



**HAL**  
open science

# On the Cauchy problem for logarithmic fractional Schrödinger equation

Rémi Carles, Fangyuan Dong

► **To cite this version:**

Rémi Carles, Fangyuan Dong. On the Cauchy problem for logarithmic fractional Schrödinger equation. 2024. hal-04538854

**HAL Id: hal-04538854**

**<https://hal.science/hal-04538854>**

Preprint submitted on 9 Apr 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# ON THE CAUCHY PROBLEM FOR LOGARITHMIC FRACTIONAL SCHRÖDINGER EQUATION

RÉMI CARLES AND FANGYUAN DONG

ABSTRACT. We consider the fractional Schrödinger equation with a logarithmic nonlinearity, when the power of the Laplacian is between zero and one. We prove global existence results in three different functional spaces: the Sobolev space corresponding to the quadratic form domain of the fractional Laplacian, the energy space, and a space contained in the operator domain of the fractional Laplacian. For this last case, a finite momentum assumption is made, and the key step consists in estimating the Lie commutator between the fractional Laplacian and the multiplication by a monomial.

## 1. INTRODUCTION

We consider the logarithmic Schrödinger equation

$$(1.1) \quad i\partial_t u - (-\Delta)^s u = \lambda \log(|u|^2) u, \quad u|_{t=0} = u_0,$$

where  $0 < s < 1$ ,  $u = u(t, x)$  represents a complex-valued function defined on  $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ , with  $d \geq 1$ . The fractional Laplacian  $(-\Delta)^s$  is defined through the Fourier transform as follows:

$$(1.2) \quad \mathcal{F}[(-\Delta)^s u](\xi) = |\xi|^{2s} \mathcal{F}u(\xi),$$

where the Fourier transform is given by

$$(1.3) \quad \mathcal{F}u(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} u(x) e^{-i\xi \cdot x} dx.$$

The fractional Laplacian  $(-\Delta)^s$  is a self-adjoint operator acting on the space  $L^2(\mathbb{R}^d)$ , characterized by a quadratic form domain  $H^s(\mathbb{R}^d)$  and an operator domain  $H^{2s}(\mathbb{R}^d)$ . The nonlocal operator  $(-\Delta)^s$  serves as the infinitesimal generators in the context of Lévy stable diffusion processes, as outlined in [2]. Fractional derivatives of the Laplacian have applications in numerous equations in mathematical physics and related disciplines, as proposed in [28, 29] in the case of linear Schrödinger equations; see also [2, 16, 22] and the associated references. Recently, there has been a strong focus on studying mathematical problems related to the fractional Laplacian purely from a mathematical perspective. Regarding specifically fractional nonlinear Schrödinger equations, important progress has been made in e.g. [4, 6, 12, 13, 18, 19, 20, 21].

The problem (1.1) does not seem to have physical motivations (so far), and was introduced in [15] as a generalization of the case  $s = 1$ , introduced in [5],

---

Partially supported by Centre Henri Lebesgue, program ANR-11-LABX-0020-0. The second author is on leave from the University of Science and Technology in Beijing, thanks to some funding from the Chinese Scholarship Council. A CC-BY public copyright license has been applied by the authors to the present document and will be applied to all subsequent versions up to the Author Accepted Manuscript arising from this submission.

and proposed in different physical contexts since (see e.g. [25, 33]). Note also that the logarithmic nonlinearity may be obtained as the limit of an homogeneous nonlinearity  $\lambda|u|^{2\sigma}u$  when  $\sigma$  goes to zero, at least when ground states are considered (case  $\lambda < 0$ ; see [31] for  $s = 1$ , [1] in the fractional case).

In [3], the author addresses the nonlinear fractional logarithmic Schrödinger equation (1.1) with  $\lambda = -1$  and  $d \geq 2$ , employing a compactness method to establish a unique global solution for the associated Cauchy problem within a suitable functional framework, inspired by [11] (for the logarithmic nonlinearity) and [13] (for the fractional Laplacian). In [32], the author investigate the existence of a global weak solution to the problem (1.1) in the case of  $\lambda = -1$ , when the space variable  $x$  belongs to some smooth bounded domain, by using a combination of potential wells theory and the Galerkin method. In this paper, we complement the approach from [3, 32] by adapting the strategy employed in [24] in the case of the standard Laplacian,  $s = 1$ .

Formally, (1.1) enjoys the the conservations of mass, angular momentum, and energy:

$$(1.4) \quad \begin{aligned} M(u(t)) &= \|u(t)\|_{L^2(\mathbb{R}^d)}^2, \\ J(u(t)) &= \operatorname{Im} \int_{\mathbb{R}^d} \bar{u}(t, x) \nabla u(t, x) dx, \\ E(u(t)) &= \frac{1}{2} \|(-\Delta)^{s/2} u(t)\|_{L^2(\mathbb{R}^d)}^2 + \frac{\lambda}{2} \int_{\mathbb{R}^d} |u(t, x)|^2 (\log |u(t, x)|^2 - 1) dx. \end{aligned}$$

The energy is well-defined in the subset of  $H^s(\mathbb{R}^d)$ ,

$$W_1^s := \left\{ u \in H^s(\mathbb{R}^d), x \mapsto |u(x)|^2 \log |u(x)|^2 \in L^1(\mathbb{R}^d) \right\}.$$

When  $s = 1$ , Hayashi and Ozawa [24] revisit the Cauchy problem for the logarithmic Schrödinger equation, constructing strong solutions in both  $H^1$  and  $W_1 = W_1^1$ . This approach deliberately avoids relying on compactness arguments, demonstrating the convergence of a sequence of approximate solutions in a complete function space. The authors in [24] also address the existence in the  $H^2$ -energy space, as discussed below.

The main contributions of this paper can be summarized as follows:

1. Construction of  $H^s$  strong solutions, without relying on the conservation of the energy.
2. Construction of solutions in the energy space  $W_1^s$ .
3. The higher  $H^{2s}$  regularity is established, by assuming some further spatial decay of the initial data.

In all cases, no sign assumption is made on  $\lambda \in \mathbb{R}$ .

**Theorem 1.1.** *Let  $\lambda \in \mathbb{R}$  and  $0 < s < 1$ . For any  $\varphi \in H^s(\mathbb{R}^d)$ , there exists a unique solution  $C(\mathbb{R}, H^s(\mathbb{R}^d))$  to (1.1) in the sense of*

$$(1.5) \quad i\partial_t u - (-\Delta)^s u = \lambda \log(|u|^2) u \quad \text{in } H^{-s}(\Omega)$$

for all bounded open sets  $\Omega \subset \mathbb{R}^d$  and all  $t \in \mathbb{R}$ , and with  $u|_{t=0} = \varphi$ . If in addition we assume  $\varphi \in W_1^s$ , this  $H^s$ -solution satisfies  $u \in (C \cap L^\infty)(\mathbb{R}, W_1^s)$  if  $\lambda < 0$  and  $u \in C(\mathbb{R}, W_1^s)$  if  $\lambda > 0$ . Moreover, the  $W_1^s$ -solution  $u$  satisfies the equation (1.5) in the sense of  $(W_1^s)^*$ , where  $(W_1^s)^*$  is the dual space of  $W_1^s$ . Finally, if

$\varphi \in H^1(\mathbb{R}^d)$ , then the solution  $u \in C(\mathbb{R}, H^s(\mathbb{R}^d))$  to (1.1) satisfies in addition  $u \in C(\mathbb{R}, H^1(\mathbb{R}^d))$ .

The next result addresses on the construction of strong solutions in  $W_2^s$ , where

$$W_2^s := \left\{ u \in H^{2s}(\mathbb{R}^d), x \mapsto u(x) \log |u(x)|^2 \in L^2(\mathbb{R}^d) \right\},$$

and this space is the natural counterpart of the space  $W_2$  of the  $H^2$ -energy space introduced in [24] for the case  $s = 1$ . Note that considering this space is interesting especially when  $s > 1/2$ , since we have seen in Theorem 1.1 that the  $H^1$  regularity is propagated, and  $H^1(\mathbb{R}^d) \subset H^{2s}(\mathbb{R}^d)$  when  $s \leq 1/2$ .

In the fractional case, it seems delicate to adapt the strategy introduced in [24], as some algebraic structure is lost. More precisely, the strategy in [24] starts by showing that  $\partial_t u \in L_{\text{loc}}^\infty(\mathbb{R}, L^2)$ , to eventually conclude that  $\Delta u \in L_{\text{loc}}^\infty(\mathbb{R}, L^2)$ . At this level of generality, this is the standard approach, as presented in e.g. [10], but the logarithmic nonlinearity actually requires some special care. The above line of reasoning needs, as an intermediary step, to know that  $u \log |u|^2 \in L_{\text{loc}}^\infty(\mathbb{R}, L^2)$ , which is by no means obvious, due to the region  $\{|u| < 1\}$  where the nonlinearity is morally sublinear. This difficulty is overcome in [24] by a beautiful algebraic identity ([24, Lemma 3.3]), whose derivation involves an integration by parts in the term

$$\text{Re}(\Delta u, u \log(|u| + \varepsilon))_{L^2} = -\text{Re} \left( \overline{u} \nabla u, \frac{\nabla |u|}{|u| + \varepsilon} \right)_{L^2} + (|\nabla u|^2, \log(|u| + \varepsilon))_{L^2}.$$

In the present case, we would face

$$\text{Re}((-\Delta)^s u, u \log(|u| + \varepsilon))_{L^2},$$

and the integration by parts would require to control a fractional derivative of  $u \log(|u| + \varepsilon)$ , at least in the case  $s < 1/2$  (for  $s > 1/2$ , one could consider the gradient again).

