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# LARGE EIGENVALUES OF THE INTENSITY-DEPENDENT RABI MODEL

RANA AL SAYED HOUSSEIN<sup>1</sup> AND LECH ZIELINSKI<sup>2</sup>

**ABSTRACT.** We prove that the spectrum of the intensity-dependent Rabi Hamiltonian consists of two eigenvalue sequences  $(E_m^+)_{m=0}^\infty, (E_m^-)_{m=0}^\infty$  satisfying a three-term asymptotic formula with the remainder estimate  $O(m^{-1} \ln m)$  when  $m$  tends to infinity. Using the three leading terms of this asymptotic formula, one obtains an approximation analogous to the generalized rotating-wave approximation for large eigenvalues of the usual quantum Rabi model.

**Keywords:** unbounded self-adjoint operators, discrete spectrum, asymptotic distribution of eigenvalues, quantum Rabi model

**Mathematics Subject Classification:** 47A75, 81Q10, 47B25, 47B36.

## 1. INTRODUCTION

The quantum Rabi model (QRM) couples a two-level system (TLS) with a quantized single-mode radiation and is considered as a particularly important model in quantum electrodynamics: we refer to [9] concerning the historical aspects of the QRM and to the review paper [32] for a list of research works and its experimental realizations.

The Hamiltonian of the QRM (see Definition 2.2), is a self-adjoint operator depending on two real parameters:  $g$  (the coupling constant) and  $\Delta$  (the energy separation in the TLS). Its spectrum is discrete and it is natural to consider methods of approximation of its eigenvalues. The earliest approximation method, called the *rotating-wave approximation* (RWA), was introduced in the famous paper of E. T. Jaynes and F. W. Cummings [20]. However, the RWA is a correct approximation only when  $g$  is close to 0 and  $\Delta$  close to 1 for the Hamiltonian given in Definition 2.2. The most popular idea of going beyond the limitations of RWA, bears the name of the *generalized rotating-wave approximation* (GRWA) after E. K. Irish [17]. The same idea was considered before by I. D. Feranchuk, L. I. Komarov, A. P. Ulyanenko [14], under the name of the *zeroth order approximation of the operator method* (see also [15]).

According to the GRWA, the spectrum of the quantum Rabi model, is composed of two eigenvalue sequences:  $(E_m^+)_{m=0}^\infty$  and  $(E_m^-)_{m=0}^\infty$ , satisfying  $E_m^\pm \approx E_{m,\text{GRWA}}^\pm$  and the formula (25) in [14] claims that

$$E_{m,\text{GRWA}}^\pm \approx m - g^2 \pm \frac{\cos\left(4g\sqrt{m} - \frac{\pi}{4}\right)}{\sqrt{2\pi g\sqrt{m}}} \text{ as } m \rightarrow \infty$$

A high quality of this approximation for large  $m$  was confirmed by numerical calculations performed by D. Braak, L. T. H. Nguyen, C. Reyes-Bustos and M. Wakayama [10]. It

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appears that if  $\varepsilon > 0$  then one has the large  $m$  asymptotic formula

$$E_m^\pm = m - g^2 \pm \frac{\Delta \cos(4g\sqrt{m} - \frac{\pi}{4})}{2\sqrt{2\pi g\sqrt{m}}} + O(m^{-1/2+\varepsilon}) \quad (1.1)$$

(see [3]-[4]). Moreover, the three-term asymptotic formula (1.1) allows one to recover the values of parameters of the model from its spectrum (see [5]) and was used by Z. Rudnick [27] to investigate the Braak's conjecture stated in [8].

Recent experimental developments have led to various generalizations of the model by considering the non-linear interactions between the TLS and the quantum field. In particular, a non-linear analogue of the Jaynes-Cummings approximation, was proposed by B. Buck, C. V. Sukumar [11] and its generalization, called the intensity-changing Rabi model, was considered e.g. in the works of B. M. Rodriguez-Lara [25] and L. Duan [12]. The Hamiltonian of the intensity-changing Rabi model is given in Definition 2.3 and physical realizations of this model were considered e.g. by means of classically simulated arrays of coupled waveguides in [24] or a trapped-ion setup driven by a superposition of field modes in [31] (see also [21]).

The purpose of this paper is to investigate large eigenvalues of the intensity-changing Rabi model. Our main result is the three-term asymptotic formula stated in Theorem 2.4. Its three leading terms describe an approximation analogous to the GRWA for large eigenvalues of the usual quantum Rabi model. Moreover, the result of Theorem 2.4 can be used to recover the values of parameters of the model from its spectrum and to investigate the spacing of consecutive eigenvalues similarly as in [10] and [27].

At the end of this introduction, we give a brief overview of earlier results. We refer to [28] and [14], for the earliest works on large eigenvalues of the QRM in physics. The QRM has a parity property, allowing one to express its Hamiltonian as a direct sum  $J_- \oplus J_+$  of two Jacobi operators, i.e. operators defined by infinite tridiagonal matrices acting in  $\ell^2(\mathbb{N})$  (see [29], [5]). A mathematical study of large eigenvalues of Jacobi matrices was initiated by J. Janas and S. Naboko in the paper [19], which contains fundamental ideas of the method of approximate diagonalizations.

