i \int_{x_1}^{x_2} \frac{\beta_n g}{\sigma} u^n \overline{u_x^n} dx \right) - 2\Re \left(\int_{x_1}^{x_2} \frac{1}{\sigma} f_n^2 g \overline{u_x^n} dx \right) \\ &+ 2\Re \left(i \int_{x_1}^{x_2} \left(\frac{g}{\sigma} \right)' f_n^1 \beta_n \overline{u_x^n} dx \right) + 2\Re \left(i \int_{x_1}^{x_2} \frac{g}{\sigma} (f_n^1)_x \beta_n \overline{u_x^n} dx \right) - 2\Re \left(\int_{x_1}^{x_2} \frac{1}{\sigma} \chi_{(x_1, x_2)} f_n^1 g \overline{u_x^n} dx \right). \end{aligned}$$

Step 3. The aim of this step is to show that $|\mathbf{A}(n)| = o(1)$ and $\sum_{i=1}^2 \frac{2}{\sigma(x_i)} |f_n^1(x_i)|^2 + |\mathbf{G}(n)| = o(1)$. For

$\mathbf{A}(n)$, following similar arguments as in Step 2 of Lemma 4.5, we have that $|\mathbf{A}(n)| = o(1)$. Moreover, since $\|F_n\|_{\mathcal{H}} = o(1)$, we have

$$|f_n^1(x_i)| \leq \sqrt{1-x_i} \left(\int_{x_i}^1 |(f_n^1)_x|^2 dx \right)^{\frac{1}{2}} \leq \sqrt{1-x_i} \max_{x \in [x_1, 1]} \frac{1}{\sqrt{\eta}} \left(\int_{x_i}^1 \eta |(f_n^1)_x|^2 dx \right)^{\frac{1}{2}} = o(1), \quad i = 1, 2.$$

Thus, we obtain

$$(5.7) \quad \sum_{i=1}^2 \frac{2}{\sigma(x_i)} |f_n^1(x_i)|^2 = o(1).$$

Now, turning to $\mathbf{G}(n)$, we will omit the detailed calculations since they are similar to the ones in Step 2 of Lemma 4.5, thus we can deduce that $|\mathbf{G}(n)| = o(1)$.

Step 4. The aim of this step is to prove the estimations in (5.3). From the first step and applying Young's inequality, we have

$$(5.8) \quad \int_{x_1}^{x_2} \eta |u_x^n|^2 dx \leq \sum_{i=1}^2 \left(\frac{\eta(x_i)}{2\mathbf{T}} |u_x^n(x_i)|^2 + \frac{\mathbf{T}}{2} \eta(x_i) |u^n(x_i)|^2 \right) + o(1).$$

From (5.5) and the definition of σ , it follows

$$\begin{cases} \sum_{i=1}^2 \eta(x_i) |u_x^n(x_i)|^2 \leq \mathbf{T} \int_{x_1}^{x_2} \eta |u_x^n|^2 dx + o(1), \\ \sum_{i=1}^2 \eta(x_i) |u^n(x_i)|^2 \leq \frac{2 \max(a(x_1), a(x_2))}{\beta_n^2} \mathbf{T} \int_{x_1}^{x_2} \eta |u_x^n|^2 dx + \frac{o(1)}{\beta_n^2}. \end{cases}$$

Inserting the above inequalities into (5.8), using the fact that β_n tends to infinity and $\|U^n\|_{\mathcal{H}} = 1$, one has

$$\frac{1}{2} \int_{x_1}^{x_2} \eta |u_x^n|^2 dx \leq \frac{\max(a(x_1), a(x_2))}{\beta_n^2} \mathbf{T}^2 \int_{x_1}^{x_2} \eta |u_x^n|^2 dx + o(1) = o(1).$$

Now, for the second estimation in (5.3), we use the equivalence between the norms given in Proposition 2.3 and the thesis follows. \square

Lemma 5.4. Assume Hypothesis 4.1. Then, the solution $(u, v)^\top \in D(\mathcal{A})$ of (3.3)-(3.4) satisfies

$$(5.9) \quad \int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx = o(1) \quad \text{and} \quad \int_0^1 \eta |u_x^n|^2 dx - \int_0^1 \frac{\lambda}{\sigma d} |u^n|^2 dx = o(1).$$

