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Optimal control of a single drive car

Fatoumata Dao, Éric Busvelle

Abstract—In this paper, we consider a two-wheel Dubins' car with only one motor. This motor drive both wheels in forward direction but only one wheel, the freewheel, in backward direction. We will calculate the time optimal synthesis. We built such a robot in order to illustrate optimal trajectories...

I. INTRODUCTION

Pontryagin maximum principle (PMP) is an elegant method to solve time optimal control problems, especially in small dimensions. The problem has been completely solved for systems in dimension 2 in [1] where authors provide a general method to build the optimal synthesis. It is no more the case in dimension 3 and many problems remain unsolved from a practical point of view, even if PMP is supposed to give all the necessary tools. In this paper, we consider a three-dimensional system, close to the classical model of the unicycle. The constant speed version, the Dubins' car, introduced by Dubins [2] has been extensively studied as a good non trivial example of nonholonomic system to illustrate the use of Pontryagin maximum principle [4], [5]. In this article, another nonholonomic version of the unicycle is considered. Even if we built such a robot (see Section II-D for a short description of our conception and several ways to test our solutions on existing robot), this problem should be considered an academic problem. Nevertheless, the reduction of dimension 3 to dimension 2 (similar to [3] where the idea of this paper can be found) and the non convexity of the control set are two difficulties that complicate the problem. Therefore, the optimal synthesis (in dimension 2 and so also in dimension 3) is not so obvious.

In the following, we will consider a two-wheel car with a disk shape where one wheel (the left wheel in our case) is completely attached to the motor axis and the second wheel (the right wheel) is a freewheel (the rear wheel of a bike). This last one is driven by the motor axis (practically, we use a dual axis motor) in the forward direction but does not turn in the backward direction.

We solve the problem consisting in reaching the origin in minimal time. Reaching the origin means the barycenter of the robot being at the origin, whatever its orientation. The degree of freedom of this kind of robot is only one in a three dimensional space (position and orientation) and the target is of codimension one.

II. DEVELOPMENT OF THE MODEL

In this section, we develop the (very simple) model of our robot. We will write the equations of motion of the robot

Corresponding author : busvelle@lis-lab.fr
LIS, UMR CNRS 7020, Université de Toulon, Aix Marseille Univ., France

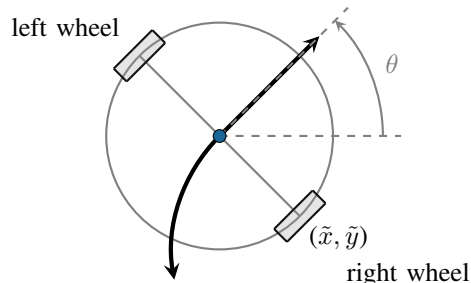


Fig. 1: Coordinates of the two-wheel robot and its possible moves

taking as coordinates the point of contact of the right wheel of the robot (\tilde{x}, \tilde{y}) (instead of the barycenter of the robot, as usual) and its orientation θ .

We denote

- R the radius of each wheel;
- ω the rotation speed of the motor; $\omega \in [0, \omega_{\max}]$;
- L the distance between left and right wheel.

In the forward direction, both wheels are driven by the motor at the same speed and the robot moves in a straight line, as does the right wheel :

$$\begin{cases} \frac{d\tilde{x}}{d\tau} = R\omega \cos \theta \\ \frac{d\tilde{y}}{d\tau} = R\omega \sin \theta \\ \frac{d\theta}{d\tau} = 0 \end{cases}$$

In the backward direction, only the left wheel is driven and the right wheel does not move. The robot turns around its right wheel.

$$\begin{cases} \frac{d\tilde{x}}{d\tau} = 0 \\ \frac{d\tilde{y}}{d\tau} = 0 \\ \frac{d\theta}{d\tau} = \frac{1}{L}R\omega \end{cases}$$