To overcome this issue, we adopt the approach considered in [7] for the case  $s = 1$ , and rely on some finite momentum assumption. For  $0 < \alpha \leq 1$ , we have

$$\mathcal{F}(H^\alpha) = \left\{ u \in L^2(\mathbb{R}^d), x \mapsto \langle x \rangle^\alpha u(x) \in L^2(\mathbb{R}^d) \right\},$$

where  $\langle x \rangle := \sqrt{1 + |x|^2}$ , and this space is equipped with the norm

$$\|u\|_{\mathcal{F}(H^\alpha)} := \|\langle x \rangle^\alpha u(x)\|_{L^2(\mathbb{R}^d)}.$$

Denote, for  $\alpha > 0$ ,  $X_\alpha^{2s} := H^{2s} \cap \mathcal{F}(H^\alpha)$ : for any  $\alpha > 0$ ,  $X_\alpha^{2s} \subset W_2^s$ , as can be seen from the estimate, valid for any  $\delta \in (0, 1)$ ,

$$(1.6) \quad |u \log(|u|^2)| \lesssim |u|^{1-\delta} + |u|^{1+\delta}.$$

**Theorem 1.2.** *Let  $\lambda \in \mathbb{R}$ ,  $0 < s < 1$ . Consider  $0 < \alpha < 2s$  with  $\alpha \leq 1$ . For any  $\varphi \in X_\alpha^{2s} = H^{2s} \cap \mathcal{F}(H^\alpha)$ , there exists a unique solution  $u \in C_w \cap L_{\text{loc}}^\infty(\mathbb{R}, X_\alpha^{2s})$  to (1.1) in the sense of*

$$(1.7) \quad i\partial_t u - (-\Delta)^s u = \lambda \log(|u|^2) u \quad \text{in } L^2(\Omega),$$

for all bounded open sets  $\Omega \subset \mathbb{R}^d$  and a.e.  $t \in \mathbb{R}$ , with  $u|_{t=0} = \varphi$ . Moreover, when  $\lambda < 0$ ,  $u \in C(\mathbb{R}, X_\alpha^{2s})$  and (1.7) holds in  $L^2(\mathbb{R}^d)$  and for all  $t \in \mathbb{R}$ .

The new difficulty in proving the above result, compared to the case  $s = 1$ , lies in the fact that the Lie bracket  $[(-\Delta)^s, \langle x \rangle^\alpha]$  requires some extra care; see Lemma 2.3.

We underline the fact that we do not know whether the  $H^\sigma$  regularity is propagated by the flow of (1.1), where  $\sigma = \max(1, 2s)$ , like in the case of the regular Laplacian  $s = 1$ .

*Notations.*

- $\int f$  is employed in place of  $\int_{\mathbb{R}^d} f(x)dx$ .
- The inner product in  $L^2$  is denoted by

$$(f, g)_{L^2} = \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx = \int f\bar{g}.$$

- Let  $C(I, X)$  (resp.  $C_w(I, X)$ ) be the space strongly (resp. weakly) continuous functions from interval  $I(\subseteq \mathbb{R})$  to  $X$ .
- Abbreviated notation: for  $T > 0$ , we write

$$C_T(X) = C([-T, T], X), \quad L_T^\infty(X) = L^\infty((-T, T), X),$$

- $A \lesssim B$  represents the inequality  $A \leq CB$  with some constant  $C > 0$ .

*Content.* The rest of the paper is organized as follows. In Section 2, we collect lemmas which are of constant use in this paper. Section 3 is dedicated to the study of the Cauchy problem for (1.1) in both  $H^s$  and  $W_s^1$ , proving Theorem 1.1. In Section 4, we consider higher regularity and prove Theorem 1.2.

## 2. USEFUL LEMMAS

The following lemma is a generalization of the inequality proven initially by Cazenave and Haraux [11] in the case  $\varepsilon = \mu = 0$ :

**Lemma 2.1** ([24], Lemma A.1). *For all  $u, v \in \mathbb{C}$  and  $\varepsilon, \mu \geq 0$ , we have*

$$(2.1) \quad |\operatorname{Im}(u \log(|u| + \varepsilon) - v \log(|v| + \mu))(\bar{u} - \bar{v})| \leq |u - v|^2 + |\varepsilon - \mu||u - v|.$$

We will also use several times the fractional Leibniz rule. We state a simplified version of a result from [30], by using the fact that BMO contains  $L^\infty$ , and considering only the  $L^2$  setting.

**Lemma 2.2** ([30], Corollary 1.4). *For  $\sigma > 0$ , let  $A^\sigma$  be a differential operator such that its symbol  $\widehat{A^\sigma}(\xi)$  is homogeneous of degree  $\sigma$  and  $\widehat{A^\sigma}(\xi) \in C^\infty(\mathbb{S}^{d-1})$ .*

- If  $0 < \sigma < 1$ ,

$$\|A^\sigma(fg) - gA^\sigma f\|_{L^2} \lesssim \|f\|_{L^2}\|(-\Delta)^{\sigma/2}g\|_{L^\infty}.$$

- If  $1 \leq \sigma < 2$ ,

$$\|A^\sigma(fg) - gA^\sigma f - \nabla g \cdot A^{\sigma, \nabla} f\|_{L^2} \lesssim \|f\|_{L^2}\|(-\Delta)^{\sigma/2}g\|_{L^\infty}.$$

where  $\widehat{A^{\sigma, \nabla}}(\xi) = -i\nabla_\xi \left( \widehat{A^\sigma}(\xi) \right)$ .

We recall that the characterization of the  $H^s$  norm, for  $0 < s < 1$ , can be expressed as follows (see e.g. [17]):

$$\begin{aligned} \|f\|_{H^s}^2 &= \|f\|_{L^2}^2 + \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|f(x) - f(y)|^2}{|x - y|^{d+2s}} dy dx \\ &= \|f\|_{L^2}^2 + \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|f(x+y) - f(x)|^2}{|y|^{d+2s}} dy dx. \end{aligned}$$

We also have, for  $0 < s < 1$  and  $f \in \mathcal{S}(\mathbb{R}^d)$  (see e.g. [17]),

$$(2.2) \quad (-\Delta)^s f(x) = c(d, s) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{f(x+y) + f(x-y) - 2f(x)}{|y|^{d+2s}} dy dx,$$

for some constant  $c(d, s)$  whose exact value is irrelevant here.

The following lemma will be crucial in the proof of Theorem 1.2.

**Lemma 2.3.** *Let  $0 < s < 1$ . If  $0 < \alpha < 2s$  and  $\alpha \leq 1$ , then the commutator  $[(-\Delta)^s, \langle x \rangle^\alpha]$  is continuous from  $H^s(\mathbb{R}^d)$  to  $L^2(\mathbb{R}^d)$ .*

*Proof.* The proof relies on the fractional Leibniz rule stated in Lemma 2.2, with  $A^\sigma = (-\Delta)^s$ , hence  $\sigma = 2s$ . Fix  $f \in C_c^\infty(\mathbb{R}^d)$ , and let  $g(x) = \langle x \rangle^\alpha$ .

We first show that under the assumptions of the lemma,  $(-\Delta)^s g \in L^\infty(\mathbb{R}^d)$ , by using the characterization (2.2). In the region  $\{|y| \geq 1\}$ , we write, since  $0 < \alpha \leq 1$ ,

$$|\langle x \pm y \rangle^\alpha - \langle x \rangle^\alpha| \leq |\langle x \pm y \rangle - \langle x \rangle|^\alpha \lesssim |y|^\alpha,$$

hence

$$\left| \int_{|y| \geq 1} \frac{\langle x+y \rangle^\alpha + \langle x-y \rangle^\alpha - 2\langle x \rangle^\alpha}{|y|^{d+2s}} dy \right| \lesssim \int_{|y| \geq 1} \frac{|y|^\alpha}{|y|^{d+2s}} dy < \infty,$$

provided that  $\alpha < 2s$ . In the ball  $\{|y| < 1\}$ , Taylor's formula yields

$$\langle x+y \rangle^\alpha + \langle x-y \rangle^\alpha - 2\langle x \rangle^\alpha = \langle \nabla^2 g(x) y, y \rangle + \mathcal{O}(|y|^3),$$

where the remainder is uniform in  $x \in \mathbb{R}^d$ , as the third order derivatives of  $g$  are bounded. Also, the Hessian of  $g$  is bounded since  $|\nabla^2 g(x)| \lesssim \langle x \rangle^{\alpha-2}$ , and

$$\left| \int_{|y| \leq 1} \frac{\langle x+y \rangle^\alpha + \langle x-y \rangle^\alpha - 2\langle x \rangle^\alpha}{|y|^{d+2s}} dy \right| \lesssim \int_{|y| \leq 1} \frac{|y|^2}{|y|^{d+2s}} dy < \infty,$$

since  $s < 1$ .