Proof. By Lemma 3.2, we have that (3.11) holds when $\beta_n \rightarrow \infty$, as $n \rightarrow \infty$. Firstly, we need to prove that

$$(5.10) \quad |\mathbf{C}(n)| = o(1) \quad \text{and} \quad |\mathbf{D}(n)| = o(1).$$

Starting with $\mathbf{C}(n)$, using similar arguments and calculations as in Lemma 4.6, taking into account that here $\beta_n \rightarrow \infty$, using the Cauchy Schwarz and the Hardy Poincaré inequalities, $\|U^n\|_{\mathcal{H}} = 1$, $\|F_n\|_{\mathcal{H}} = o(1)$ and the second limit in (5.2), we obtain

$$(5.11) \quad \left| \Re \left(i \int_0^1 \frac{\chi_{(x_1, x_2)}}{\sigma} \beta_n u^n \overline{\varphi u_x^n} dx \right) \right| \leq \max_{x \in [x_1, x_2]} \frac{1}{\sqrt{a}} \left(\int_{x_1}^{x_2} \frac{1}{\sigma} |\beta_n u^n|^2 dx \right)^{\frac{1}{2}} \left(\int_{x_1}^{x_2} \eta |u_x^n|^2 dx \right)^{\frac{1}{2}} = o(1),$$

and

$$(5.12) \quad \left| \Re \left(\int_0^1 \chi_{(x_1, x_2)} f_n^1 \frac{\varphi}{\sigma} \overline{u_x^n} dx \right) \right| \leq \max_{x \in [x_1, x_2]} \frac{1}{\sqrt{a}} \sqrt{\tilde{C}_{HP}} \left(\int_{x_1}^{x_2} \eta |(f_n^1)_x|^2 dx \right)^{\frac{1}{2}} \left(\int_{x_1}^{x_2} \eta |u_x^n|^2 dx \right)^{\frac{1}{2}} = o(1).$$

Employing analogous computations to the ones in Lemma 4.6 and considering the definition of φ , we obtain

$$(5.13) \quad \left| 2\Re \left(\int_0^1 f^2 \frac{\varphi}{\sigma} \overline{u_x^n} dx \right) \right| = o(1).$$

Now, consider the term $2\Re \left(\int_0^1 i \beta_n f_n^1 \frac{\varphi}{\sigma} \overline{u_x^n} dx \right)$. Integrating by parts and using the definition of φ we obtain

$$(5.14) \quad \begin{aligned} 2\Re \left(\int_0^1 i \beta_n f_n^1 \frac{\varphi}{\sigma} \overline{u_x^n} dx \right) &= -2\Re \left(i \int_0^{x_2} \frac{x}{\sigma} \varphi_1(f_n^1)_x \beta_n \overline{u^n} dx + i \int_{x_1}^1 \frac{(x-1)}{\sigma} \varphi_2(f_n^1)_x \beta_n \overline{u^n} dx \right) \\ &- 2\Re \left(i \int_0^1 \frac{\varphi'}{\sigma} f_n^1 \beta_n \overline{u^n} dx - i \int_0^1 \frac{\varphi}{\sigma} \left(\frac{a' - b}{a} \right) f_n^1 \beta_n \overline{u^n} dx \right) - 2\beta_n \lim_{x \rightarrow 0} \Re \left[i \frac{f_n^1 \varphi}{\sigma} \overline{u^n} \right]. \end{aligned}$$

Consider the first two terms in the right hand side of the above equation; using the monotonicity of $\frac{x}{\sqrt{a}}$ and (3.6), we can estimate them as

$$\left| \Re \left(i \int_0^{x_2} \frac{x}{\sigma} \varphi_1(f_n^1)_x \beta_n \overline{u^n} dx + i \int_{x_1}^1 \frac{(x-1)}{\sigma} \varphi_2(f_n^1)_x \beta_n \overline{u^n} dx \right) \right| \leq \kappa_1 \left(\int_0^1 \eta |(f_n^1)_x|^2 dx \right)^{\frac{1}{2}} \left(\int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx \right)^{\frac{1}{2}} = o(1),$$

where $\kappa_1 = \max \left\{ \frac{x_2}{\sqrt{a(x_2)}}, (1-x_1) \max_{x \in [x_1, 1]} \frac{1}{\sqrt{a}} \right\}$. For the third and the fourth terms in the right hand side of (5.14), using (3.6), the definition of φ and Hypothesis 4.1, we have