The following model summarize both modes using two controls denoted u^+ and u^- . Let us define the admissible control space

$$U_{\text{adm}} = \{(u^+, u^-) \in [0, 1] \times [-1, 0]; u^+ u^- = 0\}$$

where the condition $u^+ u^- = 0$ means that either $u^+ = 0$ or $u^- = 0$. The system can be written

$$\begin{cases} \frac{d\tilde{x}}{d\tau} = u^+ R\omega_{\max} \cos \theta \\ \frac{d\tilde{y}}{d\tau} = u^+ R\omega_{\max} \sin \theta \\ \frac{d\theta}{d\tau} = -\frac{1}{L}u^- R\omega_{\max} \end{cases}$$

Next, we perform a change of time by setting $t = R\omega_{\max}\tau$ in order to remove the velocity constant $R\omega_{\max}$. The system

can be written more friendly

$$\begin{cases} \frac{d\tilde{x}}{dt} = u^+ \cos \theta \\ \frac{d\tilde{y}}{dt} = u^+ \sin \theta \\ \frac{d\theta}{dt} = -\frac{1}{L}u^- \end{cases}$$

Let us express the target in these coordinates. Clearly, the robot reach the target if the right wheel is on the circle centered at the origin, with diameter L , the left wheel being on the opposite point on the circle. More explicitly, the target is defined by

$$\tilde{N} = \left\{ \left(\frac{L}{2} \sin \theta, -\frac{L}{2} \cos \theta, \theta \right), \theta \in [0, 2\pi[\right\}$$

A. Change of coordinates

Our problem is stated in the three-dimensional space $(\tilde{x}, \tilde{y}, \theta)$ but we will see that it can be reduced as a problem in two-dimension by considering the following change of coordinates :

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

This change of coordinates means that we will consider all the moves with respect to a frame attached to the robot. In these coordinates, the system becomes

$$\begin{cases} \frac{dx}{dt} = u^+ - \frac{1}{L}u^-y \\ \frac{dy}{dt} = \frac{1}{L}u^-x \end{cases}$$

so that x and y does not depend on θ anymore.

In these new coordinates, the target can be written hopefully

$$\tilde{N} = \left\{ \left(0, -\frac{L}{2} \right) \right\}$$

B. Optimal control problem

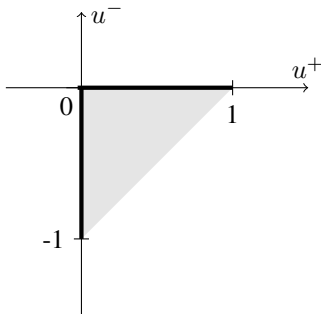


Fig. 2: Admissible control (thick line) and convexified control set

Thanks to our change of coordinates, the problem is now a two-dimensional problem and the target is a single point. However, the set of admissible controls is still not convex. In order to apply the classical Pontryagin maximum principle, we will use the new control set :

$$\text{conv}(U_{\text{adm}}) = \left\{ (u^+, u^-) \in [0, 1] \times [-1, 0]; u^+ - u^- \leq 1 \right\}$$

The Hamiltonian of the problem is

$$\begin{aligned} \mathcal{H}(x, y, p_x, p_y) &= p_x \left(u^+ - \frac{1}{L}u^-y \right) + p_y \left(\frac{1}{L}u^-x \right) \\ &= p_x u^+ + \frac{1}{L}(xp_y - yp_x)u^- \end{aligned} \quad (1)$$

and the adjoint system is

$$\begin{cases} \frac{dp_x}{dt} = -\frac{1}{L}u^-p_y \\ \frac{dp_y}{dt} = \frac{1}{L}u^-p_x \end{cases} \quad (2)$$

We will apply the Pontryagin maximum principle for the set of controls in $\text{conv}(U_{\text{adm}})$.