**First case:**  $0 < s < 1/2$ . In view of the first case in Lemma 2.2,

$$\| [(-\Delta)^s, \langle x \rangle^\alpha] f \|_{L^2} \lesssim \|f\|_{L^2} \|(-\Delta)^s g\|_{L^\infty} \lesssim \|f\|_{L^2},$$

and  $[(-\Delta)^s, \langle x \rangle^\alpha]$  is continuous from  $L^2(\mathbb{R}^d)$  to  $L^2(\mathbb{R}^d)$ .

**Second case:**  $1/2 \leq s < 1$ . In view of the second case in Lemma 2.2,

$$\| [(-\Delta)^s, \langle x \rangle^\alpha] f \|_{L^2} \lesssim \|\nabla g \cdot A^{2s, \nabla} f\|_{L^2} + \|f\|_{L^2} \|(-\Delta)^s g\|_{L^\infty}.$$

In view of the definition of  $A^{2s, \nabla}$ , with  $A^{2s} = (-\Delta)^s$ ,

$$\|A^{2s, \nabla} f\|_{L^2} \lesssim \|f\|_{\dot{H}^{2s-1}} \lesssim \|f\|_{H^s},$$

since  $0 < s < 1$ . Recalling that since  $\alpha \leq 1$ ,  $\nabla g \in L^\infty$ , the lemma is proved.  $\square$

### 3. THE CAUCHY PROBLEM IN $H^s$ AND THE ENERGY SPACE

In this section, we prove Theorem 1.1, by resuming the strategy of [24], which requires very few adaptations to treat this fractional case (essentially, the fractional Leibniz rule).

**3.1. Approximate problems.** For  $\varepsilon > 0$ , we consider the approximate equation

$$(3.1) \quad i\partial_t u_\varepsilon - (-\Delta)^s u_\varepsilon = 2\lambda u_\varepsilon \log(|u_\varepsilon| + \varepsilon), \quad u_\varepsilon(0, x) = \varphi(x).$$

We set

$$g(u) = 2u \log |u|, \quad g_\varepsilon(u) = 2u \log(|u| + \varepsilon).$$

For  $\sigma \geq 0$  we have

$$\int_0^\sigma g_\varepsilon(\tau) d\tau = \frac{1}{2}\sigma^2 \log((\sigma + \varepsilon)^2) - \frac{1}{2} \int_0^\sigma \frac{2\tau^2}{\tau + \varepsilon} d\tau.$$

We define  $G_\varepsilon(u)$  by

$$G_\varepsilon(u) = \frac{1}{2} \int |u|^2 \log((|u| + \varepsilon)^2) - \frac{1}{2} \int \mu_\varepsilon(|u|), \quad \text{for } u \in H^s(\mathbb{R}^d),$$

where

$$\mu_\varepsilon(\sigma) := \int_0^\sigma \frac{2\tau^2}{\tau + \varepsilon} d\tau \quad \text{for } \sigma \geq 0.$$

We define  $E_\varepsilon(u)$  by

$$(3.2) \quad \begin{aligned} E_\varepsilon(u) &= \frac{1}{2} \int |(-\Delta)^{s/2} u|^2 + \lambda G_\varepsilon(u) \\ &= \frac{1}{2} \int |(-\Delta)^{s/2} u|^2 + \frac{\lambda}{2} \int |u|^2 \log((|u| + \varepsilon)^2) - \frac{\lambda}{2} \int \mu_\varepsilon(|u|) \end{aligned}$$

**Lemma 3.1.** *Let  $\varphi \in H^s(\mathbb{R}^d)$  and  $\varepsilon > 0$ . Then (3.1) possesses a unique solution*

$$u_\varepsilon \in C(\mathbb{R}, H^s(\mathbb{R}^d)) \cap C^1(\mathbb{R}, H^{-s}(\mathbb{R}^d)).$$

*Moreover, the mass and energy are conserved: for all  $t \in \mathbb{R}$ ,*

$$\|u_\varepsilon(t)\|_{L^2}^2 = \|\varphi\|_{L^2}^2, \quad E_\varepsilon(u_\varepsilon(t)) = E_\varepsilon(\varphi).$$

*Proof.* Unlike in the case of the regular Laplacian,  $s = 1$ , it seems delicate to invoke Strichartz estimates independently of the space dimension  $d$  in order to solve (3.1) in  $H^s$ , since a loss of regularity is present when  $0 < s < 1$ , see [14], and [23, 26]. We rather adopt the approach of [13], which in turn resumes the arguments from [10]. A key step is to check that, for a given  $T > 0$ , (3.1) has a most one (weak) solution  $u_\varepsilon \in L_T^\infty H^s \cap W_T^{1,\infty} H^{-s}$ . By interpolation, such a solution belongs to  $C_T L^2$ , and if  $u_\varepsilon, v_\varepsilon$  are two such solutions,  $u_\varepsilon - v_\varepsilon$  solves

$$(i\partial_t - (-\Delta)^s)(u_\varepsilon - v_\varepsilon) = \lambda(u_\varepsilon \log(|u_\varepsilon| + \varepsilon) - v_\varepsilon \log(|v_\varepsilon| + \varepsilon)).$$

We then proceed with the usual argument for  $L^2$  estimates in Schrödinger equations: multiply by  $\bar{u}_\varepsilon - \bar{v}_\varepsilon$ , integrate over  $\mathbb{R}^d$ , and take the imaginary part. The term involving the fractional Laplacian vanishes by self-adjointness, and the nonlinear term is controlled thanks to Lemma 2.1 (with  $\mu = \varepsilon$ ), so we get

$$\frac{d}{dt} \|u_\varepsilon - v_\varepsilon\|_{L^2}^2 \lesssim \|u_\varepsilon - v_\varepsilon\|_{L^2}^2,$$

hence  $u_\varepsilon \equiv v_\varepsilon$  by Gronwall Lemma, since  $\|u_\varepsilon(t) - v_\varepsilon(t)\|_{L^2}$  is continuous at  $t = 0$ . The existence of such a weak solution is given by [10, Theorem 3.3.5], which is

readily adapted to the case of the fractional Laplacian, and since we note that for fixed  $\varepsilon > 0$ , there exists a function  $C^\varepsilon(\cdot)$  such that if  $\|u\|_{H^s}, \|v\|_{H^s} \leq M$ , then

$$\|g_\varepsilon(u) - g_\varepsilon(v)\|_{H^{-s}} \leq \|g_\varepsilon(u) - g_\varepsilon(v)\|_{L^2} \leq C^\varepsilon(M)\|u - v\|_{H^s}.$$

With the above uniqueness property, we can resume the proof of [10, Theorem 3.3.9] and [10, Theorem 3.4.1] for the globalization, since, as we have, for every  $\delta > 0$ ,

$$\| |u|^2 \log(|u| + \varepsilon) - \mu_\varepsilon(|u|) \| \leq C_{\varepsilon, \delta} |u|^{2+\delta} + |u|,$$

Gagliardo-Nirenberg and Young inequalities yield

$$|\lambda G_\varepsilon(u)| \leq \frac{1}{4} \|u\|_{H^s}^2 + C(\|u\|_{L^2}),$$

so we obtain the lemma.  $\square$

**3.2. Construction of weak  $H^s$  solution.** We initially establish a uniform estimate for approximate solutions within the  $H^s$  space.

