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NON-LOCALLY DISCRETE ACTIONS ON THE CIRCLE WITH AT MOST N FIXED POINTS

CHRISTIAN BONATTI, JOÃO CARNEVALE, AND MICHELE TRIESTINO

ABSTRACT. A subgroup of $\text{Homeo}_+(\mathbb{S}^1)$ is Möbius-like if every element is conjugate to an element of $\text{PSL}(2, \mathbb{R})$. In general, a Möbius-like subgroup of $\text{Homeo}_+(\mathbb{S}^1)$ is not necessarily (semi-)conjugate to a subgroup of $\text{PSL}(2, \mathbb{R})$, as discovered by N. Kovačević [Trans. Amer. Math. Soc. **351** (1999), 4823–4835]. Here we determine simple dynamical criteria for the existence of such a (semi-)conjugacy. We show that Möbius-like subgroups of $\text{Homeo}_+(\mathbb{S}^1)$ which are elementary (namely, preserving a Borel probability measure), are semi-conjugate to subgroups of $\text{PSL}(2, \mathbb{R})$. On the other hand, we provide an example of elementary subgroup of $\text{Diff}_+^\infty(\mathbb{S}^1)$ satisfying that every non-trivial element fixes at most 2 points, which is not isomorphic to any subgroup of $\text{PSL}(2, \mathbb{R})$. Finally, we show that non-elementary, non-locally discrete subgroups acting with at most N fixed points are conjugate to a dense subgroup of some finite central extension of $\text{PSL}(2, \mathbb{R})$.

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1. INTRODUCTION

Let X be a topological space, and $G \leq \text{Homeo}(X)$ a subgroup of homeomorphisms of X . Given $N \in \mathbb{N}$, we say that G *has at most N fixed points* if every non-trivial element of G has at most N fixed points. When $N = 0$, this means that the action of G on X is free. A natural problem is to characterize actions with at most N fixed points:

Question 1.1. *Given $N \in \mathbb{N}$, which subgroups of $\text{Homeo}(X)$ have at most N fixed points?*

Here we will address this problem for the case when $X = \mathbb{S}^1$ is the circle (the case $N = 2$ and $X = \mathbb{R}$ has been discussed by the second author in [5], extending classical work of Hölder and Solodov). To simplify the discussion, we will rather consider subgroups of $\text{Homeo}_+(\mathbb{S}^1)$, the group of order-preserving homeomorphisms of the circle, which is of index 2 in $\text{Homeo}(\mathbb{S}^1)$. The classical group $\text{PSL}(2, \mathbb{R})$ of Möbius transformations acts on the circle with at most 2 fixed points. This group may be seen in two ways:

- it is the group of projective transformations of the projective line \mathbb{RP}^1 ;
- it is also the group of isometries of the hyperbolic disc \mathbb{D} , which acts on its circle at infinity.

For any integer $k \geq 1$, the k -fold central extension

$$1 \rightarrow \mathbb{Z}_k \rightarrow \text{PSL}^{(k)}(2, \mathbb{R}) \rightarrow \text{PSL}(2, \mathbb{R})$$

also acts on the circle (which is homeomorphic to its k -fold covering), and it gives a natural example of group acting with at most $2k$ fixed points. Somehow we want to understand to what extent groups with at most $2k$ fixed points are comparable to subgroups of $\text{PSL}^{(k)}(2, \mathbb{R})$. The case $k = 1$ is already interesting on its own. For this, we say that a group G is *Möbius-like* if every element is individually conjugate to an element in $\text{PSL}(2, \mathbb{R})$. In practice, this means that any element in G is either conjugate to a rotation (*elliptic*), or it admits exactly one fixed point (*parabolic*), or it admits exactly two fixed points, one attracting and one repelling (*hyperbolic*). On the other hand, a basic example of subgroup of $\text{Homeo}(\mathbb{S}^1)$ with at most 2 fixed points but not Möbius-like,

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is the cyclic subgroup generated by a homeomorphism with exactly two fixed points, but which is not (topologically) hyperbolic: in this case both fixed points are (topologically) *parabolic*, namely repelling on one side and attracting on the other.

Several works have exhibited conditions under which Möbius-like groups are conjugate to subgroups of $\mathrm{PSL}(2, \mathbb{R})$. Notably, a major result in this context has been obtained by Tukia [20] (in the torsion-free case), and then independently extended by Gabai [8] and Casson and Jungreis [6], after which subgroups of $\mathrm{PSL}(2, \mathbb{R})$ are dynamical determined through the notion of convergence groups. Let us also mention the more recent work of Baik [3], describing a characterization of subgroups of $\mathrm{PSL}(2, \mathbb{R})$ in terms of invariant laminations on the circle. In [15], Kovačević proved that a Möbius-like subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ with a global fixed point is always semi-conjugate to a subgroup of $\mathrm{PSL}(2, \mathbb{R})$. We extend this result to *elementary* subgroups, that is, subgroups of $\mathrm{Homeo}_+(\mathbb{S}^1)$ preserving a Borel probability measure on \mathbb{S}^1 . Concretely, a subgroup $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ is elementary if either it admits a finite orbit or it is (continuously) semi-conjugate to a subgroup of rotations (see the discussion in §3.1, or Ghys [9, Proposition 6.17]).

Remark 1.2. This terminology is coherent with the usual notion of elementary subgroup of $\mathrm{PSL}(2, \mathbb{R})$, which is a subgroup whose action on $\overline{\mathbb{D}}$ has a finite orbit. Concretely, an elementary subgroup of $\mathrm{PSL}(2, \mathbb{R})$ is either conjugate in $\mathrm{PSL}(2, \mathbb{R})$ to a subgroup of rotations, or has a finite orbit on the circle at infinity (which has to be either a fixed point or a periodic orbit of order 2); see Kim, Koberda, and Mj [13, Proposition 2.1].

In practice, compared to the work of Kovačević, we also consider the case of periodic orbit of order 2, which is a bit harder than the other cases.

Theorem A. *If $G < \mathrm{Homeo}_+(\mathbb{S}^1)$ is an elementary, Möbius-like subgroup, then G is continuously semi-conjugate to an elementary subgroup of $\mathrm{PSL}(2, \mathbb{R})$ and, moreover, the corresponding morphism $G \rightarrow \mathrm{PSL}(2, \mathbb{R})$ is injective.*

Our proof is a modern revised version of that from [15], which is somehow difficult to digest as Kovačević needs to prove Solodov’s theorem along way. Originally presented in [19, Theorem 3.21], this is now a classical result in the subject (see Ghys [9, Theorem 6.12] and Farb–Franks [7, Theorems 1.3 and 1.4]). For further reference, it is more convenient to add a first item to its usual statement, which is a direct consequence of Hölder’s theorem [9, Theorem 6.10].

Theorem 1.3 (Solodov). *Let $G < \mathrm{Homeo}_+(\mathbb{R})$ be a subgroup with at most 1 fixed point. Then*

- *either the action of G admits a unique fixed point p , in which case the restriction of the action of G to $(-\infty, p)$ (respectively, to $(p, +\infty)$) is continuously semi-conjugate to an action by translations, and the corresponding morphism $G \rightarrow \mathbb{R}$ is injective, or*
- *the action of G is continuously semi-conjugate to an action by affine transformations, and the corresponding morphism $G \rightarrow \mathrm{Aff}_+(\mathbb{R})$ is injective.*

In the opposite direction, Kovačević [14] showed that the above result fails for general Möbius-like groups.