It is almost clear that optimal trajectories for the initial system are a concatenation of forward arc f: $(u^+, u^-) = (1, 0)$ and backward arc b: $(u^+, u^-) = (0, -1)$. Indeed, ad absurdum, an arc which would not be traveled at maximum speed could be replaced by the same arc at maximum speed so that the length of the trajectory will be shorter. This remark is less obvious if we only suppose u in the convex set. Therefore, our first result is a necessary preliminary result.

Proposition 1: The system has no optimal singular arc, and optimal trajectories are concatenation of arc such that $(u^+, u^-) = (1, 0)$ or $(u^+, u^-) = (0, -1)$

Proof: First of all, we prove that extremals are restricted to the three vertices of the triangle U_{adm} , see Fig. 2.

Denoting $q = \frac{1}{L}(xp_y - yp_x)$, let us remark that:

$$\frac{dq}{dt} = \frac{1}{L} \frac{d}{dt}(xp_y - yp_x) = \frac{1}{L}u^+p_y$$

Using (1), a singular arc is such that either $p_x = 0$ or $q = 0$ or $p_x = -q$

- 1) First case : $p_x = 0$. Along the singular axe, using (2), and since p_y can not be zero, it means that $u^- = 0$. Since $q = \frac{1}{L}xp_y$ and $\frac{dq}{dt} = -u^+p_y$ with $p_y \neq 0$, either $u^+ = 0$ or $u^+ = 1$.
- 2) Second case : $q = 0$. Since $\frac{dq}{dt} = \frac{1}{L}u^+p_y$, we have two subcases:
 - $p_y = 0$ which implies $u^- = 0$ and $p_x \neq 0$ and therefore u^+ is bang bang, 0 or 1.
 - $u^+ = 0$ so either $u^- = 0$ or $u^- \neq 0$ and in the last case, (p_x, p_y) is rotating and then $\frac{d}{dt}p_x \neq 0$ so u^+ is bang bang.
- 3) Third (and last) case : $p_x = q$

$$\frac{d}{dt}(p_x + q) = \frac{1}{L}p_y(u^+ - u^-)$$

so either $p_y = 0$ and we are back to the second case, either $u^+ - u^- = 0 \Rightarrow u^+ = u^- = 0$

We find three possible extremals but $(u^+, u^-) = (0, 0)$ is clearly not optimal. ■

Remark 1: From now and thanks to Proposition 1, we suggest a new parametrization of the control, by introducing the control set $U = [-1, 1]$ and for any $u \in U$:

$$u^+ = \begin{cases} u & \text{if } u > 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad u^- = \begin{cases} u & \text{if } u < 0 \\ 0 & \text{otherwise} \end{cases}$$

An arc with control $u = 1$ is called a forward arc (**f**) and an arc with control $u = -1$ is called a backward arc (**b**).

We will now calculate the optimal synthesis and prove this using reverse time integration.

C. Optimal synthesis

We consider the system in reverse time and we denote the derivative in reverse time using dot notation

$$\begin{cases} \dot{x} = -u^+ + \frac{1}{L}u^-y \\ \dot{y} = -\frac{1}{L}u^-x \end{cases}$$

starting from the target $(x, y) = (0, -\frac{L}{2})$ and we set $(p_x(t), p_y(t)) = (\cos \beta(t), \sin \beta(t))$ where $\beta(0) = \beta_0$ and β_0 is a parameter, $\beta_0 \in [0, 2\pi[$. Using (2) in reverse time, we obtain

$$\dot{\beta} = -\frac{1}{L}u^-$$

hence (p_x, p_y) is a counterclockwise rotating vector if $u^- \neq 0$ and remains constant if $u^- = 0$.

Recall that $q(0) = \frac{1}{2}p_x(0)$ and $\dot{q} = -\frac{1}{L}u^+p_y$. We also denote $\mathcal{C} = p_x + q$, we have $\mathcal{C}(0) = \frac{3}{2}\cos \beta_0$ and $\dot{\mathcal{C}} = -\frac{1}{L}p_y(u^+ - u^-) = -\frac{1}{L}p_y$ (see (3)).