**Lemma 3.2.** *Let  $0 < \alpha \leq 1$  and  $\varphi \in H^s$ . For all  $t \in \mathbb{R}$  we have*

$$(3.3) \quad \left\| (-\Delta)^{s/2} u_\varepsilon(t) \right\|_{L^2}^2 \leq e^{4|\lambda||t|} \|(-\Delta)^{s/2} \varphi\|_{L^2}^2.$$

*Proof.* We resume the energy estimate from [8]: in view of the conservation of the  $L^2$ -norm,

$$\begin{aligned} & \frac{d}{dt} \|u_\varepsilon(t)\|_{H^s(\mathbb{R}^d)}^2 \\ &= 2 \operatorname{Re} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \overline{(u_\varepsilon(t, x+y) - u_\varepsilon(t, x))} \partial_t (u_\varepsilon(t, x+y) - u_\varepsilon(t, x)) \frac{dx dy}{|y|^{d+2\alpha}} \\ &= -2 \operatorname{Im} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \overline{(u_\varepsilon(t, x+y) - u_\varepsilon(t, x))} (-\Delta)^s (u_\varepsilon(t, x+y) - u_\varepsilon(t, x)) \frac{dx dy}{|y|^{d+2\alpha}} \\ & \quad - 4\lambda \operatorname{Im} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \overline{(u_\varepsilon(t, x+y) - u_\varepsilon(t, x))} (g_\varepsilon(u_\varepsilon(t, x+y)) - g_\varepsilon(u_\varepsilon(t, x))) \frac{dx dy}{|y|^{d+2\alpha}}. \end{aligned}$$

Here, the first term on the right-hand side of the equation vanishes, since  $(-\Delta)^s$  is self-adjoint, and so the imaginary part of the integral in  $x$  is zero. By applying Lemma 2.1 with  $\mu = \varepsilon$ , we obtain

$$\begin{aligned} & \frac{d}{dt} \|u_\varepsilon(t)\|_{H^s(\mathbb{R}^d)}^2 \\ & \leq 4|\lambda| \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left| \operatorname{Im} \left[ \overline{(u_\varepsilon(t, x+y) - u_\varepsilon(t, x))} (g_\varepsilon(u_\varepsilon(t, x+y)) - g_\varepsilon(u_\varepsilon(t, x))) \right] \right| \frac{dx dy}{|y|^{d+2\alpha}} \\ & \leq 4|\lambda| \iint_{\mathbb{R}^d \times \mathbb{R}^d} |u_\varepsilon(t, x+y) - u_\varepsilon(t, x)|^2 \frac{dx dy}{|y|^{d+2\alpha}} \leq 4|\lambda| \|u_\varepsilon(t)\|_{H^s(\mathbb{R}^d)}^2. \end{aligned}$$

Gronwall Lemma then yields

$$\|u_\varepsilon(t)\|_{H^s(\mathbb{R}^d)}^2 \leq e^{4|\lambda|t} \|\varphi\|_{H^s(\mathbb{R}^d)}^2 \quad \text{for all } t \in \mathbb{R},$$

hence the lemma.  $\square$

It follows from Lemma 3.1 and (3.3) that for any  $T > 0$  we have

$$(3.4) \quad M_T := \sup_{0 < \varepsilon < 1} \|u_\varepsilon\|_{L_T^\infty(H^s)} \leq C(T, \|\varphi\|_{H^s}).$$



Next we prove that  $\{u_\varepsilon\}_{0 < \varepsilon < 1}$  forms a Cauchy sequence in  $C_T(L^2_{\text{loc}}(\mathbb{R}^d))$  as  $\varepsilon \downarrow 0$  for any  $T > 0$ . Take a function  $\zeta \in C_c^\infty(\mathbb{R}^d)$  satisfying

$$\zeta(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 2, \end{cases} \quad 0 \leq \zeta(x) \leq 1 \quad \text{for all } x \in \mathbb{R}^d.$$

For  $R > 0$  we set  $\zeta_R := \zeta(x/R)$ . For  $\varepsilon, \mu \in (0, 1)$ , utilizing (3.1), (2.1), and (3.4), a direct computation indicates that

$$\begin{aligned} \frac{d}{dt} \|\zeta_R(u_\varepsilon - u_\mu)\|_{L^2}^2 &= 2 \operatorname{Im} (i\zeta_R^2 \partial_t (u_\varepsilon - u_\mu), u_\varepsilon - u_\mu) \\ &= 2 \operatorname{Im} (\zeta_R^2 (-\Delta)^s (u_\varepsilon - u_\mu), u_\varepsilon - u_\mu) \\ &\quad + 4\lambda \operatorname{Im} (\zeta_R^2 (u_\varepsilon \log(|u_\varepsilon| + \varepsilon) - u_\mu \log(|u_\mu| + \mu)), u_\varepsilon - u_\mu). \end{aligned}$$

The first term on the right hand side is estimated thanks to the fractional Leibniz rule recalled in (the first case of) Lemma 2.2, since

$$\operatorname{Im} \left( (-\Delta)^{s/2} (u_\varepsilon - u_\mu), \zeta_R^2 (-\Delta)^{s/2} (u_\varepsilon - u_\mu) \right) = 0,$$

by:

$$\begin{aligned} &\left| \operatorname{Im} \left( (-\Delta)^{s/2} (u_\varepsilon - u_\mu), (-\Delta)^{s/2} (\zeta_R^2 (u_\varepsilon - u_\mu)) \right) \right| \\ &\lesssim \|u_\varepsilon - u_\mu\|_{\dot{H}^s} \|u_\varepsilon - u_\mu\|_{L^2} \|(-\Delta)^{s/2} (\zeta_R^2)\|_{L^\infty}. \end{aligned}$$

The estimate  $\|(-\Delta)^{s/2} \zeta_R^2\|_{L^\infty} \lesssim 1/R^s$  follows by homogeneity (using e.g. Fourier transform), and thus

$$\begin{aligned} \frac{d}{dt} \|\zeta_R(u_\varepsilon - u_\mu)\|_{L^2}^2 &\leq \frac{C}{R^s} \|u_\varepsilon - u_\mu\|_{\dot{H}^s} \|u_\varepsilon - u_\mu\|_{L^2} \\ &\quad + 4|\lambda| \left( \|\zeta_R(u_\varepsilon - u_\mu)\|_{L^2}^2 + |\varepsilon - \mu| \|\zeta_R^2(u_\varepsilon - u_\mu)\|_{L^1} \right). \end{aligned}$$

Gronwall Lemma implies

$$(3.5) \quad \|\zeta_R(u_\varepsilon - u_\mu)(t)\|_{L^2}^2 \leq e^{4|\lambda|T} \left( \frac{C(M_T)}{R^s} + |\varepsilon - \mu| |B_{2R}|^{1/2} \|\varphi\|_{L^2} \right),$$

for all  $t \in [-T, T]$ , where we have used

$$\|\zeta_R^2(u_\varepsilon - u_\mu)\|_{L^1} \leq \|u_\varepsilon - u_\mu\|_{L^2(B_{2R})} \leq 2|B_{2R}|^{1/2} \|\varphi\|_{L^2}.$$

We now fix  $R_0 > 0$  and take  $R \in (R_0, \infty)$  as a parameter. It follows from (3.5) that

$$\|u_\varepsilon - u_\mu\|_{C_T(L^2(B_{R_0}))}^2 \leq \|\zeta_R(u_\varepsilon - u_\mu)\|_{C_T(L^2)}^2 \leq C(T, \|\varphi\|_{H^s}) \left( \frac{1}{R^s} + |\varepsilon - \mu| |B_{2R}|^{1/2} \right),$$

which yields

$$\limsup_{\varepsilon, \mu \downarrow 0} \|u_\varepsilon - u_\mu\|_{C_T(L^2(B_{R_0}))}^2 \leq \frac{C(T, \|\varphi\|_{H^s})}{R^s} \xrightarrow{R \rightarrow \infty} 0.$$

As  $R_0 > 0$  is arbitrary, we conclude that the sequence  $\{u_\varepsilon\}_{0 < \varepsilon < 1}$  constitutes a Cauchy sequence in  $C_T(L^2_{\text{loc}}(\mathbb{R}^d))$ . When combining this with Lemma 3.1, this entails that there exists a function  $u \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^d))$  such that

$$(3.6) \quad u_\varepsilon \rightarrow u \quad \text{in } C_T(L^2_{\text{loc}}(\mathbb{R}^d)) \quad \text{as } \varepsilon \downarrow 0,$$

for all  $T > 0$ .

**Lemma 3.3.**  $u \in L_{\text{loc}}^\infty(\mathbb{R}, H^s(\mathbb{R}^d))$  and

$$(3.7) \quad u_\varepsilon(t) \rightharpoonup u(t) \quad \text{in } H^s(\mathbb{R}^d) \quad \text{for all } t \in \mathbb{R}.$$

*Proof.* First it follows from (3.6) that

$$(3.8) \quad u_\varepsilon(t) \rightharpoonup u(t) \quad \text{in } L^2(\mathbb{R}^d) \quad \text{for all } t \in \mathbb{R}.$$

To prove  $u \in L_{\text{loc}}^\infty(\mathbb{R}, H^s(\mathbb{R}^d))$ , we use the characterization of  $H^s$  functions by duality. For any  $\psi \in C_c^\infty(\mathbb{R}^d)$  and  $t \in [-T, T]$  we obtain from (3.4) that

$$\left| \int u_\varepsilon(t)(-\Delta)^{s/2}\psi \right| = \left| \int (-\Delta)^{s/2}u_\varepsilon(t)\psi \right| \leq \|u_\varepsilon(t)\|_{\dot{H}^s} \|\psi\|_{L^2} \leq M_T \|\psi\|_{L^2}.$$

Then it follows from (3.8) that

$$\left| \int u(t)(-\Delta)^{s/2}\psi \right| \leq M_T \|\psi\|_{L^2} \quad \text{for all } t \in [-T, T].$$

We infer that for all  $t \in [-T, T]$ ,

$$u(t) \in H^s(\mathbb{R}^d) \quad \text{and} \quad \|(-\Delta)^{s/2}u(t)\|_{L^2} \leq M_T,$$

hence  $u \in L_{\text{loc}}^\infty(\mathbb{R}, H^s(\mathbb{R}^d))$ . Also, in view of (3.8),

$$\int (-\Delta)^{s/2}u_\varepsilon(t)\psi \rightarrow \int (-\Delta)^{s/2}u(t)\psi,$$

for any  $\psi \in C_c^\infty(\mathbb{R}^d)$  and  $t \in \mathbb{R}$ . Using (3.4) and a density argument, we deduce that

$$(-\Delta)^{s/2}u_\varepsilon(t) \rightharpoonup (-\Delta)^{s/2}u(t) \quad \text{in } L^2(\mathbb{R}^d), \quad \text{for all } t \in \mathbb{R},$$

hence the lemma.  $\square$

Next, we prove that the convergence of the nonlinear term.