Theorem 1.4 (Kovačević). *There exist finitely generated (even finitely presented) Möbius-like subgroups of $\mathrm{Homeo}_+(\mathbb{S}^1)$ whose action is minimal (every orbit is dense) but not conjugate into $\mathrm{PSL}(2, \mathbb{R})$.*

When a subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ has at most 2 fixed points but is not semi-conjugate into $\mathrm{PSL}(2, \mathbb{R})$ (or not even Möbius-like), one may still wonder whether it is *abstractly* isomorphic to a subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Building on the proof of Theorem A, we settle this problem for the negative.

Theorem B. *There exists a finitely generated, elementary group of smooth (C^∞) circle diffeomorphisms, with at most 2 fixed points, and which is not isomorphic to any subgroup of $\mathrm{PSL}(2, \mathbb{R})$.*

For the next main result, we will consider the following topological condition (inspired by work of Rebelo [18] in the real-analytic setting). To state it, recall (see [9, Proposition 5.6]) that any subgroup $G \leq \text{Homeo}_+(\mathbb{S}^1)$ admits a non-empty G -invariant compact subset $\Lambda \subset \mathbb{S}^1$ which is minimal with respect to inclusion, called a *minimal invariant subset*; moreover, if G does not have any periodic orbit, the minimal invariant subset is always unique, and it is either the whole circle or a Cantor set. In particular, when $G \leq \text{Homeo}_+(\mathbb{S}^1)$ is a non-elementary subgroup, we can refer to the minimal invariant subset of G . We also say that a non-empty interval $I \subset \mathbb{S}^1$ is *wandering* (for G) if $\Lambda \cap I = \emptyset$.

Definition 1.5. A non-elementary subgroup $G \leq \text{Homeo}_+(\mathbb{S}^1)$ of circle homeomorphisms is *locally discrete* if for every non-wandering interval $I \subset \mathbb{S}^1$, the identity is isolated among the collection of restrictions $\{g|_I : g \in G\} \subset C^0(I; \mathbb{S}^1)$, with respect to the C^0 topology.

It is clear that if a subgroup is locally discrete, then it is discrete in the usual sense. In fact, we will see that in the C^0 topology, the two properties are equivalent (Lemma 4.5), whereas in the real-analytic setting of Rebelo [18], the equivalence is an open problem (see the discussion in Alvarez *et al.* [1, Remark 1.9]). The condition of considering non-wandering intervals only in the definition sounds technical, but it is actually the appropriate one (see Proposition 4.2). For non-locally discrete groups with at most N fixed points, we prove the following.

Theorem C. *Let $G < \text{Homeo}_+(\mathbb{S}^1)$ be a non-elementary, non-locally discrete subgroup with at most N fixed points. Then, there exists $k \geq 1$ such that G is conjugate to a dense subgroup of $\text{PSL}^{(k)}(2, \mathbb{R})$.*

Remark 1.6. The degree of the extension k is given by half of the largest number of fixed points for non-trivial elements in G (which is thus necessarily even).

Corollary D. *Let $G < \text{Homeo}_+(\mathbb{S}^1)$ be a non-elementary, non-locally discrete subgroup with at most 2 fixed points. Then, G is conjugate to a dense subgroup of $\text{PSL}(2, \mathbb{R})$.*

Remark 1.7. Another consequence of Theorem C is that, if moreover the group G is finitely generated, it necessarily contains elements with irrational number (see Kim, Koberda, and Mj [13, Lemma 2.24]). Actually, a key step of the proof (see Lemma 4.10) is to show that the *closure* \overline{G} contains an element with irrational rotation number. Let us also comment that when considering finitely generated subgroups $G \leq \text{Diff}_+^1(\mathbb{S}^1)$ which are non-locally discrete with respect to the C^1 topology, the analogue conclusion is still open, even in real-analytic regularity (see Matsuda [17]).

Let us describe the structure of the paper. After introducing the main terminology in Section 2, we discuss Theorems A and B in Section 3. The last section is devoted to the proof of Theorem C. Let us give a quick outline of the proof. First of all, we remark that the action of G is minimal (Lemma 4.5). The next step is to understand which elements belong to the C^0 closure \overline{G} : we prove that \overline{G} has also at most N fixed points (Lemma 4.7) and, up to conjugacy, it contains the group of rotations $\text{SO}(2)$ (Lemma 4.10). We then conclude by using the classification by Giblin and Markovic [10] (see Theorem 4.11) of closed subgroups of $\text{Homeo}_+(\mathbb{S}^1)$ acting transitively and containing a non-constant continuous path.

2. PRELIMINARIES

2.1. Topology of the circle and of the group of circle homeomorphisms. In what follows, we let $\text{Homeo}_+(\mathbb{S}^1)$ be the group of orientation-preserving homeomorphisms of the circle. Here, the circle \mathbb{S}^1 will be considered as the one-dimensional torus \mathbb{R}/\mathbb{Z} , with its normalized Lebesgue measure. Given a Borel subset $A \subset \mathbb{S}^1$, we denote by $|A|$ its Lebesgue measure. Any $f \in \text{Homeo}_+(\mathbb{S}^1)$ lifts to a homeomorphism $F : \mathbb{R} \rightarrow \mathbb{R}$ commuting with integer translations, and this lift is defined uniquely, up to integer translations. This allows to identify the universal cover of $\text{Homeo}_+(\mathbb{S}^1)$ with the group $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ of homeomorphisms of the line commuting with integer translations.

If $x, y \in \mathbb{S}^1$ are two distinct points, we denote by (x, y) the (open) interval of points $z \in \mathbb{S}^1$ such that the triple (x, z, y) is positively ordered. When $x = y$, (x, x) is by convention, just the empty set. Similarly we denote other kind of intervals $[x, y]$, $[x, y)$, $(x, y]$. We can then define the distance $d(x, y) = \min\{|(x, y)|, |(y, x)|\}$. When $d(x, y) < 1/2$, this is the usual distance between points in any Euclidean chart containing x, y , so that we can write (with slight abuse of notation) $|x - y|$ instead of $d(x, y)$. We want to consider the C^0 topology on $\text{Homeo}_+(\mathbb{S}^1)$, and in order to quantify this, we consider the distance $d_\infty(f, g) := \sup_{x \in \mathbb{S}^1} d(f(x), g(x))$. It is well-known that the distance d_∞ on $\text{Homeo}_+(\mathbb{S}^1)$ is not complete, and for this reason we will also consider the additional distance $d_{C^0}(f, g) := d_\infty(f, g) + d_\infty(f^{-1}, g^{-1})$, which makes $\text{Homeo}_+(\mathbb{S}^1)$ complete. When $d_\infty(f, g) < 1/2$, we can take any lifts $\tilde{f}, \tilde{g} \in \text{Homeo}_\mathbb{Z}(\mathbb{R})$ of f and g , respectively, such that $|\tilde{f}(0) - \tilde{g}(0)| < 1$, and we have $d_\infty(f, g) = \|\tilde{f} - \tilde{g}\|$, where $\|\cdot\|$ is the usual uniform norm on continuous functions. Because of this, when $d_\infty(f, g) < 1/2$, we will also write $\|f - g\|$ for the distance $d_\infty(f, g)$, and tacitly make computations in a local Euclidean chart. This also justifies the following definition.

Definition 2.1. Let $f \in \text{Homeo}_+(\mathbb{S}^1)$ be a circle homeomorphism such that $d_\infty(f, \text{id}) < 1/2$. We say that f is *positive* if for every $x \in \mathbb{S}^1$, we have $f(x) \in (x, x + \frac{1}{2})$. When f^{-1} is positive, we say that f is *negative*. Given an interval $I \subset \mathbb{S}^1$, we say that f is positive (respectively, negative) on I if the previous conditions hold simply for any $x \in I$.