Considering (1) and Proposition 1, $u = 1$ iff $\mathcal{C} > 0$ and $u = -1$ otherwise (using notation from Remark 1).

Proposition 2: Depending on β_0 , optimal trajectories in direct time are either **f**, **b**, **fb**, **bf** or **fbf** (see Figure 3):

- If $0 < \beta_0 < \frac{\pi}{2}$, the optimal control is $u = 1$ followed by $u = -1$ followed by $u = 1$ (in reverse time), arcs in direct time are **fbf** or **bf** (second case) or **f** (third case).
- If $\frac{\pi}{2} < \beta_0 < \frac{3\pi}{2}$, the optimal control is $u = -1$ followed by $u = 1$ (in reverse time), arcs are **bf** or **b**.
- If $\frac{3\pi}{2} \leq \beta_0 < 2\pi$, the optimal control is $u = 1$ and there is no commutation (arc **f**).

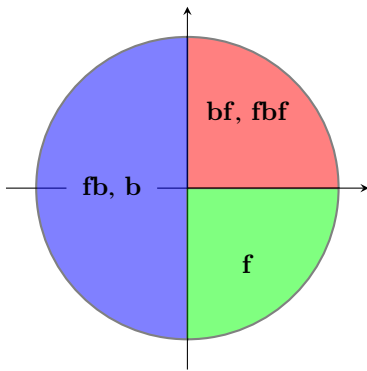


Fig. 3: Different cases depending on β_0 , $\dot{\beta} = -\frac{1}{L}u^-$

Proof: Let us begin by the simplest case $\frac{3\pi}{2} < \beta_0 < 2\pi$. Hence $p_x(0) = \cos \beta(0) > 0$, $q(0) = \frac{1}{2}\cos \beta(0) > 0$ and $u = 1$. Since $\dot{p}_x = \dot{p}_y = 0$ (since $u^- = 0$) we deduce $\dot{q} = -\frac{1}{L}u^+p_y = -\frac{1}{L}u^+\sin \beta(0) > 0$ so there is no commutation. If $\beta(0) = \frac{3\pi}{2}$ then $u = 0$ is an extremal which is not optimal. The corresponding trajectory is simply the negative abscissa axis.

The second case is $\frac{\pi}{2} < \beta_0 < \frac{3\pi}{2}$ so that $\mathcal{C}(0) = \frac{3}{2}\cos \beta_0 < 0$ and therefore $u = -1$. Moreover, $\dot{\mathcal{C}} = -\frac{1}{L}p_y = \dot{p}_x$ and therefore $\mathcal{C} - \mathcal{C}(0) = p_x - p_x(0) \Rightarrow \mathcal{C} = \cos(\beta_0 + \frac{t}{L}) + \frac{1}{2}\cos \beta_0$. There exists t_1 such that $\mathcal{C}(t_1) = 0$ and $\dot{\mathcal{C}}(t_1) = -\frac{1}{L}\sin(\beta_0 + t_1/L) > 0$. t_1 is a commutation, the control u switches to $+1$, then $\dot{\beta} = 0$ and $\dot{\mathcal{C}} = -\frac{1}{L}p_y(t_1) > 0$ and will no more nullify itself since p_y remains constant.

In positive time, trajectories are straight horizontal lines from minus infinity (corresponding to $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = 0$) to the circle of radius 1 (corresponding to $\frac{dx}{dt} = -\frac{1}{L}y$ and $\frac{dy}{dt} = \frac{1}{L}x$)