The question of the behaviour of large eigenvalues of Jacobi matrices  $J_\pm$ , was first posed by E. A. Tur [29]-[30] and it was mentioned by A. Boutet de Monvel, S. Naboko and L. O. Silva in [1]-[2]. Due to the difficulty of the problem, the papers [1]-[2] give the asymptotic estimates for a simpler class of operators ("modified Jaynes-Cummings models"). However, using the ideas of [19], E. A. Ivanovich [16] proved the two-term asymptotic formula for large eigenvalues of  $J_\pm$  with error  $O(n^{-1/16})$  and the three-term asymptotic formula was proved in [4].

In [7], a three-term formula was proved for large eigenvalues of another non-linear Rabi model, the two-photon QRM. In this paper, we use a result from [7], stated in Theorem 4.2.

## 2. MAIN RESULT

**Notations 2.1.** (a) In what follows,  $\mathbb{Z}$  is the set of integers and  $\mathbb{N} := \{n \in \mathbb{Z} : n \geq 0\}$ .

(b) We denote by  $\ell^2(\mathbb{N})$  the complex Hilbert space of square-summable sequences  $x : \mathbb{N} \rightarrow \mathbb{C}$  equipped with the scalar product

$$\langle x, y \rangle_{\ell^2(\mathbb{N})} = \sum_{m=0}^{\infty} \overline{x(m)}y(m) \quad (2.1)$$

and the norm  $\|x\|_{\ell^2(\mathbb{N})} := \langle x, x \rangle_{\ell^2(\mathbb{N})}^{1/2}$ . For  $s > 0$  we denote

$$\ell^{2,s}(\mathbb{N}) := \left\{ x \in \ell^2(\mathbb{N}) : \sum_{m=0}^{\infty} (1+m^2)^s |x(m)|^2 < \infty \right\} \quad (2.2)$$

(c) The canonical basis of  $\ell^2(\mathbb{N})$  is denoted  $\{e_n\}_{n \in \mathbb{N}}$  (i.e.  $e_n(m) = \delta_{n,m}$  for  $n, m \in \mathbb{N}$ ).

(d) The annihilation and creation operators,  $\hat{a}$  and  $\hat{a}^\dagger$ , are the linear maps  $\ell^{2,1/2}(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$  satisfying

$$\hat{a}^\dagger e_n = \sqrt{n+1} e_{n+1} \quad \text{for } n \in \mathbb{N} \quad (2.3)$$

$$\hat{a} e_0 = 0 \quad \text{and} \quad \hat{a} e_n = \sqrt{n} e_{n-1} \quad \text{for } n \in \mathbb{N} \setminus \{0\}. \quad (2.4)$$

(e) Using  $(1, 0) \in \mathbb{C}^2$  and  $(0, 1) \in \mathbb{C}^2$  as the canonical basis of the Euclidean space  $\mathbb{C}^2$ , we denote by  $\sigma_x, \sigma_z, I_2$ , the linear operators in  $\mathbb{C}^2$  defined by the matrices

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.5)$$

(f) The Hamiltonian of the single-mode radiation is the linear map  $\hat{N} : \ell^{2,1}(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$  defined by the formula

$$\hat{N} e_n = \hat{a}^\dagger \hat{a} e_n = n e_n \quad \text{for } n \in \mathbb{N}. \quad (2.6)$$

**Definition 2.2.** Assume that  $g > 0$  and  $\Delta \in \mathbb{R}$ . The Hamiltonian of the QRM is the linear map  $\mathbf{H} : \mathbb{C}^2 \otimes \ell^{2,1}(\mathbb{N}) \rightarrow \mathbb{C}^2 \otimes \ell^2(\mathbb{N})$  given by

$$\mathbf{H} = I_2 \otimes \hat{N} + g \sigma_x \otimes (\hat{a} + \hat{a}^\dagger) + \frac{\Delta}{2} \sigma_z \otimes I_{\ell^2(\mathbb{N})}. \quad (2.7)$$

**Definition 2.3.** Assume that  $\kappa \geq 0$ ,  $0 < g < \frac{1}{2}$  and  $\Delta \in \mathbb{R}$ . The Hamiltonian of the intensity-changing QRM is the linear map  $\mathbf{H}_\kappa : \mathbb{C}^2 \otimes \ell^{2,1}(\mathbb{N}) \rightarrow \mathbb{C}^2 \otimes \ell^2(\mathbb{N})$  given by

$$\mathbf{H}_\kappa = I_2 \otimes \hat{N} + g \sigma_x \otimes \left( (\hat{N} + 2\kappa)^{1/2} \hat{a} + \hat{a}^\dagger (\hat{N} + 2\kappa)^{1/2} \right) + \frac{\Delta}{2} \sigma_z \otimes I_{\ell^2(\mathbb{N})}. \quad (2.8)$$