**Lemma 3.4.** For all  $t \in \mathbb{R}$  we have

$$g_\varepsilon(u_\varepsilon(t)) \rightarrow g(u(t)) \quad \text{in } L_{\text{loc}}^2(\mathbb{R}^d) \quad \text{as } \varepsilon \downarrow 0.$$

*Proof.* We show that for any  $\Omega \subset \subset \mathbb{R}^d$  and  $t \in \mathbb{R}$ ,

$$u_\varepsilon(t) \log(|u_\varepsilon(t)| + \varepsilon) \rightarrow u(t) \log|u(t)| \quad \text{in } L^2(\Omega) \quad \text{as } \varepsilon \downarrow 0.$$

In view of [24, Lemma A.2], we know that for  $\alpha \in (0, 1)$ , there exists  $C(\alpha) > 0$  such that for all  $u, v \in \mathbb{C}$ ,  $\varepsilon \in (0, 1)$

$$\begin{aligned} |v \log(|v| + \varepsilon) - u \log|u|| &\leq \varepsilon + |u - v| + C(\alpha) \times \\ &\quad (1 + |u|^{1-\alpha} \log^+ |u| + |v|^{1-\alpha} \log^+ |v|) |u - v|^\alpha, \end{aligned}$$

where  $\log^+ x := \max(\log x, 0)$ . Hence, for any  $\delta > 0$  small, there exists  $C(\delta) > 0$  such that

$$\begin{aligned} |u_\varepsilon \log(|u_\varepsilon| + \varepsilon) - u \log|u|| &\leq \varepsilon + |u_\varepsilon - u| + C(\alpha) \times \\ &\quad \left(1 + |u_\varepsilon|^{\frac{1}{2}+\delta} + |u|^{\frac{1}{2}+\delta}\right) |u_\varepsilon - u|^{1/2}. \end{aligned}$$

Fixing  $\delta > 0$  sufficiently small so that  $H^s(\mathbb{R}^d) \subset L^{2+4\delta}(\mathbb{R}^d)$ , we have

$$\begin{aligned} \left\| |u_\varepsilon|^{\frac{1}{2}+\delta} |u_\varepsilon - u|^{1/2} \right\|_{L^2(\Omega)}^2 &= \int_\Omega |u_\varepsilon|^{1+2\delta} |u_\varepsilon - u| \leq \|u_\varepsilon\|_{L^{2+4\delta}}^{1+2\delta} \|u_\varepsilon - u\|_{L^2(\Omega)} \\ &\lesssim \|u_\varepsilon\|_{H^s}^{1+2\delta} \|u_\varepsilon - u\|_{L^2(\Omega)}. \end{aligned}$$

Therefore, the results follows from (3.4) and (3.6).  $\square$

From (3.1) it follows that for every  $\varphi \in C_c^\infty(\mathbb{R}^d)$  and every  $\phi \in C_c^1(\mathbb{R})$ ,

$$\begin{aligned} \int_{\mathbb{R}} (iu_\varepsilon, \psi)_{L^2} \phi'(t) dt &= - \int_{\mathbb{R}} \langle i\partial_t u_\varepsilon, \psi \rangle_{H^{-s}, H^s} \phi(t) dt \\ &= - \int_{\mathbb{R}} \langle (-\Delta)^s u_\varepsilon + 2\lambda u_\varepsilon \log(|u_\varepsilon| + \varepsilon), \psi \rangle_{H^{-s}, H^s} \phi(t) dt \\ &= - \int_{\mathbb{R}} \left\{ \left( (-\Delta)^{s/2} u_\varepsilon, (-\Delta)^{s/2} \psi \right)_{L^2} + (\lambda g_\varepsilon(u_\varepsilon), \psi)_{L^2} \right\} \phi(t) dt. \end{aligned}$$

From (3.7),  $u_\varepsilon(t) \rightharpoonup u(t)$  in  $H^s(\mathbb{R}^d)$ . In view of Lemma 3.4, taking the limit  $\varepsilon \downarrow 0$  yields

$$\int_{\mathbb{R}} (iu, \psi)_{L^2} \phi'(t) dt = - \int_{\mathbb{R}} \left\{ \left( (-\Delta)^{s/2} u, (-\Delta)^{s/2} \psi \right)_{L^2} + (\lambda g(u), \psi)_{L^2} \right\} \phi(t) dt.$$

It can be easily verified for any  $\Omega \subset \subset \mathbb{R}^d$ ,

$$u \in L_{\text{loc}}^\infty(\mathbb{R}, H^s(\mathbb{R}^d)) \cap W_{\text{loc}}^{s, \infty}(\mathbb{R}, H^{-s}(\Omega))$$

and

$$(3.9) \quad i\partial_t u - (-\Delta)^s u = \lambda g(u) \quad \text{in } H^{-s}(\Omega),$$

for almost all  $t \in \mathbb{R}$ .

**3.3. Uniqueness and regularity.** Following [11, Lemme 2.2.1], we have:

**Lemma 3.5.** *Assume that, for some  $T > 0$ ,  $u, v \in L_T^\infty(H^s(\mathbb{R}^d))$  solve (1.1) in the distribution sense. Then  $u = v$ .*

*Proof.* We set

$$M := \max \left\{ \|u\|_{L_T^\infty(H^s)}, \|v\|_{L_T^\infty(H^s)} \right\}.$$

As mentioned above,  $u, v$  satisfy the equation in the sense of (3.9). Resuming the cut-off function  $\zeta_R$ , and the computations from Section 3.2 (with  $u_\varepsilon$  replaced by  $u$  and  $u_\mu$  replaced by  $v$ ), Gronwall Lemma yields, like for (3.5) (with now  $\varepsilon = \mu = 0$ ),

$$\|\zeta_R(u - v)(t)\|_{L^2}^2 \leq e^{4|\lambda|T} \left( \|\zeta_R(u(0) - v(0))\|_{L^2}^2 + \frac{C(M)}{R^s} T \right) \quad \text{for all } t \in [-T, T].$$

By Fatou's Lemma,

$$\|(u - v)(t)\|_{L^2}^2 \leq \liminf_{R \rightarrow \infty} \|\zeta_R(u - v)(t)\|_{L^2}^2 \leq 0,$$

for all  $t \in [-T, T]$ . Therefore,  $u = v$  on  $[-T, T]$ .  $\square$

Continuity in time and strong  $L^2$  convergence are established like in the proof of [24, Lemma 2.10].

**Lemma 3.6.**  $u \in C_w(\mathbb{R}, H^s(\mathbb{R}^d)) \cap C(\mathbb{R}, L^2(\mathbb{R}^d))$  and

$$u_\varepsilon(t) \rightarrow u(t) \quad \text{in } L^2(\mathbb{R}^d).$$

*Proof.* First we note that  $u \in C_w(\mathbb{R}, H^s(\mathbb{R}^d))$ . Indeed this easily follows from Lemma 3.3 and  $u \in C(\mathbb{R}, L^2_{\text{loc}}(\mathbb{R}^d))$ . Next, we obtain from Lemma 3.1 and (3.8) that

$$\|u(t)\|_{L^2}^2 \leq \liminf_{\varepsilon \rightarrow 0} \|u_\varepsilon(t)\|_{L^2}^2 = \|\varphi\|_{L^2}^2 \quad \text{for all } t \in \mathbb{R}.$$

Uniqueness of solutions yields that

$$(3.10) \quad \|u(t)\|_{L^2}^2 = \|\varphi\|_{L^2}^2 \quad \text{for all } t \in \mathbb{R}$$

As  $u \in C_w(\mathbb{R}, L^2(\mathbb{R}^d))$ , we deduce that  $u \in C(\mathbb{R}, L^2(\mathbb{R}^d))$ . Since no mass is lost in the weak convergence (3.8), the convergence is strong in  $L^2$ .  $\square$