The third case is $0 \leq \beta_0 < \frac{\pi}{2}$. Since $\mathcal{C}(0) = \frac{3}{2}\cos \beta_0$, we begin by $u = 1$. Therefore, $\dot{\beta} = 0$ so by simple integration, $\mathcal{C}(t) = \frac{3}{2}\cos \beta_0 - \frac{1}{L}t\sin \beta_0$. Clearly, $t_1 = \frac{3L}{2}\cot \beta_0$ is the time of the first commutation since $\mathcal{C}(t_1) = 0$ and $\dot{\mathcal{C}}(t_1) < 0$. After time t_1 , we have $u = -1$. As in the previous case, we remark that $\dot{\mathcal{C}} = \dot{p}_x$ hence $\mathcal{C}(t) = p_x(t) - p_x(t_1) = p_x(t) - p_x(t_0) = \cos(\beta_0 + \frac{1}{L}(t - t_1)) - \cos \beta_0$. There exists a second commutation time t_2 defined by $\mathcal{C}(t_2) = 0$ and we obtain easily $\frac{1}{L}(t_2 - t_1) = 2\pi - 2\beta_0$. From t_2 , $\dot{\mathcal{C}} = -\frac{1}{L}p_y$ which is a negative two-wheels so this is the last switch.

To summarize, in positive time, these optimal trajectories are described by a horizontal straight line followed by an arc and then we are back to the first case. In order to describe explicitly the optimal synthesis, we still have to calculate the only not trivial commutation curve which is crossed at time t_2 . We just have to integrate the system (in reverse time) from $(x(0), y(0) = (0, -\frac{L}{2}))$ to $x(t_2), y(t_2)$ as a parametrized curve with respect to the parameter $\beta_0 \in [0, \frac{\pi}{2}]$.

A tedious but straightforward calculation gives

$$\begin{cases} x(\beta_0) = \frac{L}{2} \left(4 \cos \beta_0 \sin \beta_0 - 3 \frac{\cos \beta_0}{\sin \beta_0} \right) \\ y(\beta_0) = \frac{L}{2} (4 \cos^2 \beta_0 + 1) \end{cases} \quad (3)$$

with $0 < \beta_0 \leq \frac{\pi}{2}$. ■

The 2D optimal synthesis is plotted Figure 4.

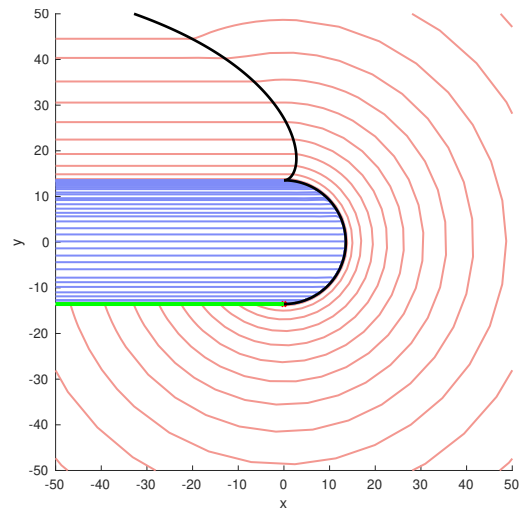


Fig. 4: Two dimensional optimal synthesis ($L = 27.1\text{cm}$)

D. Practical implementation

Figure 4 and equations (3) are sufficient to calculate the time minimum trajectory for any initial position of the robot. Indeed, the negative part of the abscissa axes (green line), the semi circle in the right part of the plane of diameter L and the curve (3) (black curves) split the plane in two domains where the left part corresponds to forward arcs and the right part to backward arcs. Therefore, at any time, if the target is at (x^*, y^*) (the right of the robot being $(0, 0)$), we have the following optimal strategy :

- 1) **fb** If $-\frac{L}{2} \leq y^* \leq \frac{L}{2}$ and $x^* < \frac{1}{2}\sqrt{L^2 - 4y^{*2}}$ then goes forward until $x^* = \frac{1}{2}\sqrt{L^2 - 4y^{*2}}$ and then goes backward until the target. See Figure 5a.
- 2) **fbf** If $\frac{L}{2} < y^* < \frac{5L}{2}$ and $x < \psi(y^*)$ (ψ being defined below, (4)) then goes forward until $x = \psi(y^*)$ then move on to step 3)
- 3) **bf** Otherwise, goes backward until the target is pointed and then goes forward until reaching the target.