Our main result is the following

**Theorem 2.4.** Assume that  $0 < g < \frac{1}{2}$ ,  $\kappa \geq 0$ ,  $\Delta \in \mathbb{R}$  and  $\mathbf{H}_\kappa$  is defined by (2.8). Then there exists  $\{v_m^+\}_{m \in \mathbb{N}} \cup \{v_m^-\}_{m \in \mathbb{N}}$ , an orthonormal basis of  $\mathbb{C}^2 \otimes \ell^2(\mathbb{N})$ , such that

$$\mathbf{H}_\kappa v_m^\pm = E_m^\pm v_m^\pm, \quad m = 0, 1, 2, \dots$$

and the eigenvalue sequences  $(E_m^+)_{m \in \mathbb{N}}$ ,  $(E_m^-)_{m \in \mathbb{N}}$ , satisfy the large  $m$  estimates

$$E_m^\pm = \sqrt{1 - 4g^2} (m + \kappa) - \kappa \pm \tau_m + O(m^{-1} \ln m) \quad (2.9)$$

with

$$\tau_m = \frac{\Delta}{2} \frac{(1 - 4g^2)^{1/4}}{\sqrt{\pi g m}} \cos \left( 4\alpha(m + \kappa) + \pi \left( \frac{1}{4} - \kappa \right) \right), \quad (2.10)$$

where

$$\alpha := \arctan \left( \sqrt{\frac{1 - 2g}{1 + 2g}} \right). \quad (2.11)$$

Our approach allows us to deduce the assertion of Theorem 2.4 from a slightly more general result: Theorem 3.3 stated in Section 3 and proved in Section 5. In Section 3 we also describe some basic properties of the considered model. In Section 4 we describe an auxiliary result from our earlier papers [6], [7]. This result is used in Section 5 to prove Theorem 3.3. In Section 6 we state the Rozenblum's estimate, which is one of ingredients of our proof.

### 3. REMARKS

**3.1. The case  $\Delta = 0$ .** The spectrum of the intensity-changing Rabi model is explicitly known if  $\Delta = 0$ . Indeed, if  $0 < g < 1/2$  and  $\kappa \geq 0$ , then (see [25], [12]) the spectrum of

$$\mathbf{H}_{0,\kappa} = I_2 \otimes \hat{N} + g\sigma_x \otimes \left( (\hat{N} + 2\kappa)^{1/2} \hat{a} + \hat{a}^\dagger (\hat{N} + 2\kappa)^{1/2} \right) \quad (3.1)$$

is composed of the sequence of double eigenvalues

$$E_m^0 = \sqrt{1 - 4g^2} (m + \kappa) - \kappa, \quad m = 0, 1, 2, \dots \quad (3.2)$$

We remark that Theorem 2.4 implies  $E_m^\pm - E_m^0 = O(m^{-1/2})$  as  $m \rightarrow \infty$  in spite of the fact that perturbation  $\mathbf{H}_\kappa - \mathbf{H}_{0,\kappa}$  is not compact.

**3.2. Parity decomposition.** An important property of the intensity-changing Rabi model, is the fact its Hamiltonian is similar to a direct sum of two Jacobi operators, i.e. operators defined by infinite tridiagonal matrices acting in  $\ell^2(\mathbb{N})$ . In what follows, we introduce a class of operators which can be analyzed by our approach similarly as the the intensity-changing Rabi model.

**Notations 3.1.** (a) If  $a : \mathbb{N} \rightarrow \mathbb{R}$ , then  $a(\hat{N}) = \text{diag}(a(j))_{j \in \mathbb{N}}$  denotes the self-adjoint operator in  $\ell^2(\mathbb{N})$  satisfying

$$a(\hat{N})e_j = a(j)e_j \text{ for } j \in \mathbb{N}. \quad (3.3)$$

(b) Assume that  $-\frac{1}{2} < g < \frac{1}{2}$ ,  $\Delta \in \mathbb{R}$  and  $a : \mathbb{N} \rightarrow \mathbb{R}$  satisfies the estimate

$$a(j) \sim gj^{1/2} \text{ as } j \rightarrow \infty. \quad (3.4)$$

Then we define  $\mathbf{H}_a$  as the linear map  $\mathbf{H}_a : \mathbb{C}^2 \otimes \ell^{2,1}(\mathbb{N}) \rightarrow \mathbb{C}^2 \otimes \ell^2(\mathbb{N})$  given by

$$\mathbf{H}_a = I_2 \otimes \hat{N} + g\sigma_x \otimes \left( a(\hat{N})\hat{a} + \hat{a}^\dagger a(\hat{N}) \right) + \frac{\Delta}{2} \sigma_z \otimes I_{\ell^2(\mathbb{N})}. \quad (3.5)$$

(c) Assume that  $-\frac{1}{2} < g < \frac{1}{2}$ ,  $\delta \in \mathbb{R}$  and  $a : \mathbb{N} \rightarrow \mathbb{R}$  satisfies (3.4). Then we define  $\hat{J}_a^\delta$  as the linear map  $\ell^{2,1}(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$  given by

$$\begin{cases} \hat{J}_a^\delta e_j = (j + \delta(-1)^j)e_j + a(j)\sqrt{j+1}e_{j+1} + a(j-1)\sqrt{j}e_{j-1}, & j \geq 1, \\ \hat{J}_a^\delta e_0 = \delta e_0 + a(0)e_1. \end{cases} \quad (3.6)$$