$$\left| \Re \left(i \int_0^1 \frac{\varphi'}{\sigma} f_n^1 \beta_n \overline{u^n} dx - i \int_0^1 \frac{\varphi}{\sigma} \left(\frac{a'-b}{a} \right) f_n^1 \beta_n \overline{u^n} dx \right) \right| \leq \kappa_2 \left(\int_0^1 \eta |(f_n^1)_x|^2 dx \right)^{\frac{1}{2}} \left(\int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx \right)^{\frac{1}{2}} = o(1),$$

$$\text{where } \kappa_2 = \max \left\{ \|\varphi'\|_\infty \sqrt{\tilde{C}_{HP}}, \sqrt{\tilde{C}_{HP}} \left(1 + \frac{K_a}{2} + \left\| x \frac{a'-b}{a} \right\|_{L^\infty(x_1, x_2)} + \left\| (x-1) \frac{a'-b}{a} \right\|_{L^\infty(x_1, x_2)} \right) \right\}.$$

To establish the last limit term in (5.14), thanks to the definition of φ , it is sufficient to show that $\lim_{x \rightarrow 0} \Re \left[i \frac{x \varphi_1 f_n^1}{\sigma} \overline{u^n} \right] = 0$. Thus, thanks to the fifth term of Lemma 3.1 and Young's inequality, we obtain

$$(5.15) \quad \lim_{x \rightarrow 0} \left| \Re \left[i \frac{x \varphi_1 f_n^1}{\sigma} \overline{u^n} \right] \right| \leq \lim_{x \rightarrow 0} \eta \varphi_1 \frac{x |u^n|^2}{2a} + \lim_{x \rightarrow 0} \eta \varphi_1 \frac{x |f_n^1|^2}{2a} = 0.$$

Hence, coming back to (5.14), we get

$$(5.16) \quad \left| 2 \Re \left(\int_0^1 i \beta_n f_n^1 \frac{\varphi}{\sigma} \overline{u_x^n} dx \right) \right| = o(1).$$

Using again the second limit in (5.2) and (5.3), the fact that $\beta_n \rightarrow \infty$ and $\text{supp } \hat{\varphi} = [x_1, x_2]$, we get

$$(5.17) \quad \begin{aligned} & \int_{x_1}^{x_2} \frac{\varphi_1 + \varphi_2}{\sigma} |\beta_n u^n|^2 dx + \int_{x_1}^{x_2} (\varphi_1 + \varphi_2) \eta |u_x^n|^2 dx + \int_{x_1}^{x_2} \frac{\lambda(\varphi_1 + \varphi_2)}{\sigma d} |u^n|^2 dx \\ & + \int_0^1 \frac{\hat{\varphi}}{\sigma} |\beta_n u^n|^2 dx + \int_0^1 \hat{\varphi} \eta |u_x^n|^2 dx + \int_0^1 \frac{\lambda \hat{\varphi}}{\sigma d} |u^n|^2 dx + \int_{x_1}^{x_2} \frac{1}{\sigma} |\beta_n u^n|^2 dx \\ & + \int_{x_1}^{x_2} \eta |u_x^n|^2 dx + \int_{x_1}^{x_2} \frac{\lambda}{\sigma d} |u^n|^2 dx = o(1). \end{aligned}$$

For the last four terms of $\mathbf{C}(n)$, using the second limit in (5.2) and (5.3), we have

$$(5.18) \quad \int_{x_1}^{x_2} \frac{\varphi}{\sigma} \left(\frac{a'-b}{a} \right) |\beta_n u^n|^2 dx \leq \left\| \frac{a'-b}{a} \right\|_{L^\infty(x_1, x_2)} \int_{x_1}^{x_2} \frac{1}{\sigma} |\beta_n u^n|^2 dx = o(1)$$

and, similarly, we can get

$$(5.19) \quad \int_{x_1}^{x_2} \frac{b}{a} \eta |u_x^n|^2 dx + \int_{x_1}^{x_2} \frac{\lambda \varphi}{\sigma d} \left(\frac{a'-b}{a} \right) |u^n|^2 dx + \int_{x_1}^{x_2} \frac{\lambda \varphi d'}{\sigma d} |u^n|^2 dx = o(1).$$

Thus, from (5.11)-(5.13), (5.16)-(5.19) we obtain that $|\mathbf{C}(n)| = o(1)$. Now, for $\mathbf{D}(n)$, employing analogous computations to the ones used for $\mathbf{C}(n)$, along with the findings from Lemmas 4.4 and 4.5, $\|U^n\|_{\mathcal{H}} = 1$, and $\|F_n\|_{\mathcal{H}} \rightarrow 0$, we conclude that $|\mathbf{D}(n)| = o(1)$. Finally, to prove (4.15), we will distinguish two cases according to the values of λ .