Remark 2.2. Observe that the definition above implies that a positive (respectively, negative) homeomorphism does not have any fixed point. Moreover, when $f, g \in \text{Homeo}_+(\mathbb{S}^1)$ are positive, the composition fg is also positive, provided that $d_\infty(f, \text{id}) + d_\infty(g, \text{id}) < 1/2$.

As we work with homeomorphisms having a prescribed bound on the number of fixed points, it is fundamental to have in mind that if $f, g \in \text{Homeo}_+(\mathbb{S}^1)$ and $x \in \mathbb{S}^1$ are such that $f(x) = g(x)$, then $g^{-1}f(x) = x$. When this occurs, we say that f and g *cross*. More specifically, $I \subset \mathbb{S}^1$, we say that f and g *cross in I* if there exists $x \in I$ such that $f(x) = g(x)$. More specifically, we say that f and g *cross hyperbolically* at x , if x is a (topologically) hyperbolic fixed point (either attracting, or repelling) for the composition fg^{-1} : equivalently, one has

$$(f(x - \eta) - g(x - \eta))(f(x + \eta) - g(x + \eta)) < 0$$

for any sufficiently small $\eta > 0$. We will also say that f and g *cross weakly hyperbolically* in an interval I if there exists an interval $J = [x, y] \subset I$ (possibly, $x = y$) such that $f|_J = g|_J$ and

$$(f(x - \eta) - g(x - \eta))(f(y + \eta) - g(y + \eta)) < 0$$

for any sufficiently small $\eta > 0$.

Remark 2.3. Having a weakly hyperbolically crossing in some open interval I is stable under sufficiently small C^0 perturbations.

We will be interested in counting the number of points for which $f(x) = g(x)$, so that we will say that f and g *cross once, twice, 3 times, and so on*.

2.2. Semi-conjugacy of group actions on the circle. A group action on a topological space X is semi-conjugate to an action of the group on a topological space Y if there exists a continuous equivariant map from X to Y . This classical notion of semi-conjugacy in dynamical systems can be slightly modified to get an equivalence relation on group actions on the line and the circle. For a detailed discussion, we refer to the monograph by Kim, Koberda, and Mj [13], from which we borrow some terminology.

Let G be a group, and $\varphi, \psi : G \rightarrow \text{Homeo}_+(\mathbb{R})$ two homomorphisms. We say that φ and ψ are *conjugate* if there exists $h \in \text{Homeo}_+(\mathbb{R})$ which is (φ, ψ) -equivariant: $h(\varphi(g)(x)) = \psi(g)(h(x))$ for any $g \in G$ and $x \in \mathbb{R}$. We say that φ and ψ are *semi-conjugate* if there exists a non-decreasing map $h : \mathbb{R} \rightarrow \mathbb{R}$ such that $|h(x)| \rightarrow \infty$ as $|x| \rightarrow \infty$, and which is (φ, ψ) -equivariant. When the map h is continuous, we recover the usual notion of continuous semi-conjugacy from φ to ψ .

Similarly, when $\varphi, \psi : G \rightarrow \text{Homeo}_+(\mathbb{S}^1)$, we say that φ and ψ are conjugate if there exists a (φ, ψ) -equivariant circle homeomorphism. However, for extending the notion of semi-conjugacy to this situation, we have to pass through actions on the line: using the cyclic central extension

$$1 \rightarrow \mathbb{Z} \rightarrow \text{Homeo}_{\mathbb{Z}}(\mathbb{R}) \rightarrow \text{Homeo}_+(\mathbb{S}^1) \rightarrow 1,$$

we say that $\varphi, \psi : G \rightarrow \text{Homeo}_+(\mathbb{S}^1)$ are semi-conjugate if we can find a cyclic central extension

$$1 \rightarrow \mathbb{Z} \rightarrow \tilde{G} \rightarrow G \rightarrow 1,$$

and semi-conjugate homomorphisms $\tilde{\varphi}, \tilde{\psi} : \tilde{G} \rightarrow \text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ (in the previous sense), which send any integer $n \in \mathbb{Z} \subset \tilde{G}$ to the translation by n , and lift the homomorphisms φ and ψ , respectively: in short, the following diagram must commute

$$\begin{array}{ccccccc} & & \tilde{G} & \longrightarrow & G & & \\ & & \downarrow \tilde{\varphi}, \tilde{\psi} & & \downarrow \varphi, \psi & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \text{Homeo}_{\mathbb{Z}}(\mathbb{R}) & \longrightarrow & \text{Homeo}_+(\mathbb{S}^1) \longrightarrow 1 \end{array}$$

When $\tilde{\varphi}$ is continuously semi-conjugate to $\tilde{\psi}$, we say that φ is continuously semi-conjugate to ψ . Note that this does not coincide with usual notion in dynamical systems, as we only allow continuous semi-conjugacies through degree-1 maps of the circle.

With abuse of terminology, we will also say that the subgroups $\varphi(G), \psi(G)$ are (continuously) (semi-)conjugate, and more generally, we say that two subgroups of homeomorphisms G_1 and G_2 are (continuously) (semi-)conjugate, if there exist a group G and surjective homomorphisms $\varphi_i : G \rightarrow G_i$ (for $i \in \{1, 2\}$) which are (continuously) (semi-)conjugate.

3. ELEMENTARY GROUPS

3.1. Basic results. Recall from the introduction that a subgroup G of $\text{Homeo}_+(\mathbb{S}^1)$ is *elementary* if its action preserves a Borel probability measure. Recall from the introduction that an elementary subgroup $G < \text{Homeo}_+(\mathbb{S}^1)$, either admits a finite orbit or is semi-conjugate to a subgroup of rotations. We assume that the reader is familiar with the notion of rotation number for circle homeomorphisms. It is well-known that given a circle homeomorphism $g \in \text{Homeo}_+(\mathbb{S}^1)$, we can compute its rotation number $\text{rot}(g) \in \mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$ by taking *any* g -invariant Borel probability measure μ and point $x \in \mathbb{S}^1$, and measure how much x is displaced: $\text{rot}(g) = \mu[x, g(x)]$. Therefore, if $G < \text{Homeo}_+(\mathbb{S}^1)$ is an elementary subgroup, with invariant Borel probability measure μ , then one has

$$\text{rot}(fg) = \mu[x, fg(x)] = \mu[x, g(x)] + \mu[g(x), fg(x)] = \text{rot}(g) + \text{rot}(f),$$

which means that the restriction to G of the function rotation number defines a homomorphism. More precisely, it gives a homomorphism $\rho_G : G \rightarrow \text{SO}(2)$ defined by $g \mapsto R_{\text{rot}(g)}$ (which does not depend on the choice of the invariant measure μ), and the actions of G and $\rho_G(G)$ are semi-conjugate (according to the terminology introduced in the previous section). Moreover, if the action of G has no finite orbit, then G is continuously semi-conjugate to $\rho_G(G)$ (see Ghys [9, Proposition 6.17]). We have the following basic result.

Lemma 3.1. *Let $G < \text{Homeo}_+(\mathbb{S}^1)$ be a subgroup with an invariant Borel probability measure μ on \mathbb{S}^1 . Then, the kernel of the homomorphism $\rho_G : G \rightarrow \text{SO}(2)$ fixes $\text{supp}(\mu)$ pointwise.*

Proof. Take $x \in \text{supp}(\mu)$. If $g \in G$ is such that $g(x) \neq x$, then $\mu[x, g(x)] \neq 0$ and thus $\text{rot}(g) \neq 0$. \square

Let us give a straightforward application in the case when the elementary subgroup has at most N fixed points.