We observe that the associated matrix  $\left( \langle e_j, \hat{J}_a^\delta e_k \rangle_{\ell^2(\mathbb{N})} \right)_{j,k \in \mathbb{N}}$  has the form

$$\begin{pmatrix} \delta & a(0)\sqrt{1} & 0 & 0 & 0 \\ a(0)\sqrt{1} & 1 - \delta & a(1)\sqrt{2} & 0 & 0 \\ 0 & a(1)\sqrt{2} & 2 + \delta & a(2)\sqrt{3} & 0 \\ 0 & 0 & a(2)\sqrt{3} & 3 - \delta & a(3)\sqrt{4} \\ 0 & 0 & 0 & a(3)\sqrt{4} & 4 + \delta \\ & & & & \ddots \end{pmatrix} \quad (3.7)$$

**Lemma 3.2.** *Assume that  $-\frac{1}{2} < g < \frac{1}{2}$  and  $a : \mathbb{N} \rightarrow \mathbb{R}$  satisfies (3.4).*

(a) *If  $\delta \in \mathbb{R}$  and  $\hat{J}_a^\delta$  is defined by Notation 3.1(c), then  $\hat{J}_a^\delta$  is self-adjoint, bounded from below and has discrete spectrum.*

(b) *If  $\Delta \in \mathbb{R}$  and  $\mathbf{H}_a$  is defined by Notation 3.1(b), then  $\mathbf{H}_a$  is similar to the direct sum  $\hat{J}_a^{-\Delta/2} \oplus \hat{J}_a^{\Delta/2}$ .*

*Proof.* (a) It suffices to prove the statement for  $\delta = 0$ . The off-diagonal entries of the matrix (3.7) are of the form  $b(j) = a(j)\sqrt{j+1}$  and the assumption (3.4) ensures  $b(j) \sim gj$  as  $j \rightarrow \infty$ . Therefore

$$\lim_{j \rightarrow \infty} \frac{j^2}{b(j)^2 + b(j-1)^2} = \frac{1}{2g^2} \quad (3.8)$$

and  $-\frac{1}{2} < g < \frac{1}{2} \Rightarrow \frac{1}{2g^2} > 2$ . Since the value of the limit (3.8) is strictly greater than 2, the criterion of Janas-Naboko (see e.g. [19, Section 1]) implies that  $\hat{J}_a^0$  is self-adjoint, bounded from below and has discrete spectrum.

(b) We can write the basis of  $\mathbb{C}^2 \otimes \ell^2(\mathbb{N})$  in the form  $\mathcal{B}_- \cup \mathcal{B}_+$  with

$$\begin{aligned} \mathcal{B}_- &= \{(0, 1) \otimes e_{2j} : j \in \mathbb{N}\} \cup \{(1, 0) \otimes e_{2j+1} : j \in \mathbb{N}\} \\ \mathcal{B}_+ &= \{(1, 0) \otimes e_{2j} : j \in \mathbb{N}\} \cup \{(0, 1) \otimes e_{2j+1} : j \in \mathbb{N}\} \end{aligned}$$

and introduce the corresponding orthogonal decomposition  $\mathbb{C}^2 \otimes \ell^2(\mathbb{N}) = \mathcal{H}_- \oplus \mathcal{H}_+$ , i.e.  $\mathcal{B}_\pm$  is a basis of  $\mathcal{H}_\pm$ . Let  $U_\pm : \ell^2(\mathbb{N}) \rightarrow \mathcal{H}_\pm$  denote the unitary operator defined by

$$\begin{aligned} U_- e_{2j} &= (0, 1) \otimes e_{2j}, & U_- e_{2j+1} &= (1, 0) \otimes e_{2j+1}, \\ U_+ e_{2j} &= (1, 0) \otimes e_{2j}, & U_+ e_{2j+1} &= (0, 1) \otimes e_{2j+1}. \end{aligned}$$

It remains to observe that  $\mathbf{H}_a = H_a^- \oplus H_a^+$  holds with  $H_a^\pm = U_\pm J_a^{\pm\Delta/2} U_\pm^{-1}$ .  $\square$

**3.3. Three-term asymptotic formula for  $\mathbf{H}_a$ .** In Section 5 we will prove

**Theorem 3.3.** *Let  $\Delta \in \mathbb{R}$ ,  $\kappa \geq 0$  and  $0 < g < \frac{1}{2}$ . Assume that  $\mathbf{H}_a$  is defined by (3.5) with  $a : \mathbb{N} \rightarrow \mathbb{R}$  satisfying the estimate*

$$a(j) = gj^{1/2}(1 + \kappa j^{-1} + O(j^{-2})) \text{ as } j \rightarrow \infty. \quad (3.9)$$

*Then there exists  $\{v_m^+\}_{m \in \mathbb{N}} \cup \{v_m^-\}_{m \in \mathbb{N}}$ , an orthonormal basis of  $\mathbb{C}^2 \otimes \ell^2(\mathbb{N})$ , such that*

$$\mathbf{H}_a v_m^\pm = E_m^\pm v_m^\pm, \quad m = 0, 1, 2, \dots$$

*and the eigenvalue sequences  $(E_m^+)_{m \in \mathbb{N}}$ ,  $(E_m^-)_{m \in \mathbb{N}}$ , satisfy the large  $m$  estimates (2.9) with  $\tau_m$  given by (2.10)-(2.11).*

Since  $a(j) = g(j+2\kappa)^{1/2}$  satisfies (3.9), it is clear that Theorem 3.3 is a generalization of Theorem 2.4.