**Lemma 3.7.**  $u \in C(\mathbb{R}, H^s(\mathbb{R}^d))$ .

*Proof.* We just need to show the continuity  $t \mapsto u(t) \in H^s(\mathbb{R}^d)$  at  $t = 0$ . It follows from (3.3), (3.7), and the weak lower semicontinuity of the norm that

$$\|u(t)\|_{\dot{H}^s}^2 \leq e^{4|\lambda||t|} \|\varphi\|_{\dot{H}^s}^2.$$

Passing to the limit as  $t \rightarrow 0$  we have

$$\limsup_{t \rightarrow 0} \|u(t)\|_{\dot{H}^s}^2 \leq \|\varphi\|_{\dot{H}^s}^2.$$

On the other hand, it follows from the weak continuity  $t \mapsto u(t) \in H^s(\mathbb{R}^d)$  at  $t = 0$  that

$$\|\varphi\|_{\dot{H}^s}^2 \leq \liminf_{t \rightarrow 0} \|u(t)\|_{\dot{H}^s}^2.$$

So we obtain

$$\lim_{t \rightarrow 0} \|u(t)\|_{\dot{H}^s}^2 = \|\varphi\|_{\dot{H}^s}^2.$$

Therefore, the weak convergence in (3.7) is actually strong.  $\square$

**3.4. Construction of solutions in  $W_1^s$ .** We now assume that  $\varphi \in W_1^s \subset H^s(\mathbb{R}^d)$ . From the dominated convergence theorem we have

$$E_\varepsilon(\varphi) \rightarrow E(\varphi) \quad \text{as } \varepsilon \downarrow 0,$$

recalling that  $E_\varepsilon(\varphi)$  and  $E(\varphi)$  are defined in (3.2) and (1.4), respectively. Let  $\theta \in C_c^1(\mathbb{C}, \mathbb{R})$  satisfying

$$\theta(z) = \begin{cases} 1 & \text{if } |z| \leq 1/4, \\ 0 & \text{if } |z| \geq 1/2, \end{cases} \quad 0 \leq \theta(z) \leq 1 \quad \text{for } z \in \mathbb{C},$$

and set, for  $\varepsilon > 0$ ,

$$F_{1\varepsilon}(u) = \theta(u)|u|^2 \log(|u| + \varepsilon), \quad F_{2\varepsilon}(u) = (1 - \theta(u))|u|^2 \log(|u| + \varepsilon), \\ F_1(u) = \theta(u)|u|^2 \log(|u|^2), \quad F_2(u) = (1 - \theta(u))|u|^2 \log(|u|^2).$$

In the subsequent discussion, we confine the range of  $\varepsilon$  to  $(0, 1/2)$ . The energy expressed in equation (3.1) is denoted as

$$E_\varepsilon(u) = \frac{1}{2} \int |(-\Delta)^{s/2} u|^2 + \frac{\lambda}{2} \int F_{1\varepsilon}(u) + \frac{\lambda}{2} \int F_{2\varepsilon}(u) - \frac{\lambda}{2} \int \mu_\varepsilon(|u|).$$

Taking  $\delta > 0$  sufficiently small,

$$(3.11) \quad \int |F_2(u)| \lesssim \int |u|^{2+\delta} \lesssim \left( \|u\|_{L^2}^{1-\eta} \|u\|_{\dot{H}^s}^\eta \right)^{2+\delta}, \quad \eta = \frac{d}{s} \left( \frac{1}{2} - \frac{1}{2+\delta} \right) \in (0, 1).$$

In particular,

$$\text{for } u \in H^s(\mathbb{R}^d), \quad u \in W_1^s \iff \int |F_1(u)| < \infty.$$

**Lemma 3.8.** *For all  $t \in \mathbb{R}$  we have, as  $\varepsilon \rightarrow 0$ ,*

$$\int \mu_\varepsilon(|u_\varepsilon(t)|) \rightarrow \int |u(t)|^2, \quad \int F_{2\varepsilon}(u_\varepsilon(t)) \rightarrow \int F_2(u(t)).$$

The proof of this lemma is found in [24, Lemma 2.13], and relies on the observation that for any  $\delta \in (0, 1)$  there exists  $C(\delta) > 0$  such that

$$|F_{2\varepsilon}(z) - F_2(w)| \leq C(\delta) (|z|^{1+\delta} + |w|^{1+\delta}) |z - w| \quad \text{for all } z, w \in \mathbb{C}.$$

**Proposition 3.9.** *Let  $\lambda < 0$ . Then,  $u \in (C \cap L^\infty)(\mathbb{R}, W_1^s)$  and  $E(u(t)) = E(\varphi)$  for all  $t \in \mathbb{R}$ .*

*Proof.* For  $\varepsilon \in (0, 1/2)$ , we have  $F_{1\varepsilon}(u) \leq 0$ , and we can rewrite the first two terms  $E_\varepsilon(u)$  as

$$\frac{1}{2} \int |(-\Delta)^{s/2} u|^2 + \frac{\lambda}{2} \int F_{1\varepsilon}(u) = \frac{1}{2} \int |(-\Delta)^{s/2} u|^2 + \frac{|\lambda|}{2} \int |F_{1\varepsilon}(u)|.$$

The weak lower semicontinuity of the norm, Fatou's lemma (for the second term), and Lemma 3.8 imply

$$\begin{aligned} \frac{1}{2} \int |(-\Delta)^{s/2} u(t)|^2 + \frac{|\lambda|}{2} \int |F_1(u(t))| &\leq \liminf_{\varepsilon \rightarrow 0} \left( E_\varepsilon(u_\varepsilon(t)) - \frac{\lambda}{2} \int F_{2\varepsilon}(u_\varepsilon(t)) + \frac{\lambda}{2} \int \mu_\varepsilon(u_\varepsilon(t)) \right) \\ &\leq E(\varphi) - \frac{\lambda}{2} \int F_2(u(t)) + \frac{\lambda}{2} \int |u(t)|^2, \end{aligned}$$

for all  $t \in \mathbb{R}$ . It implies that

$$u(t) \in W_1^s, \quad E(u(t)) \leq E(\varphi) \quad \text{for all } t \in \mathbb{R}.$$

Invoking Lemma 3.6, we obtain that the conservation of the energy

$$(3.12) \quad E(u(t)) = E(\varphi) \quad \text{for all } t \in \mathbb{R}.$$

From inequality (3.11) with  $(2 + \delta)\eta < 2$ , and the identity (3.10) we get

$$\int |(-\Delta)^{s/2} u(t)|^2 + \int |F_1(u(t))| \leq C(E(\varphi), \|\varphi\|_{L^2}),$$

for all  $t \in \mathbb{R}$ . Therefore we deduce that

$$u \in L^\infty(\mathbb{R}, H^s(\mathbb{R}^d)) \quad \text{and} \quad t \mapsto \int |u(t)|^2 \log(|u(t)|^2) \in L^\infty(\mathbb{R}),$$

and thus  $u \in L^\infty(\mathbb{R}, W_1^s)$ . Moreover, from (3.12) and Lemma (3.7), we know that

$$t \mapsto \int |u(t)|^2 \log(|u(t)|^2) \in C(\mathbb{R}) \iff u \in C(\mathbb{R}, W_1^s),$$

which completes the proof.  $\square$

**Proposition 3.10.** *Let  $\lambda > 0$ . Then,  $u \in C(\mathbb{R}, W_1^s)$ .*

*Proof. Step 1.* We show that  $u \in L_{\text{loc}}^\infty(\mathbb{R}, W_1^s)$ . It follows from (3.2) and (3.12) that for any  $T > 0$  and  $t \in [-T, T]$ ,

$$\begin{aligned} \frac{|\lambda|}{2} \int |F_{1\varepsilon}(u_\varepsilon(t))| &= -\frac{\lambda}{2} \int F_{1\varepsilon}(u_\varepsilon(t)) \\ &= -E_\varepsilon(u_\varepsilon(t)) + \frac{1}{2} \int |(-\Delta)^{s/2} u_\varepsilon(t)|^2 \\ &\quad + \frac{\lambda}{2} \left( \int F_{2\varepsilon}(u_\varepsilon(t)) - \int \mu_\varepsilon(|u_\varepsilon(t)|) \right). \end{aligned}$$

Fatou's Lemma and (3.4) imply

$$\frac{|\lambda|}{2} \int |F_1(u(t))| \leq \liminf_{\varepsilon \rightarrow 0} \frac{|\lambda|}{2} \int |F_{1\varepsilon}(u_\varepsilon(t))| \leq -E(\varphi) + C(M_T),$$

for all  $t \in [-T, T]$ . This entails

$$t \mapsto \int |u(t)|^2 \log(|u(t)|^2) \in L_{\text{loc}}^\infty(\mathbb{R}),$$

hence the claim.