Lemma 3.2. *Let $G < \text{Homeo}_+(\mathbb{S}^1)$ be a subgroup with at most N fixed points, preserving a Borel probability measure μ on \mathbb{S}^1 , whose support contains at least $N+1$ points. Then, the homomorphism $\rho_G : G \rightarrow \text{SO}(2)$ is injective, and G is continuously semi-conjugate to $\rho_G(G)$.*

In particular, G is isomorphic to a subgroup of $\text{SO}(2)$, and thus abelian.

Proof. If there was an element in the kernel, by Lemma 3.1, it would fix the support of μ , and thus at least $N+1$ points. As we are assuming that G acts with at most N fixed points, this gives that the kernel is trivial. Moreover, if the action of G has a finite orbit, then the image $\rho_G(G)$ is finite, and so is G . In this case, we have that G is conjugate to $\rho_G(G)$ (see e.g. Herman [11, §II.6]). If the action of G has not finite orbit, then by [9, Proposition 6.17] we get that G is continuously semi-conjugate to $\rho_G(G)$. \square

3.2. Möbius-like elementary groups. Here we discuss the first result, namely Theorem A, which states that elementary, Möbius-like subgroups of $\text{Homeo}_+(\mathbb{S}^1)$ are continuously semi-conjugate to subgroups of $\text{PSL}(2, \mathbb{R})$.

Proof of Theorem A. Let ν be a Borel probability measure preserved by the action of G on the circle. If $\text{supp}(\nu)$ contains at least 3 points, then Lemma 3.2 gives that G is continuously semi-conjugate to the subgroup of rotations $\rho_G(G) \leq \text{SO}(2) < \text{PSL}(2, \mathbb{R})$, and actually isomorphic.

Thus, from now on, we can assume that the action of G has a finite orbit, which is either a fixed point or a pair of points. Assume first that there is a unique global fixed point for G . Then, by Solodov's theorem (Theorem 1.3), G is continuously semi-conjugate to a subgroup of affine transformations, and the corresponding morphism $G \rightarrow \text{Aff}_+(\mathbb{R}) < \text{PSL}(2, \mathbb{R})$ is injective. Assume next that G has two global fixed points $p, q \in \mathbb{S}^1$. After Hölder's theorem (Theorem 1.3), the restrictions of the actions of G to the two connected components of $\mathbb{S}^1 \setminus \{p, q\}$ are both continuously semi-conjugate to actions by translations, and moreover the corresponding induced homomorphisms $G \rightarrow \mathbb{R}$ and $G \rightarrow \mathbb{R}$ are injective. Therefore, after a continuous global semi-conjugacy (sending p and q to the points 0 and ∞ of $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$), we can identify G with a Möbius-like subgroup Γ of the group $\Lambda \cong \mathbb{R}^2$ of homeomorphisms $f : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$ for which there exist $\alpha = \alpha(f), \beta = \beta(f) \in \mathbb{R}$ such that

$$f(x) = \begin{cases} e^\alpha x & \text{for } x \in [0, \infty], \\ e^\beta x & \text{for } x \in [\infty, 0]. \end{cases}$$

From now on, we will work with Γ instead of G . Take a non-trivial element $f \in \Gamma$, and replacing f by f^{-1} if necessary, we assume $\alpha(f), \beta(f) > 0$. Up to replace Γ by the subgroup of Λ obtained by conjugating Γ by the homeomorphism

$$h(x) = \begin{cases} x^{1/\alpha(f)} & \text{for } x \in [0, \infty], \\ -(-x)^{1/\beta(f)} & \text{for } x \in [\infty, 0], \end{cases}$$

we can assume that $\alpha(f) = \beta(f) = 1$.

We claim that also for every other $g \in \Gamma$, we have $\alpha(g) = \beta(g)$. Indeed, if for some $g \in \Gamma$, $\alpha = \alpha(g)$ and $\beta = \beta(g)$ satisfy $\alpha - \beta \neq 0$, then there exists an integer $N \in \mathbb{Z}$ such that $N(\alpha - \beta) > 1$ and so, there exists a second integer $M \in \mathbb{Z}$ such that $N\alpha > M > N\beta$. Therefore, $N\alpha - M > 0$ and $N\beta - M < 0$ and so for the element $\gamma = g^N f^{-M} \in \Gamma$ one has $\alpha(\gamma) > 0$ and $\beta(\gamma) < 0$. This contradicts the assumption that Γ is Möbius-like. Now, since the subgroup of elements $g \in \Lambda$ with $\alpha(g) = \beta(g)$ is a subgroup of $\text{PSL}(2, \mathbb{R})$ (it is the stabilizer of the points 0 and ∞), we conclude that Γ is in $\text{PSL}(2, \mathbb{R})$.

For the last case, we will assume that G has a finite orbit of order 2, and denote by ν the corresponding invariant probability measure. After Lemma 3.1, we have a short exact sequence

$$1 \rightarrow G_0 \rightarrow G \rightarrow \mathbb{Z}_2 \rightarrow 1,$$

where we write $G_0 = \ker(\rho_G)$. By the previous case (two global fixed points), G_0 is isomorphic and continuously semi-conjugate to a subgroup of the stabilizer of 0 and ∞ in $\text{PSL}(2, \mathbb{R})$, which is

isomorphic to \mathbb{R} . By continuity of the rotation number, the closure \overline{G} in $\text{Homeo}_+(\mathbb{S}^1)$ also fits a short exact sequence

$$1 \rightarrow \overline{G_0} \rightarrow \overline{G} \rightarrow \mathbb{Z}_2 \rightarrow 1,$$

and acts on the circle fixing the same two fixed points as G_0 . In particular, it is left-orderable, and actually order-isomorphic to either \mathbb{Z} or \mathbb{R} , with the standard order (or the reflected one). Let $a \in G$ be an element of the group with $\text{rot}(a) = 1/2$. Observe that if $a^2 \neq \text{id}$ then a^2 fixes the 2 atoms of ν and no further point, so they are both parabolic fixed points. This contradicts the Möbius-like assumption. Thus, we must have $a^2 = \text{id}$. Therefore the exact sequence splits, so that $\overline{G} \cong \overline{G_0} \rtimes_A \mathbb{Z}_2$, where A is the involution of $\text{Aut}(\overline{G_0})$ defined by the conjugacy by a (clearly this also gives the isomorphism $G \cong G_0 \rtimes_A \mathbb{Z}_2$, where we keep denoting by A the restriction of the involution to G_0). We denote by $-\text{id} \in \text{Aut}(\overline{G_0})$ the involution that sends any element to its inverse.

Claim 1. *We have $A = -\text{id}$, thus $G \cong G_0 \rtimes_{-\text{id}} \mathbb{Z}_2$.*

Proof of claim. Note that A acts as an order-automorphism of order 2 on $\overline{G_0}$, which is isomorphic to either \mathbb{Z} or \mathbb{R} . Hence it must be $A \in \{\text{id}, -\text{id}\}$. However, if $A = \text{id}$, then any non-trivial element of G_0 is centralized by a , and therefore its two fixed points should have the same dynamical nature (both parabolic), contradicting the Möbius-like assumption. We conclude that $aga^{-1} = g^{-1}$ for any element of $\overline{G_0}$. \square