#### 4. AN AUXILIARY PROBLEM IN $\ell^2(\mathbb{Z})$

**Notations 4.1.** (a) We denote by  $\ell^2(\mathbb{Z})$  the complex Hilbert space of square-summable sequences  $x : \mathbb{Z} \rightarrow \mathbb{C}$  equipped with the scalar product

$$\langle x, y \rangle_{\ell^2(\mathbb{Z})} = \sum_{k \in \mathbb{Z}} \overline{x(k)} y(k) \quad (4.1)$$

and the norm  $\|x\|_{\ell^2(\mathbb{Z})} := \langle x, x \rangle_{\ell^2(\mathbb{Z})}^{1/2}$ . For  $s > 0$  we denote

$$\ell^{2,s}(\mathbb{Z}) := \{x \in \ell^2(\mathbb{Z}) : \|x\|_{\ell^{2,s}(\mathbb{Z})} < \infty\}$$

where

$$\|x\|_{\ell^{2,s}(\mathbb{Z})} := \left( \sum_{k \in \mathbb{Z}} (1+k^2)^s |x(k)|^2 \right)^{1/2} \quad (4.2)$$

(b) The canonical basis of  $\ell^2(\mathbb{Z})$  is denoted  $\{\tilde{e}_j\}_{j \in \mathbb{Z}}$  (i.e.  $\tilde{e}_j(k) = \delta_{j,k}$  for  $j, k \in \mathbb{Z}$ ).

(c) If  $\gamma, \delta$  and  $g$  are fixed real numbers, then we define  $\tilde{J}_\gamma^\delta$  as the linear map  $\ell^{2,1}(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$  given by the formula

$$\tilde{J}_\gamma^\delta \tilde{e}_j = (j + \delta(-1)^j) \tilde{e}_j + g(j + \gamma) \tilde{e}_{j+1} + g(j - 1 + \gamma) \tilde{e}_{j-1} \text{ for } j \in \mathbb{Z}. \quad (4.3)$$

The associated matrix  $\left( \langle \tilde{e}_j, \tilde{J}_\gamma^\delta \tilde{e}_k \rangle_{\ell^2(\mathbb{Z})} \right)_{j,k \in \mathbb{Z}}$  has the form

$$\begin{pmatrix} \ddots & & & & & & & & & & \\ & -2 + \delta & g(-2 + \gamma) & 0 & 0 & 0 & & & & & \\ & g(-2 + \gamma) & -1 - \delta & g(-1 + \gamma) & 0 & 0 & & & & & \\ & 0 & g(-1 + \gamma) & \delta & g\gamma & 0 & & & & & \\ & 0 & 0 & g\gamma & 1 - \delta & g(1 + \gamma) & & & & & \\ & 0 & 0 & 0 & g(1 + \gamma) & 2 + \delta & & & & & \\ & & & & & & \ddots & & & & \end{pmatrix}$$

If  $\delta = 0$ , then the operator  $\tilde{J}_\gamma^0$  belongs to the class of operators studied in Edward [13]. In particular, Edward proves that if  $-\frac{1}{2} < g < 1/2$ , then the spectrum of  $\tilde{J}_\gamma^0$  is composed of the eigenvalue sequence  $\{\beta j + (\gamma - \frac{1}{2})(\beta - 1)\}_{j \in \mathbb{Z}}$ , where

$$\beta := \sqrt{1 - 4g^2}. \quad (4.4)$$

Our proof of Theorem 3.3 (and 2.4) will use the following result from [6]:

**Theorem 4.2.** *Let  $\tilde{J}_\gamma^\delta$  be given by (4.3). If  $0 < g < 1/2$  then the spectrum of  $\tilde{J}_\gamma^\delta$  is composed of a non-decreasing sequence of eigenvalues  $\{\lambda_j(\tilde{J}_\gamma^\delta)\}_{j \in \mathbb{Z}}$  which can be labeled so that*

$$\lambda_j(\tilde{J}_\gamma^\delta) = \beta j + \left(\gamma - \frac{1}{2}\right)(\beta - 1) + \mathfrak{r}_\gamma^\delta(j) + O(j^{-1} \ln j) \text{ as } j \rightarrow \infty \quad (4.5)$$

and

$$\mathfrak{r}_\gamma^\delta(j) = \delta \left( \frac{\beta}{2\pi g j} \right)^{1/2} \cos(4\alpha j + \hat{\theta}_\gamma), \quad (4.6)$$

where  $\beta$  is given by (4.4),  $\alpha$  by (2.11) and

$$\hat{\theta}_\gamma = \left(\gamma - \frac{1}{2}\right)(4\alpha - \pi) + \frac{\pi}{4}. \quad (4.7)$$