where ψ is obtained by eliminating the parameter β_0 in (3) :

$$\psi(y) = \sqrt{12yL - 5L^2 - 4y^2} \frac{L - y}{5L - 2y} \quad (4)$$

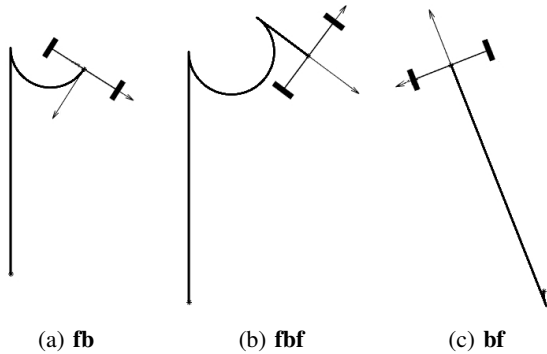


Fig. 5: Three optimal paths : **fb**, **fbf** and **bf**

In order to illustrate this algorithm with an actual robot, we can use any robotic platform such as a Turtlebot which has two wheels with two independant motors. Just ban one of the motor from running in one direction. Another solution is to use your child's Dickie Toys Rc Single Drive 1:32, which is the only car satisfying our control constraints (it is dedicated to young children and the RC has only two buttons : straight forward and turn backward). The last solution is to build a robot with a freewheel and a dual axes motor as explained in the beginning. This is what we have done using a Raspberry Pi for implementing the algorithm and a camera in order to find the target of the floor, see Figure 6. When the target is not in the camera field, the optimal strategy is to go backward until the robot point on the target.

III. CONCLUSION

Although the initial problem is a three-dimensional problem, we proved that it can be reduced to a two-dimensional problem. The set of admissible control is non convex and this difficulty can be solved by considering the convexified

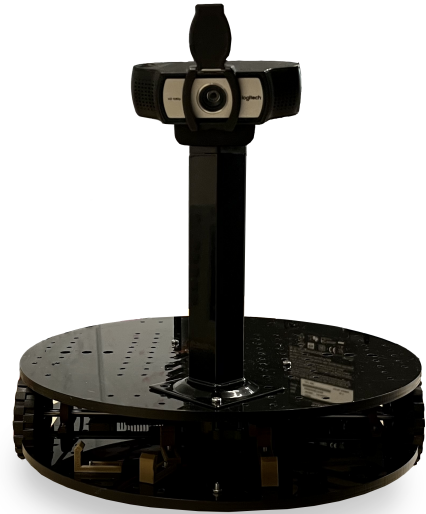


Fig. 6: The single drive robot built with the support of the department of electrical engineering and industrial computing of the university institute of technology of Toulon (IUT GEII)

control set and by verifying that the optimal control remains in the control set. A simpler choice would be to admit that optimal controls are a concatenation of forward and backward arcs so that the convexified control set is a segment instead of a triangle. This extra assumption is natural and will be presented during the talk in the conference.

REFERENCES

- [1] Boscain, Ugo, and Benedetto Piccoli. Optimal synthesis for control systems on 2-D manifolds. Vol. 43. Springer Science & Business Media, 2003.
- [2] Dubins, Lester E. "On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents." American Journal of mathematics 79, no. 3 (1957): 497-516.
- [3] Maillot, Thibault, Ugo Boscain, Jean-Paul Gauthier, and Ulysse Serres. "Lyapunov and minimum-time path planning for drones." Journal of Dynamical and Control Systems 21 (2015): 47-80.
- [4] Reeds, James, and Lawrence Shepp. "Optimal paths for a car that goes both forwards and backwards." Pacific journal of mathematics 145, no. 2 (1990): 367-393.
- [5] Bill Triggs. Motion Planning for Nonholonomic Vehicles: An Introduction. 1993, <https://inria.hal.science/inria-00548415/file/Triggs-nonhol193.pdf>