Let us now see that G is continuously semi-conjugate to a subgroup of $\text{PSL}(2, \mathbb{R})$. Let $p, q = a(p) \in \mathbb{S}^1$ be the finite orbit, and take a G_0 -invariant atomless Radon measure μ on (p, q) (coming from Hölder's theorem). When $y < x$, we will use the convention $\mu[x, y] = -\mu(y, x]$. With such notation, we have that for any $f \in G_0$, there exists a constant $\alpha(f) \in \mathbb{R}$ such that $\mu[x_0, f(x)] = \mu[x_0, x] + \alpha(f)$ for any choice of $x_0, x \in (p, q)$. The function $\alpha : G_0 \rightarrow \mathbb{R}$ is an injective homomorphism, and it actually corresponds, up to a scalar multiple, to the $\alpha(f)$ previously considered. Now, fix a point $x_0 \in (p, q)$, and consider the function $h : \mathbb{S}^1 \rightarrow \mathbb{RP}^1$ defined by

$$h(x) = \begin{cases} 0 & \text{if } x = p, \\ e^{\mu[x_0, x]} & \text{if } x \in (p, q), \\ \infty & \text{if } x = q, \\ -e^{-\mu[x_0, a(x)]} & \text{if } x \in (q, p). \end{cases}$$

The function h is continuous and of degree 1. Let us check that it defines a semi-conjugacy from G to the subgroup of $\text{PSL}(2, \mathbb{R})$, generated by the maps

$$A_f(t) = e^{\alpha(f)}t \quad (\text{for } f \in G_0),$$

and $I(t) = -1/t$. Indeed, if $x \in (p, q)$, then we have

$$h(a(x)) = -e^{-\mu[x_0, a(a(x))]} = -1/e^{\mu[x_0, x]} = -1/h(x) = I(h(x)),$$

and for $f \in G_0$,

$$h(f(x)) = e^{\mu[x_0, f(x)]} = e^{\mu[x_0, x] + \alpha(f)} = e^{\alpha(f)}h(x) = A_f(h(x)).$$

When $x \in (q, p)$, then we have

$$h(a(x)) = -e^{\mu[x_0, a(x)]} = -1/(-e^{-\mu[x_0, x]}) = -1/h(x) = I(h(x)),$$

and for $f \in G_0$,

$$h(f(x)) = -e^{-\mu[x_0, af(x)]} = -e^{-\mu[x_0, f^{-1}a(x)]} = -e^{-\mu[x_0, a(x)] - \alpha(f^{-1})} = e^{\alpha(f)}h(x) = A_f(h(x)).$$

This concludes the proof. \square

3.3. A group of smooth diffeomorphisms which is not isomorphic to any Möbius group. Here we give an example of group of C^∞ circle diffeomorphisms, with at most 2 fixed points, but which is not isomorphic to any subgroup of $\mathrm{PSL}(2, \mathbb{R})$.

Proof of Theorem B. For the construction of this example we will consider the following maps with respect to the projective coordinates of the circle: fix $\lambda, \mu > 1$ such that $\log \lambda$ and $\log \mu$ are linearly independent over \mathbb{Q} , and set

$$f(x) = \begin{cases} \lambda x & \text{for } x \in [0, \infty], \\ \mu x & \text{for } x \in [\infty, 0]. \end{cases}$$

For convenience, set $g = R_{\frac{1}{2}} f R_{\frac{1}{2}}$. It is clear that f and g generate a rank 2 abelian free group. Moreover, conjugation by the rotation $R_{\frac{1}{2}}$ defines an action of \mathbb{Z}_2 on such \mathbb{Z}^2 given by the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(with respect to the basis f, g). In other terms, the group $G = \langle f, R_{\frac{1}{2}} \rangle$ is isomorphic to the semi-direct product $\mathbb{Z}^2 \rtimes_A \mathbb{Z}_2$. By construction, one can observe that G acts with at most 2 fixed points.

Claim. *The group G is not isomorphic to any subgroup of $\mathrm{PSL}(2, \mathbb{R})$.*

Proof of claim. By contradiction, let us assume G is isomorphic to some subgroup $H \leq \mathrm{PSL}(2, \mathbb{R})$. Then $H \cong \mathbb{Z}^2 \rtimes_A \mathbb{Z}_2$ is virtually abelian, and therefore its action on \mathbb{S}^1 preserves a Borel probability measure. We conclude that H is elementary and Möbius-like. In particular, it satisfies the assumptions of Theorem A. Let us go through the cases discussed in its proof. If H preserves an atomless Borel probability measure, that it must be abelian by Lemma 3.2, but this is a contradiction. In the case H has fixed points, then by Solodov's theorem, H is isomorphic to a subgroup of $\mathrm{Aff}_+(\mathbb{R})$, and therefore torsion free. This is again a contradiction. Therefore we are left with the case of order-2 orbit, so by Claim 1, we have

$$(3.1) \quad H \cong \mathbb{Z}^2 \rtimes_{-\mathrm{id}} \mathbb{Z}_2 = \langle f, g, a \mid [f, g] = \mathrm{id}, a^2 = \mathrm{id}, afa^{-1} = f^{-1}, aga^{-1} = g^{-1} \rangle.$$

However, this contradicts the fact that

$$(3.2) \quad H \cong \mathbb{Z}^2 \rtimes_A \mathbb{Z}_2 = \langle f, g, a \mid [f, g] = \mathrm{id}, a^2 = \mathrm{id}, afa^{-1} = g \rangle,$$

because the the semi-direct products at lines (3.1) and (3.2) have different abelianizations (as one easily checks from the presentations: (3.1) gives $(\mathbb{Z}_2)^3$, while (3.2) gives $\mathbb{Z} \times \mathbb{Z}_2$). This ends the proof of the claim. \square

Finally, conjugating G by a suitable C^∞ homeomorphism which is infinitely flat at 0 and ∞ , we can embed G into $\mathrm{Diff}^\infty(\mathbb{S}^1)$. This proves Theorem B. \square

Remark 3.3. An example of a *minimal* finitely generated group of circle homeomorphisms, with at most 2 fixed points, and which is not isomorphic to any subgroup of $\mathrm{PSL}(2, \mathbb{R})$, can be built by taking a free product of the group G from Theorem B with itself, obtained by blowing-up a free orbit. This kind of construction is inspired by Kovačević's work [14], and it is detailed in the second author's PhD thesis: indeed, let G_1 and G_2 be two copies of the group G , considered as acting on distinct circles Γ_1 and Γ_2 , and choose any two points $x \in \Gamma_1$ and $y \in \Gamma_2$ with trivial stabilizer in G_1 and G_2 , respectively. Then, [4, Theorem D] gives that the *amalgamated product* $(G_1, x) \star (G_2, y)$ (properly defined after [4, Theorem 4.7]; note that in this case it is simply a *free* product) is a subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ acting minimally, and with at most 2 fixed points.

4. NON-LOCALLY DISCRETE GROUPS WITH AT MOST N FIXED POINTS

In this section we discuss Theorem C, about non-locally discrete groups with at most N fixed points. We start this section with an example that justifies our definition of non-local discreteness (Definition 1.5).

We first need a result that can be seen as a consequence of the ‘‘Projective Baumslag Lemma’’ [13, Lemma 3.4].