Case 1. If $\lambda > 0$:

By (3.11) we deduce that

$$\begin{aligned} & \left(1 + \frac{K_a}{2} - \mathcal{M}_0 - \mathcal{M}_1 \right) \int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx + \left(1 - \frac{K_a}{2} - \mathcal{N}_0 - \mathcal{N}_1 \right) \int_0^1 \eta |u_x^n|^2 dx \\ & + \left(1 + \frac{K_a}{2} - \mathcal{M}_0 - \mathcal{M}_1 - \mathcal{D}_0 - \mathcal{D}_1 \right) \int_0^1 \frac{\lambda}{\sigma d} |u^n|^2 dx = o(1). \end{aligned}$$

Moreover, by Hypothesis 4.1 and using similar arguments as in Lemma 4.6 for the case when $\lambda > 0$, we can conclude that

$$\int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx = o(1), \quad \text{and} \quad \int_0^1 \eta |u_x^n|^2 dx = o(1).$$

Case 2. If $\lambda \leq 0$:

By (3.11) we have that

$$(5.20) \quad \begin{aligned} & \left(1 + \frac{K_a}{2} - \mathcal{M}_0 - \mathcal{M}_1\right) \int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx + \left(1 - \frac{K_a}{2} - \mathcal{N}_0 - \mathcal{N}_1\right) \int_0^1 \eta |u_x^n|^2 dx + \\ & \left(1 + \frac{K_a}{2} + \mathcal{M}_0 + \mathcal{M}_1 + \mathcal{D}_0 + \mathcal{D}_1\right) \int_0^1 \frac{\lambda}{\sigma d} |u^n|^2 dx = o(1). \end{aligned}$$

Moreover, by the Hardy-Poincaré inequality given in Proposition 2.2, by Hypothesis 4.1, the fact that $\lambda < 0$ and $K_a + 2K_d \leq 2$ and proceeding as in Lemma 4.6 for the case $\lambda < 0$, we obtain that

$$\left(1 + \frac{K_a}{2} - \mathcal{M}_0 - \mathcal{M}_1\right) \int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx + \left(1 - \frac{K_a}{2} - \mathcal{N}_0 - \mathcal{N}_1 + \left(4 + \left\| \frac{d'}{d} \right\|_{L^\infty(x_2, 1)}\right) \lambda C_{HP}\right) \int_0^1 \eta |u_x^n|^2 dx = o(1).$$

Therefore, by Hypothesis (4.1), we conclude that

$$\int_0^1 \frac{1}{\sigma} |\beta_n u^n|^2 dx = o(1) \quad \text{and} \quad \int_0^1 \eta |u_x^n|^2 dx = o(1)$$

Using the equivalence between the norms given in Proposition 4.3, we obtain the last estimation in (5.9). \square

Proof. [Proof of Theorem 5.1.] Thanks to Lemma 5.4, we deduce that $\|U\|_{\mathcal{H}} = o(1)$, which contradicts $\|U\|_{\mathcal{H}} = 1$. Consequently, the second condition in (E) holds and the proof is thus complete. \square

APPENDIX A. EXAMPLES

Here, we provide some examples of functions a , b , and d that satisfy the conditions required in Hypothesis 4.1. It's worth noting that exploring these examples imposes certain constraints on the selection of x_1 and x_2 .

Example A.1

When a is (WD) and $\lambda > 0$: (see Figure 1)

Consider $a(x) = 2\sqrt{x}$, $b(x) = 1$ and $d(x) = \frac{1}{4}x^{\frac{1}{8}}$. In order to have that the functions a and b fulfill Hypothesis 4.1, it is necessary that the following conditions

$$h_1(x) := \frac{4(1-x_2)(1-\sqrt{x_2})+1}{x_2} - 4 < 0 \quad \text{and} \quad x_1 < \left(\frac{3}{2} - \frac{1-x_2}{\sqrt{x_2}}\right)^2$$

are satisfied. Let $\alpha_1 := \left(-\frac{\sqrt{7}\cos\left(\frac{\arctan\left(\frac{9\sqrt{271}}{3}\right)}{3}\right)}{3} + \frac{2}{3} + \frac{\sqrt{7}\sqrt{3}\sin\left(\frac{\arctan\left(\frac{9\sqrt{271}}{3}\right)}{3}\right)}{3}\right)^2 \approx 0.43917$. Thus, it is sufficient to take $\alpha_1 < x_2 < 1$ and $x_1 < x_2$ so that the above inequalities are satisfied.