*Proof.* See [7, Theorem 2.3].  $\square$

## 5. PROOF OF THEOREM 3.3

### 5.1. Step 1 of the proof of Theorem 3.3.

**Notations 5.1.** Let  $\gamma, \delta$  be real numbers and  $0 < g < 1/2$ .

(a) We define  $\hat{J}_\gamma^\delta$  as the linear map  $\ell^{2,1}(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$  given by

$$\begin{cases} \hat{J}_\gamma^\delta e_j = (j + \delta(-1)^j) e_j + g(j + \gamma) e_{j+1} + g(j - 1 + \gamma) e_{j-1} \text{ for } j \geq 1, \\ \hat{J}_\gamma^\delta e_0 = \delta e_0 + g\gamma e_1. \end{cases} \quad (5.1)$$

(b) We observe that  $\hat{J}_\gamma^\delta = \hat{J}_a^\delta$  if  $\hat{J}_a^\delta$  is defined by (3.6) with  $a(j) = g(j + \gamma)(j + 1)^{-1/2}$ . Therefore  $\hat{J}_\gamma^\delta$  is self-adjoint, bounded from below and has discrete spectrum. The non-decreasing sequence of eigenvalues of  $\hat{J}_\gamma^\delta$  will be denoted  $\{\lambda_n(\hat{J}_\gamma^\delta)\}_{n \in \mathbb{N}}$ .

**Lemma 5.2.** *Let  $\gamma, \delta$  be real numbers,  $0 < \gamma < 1/2$  and let  $\hat{J}_\gamma^\delta$  be given by (5.1).*

(a) *If  $\delta = 0$  and  $N \geq 1$ , then the estimate*

$$\lambda_n(\hat{J}_\gamma^0) = d_{n,\gamma} + O(n^{-N}) \text{ as } n \rightarrow \infty \quad (5.2)$$

*holds with*

$$d_{n,\gamma} := \beta n + \left(\gamma - \frac{1}{2}\right)(\beta - 1), \quad (5.3)$$

*where  $\beta$  is given by (4.4), i.e.  $\beta = \sqrt{1 - 4g^2}$ .*

(b) *If  $\hat{J}_a^\delta$  is defined by (3.6) and  $a$  satisfies (3.9), then*

$$\lambda_n(\hat{J}_a^\delta) = \lambda_n(\hat{J}_{\kappa+1/2}^\delta) + O(n^{-1}) \text{ as } n \rightarrow \infty \quad (5.4)$$

*Proof.* (a) See J. Janas, M. Malejki [18, Theorem 3.4].

(b) Assume that  $a$  satisfies (3.9) and denote  $b(j) := a(j)\sqrt{j+1}$ . Then

$$\begin{aligned} b(j) &= gj^{1/2}(1 + \kappa j^{-1} + O(j^{-2}))j^{1/2}(1 + \frac{1}{2}j^{-1} + O(j^{-2})) \\ &= g(j + \kappa + \frac{1}{2}) + O(j^{-1}) \text{ as } j \rightarrow \infty \end{aligned}$$

If  $j \geq 1$  then  $(\hat{J}_a^\delta - \hat{J}_{\kappa+1/2}^\delta)e_j = r(j)e_{j+1} + r(j-1)e_{j-1}$  holds with

$$r(j) = b(j) - g(j + \kappa + \frac{1}{2}) = O(j^{-1})$$

and implies that the operator  $\hat{J}_a^\delta - \hat{J}_{\kappa+1/2}^\delta$  is bounded  $\ell^2(\mathbb{N}) \rightarrow \ell^{2,1}(\mathbb{N})$ . Assume that  $c \in \mathbb{R}$  is not in the spectrum of  $\hat{J}_a^\delta$  or  $\hat{J}_{\kappa+1/2}^\delta$ . Then

$$|\hat{J}_a^\delta - c|(\hat{J}_a^\delta - \hat{J}_{\kappa+1/2}^\delta) \quad \text{and} \quad |\hat{J}_{\kappa+1/2}^\delta - c|(\hat{J}_a^\delta - \hat{J}_{\kappa+1/2}^\delta)$$

are bounded operators in  $\ell^2(\mathbb{N})$  and applying the Rozenblum's theorem with  $\alpha = -1$  (see Section 6), we obtain

$$\lambda_n(\hat{J}_a^\delta) - \lambda_n(\hat{J}_{\kappa+1/2}^\delta) = O(|\lambda_n(\hat{J}_{\kappa+1/2}^\delta) - c|^{-1}) \text{ as } n \rightarrow \infty \quad (5.5)$$

However, (5.2)-(5.3) and the min-max principle imply

$$\lambda_n(\hat{J}_{\kappa+1/2}^\delta) = \beta n + O(1), \quad (5.6)$$

hence the right-hand side of (5.5) is  $O(n^{-1})$ .  $\square$

## 5.2. Step 2 of the proof of Theorem 3.3.

**Notations 5.3.** (a) If  $\mathcal{J} \subset \mathbb{Z}$  then  $\ell^2(\mathcal{J})$  is identified with the closed subspace of  $\ell^2(\mathbb{Z})$  generated by  $\{\tilde{e}_j\}_{j \in \mathcal{J}}$ , i.e. with  $\{x \in \ell^2(\mathbb{Z}) : x(k) = 0 \text{ for } k \in \mathbb{Z} \setminus \mathcal{J}\}$ . Thus we can write

$$\ell^2(\mathbb{Z}) = \ell^2(\mathbb{Z} \setminus \mathbb{N}) \oplus \ell^2(\mathbb{N}). \quad (5.7)$$

(b) If  $\mathcal{J} \subset \mathbb{Z}$  then  $\Pi_{\mathcal{J}}$  denotes the orthogonal projection  $\ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathcal{J})$ .