Lemma 4.1. *Let $A \leq \mathrm{PSL}(2, \mathbb{R})$ be a countable subgroup. Then, there exists a countable set $D \subset \mathrm{SO}(2)$ such that for every rotation $R_\rho \in \mathrm{SO}(2) \setminus D$, the subgroup $\langle A, R_\rho \rangle$ is isomorphic to the free product $A * \langle R_\rho \rangle$.*

Proof. The subgroup $\mathrm{SO}(2)$ is a maximal abelian subgroup of $\mathrm{PSL}(2, \mathbb{R})$, acting without fixed points. Let

$$w = (k, g_1, \dots, g_k, m_1, \dots, m_k)$$

be a choice of an integer $k \geq 1$, non-trivial elements $g_1, \dots, g_k \in A$, and finitely many integers $m_1, \dots, m_k \in \mathbb{Z} \setminus \{0\}$ (so, there are countably many such choices). By [13, Lemma 3.4], for any such w , there exists a finite subset $D_w \subset \mathrm{SO}(2)$ such that for any $R_\rho \in \mathrm{SO}(2) \setminus D_w$ the composition

$$\phi(w) = g_1 R_\rho^{m_1} \dots g_k R_\rho^{m_k}$$

is non-trivial (more precisely, the square of its trace is different from 4). We can then take $D = \bigcup_w D_w$, which is a countable union of finite subsets, and hence countable. \square

Proposition 4.2. *There exists a finitely generated non-elementary subgroup $G < \mathrm{Homeo}_+(\mathbb{S}^1)$ with at most 2 fixed points, which is not Möbius-like, and for which there exists a wandering interval $I \subset \mathbb{S}^1$ such that the image of $\mathrm{Stab}(G, I)$ is non-discrete in $\mathrm{Homeo}_+(I)$.*

Proof. First, we will construct a non-elementary subgroup of $\mathrm{PSL}(2, \mathbb{R})$ with the stabilizer of a point $p \in \mathbb{S}^1$ being parabolic and isomorphic to \mathbb{Z}^2 . For this, let T_α, T_β be two parabolic elements of $\mathrm{PSL}(2, \mathbb{R})$ fixing the same point $p \in \mathbb{S}^1$, such that the subgroup $T = \langle T_\alpha, T_\beta \rangle$ is free abelian of rank 2. After Lemma 4.1, we can find an irrational rotation $R_\rho \in \mathrm{SO}(2)$ such that $F = \langle T, R_\rho \rangle$ is isomorphic to the free product $T * \langle R_\rho \rangle \cong \mathbb{Z}^2 * \mathbb{Z}$. Observe that the stabilizer of the point p has not changed, that is $\mathrm{Stab}(F, p) = \mathrm{Stab}(T, p) = T$. Indeed, for every element $g \in \mathrm{PSL}(2, \mathbb{R})$ that fixes the point p , we have that the parabolic element $gT_\alpha g^{-1}$ fixes p and so it commutes with T_α ; so if $\mathrm{Stab}(F, p) \neq T$, we could then find an element $g \in F \setminus T$ such that $[gT_\alpha g^{-1}, T_\alpha] = \mathrm{id}$. This is not possible after our choice of R_ρ . Finally, note that F is a non-elementary subgroup of $\mathrm{PSL}(2, \mathbb{R})$ with a parabolic stabilizer T of the point p isomorphic to \mathbb{Z}^2 .

For the second step of the construction, we *blow-up* the action of F at the orbit of p (see for instance Kim and Koberda [12]), in such a way that the action of the stabilizer T on the interval I inserted at p is conjugate to the original action of T on $\mathbb{S}^1 \setminus \{p\} \cong \mathbb{R}$ (which is a minimal action by translations). Choosing the good orientation for this T -action on I , we have that any non-trivial element of T acts on the new circle with 2 parabolic fixed points. We call G the resulting subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$, which is abstractly isomorphic to $F \cong \mathbb{Z}^2 * \mathbb{Z}$. It is not difficult to verify that any non-trivial element of G has at most 2 fixed points (we do not have changed the action on the complement of the F -orbit of p , and conjugates of T are now acting with at most 2 fixed points). Therefore, G is a non-elementary group of circle homeomorphisms, with at most 2 fixed points. The subgroup $\mathrm{Stab}(G, I) \cong T$ acts on I minimally by translations, and thus its image in $\mathrm{Homeo}_+(I)$ is non-discrete. \square

It turns out that non-discreteness and non-local discreteness are equivalent for non-elementary subgroups. To see this, we need a fundamental structural result for non-elementary subgroups. To state it, we say that the action of a subgroup $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ is *proximal* if for every non-empty open intervals $I, J \subset \mathbb{S}^1$, there exists an element $g \in G$ such that $g(I) \subset J$. Note that if the

action is proximal, then it is automatically minimal, and G non-elementary. Being proximal is not invariant under semi-conjugacy, since if J is a wandering interval and I is not, there is no way we can send I inside J . For this reason, we also say that the action of a non-elementary subgroup G is proximal *in restriction to the minimal invariant subset* if the previous statement holds only for intervals $J \subset \mathbb{S}^1$ which are non-wandering (that is, intersecting the minimal invariant subset). This property is preserved under semi-conjugacy, and when the action is minimal, we recover the usual notion of proximal action. In particular, any non-elementary subgroup G which is proximal in restriction to the minimal invariant subset is (continuously) semi-conjugate to a proximal action. As a consequence, if G is proximal in restriction to the minimal invariant subset, then it is automatically non-elementary. The following fundamental result can be deduced from the work of Antonov [2], although it has been unknown to experts for a long time (see e.g. Ghys [9, §5.2]). Our statement is very close to that appearing in the work of Malyutin [16, Theorem 1].

Theorem 4.3 (Antonov). *Let $G \leq \text{Homeo}_+(\mathbb{S}^1)$ be a non-elementary subgroup. Then there exists a finite order element $\gamma \in \text{Homeo}_+(\mathbb{S}^1)$ which commutes with every $g \in G$, and such that the induced action of G on the quotient $\mathbb{S}^1/\langle\gamma\rangle$ is proximal in restriction to the minimal invariant subset.*

Remark 4.4. In the setting of Theorem 4.3, several dynamical properties are equivalent for the actions of G on \mathbb{S}^1 and $\mathbb{S}^1/\langle\gamma\rangle$:

- (local) discreteness,
- minimality,
- all non-trivial elements have a uniformly bounded number of fixed points (because if γ is of order d , and an element $g \in G$ acts on $\mathbb{S}^1/\langle\gamma\rangle$ with k fixed points, then g^d acts on \mathbb{S}^1 with dk fixed points).

In particular, the last argument gives that if an element $g \in G$ acts on $\mathbb{S}^1/\langle\gamma\rangle$ without fixed points, then it also acts without fixed points on \mathbb{S}^1 .

We will use this correspondence several times in the rest of the section to work under the more convenient assumption that the action of G on \mathbb{S}^1 is proximal in restriction to the minimal set.

Lemma 4.5. *If $G \leq \text{Homeo}_+(\mathbb{S}^1)$ is a non-elementary, non-locally discrete subgroup, then G is non-discrete.*

Moreover, if G has at most N fixed points, its action on the circle is minimal.

Proof. Because of the principle in Remark 4.4, we can assume that the action of G on \mathbb{S}^1 is proximal in restriction to the minimal invariant subset. Since G is non-locally discrete, there exists a non-wandering open interval $I \subset \mathbb{S}^1$, and a sequence of non-trivial elements $(g_n)_{n \in \mathbb{N}} \subset G$, such that $g_n|_I \rightarrow \text{id}|_I$. If $\bar{I} = \mathbb{S}^1$, we immediately get that G is non-discrete. Otherwise, fix $\eta \in (0, \frac{1}{2})$, and a non-wandering open interval $J = (a, b) \subset \mathbb{S}^1$ with $|J| < \frac{\eta}{2}$. By proximality in restriction to the minimal invariant subset, we can find an element $f \in G$ such that $f(\mathbb{S}^1 \setminus \bar{I}) \subset J$. By continuity of composition, we have that the sequence of conjugates $h_n := f g_n f^{-1}$ converges to the identity in restriction to $f(I)$. In particular, we can fix $n \in \mathbb{N}$ such that $|h_n(x) - x| < \frac{\eta}{2}$ for every $x \in f(I)$. On the other hand, if $x \in J = (a, b)$, as $a, b \in f(I)$, we have

$$a - \frac{\eta}{2} \leq h_n(a) \leq h_n(x) \leq h_n(b) \leq b + \frac{\eta}{2},$$

and since $|J| < \frac{\eta}{2}$, this gives

$$- \eta < (a - x) - \frac{\eta}{2} \leq h_n(x) - x \leq (b - x) + \frac{\eta}{2} < \eta.$$

As $\mathbb{S}^1 = f(I) \cup J$, we conclude that $|h_n(x) - x| < \eta$ for any $x \in \mathbb{S}^1$, as wanted.