**Step 2.** We show that  $u \in C(\mathbb{R}, W_1^s)$ . We check that the map  $t \mapsto \int F_2(u(t))$  is continuous, and then we need to show that so is  $t \mapsto \int F_1(u(t))$ . As in the proof of Lemma 3.7, we consider continuity at  $t = 0$  only. Resuming the computation for the preceding paragraph, we derive

$$\begin{aligned} \frac{|\lambda|}{2} \int |F_{1\varepsilon}(u_\varepsilon(t))| &= -E_\varepsilon(u_\varepsilon(t)) + \frac{1}{2} \int |(-\Delta)^{s/2} u_\varepsilon(t)|^2 \\ &\quad + \frac{\lambda}{2} \left( \int F_{2\varepsilon}(u_\varepsilon(t)) - \int \mu_\varepsilon(|u_\varepsilon(t)|) \right) \\ &\leq -E_\varepsilon(\varphi) + \frac{1}{2} e^{4|\lambda||t|} \|(-\Delta)^{s/2} \varphi\|_{L^2}^2 \\ &\quad + \frac{\lambda}{2} \left( \int F_{2\varepsilon}(u_\varepsilon(t)) - \int \mu_\varepsilon(|u_\varepsilon(t)|) \right). \end{aligned}$$

In view of Fatou's Lemma and Lemma 3.8, we infer

$$\frac{|\lambda|}{2} \int |F_1(u(t))| \leq -E(\varphi) + \frac{1}{2} e^{4|\lambda||t|} \|(-\Delta)^{s/2} \varphi\|_{L^2}^2 + \frac{\lambda}{2} \int F_2(u(t)) - \frac{\lambda}{2} \int |u(t)|^2.$$

Passing to the limit  $t \rightarrow 0$  yields

$$\begin{aligned} \limsup_{t \rightarrow 0} \frac{|\lambda|}{2} \int |F_1(u(t))| &\leq -E(\varphi) + \frac{1}{2} \|(-\Delta)^{s/2} \varphi\|_{L^2}^2 + \frac{\lambda}{2} \int F_2(\varphi) - \frac{\lambda}{2} \int |\varphi|^2 \\ &= -\frac{\lambda}{2} \int F_1(\varphi) = \frac{|\lambda|}{2} \int |F_1(\varphi)|. \end{aligned}$$

Thanks to Fatou's Lemma,

$$\int |F_1(\varphi)| \leq \liminf_{t \rightarrow 0} \int |F_1(u(t))|,$$

hence the proposition.  $\square$

Since regardless of the sign of  $\lambda$ ,  $u \in C(\mathbb{R}, W_1^s)$ , arguing like in the proof of [9, Lemma 2.6], we infer

$$i\partial_t u - (-\Delta)^s u = \lambda u \log(|u|^2) \quad \text{in } (W_1^s)^*.$$

**3.5. The  $H^1$  case.** To conclude the proof of Theorem 1.1, we now assume  $\varphi \in H^1(\mathbb{R}^d)$ . Since  $0 < s < 1$ , we already know that (1.1) has a unique solution  $u \in C(\mathbb{R}, H^s(\mathbb{R}^d))$ . We note that the solution  $u_\varepsilon$  to (3.1) is bounded in  $H^1(\mathbb{R}^d)$ , uniformly on any time interval  $[-T, T]$  and in  $\varepsilon \in (0, 1]$ . Indeed, applying the gradient to (3.1) yields

$$i\partial_t \nabla u_\varepsilon - (-\Delta)^s \nabla u_\varepsilon = 2\lambda \nabla u_\varepsilon \log(|u_\varepsilon| + \varepsilon) + 2\lambda \frac{u_\varepsilon}{|u_\varepsilon| + \varepsilon} \nabla |u_\varepsilon|,$$

and the standard  $L^2$  estimate readily provides

$$\frac{d}{dt} \|\nabla u_\varepsilon\|_{L^2}^2 \leq 4|\lambda| \|\nabla u_\varepsilon\|_{L^2} \|\nabla |u_\varepsilon|\|_{L^2} \leq 4|\lambda| \|\nabla u_\varepsilon\|_{L^2}^2.$$

The conclusion of Theorem 1.1 then follows from the same arguments as above, when we proved that  $u \in C(\mathbb{R}, H^s(\mathbb{R}^d))$ .

#### 4. THE CAUCHY PROBLEM IN THE $H^{2s}$ REGULARITY

In this section we show that if  $\varphi \in X_\alpha^{2s} = H^{2s} \cap \mathcal{F}(H^\alpha)$ , then the solution  $u \in C(\mathbb{R}, H^s)$  provided by Theorem 1.1 actually belongs to  $C_w \cap L_{\text{loc}}^\infty(\mathbb{R}, X_\alpha^{2s})$  (note the obvious relation  $X_\alpha^{2s} \subset H^s$ ).

The strategy is inspired by the classical one in the case of the nonlinear Schrödinger equation, when  $H^2$  regularity is addressed, see [27] (see also [10]): we first prove that  $\partial_t u \in L_{\text{loc}}^\infty(\mathbb{R}, L^2)$ , and eventually use the equation, (1.1), to conclude that  $(-\Delta)^s u \in L_{\text{loc}}^\infty(\mathbb{R}, L^2)$ . The intermediate step consists in considering the nonlinearity, to show that  $u \log |u|^2 \in L_{\text{loc}}^\infty(\mathbb{R}, L^2)$ : due to the singularity of the logarithm at the origin, this is by no means obvious (in particular, the information  $u \in C(\mathbb{R}, H^s)$  and the Sobolev embedding do not suffice to conclude). The first step is indeed:

**Lemma 4.1.** *Let  $\alpha > 0$ ,  $\varphi \in X_\alpha^{2s}$ , and, for  $\varepsilon > 0$ ,  $u_\varepsilon$  solve (3.1). For all  $t \in \mathbb{R}$  we have*

$$\|\partial_t u_\varepsilon(t)\|_{L^2}^2 \leq e^{4|\lambda|t} \|\partial_t u_\varepsilon(0)\|_{L^2}^2,$$

and there exists a map  $C$  independent of  $\varepsilon \in (0, 1)$  such that

$$\|\partial_t u_\varepsilon(0)\|_{L^2} \leq C (\|\varphi\|_{H^{2s}}, \|\langle x \rangle^\alpha \varphi\|_{L^2}).$$

*Proof.* For the first part of the lemma, we compute

$$\begin{aligned} \frac{d}{dt} \|\partial_t u_\varepsilon\|_{L^2}^2 &= 2 \operatorname{Re} (\partial_t^2 u_\varepsilon, \partial_t u_\varepsilon) \\ &= -2 \operatorname{Im} (\partial_t \{(-\Delta)^s u_\varepsilon + 2\lambda u_\varepsilon \log(|u_\varepsilon| + \varepsilon)\}, \partial_t u_\varepsilon) \\ &= -4\lambda \operatorname{Im} \left( \frac{u_\varepsilon}{|u_\varepsilon| + \varepsilon} \partial_t |u_\varepsilon|, \partial_t u_\varepsilon \right) \leq 4|\lambda| \|\partial_t u_\varepsilon(t)\|_{L^2}^2, \end{aligned}$$

hence the announced inequality by Gronwall Lemma. Now in view of (3.1),

$$\begin{aligned} \|\partial_t u_\varepsilon(0)\|_{L^2} &\leq \|(-\Delta)^s u_\varepsilon(0)\|_{L^2} + 2|\lambda| \|u_\varepsilon(0) \log(|u_\varepsilon(0)| + \varepsilon)\|_{L^2} \\ &\leq \|\varphi\|_{H^{2s}} + 2|\lambda| \|\varphi \log(|\varphi| + \varepsilon)\|_{L^2}. \end{aligned}$$

For  $\delta > 0$ ,

$$|\varphi \log(|\varphi| + \varepsilon)| \lesssim |\varphi| ((|\varphi| + \varepsilon)^{-\delta} + (|\varphi| + \varepsilon)^\delta) \lesssim |\varphi|^{1-\delta} + |\varphi|(|\varphi|^\delta + 1),$$

and, provided that  $\delta > 0$  is sufficiently small (in terms of  $s$  and  $\alpha$ ),

$$\|\varphi|^{1-\delta}\|_{L^2} \lesssim \|\langle x \rangle^\alpha \varphi\|_{L^2}^{1-\delta}, \quad \|\varphi(|\varphi|^\delta + 1)\|_{L^2} \lesssim \|\varphi\|_{H^{2s}}^{1+\delta} + \|\varphi\|_{L^2},$$

hence the lemma.  $\square$

Combined with (3.4),

$$(4.1) \quad N_T := \sup_{\varepsilon \in (0,1)} \left( \|u_\varepsilon\|_{C_T(H^s)} + \|\partial_t u_\varepsilon\|_{C_T(L^2)} \right) \leq C(T, \|\varphi\|_{X_\alpha^{2s}}).$$

The unique solution  $u \in C(\mathbb{R}, H^s(\mathbb{R}^d))$  to (1.1) was constructed in Section 3, obtained as the limit of  $u_\varepsilon$  as  $\varepsilon \rightarrow 0$ , and we deduce from (4.1) that

$$u \in W_{\text{loc}}^{1,\infty}(\mathbb{R}, L^2(\mathbb{R}^d)), \quad \partial_t u_\varepsilon(t) \rightharpoonup \partial_t u(t) \quad \text{in } L^2(\mathbb{R}^d).$$