(c) We identify  $\tilde{e}_j$  and  $e_j$  for  $j \geq 0$ .



**Lemma 5.4.** *Let  $\gamma, \delta$  be real numbers and  $0 < g < 1/2$ . Let  $\tilde{J}_\gamma^\delta$  and  $\hat{J}_\gamma^\delta$  be given by Notations 4.1 and 5.1. If  $\{\lambda_n(\hat{J}_\gamma^\delta)\}_{n \in \mathbb{N}}$  is the non-decreasing sequence of eigenvalues of  $\hat{J}_\gamma^\delta$  and  $\{\lambda_j(\tilde{J}_\gamma^\delta)\}_{j \in \mathbb{Z}}$  is the non-decreasing sequence of eigenvalues satisfying (4.5), then there exists  $\kappa \in \mathbb{Z}$  such that*

$$\lambda_n(\hat{J}_\gamma^\delta) = \lambda_{n+\kappa}(\tilde{J}_\gamma^\delta) + O(n^{-1}) \text{ as } n \rightarrow \infty.$$

*Proof.* Denote  $\check{J}_\gamma^\delta := \Pi_{\mathbb{Z} \setminus \mathbb{N}} \tilde{J}_\gamma^\delta|_{\ell^2(\mathbb{Z} \setminus \mathbb{N})}$ . Then, reasoning as in the proof of Lemma 3.2(a), we find that  $-\check{J}_\gamma^\delta$  is self-adjoint, bounded from below and has discrete spectrum. Moreover the operator  $R_\gamma^\delta := \tilde{J}_\gamma^\delta - (\check{J}_\gamma^\delta \oplus \hat{J}_\gamma^\delta)$  is self-adjoint and satisfies  $R_\gamma^\delta \tilde{e}_j = 0$  if  $|j| > 1$ . Assume that  $c \in \mathbb{R}$  is not in the spectrum of  $\tilde{J}_\gamma^\delta$  or  $\check{J}_\gamma^\delta \oplus \hat{J}_\gamma^\delta$ . Then

$$|\tilde{J}_\gamma^\delta - c|R_\gamma^\delta \quad \text{and} \quad |\check{J}_\gamma^\delta \oplus \hat{J}_\gamma^\delta - c|R_\gamma^\delta$$

are bounded operators in  $\ell^2(\mathbb{Z})$  and applying the Rozenblum's theorem with  $\alpha = -1$  (see Section 6), we can find an eigenvalue sequence  $\{\lambda_j(\check{J}_\gamma^\delta \oplus \hat{J}_\gamma^\delta)\}_{j \in \mathbb{Z}}$  such that

$$\lambda_j(\tilde{J}_\gamma^\delta) - \lambda_j(\check{J}_\gamma^\delta \oplus \hat{J}_\gamma^\delta) = O(|\lambda_j(\tilde{J}_\gamma^\delta) - c|^{-1}) \text{ as } |j| \rightarrow \infty \quad (5.8)$$

and the right-hand side of (5.8) is  $O(j^{-1})$  due to (4.5). We observe that the spectrum of  $\check{J}_\gamma^\delta \oplus \hat{J}_\gamma^\delta$  is composed of eigenvalues of  $\check{J}_\gamma^\delta$  and  $\hat{J}_\gamma^\delta$ . Moreover, the spectrum of  $\check{J}_\gamma^\delta$  is included in a certain half-line  $(-\infty, C_0]$ . Therefore, choosing  $n_0 \in \mathbb{N}$  large enough, we ensure  $\lambda_{n_0}(\hat{J}_\gamma^\delta) = \lambda_{n_0+\kappa}(\check{J}_\gamma^\delta \oplus \hat{J}_\gamma^\delta)$  for a certain  $\kappa \in \mathbb{Z}$  and

$$n \geq n_0 \Rightarrow \lambda_n(\hat{J}_\gamma^\delta) = \lambda_{n+\kappa}(\check{J}_\gamma^\delta \oplus \hat{J}_\gamma^\delta). \quad (5.9)$$