For the second part of the statement, assume the action admits an invariant Cantor set $\Lambda \subset \mathbb{S}^1$. For any given $\varepsilon > 0$, every element $g \in G$ which is ε -close to the identity must fix every gap of Λ (that is, any connected component of the complement of Λ) whose size exceeds ε . For sufficiently small ε , this gives that g fixes more than N points, and thus $g = \text{id}$. \square

Lemma 4.6. *Let $G < \text{Homeo}_+(\mathbb{S}^1)$ be a non-elementary, non-discrete subgroup with at most N fixed points. Then, there exists a sequence of fixed-point-free elements in G converging to the identity.*

Proof. After Lemma 4.5, we know that the action of G on \mathbb{S}^1 is minimal. Using again the correspondence in Remark 4.4, we can assume that G is proximal. Under this additional assumption, we can take a sequence of non-trivial elements converging to the identity whose fixed points are all contained in an arbitrarily small interval:

Claim 1. *For every interval $J \subset \mathbb{S}^1$ and every $\varepsilon > 0$, there exists an element $g \in G$ which is ε -close to the identity and has no fixed points in the interval J .*

Proof of claim. Let $(g_n)_{n \in \mathbb{N}} \subset G$ be a sequence of non-trivial elements such that $g_n \rightarrow \text{id}$. If $(g_n)_{n \in \mathbb{N}} \subset G$ contains a subsequence without fixed points, there is nothing to do. Otherwise, by taking a subsequence, we can assume that the fixed points of g_n are converging to points $p_1, \dots, p_M \in \mathbb{S}^1$, for some $M \leq N$. Let $J \subset \mathbb{S}^1$ be any closed interval. By proximality of G , there exists an element $k \in G$ which sends p_1, \dots, p_M to the complement of J . Now, by choosing g_n close enough to the identity, we can assume that $kg_nk^{-1} \in G$ is ε -close to the identity, and that this element only fixes points which are in the complement of the interval J . \square

Throughout the rest of the proof, we use the distance d_∞ on $\text{Homeo}_+(\mathbb{S}^1)$ to quantify how close elements are. Recall that this can be locally computed by using the uniform norm $\|\cdot\|$. We assume by contradiction that for a fixed (small) $\varepsilon > 0$, every element ε -close to the identity has fixed points. After Claim 1, we can define recursively a sequence of nested intervals $(J_n) \subset \mathbb{S}^1$ with the following properties.

- (1) The sequence (J_n) is shrinking to a point $p \in \mathbb{S}^1$, namely $\bigcap_n J_n = \{p\}$.
- (2) For any $n \in \mathbb{N}$, there exists an element $g_n \in G$ such that $\|g_n - \text{id}\| \leq \frac{\varepsilon}{2}$ and g_n is positive on the complement of J_n , with $g_n(J_n) = J_n$.

Claim 2. *For any $n \in \mathbb{N}$ such that $|J_n| < 1 - \frac{\varepsilon}{2}$, there exists $m \in \mathbb{N}$ such that $\frac{\varepsilon}{4} < \|g_n^m - \text{id}\| \leq \frac{\varepsilon}{2}$.*

Proof of claim. Indeed, if we assume that g_n is $\frac{\varepsilon}{4}$ -close to the identity (otherwise $m = 1$ works), since g_n has fixed points only in J_n , for a sufficiently large power $m \in \mathbb{N}$, the distance of g_n^m to the identity will be larger than $\frac{\varepsilon}{2}$. Therefore, there exists $m_0 \in \mathbb{N}$ such that $g_n^{m_0}$ is not $\frac{\varepsilon}{2}$ -close to the identity, but g_n^m is $\frac{\varepsilon}{2}$ -close to the identity for every $0 \leq m < m_0$. The point is that $g_n^{m_0-1}$ is $\frac{\varepsilon}{2}$ -close to the identity, but it is not $\frac{\varepsilon}{4}$ -close: indeed, there exists $x \in \mathbb{S}^1$ such that

$$\frac{\varepsilon}{2} < |g_n^{m_0}(x) - x| < |g_n(g_n^{m_0-1}(x)) - g_n^{m_0-1}(x)| + |g_n^{m_0-1}(x) - x| < \frac{\varepsilon}{4} + |g_n^{m_0-1}(x) - x|.$$

This proves the claim. \square

From now on, after Claim 2, we can and will assume that $\frac{\varepsilon}{4} < \|g_n - \text{id}\| \leq \frac{\varepsilon}{2}$. Hence, when n is sufficiently large so that $|J_n| < \frac{\varepsilon}{4}$, we can find a point $x_n \in \mathbb{S}^1 \setminus J_n$ such that $g_n(x_n) > x_n + \frac{\varepsilon}{4}$. Consider the interval $I_n \subset \mathbb{S}^1$ defined by $I_n := (x_n + \frac{\varepsilon}{12}, x_n + \frac{\varepsilon}{6})$. After passing to a subsequence, we can assume that x_n converges to a point $x \in \mathbb{S}^1$, and for $n_0 \in \mathbb{N}$ large enough we have that $I := \bigcap_{n \geq n_0} I_n$ is a non-trivial interval. With such choices, for every $y \in I$ and $n \geq n_0$ we have $x_n + \frac{\varepsilon}{6} > y$ and

$$(4.1) \quad g_n(y) > g_n(x_n) > x_n + \frac{\varepsilon}{4} > y + \frac{\varepsilon}{12}.$$

On the other hand, for every $n \in \mathbb{N}$, the interval J_n is disjoint from the interval $[x_n, x_n + \frac{\varepsilon}{4}]$, so that by choosing $n_1 \geq n_0$ such that $|x_n - x| < \frac{\varepsilon}{48}$ for any $n \geq n_1$, we have that the intervals I and J_n are at least $\frac{\varepsilon}{24}$ -apart for any such n . We deduce that the union $J := \bigcup_{n \geq n_1} J_n$ and I are at least $\frac{\varepsilon}{24}$ -apart. After Claim 1, we can take an element $f \in G$ which is $\frac{\varepsilon}{2}$ -close to the identity, and negative on the complement of I . As J and I are separated, we can find $\delta > 0$ such that $f(y) < y - \delta$ for every $y \in J$.

Take $m > n_1$ sufficiently large such that $|J_m| < \delta$; we claim that the element $f^{-1}g_m \in G$ is ε -close to the identity and it has no fixed points in the circle. Indeed, since f is negative on the complement of I and g_m is positive on the complement of J_m , it is clear that g_m does not cross f in the complement of $J_m \cup I$. Now, we have $|I| \leq |I_m| = \frac{\varepsilon}{12}$, so that from the inequality (4.1) we deduce that g_m does not cross f in I . Similarly, the size of the interval J_m is smaller than δ and $f(y) < y - \delta$ for every $y \in J \supset J_m$, which implies that g_m does not cross f in J_m . Therefore, g_m does not cross f in the whole circle \mathbb{S}^1 , which implies that the element $f^{-1}g_m$ has no fixed points in \mathbb{S}^1 . Finally, as $f^{-1}g_m$ is the composition of two elements $\frac{\varepsilon}{2}$ -close to the identity, this element is ε -close to the identity, as desired. \square

Given a subgroup $G \leq \text{Homeo}_+(\mathbb{S}^1)$, we denote by \overline{G} its closure in $\text{Homeo}_+(\mathbb{S}^1)$ with respect to the C^0 topology, which is still a subgroup of $\text{Homeo}_+(\mathbb{S}^1)$.