Example A.2

When a is (SD) and $\lambda > 0$: (see Figure 2)

Consider $a(x) = x\sqrt{x}$, $b(x) = \frac{1}{8}x$ and $d(x) = \frac{1}{8}x^{\frac{1}{16}}$. In order to have that the functions a and b fulfill Hypothesis 4.1, it is necessary that the following conditions

$$h_2(x) := (1-x_2) \left(\frac{3}{2x_2} - \frac{1}{8\sqrt{x_2}}\right) + \frac{1}{16x_2} - \frac{1}{8} < 0 \quad \text{and} \quad x_1 < \left(2 - \frac{1-x_2}{\sqrt{x_2}}\right)^2$$

are satisfied. Taking $\alpha_2 := \left(-\frac{2\sqrt{43}\cos\left(\frac{\arctan\left(\frac{3\sqrt{1304751}}{8347}\right)}{3}\right)}{3} + \frac{13}{3} + \frac{2\sqrt{3}\sqrt{43}\sin\left(\frac{\arctan\left(\frac{3\sqrt{1304751}}{8347}\right)}{3}\right)}{3}\right)^2 \approx 0.958$, we

can choose $\alpha_2 < x_2 < 1$, $x_1 < x_2$, so that the above inequalities are satisfied.

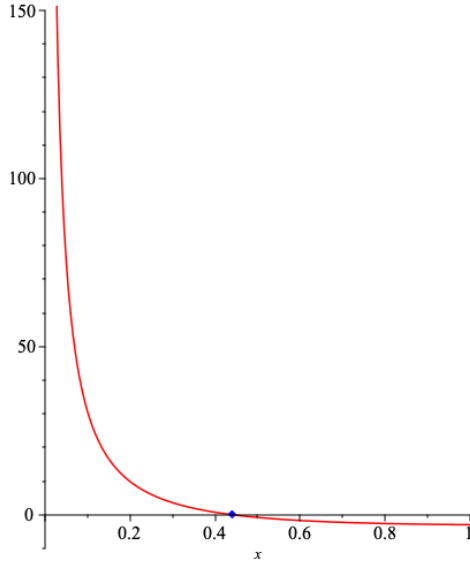


FIGURE 1. a is (WD) and $\lambda > 0$

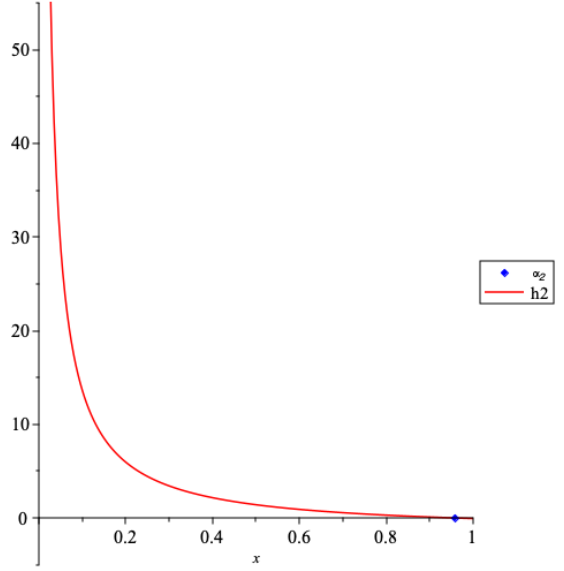


FIGURE 2. a is (SD) and $\lambda > 0$

Example A.3

When a is (WD) and $\lambda < 0$: (see Figure 3)