We complete the proof, combining (5.8) and (5.9).  $\square$

### 5.3. Step 3 of the proof of Theorem 3.3.

**Lemma 5.5.** *The assertion of Lemma 5.4 holds with  $\kappa = 0$ .*

*Proof.* In this proof we fix  $\gamma$  and consider  $\delta \rightarrow \kappa(\delta)$  as a function  $\mathbb{R} \rightarrow \mathbb{Z}$ , satisfying

$$|\lambda_n(\hat{J}_\gamma^\delta) - \lambda_{n+\kappa(\delta)}(\tilde{J}_\gamma^\delta)| \leq C_\delta n^{-1} \text{ for } n \geq 1, \quad (5.10)$$

where  $C_\delta$  is a constant independent of  $n$ . We observe that, combining Theorem 4.2 with (5.10), we get

$$|\lambda_n(\hat{J}_\gamma^\delta) - d_{n+\kappa(\delta), \gamma}| \leq C'_\delta n^{-1/2} \text{ for } n \geq 1, \quad (5.11)$$

where  $C'_\delta$  is a constant independent of  $n$  and

$$d_{n, \gamma} := \beta n + \left(\gamma - \frac{1}{2}\right)(\beta - 1). \quad (5.12)$$

We recall that Lemma 5.2 ensures  $\kappa(0) = 0$ . Consider  $\delta' \in \mathbb{R}$ . Then the min-max principle ensures

$$|\lambda_n(\hat{J}_\gamma^\delta) - \lambda_n(\hat{J}_\gamma^{\delta'})| \leq |\delta - \delta'|. \quad (5.13)$$

Using (5.13), (5.11) and an analogical estimate for  $\delta'$ , we obtain

$$|d_{n+\kappa(\delta), \gamma} - d_{n+\kappa(\delta'), \gamma}| \leq (C'_\delta + C'_{\delta'})n^{-1/2} + |\delta - \delta'| \quad (5.14)$$

and consequently

$$\limsup_{n \rightarrow \infty} |d_{n+\kappa(\delta), \gamma} - d_{n+\kappa(\delta'), \gamma}| \leq |\delta - \delta'|. \quad (5.15)$$

However, (5.12) gives

$$\lim_{n \rightarrow \infty} |d_{n+\kappa(\delta), \gamma} - d_{n+\kappa(\delta'), \gamma}| = |\kappa(\delta) - \kappa(\delta')| \beta \quad (5.16)$$

and, combining (5.16) with (5.15), we find  $|\kappa(\delta) - \kappa(\delta')| \leq \beta^{-1} |\delta - \delta'|$ . Thus,  $\kappa : \mathbb{R} \rightarrow \mathbb{Z}$  is locally constant, hence  $\kappa(\delta) = \kappa(0)$  and  $\kappa(0) = 0$ .  $\square$

**5.4. End of the proof of Theorem 3.3.** Combining Lemma 5.5 with Theorem 4.2, we obtain

$$\lambda_n(\hat{J}_\gamma^\delta) = d_{n, \gamma} + \mathfrak{r}_\gamma^\delta(n) + O(n^{-1} \ln n) \text{ as } n \rightarrow \infty \quad (5.17)$$

with  $\mathfrak{r}_\gamma^\delta$  given by (4.6)-(4.7). Using Lemma 5.2(b) and (5.17), we find

$$\lambda_n(\hat{J}_a^\delta) = d_{n, \kappa+1/2} + \mathfrak{r}_{\kappa+1/2}^\delta(n) + O(n^{-1} \ln n) \text{ as } n \rightarrow \infty. \quad (5.18)$$

To complete the proof, we observe that the parity decomposition (see Lemma 3.2(b)) gives the assertion of Theorem 3.3 with  $E_m^\pm = \lambda_m(\hat{J}_a^{\pm\Delta/2})$  and we obtain (2.9)-(2.10), using (5.18) with  $\delta = \pm\Delta/2$ .

## 6. ROZENBLUM'S ESTIMATE

**Theorem 6.1** (G. V. Rozenblum). *Let  $A$  and  $B$  be unbounded self-adjoint operators with discrete spectrum in a separable Hilbert space  $\mathcal{H}$  such that  $0$  not in the spectrum of  $A$  or  $B$ . Assume that there exists  $\alpha < 1$  such that*

$$|A|^{-\alpha}(A - B) \text{ and } |B|^{-\alpha}(A - B) \quad (6.1)$$

are bounded operators in  $\mathcal{H}$ .

(a) *If  $A$  and  $B$  are bounded from below and  $\{\lambda_n(A)\}_{n \in \mathbb{N}}$ ,  $\{\lambda_n(B)\}_{n \in \mathbb{N}}$  are non-decreasing eigenvalue sequences of  $A$  and  $B$  (counting the multiplicities), then*

$$\lambda_n(A) - \lambda_n(B) = O(|\lambda_n(A)|^\alpha) \text{ as } n \rightarrow \infty. \quad (6.2)$$

(b) *If  $A$  and  $B$  are not semi-bounded and  $\{\lambda_j(A)\}_{j \in \mathbb{Z}}$  is an eigenvalue sequence of  $A$  (counting the multiplicities), then we can find  $\{\lambda_j(B)\}_{j \in \mathbb{Z}}$ , an eigenvalue sequence of  $B$  (counting the multiplicities), so that*

$$\lambda_j(A) - \lambda_j(B) = O(|\lambda_j(A)|^\alpha) \text{ as } |j| \rightarrow \infty. \quad (6.3)$$

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