As announced above, the next step consists in showing that  $u \log |u|^2$  belongs to  $L_{\text{loc}}^\infty(\mathbb{R}, L^2)$ . Using the same estimates as in the proof of Lemma 4.1, it suffices to prove the following result:

**Lemma 4.2.** *Let  $0 < s < 1$ ,  $0 < \alpha < 2s$  with  $\alpha \leq 1$ , and  $\varphi \in X_\alpha^{2s}$ . Then the solution  $u \in C(\mathbb{R}, H^s)$  provided by Theorem 1.1 also belongs to  $C_w \cap L_{\text{loc}}^\infty(\mathbb{R}, \mathcal{F}(H^\alpha))$ .*

*Proof.* Let  $\varepsilon > 0$ : multiplying (3.1) by  $\langle x \rangle^\alpha$ , we find

$$i\partial_t(\langle x \rangle^\alpha u_\varepsilon) - \langle x \rangle^\alpha (-\Delta)^s u_\varepsilon = 2\lambda \langle x \rangle^\alpha u_\varepsilon \log(|u_\varepsilon| + \varepsilon),$$

which can be rewritten as

$$i\partial_t(\langle x \rangle^\alpha u_\varepsilon) - (-\Delta)^s(\langle x \rangle^\alpha u_\varepsilon) = 2\lambda \langle x \rangle^\alpha u_\varepsilon \log(|u_\varepsilon| + \varepsilon) - [(-\Delta)^s, \langle x \rangle^\alpha]u_\varepsilon.$$

Multiplying the above equation by  $\langle x \rangle^\alpha \bar{u}_\varepsilon$ , integrating over  $\mathbb{R}^d$  and taking the imaginary part, we obtain, since  $(-\Delta)^s$  is self-adjoint,

$$\frac{d}{dt} \|\langle x \rangle^\alpha u_\varepsilon\|_{L^2}^2 \leq 2 \|\langle x \rangle^\alpha u_\varepsilon\|_{L^2} \| [(-\Delta)^s, \langle x \rangle^\alpha] u_\varepsilon \|_{L^2}.$$

The last factor is estimated thanks to Lemma 2.3: for  $T > 0$  and  $t \in [-T, T]$ ,

$$\| [(-\Delta)^s, \langle x \rangle^\alpha] u_\varepsilon(t) \|_{L^2} \lesssim \|u_\varepsilon(t)\|_{H^s} \lesssim M_T \lesssim N_T.$$

Gronwall Lemma implies that  $u_\varepsilon$  is uniformly bounded in  $L_T^\infty \mathcal{F}(H^\alpha)$ , and the lemma follows by the same arguments as in Section 3.  $\square$

As explained above, we conclude that  $(-\Delta)^s u \in C_w \cap L_{\text{loc}}^\infty(\mathbb{R}, L^2)$ , and Theorem 1.2 follows, keeping Lemma 4.2 in mind.

## REFERENCES

- [1] X. An and X. Yang. Convergence from power-law to logarithm-law in nonlinear fractional Schrödinger equations. *J. Math. Phys.*, 64(1):Paper No. 011506, 13, 2023.
- [2] D. Applebaum. Lévy processes—from probability to finance and quantum groups. *Notices of the AMS*, 51(11):1336–1347, 2004.
- [3] A. H. Ardila. Existence and stability of standing waves for nonlinear fractional Schrödinger equation with logarithmic nonlinearity. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 155:52–64, 2017.
- [4] S. Bhattarai. Existence and stability of standing waves for nonlinear Schrödinger systems involving the fractional Laplacian. *arXiv preprint arXiv:1604.01718*, 2016.
- [5] I. Białyński-Birula and J. Mycielski. Nonlinear wave mechanics. *Ann. Physics*, 100(1-2):62–93, 1976.
- [6] X. Cabré and Y. Sire. Nonlinear equations for fractional Laplacians. I: Regularity, maximum principles, and Hamiltonian estimates. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 31(1):23–53, 2014.
- [7] R. Carles and I. Gallagher. Universal dynamics for the defocusing logarithmic Schrödinger equation. *Duke Math. J.*, 167(9):1761–1801, 2018.



- [8] R. Carles, M. Hayashi, and T. Ozawa. Low regularity solutions to the logarithmic Schrödinger equation. *Pure Appl. Anal.*, 2024. To appear. Archived at <https://arxiv.org/abs/2311.01801>.
- [9] T. Cazenave. Stable solutions of the logarithmic Schrödinger equation. *Nonlinear Anal.*, 7(10):1127–1140, 1983.
- [10] T. Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2003.
- [11] T. Cazenave and A. Haraux. Équations d'évolution avec non linéarité logarithmique. *Ann. Fac. Sci. Toulouse Math. (5)*, 2(1):21–51, 1980.
- [12] Y. Cho, H. Hajaiej, G. Hwang, and T. Ozawa. On the Cauchy problem of fractional Schrödinger equation with Hartree type nonlinearity. *Funkcialaj Ekvacioj*, 56(2):193–224, 2013.
- [13] Y. Cho, H. Hajaiej, G. Hwang, and T. Ozawa. On the orbital stability of fractional Schrödinger equations. *Commun. Pure Appl. Anal.*, 13(3):1267–1282, 2014.
- [14] Y. Cho, T. Ozawa, and S. Xia. Remarks on some dispersive estimates. *Commun. Pure Appl. Anal.*, 10(4):1121–1128, 2011.
- [15] P. d'Avenia, M. Squassina, and M. Zenari. Fractional logarithmic Schrödinger equations. *Mathematical Methods in the Applied Sciences*, 38(18):5207–5216, 2015.
- [16] A. David. *Levy processes and stochastic calculus. Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge*, 2004.
- [17] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.
- [18] V. D. Dinh. Well-posedness of nonlinear fractional Schrödinger and wave equations in Sobolev spaces. *Int. J. Appl. Math.*, 31:483–525, 2018.
- [19] B. Guo, Y. Han, and J. Xin. Existence of the global smooth solution to the period boundary value problem of fractional nonlinear Schrödinger equation. *Applied Mathematics and Computation*, 204(1):468–477, 2008.
- [20] B. Guo and D. Huang. Existence and stability of standing waves for nonlinear fractional Schrödinger equations. *Journal of Mathematical Physics*, 53(8), 2012.
- [21] B. Guo and Z. Huo. Global well-posedness for the fractional nonlinear Schrödinger equation. *Communications in Partial Differential Equations*, 36(2):247–255, 2010.
- [22] X. Guo and M. Xu. Some physical applications of fractional Schrödinger equation. *Journal of mathematical physics*, 47(8), 2006.
- [23] Z. Guo and Y. Wang. Improved Strichartz estimates for a class of dispersive equations in the radial case and their applications to nonlinear Schrödinger and wave equations. *J. Anal. Math.*, 124:1–38, 2014.
- [24] M. Hayashi and T. Ozawa. The Cauchy problem for the logarithmic Schrödinger equation revisited. Preprint, <https://arxiv.org/abs/2309.01695>, 2023.
- [25] E. F. Hefter. Application of the nonlinear Schrödinger equation with a logarithmic inhomogeneous term to nuclear physics. *Physical Review A*, 32(2):1201, 1985.
- [26] Y. Hong and Y. Sire. On fractional Schrödinger equations in Sobolev spaces. *Commun. Pure Appl. Anal.*, 14(6):2265–2282, 2015.
- [27] T. Kato. On nonlinear Schrödinger equations. *Ann. IHP (Phys. Théor.)*, 46(1):113–129, 1987.
- [28] N. Laskin. Fractional quantum mechanics and Lévy path integrals. *Phys. Lett. A*, 268(4-6):298–305, 2000.
- [29] N. Laskin. Fractional Schrödinger equation. *Phys. Rev. E (3)*, 66(5):056108, 7, 2002.
- [30] D. Li. On Kato-Ponce and fractional Leibniz. *Rev. Mat. Iberoam.*, 35(1):23–100, 2019.
- [31] Z.-Q. Wang and C. Zhang. Convergence from power-law to logarithm-law in nonlinear scalar field equations. *Arch. Ration. Mech. Anal.*, 231(1):45–61, 2019.
- [32] H. Zhang and Q. Hu. Existence of the global solution for fractional logarithmic Schrödinger equation. *Comput. Math. Appl.*, 75(1):161–169, 2018.
- [33] K. G. Zloshchastiev. Logarithmic nonlinearity in theories of quantum gravity: Origin of time and observational consequences. *Grav. Cosmol.*, 16:288–297, 2010.

UNIV RENNES, CNRS, IRMAR - UMR 6625, F-35000 RENNES, FRANCE

Email address: Remi.Carles@math.cnrs.fr

Email address: dfangyuan20@163.com