Consider $a(x) = \sqrt{x}$, $b(x) = 1$, d can be chosen as any (WD) function such that $K_a + 2K_d \leq 2$ and $\lambda = -\frac{1}{4(4 + \|\frac{d'}{d}\|_{L^\infty(x_2, 1)})C_{HP}}$. If we want that a and b fulfill Hypothesis 4.1, it is necessary that the following conditions

$$h_3(x) := \frac{(1-x_2)|1-2\sqrt{x_2}|}{2x_2} - \frac{3}{4} < 0 \quad \text{and} \quad x_1 < \left(\frac{1}{2} - \frac{1-x_2}{\sqrt{x_2}}\right)^2$$

are satisfied. Let $\alpha_3 := \left(\frac{\sqrt{73} \cos\left(\frac{\arctan\left(\frac{36\sqrt{298}}{53}\right)}{3}\right)}{12} + \frac{5}{12} + \frac{\sqrt{3}\sqrt{73} \sin\left(\frac{\arctan\left(\frac{36\sqrt{298}}{53}\right)}{3}\right)}{12}\right)^2 \approx 0.14159$. Then, it is

sufficient to consider $\alpha_3 < x_2 < 1$ and $x_1 < x_2$ so that the above inequalities are satisfied.

Example A.4

When a is (SD) and $\lambda < 0$: (see Figure 4)

Consider $a(x) = x\sqrt{x}$, $b(x) = \frac{1}{8}x$, d can be chosen as any (WD) function such that $K_a + 2K_d \leq 2$ and $\lambda = -\frac{1}{32(4 + \|\frac{d'}{d}\|_{L^\infty(x_2, 1)})C_{HP}}$. If we require that the functions a and b fulfill Hypothesis 4.1, it is necessary that the following conditions

$$h_4(x) := (1-x_2) \left(\frac{3}{2x_2} - \frac{1}{8\sqrt{x_2}}\right) - \frac{1}{4} < 0 \quad \text{and} \quad x_1 < \left(\frac{7}{4} - \frac{(1-x_2)}{\sqrt{x_2}}\right)^2$$

are satisfied. Taking $\alpha_4 := \left(-\frac{\sqrt{199} \cos\left(\frac{\arctan\left(\frac{3\sqrt{98286}}{2645}\right)}{3}\right)}{3} + \frac{14}{3} + \frac{\sqrt{3}\sqrt{199} \sin\left(\frac{\arctan\left(\frac{3\sqrt{98286}}{2645}\right)}{3}\right)}{3}\right)^2 \approx 0.8467$, it is

sufficient to consider $\alpha_4 < x_2 < 1$ and $x_1 < x_2$ so that the above inequalities are satisfied.

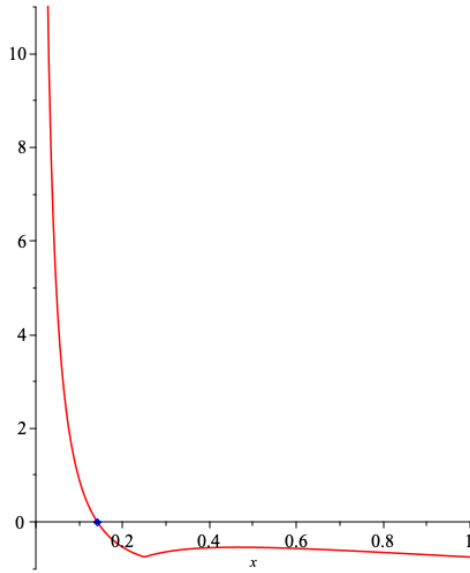


FIGURE 3. a is (WD) and $\lambda < 0$

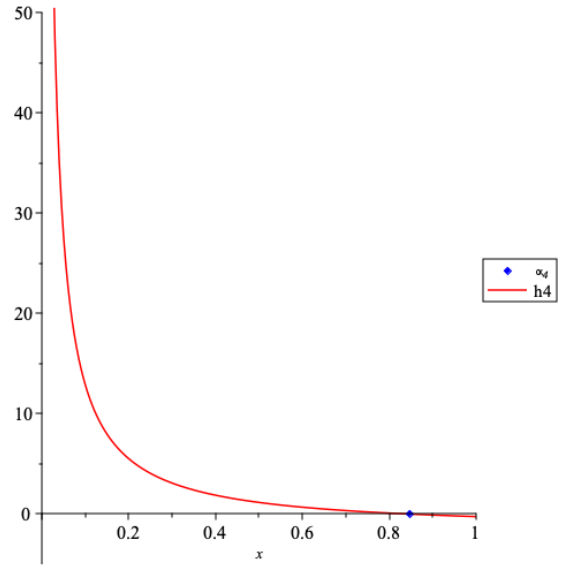


FIGURE 4. a is (SD) and $\lambda < 0$

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DECLARATION

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