Lemma 4.7. *Let $G < \text{Homeo}_+(\mathbb{S}^1)$ be a non-elementary, non-discrete subgroup with at most N fixed points. Then, its closure \overline{G} has at most N fixed points.*

Proof. As before, we use the correspondence in Remark 4.4 to work under the additional assumption that G is proximal. We then remark that any $f \in \overline{G}$ has at most $\lfloor \frac{N}{2} \rfloor$ components I of $\text{supp}(f) := \mathbb{S}^1 \setminus \text{Fix}(f)$ for which $f(x) > x$ for every $x \in I$. Indeed, arguing by contradiction, by Lemma 4.6, we can take an element $g \in G$, sufficiently close to the identity and without fixed points, so that g crosses weakly hyperbolically the element f at least $2 \left(\lfloor \frac{N}{2} \rfloor + 1 \right) \geq N + 1$ times. After Remark 2.3, we can find an element $f_0 \in G$ sufficiently close to f , which also crosses hyperbolically g at least $N + 1$ times. This contradicts the assumption that G have at most N fixed points. Repeating the argument for f^{-1} , we deduce that $\text{supp}(f)$ has at most $2 \lfloor \frac{N}{2} \rfloor \leq N$ connected components, and so does $\text{Fix}(f)$.

Assume now that \overline{G} contains a non-trivial element f such that $\text{Fix}(f)$ contains a non-empty open interval I . Let us choose such an element $f \in \overline{G}$ such that the number C of connected components of $\text{supp}(f)$ is maximal. As we are assuming that G be proximal, we can find an element $h \in G$ such that $h(\text{supp}(f)) \subset I$. Then the element $f' = fhfh^{-1} \in \overline{G}$ is such that $\text{supp}(f')$ has $2C > C$ connected components, contradicting maximality. \square

The next result we need is very general (and classical).

Lemma 4.8. *Let $f \in \text{Homeo}_+(\mathbb{S}^1)$ be a circle homeomorphism of order $q \geq 2$, with rotation number $\text{rot}(f) = \frac{p}{q}$, where $p \geq 1$ is a natural number. Then, for every $\varepsilon > 0$ there exists $\delta \in (0, 1/2)$ such that for every circle homeomorphism $g \in \text{Homeo}_+(\mathbb{S}^1)$ which is positive and δ -close to the identity, one has*

$$d_{C^0}(f, gf) < \varepsilon \quad \text{and} \quad \text{rot}(gf) \in \left(\frac{p}{q}, \frac{p}{q} + \frac{1}{q^3} \right].$$

Proof. Note that the homeomorphism f is conjugate in $\text{Homeo}_+(\mathbb{S}^1)$ to the rotation $R_{p/q}$ (see Herman [11, §II.6]) Since $\text{Homeo}_+(\mathbb{S}^1)$ is a topological group (so composition, and thus conjugacy, are continuous), and since the function rotation number is continuous on $\text{Homeo}_+(\mathbb{S}^1)$, and conjugacy-invariant, we can assume $f = R_{p/q}$, and fix $\varepsilon > 0$ and $\delta > 0$ such that

$$(4.2) \quad d_{C^0}(R_{p/q}, gR_{p/q}) < \varepsilon \quad \text{and} \quad \text{rot}(gR_{p/q}) \in \left(\frac{p}{q} - \frac{1}{q^3}, \frac{p}{q} + \frac{1}{q^3} \right]$$

for every circle homeomorphism $g \in \text{Homeo}_+(\mathbb{S}^1)$ which is δ -close to the identity.

Let us next make the basic observation that if $h, k \in \text{Homeo}_+(\mathbb{R})$ are two homeomorphisms of the real line such that $h \leq k$ (meaning that $h(x) \leq k(x)$ for any $x \in \mathbb{R}$), then for any integer $n \geq 0$, one has $h^n \leq k^n$. This can be easily checked by induction:

$$h^n(x) = h(h^{n-1}(x)) \leq h(k^{n-1}(x)) \leq k(k^{n-1}(x)) = k^n(x).$$

For homeomorphisms $h \in \text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ of the real line commuting with integer translations, the *translation number* $\tau(h)$ is defined as

$$\tau(h) = \lim_{n \rightarrow \infty} \frac{h^n(x)}{n},$$

and this limit does not depend on the choice $x \in \mathbb{R}$. Consequently, if $h, k \in \text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ are such that $h \leq k$, then $\tau(h) \leq \tau(k)$. The rotation number $\text{rot}(f)$ of a homeomorphism $f \in \text{Homeo}_+(\mathbb{S}^1)$ is classically defined by taking any lift $\tilde{f} \in \text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ and setting $\text{rot}(f) = \tau(\tilde{f}) \pmod{\mathbb{Z}}$.

For what follows, we denote by T_a the translation by $a \in \mathbb{R}$. Lift $f = R_{p/q}$ to the translation $T_{p/q}$. If $g \in \text{Homeo}_+(\mathbb{S}^1)$ is positive and δ -close to the identity, we can find some $\lambda \in (0, \delta)$ and a lift $\tilde{g} \in \text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ such that $T_{p/q} < T_{p/q+\lambda} \leq \tilde{g}T_{p/q} \leq T_{p/q+\delta}$. Taking translation numbers, we get

$$p/q + \lambda \leq \tau(\tilde{g}R_{p/q}) \leq p/q + \delta,$$

so that passing to rotation numbers, and considering condition (4.2), we get

$$\text{rot}(gR_{p/q}) \in \left[\frac{p}{q} + \lambda, \frac{p}{q} + \frac{1}{q^3} \right],$$

as desired. \square

Lemma 4.9. *Let $G < \text{Homeo}_+(\mathbb{S}^1)$ be a non-elementary, non-discrete subgroup with at most N fixed points. Then, \overline{G} contains an element with irrational rotation number.*

Proof. If the subgroup $G < \text{Homeo}_+(\mathbb{S}^1)$ has an element with irrational rotation number there is nothing to prove, because $G \subset \overline{G}$. Therefore, we will suppose that G has no element with irrational rotation number. We want to construct a converging sequence $(h_n)_{n \in \mathbb{N}} \subset G$ whose limit $h \in \overline{G}$ has irrational rotation number.

We start by choosing a sequence of elements $(f_n)_{n \in \mathbb{N}} \subset G$ without fixed points and converging to the identity, whose existence is ensured by Lemma 4.6. After changing f_n for f_n^{-1} when necessary and taking a subsequence, we can assume that $(f_n)_{n \in \mathbb{N}}$ is a sequence of positive circle homeomorphisms whose distance to the identity decreases. Let us fix also a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of positive numbers such that $\sum \varepsilon_n < 1/2$. We choose $h_0 \in G$ to be the first element f_{m_0} such that $d_{\mathcal{C}^0}(f_{m_0}, \text{id}) < \varepsilon_0$. We write $\text{rot}(h_0) = p_0/q_0$, where $p_0, q_0 \geq 1$ are natural numbers with $\text{gcd}(p_0, q_0) = 1$.

Now, let us assume by induction that $h_n \in G$ is a positive circle homeomorphism such that $d_{\mathcal{C}^0}(h_n, \text{id}) < \sum_{k=0}^n \varepsilon_k$ and of rational rotation number

$$\text{rot}(h_n) = \frac{p_n}{q_n} \in \left(\frac{p_{n-1}}{q_{n-1}}, \frac{p_{n-1}}{q_{n-1}} + \frac{1}{q_{n-1}^3} \right],$$

where $p_n, q_n \geq 1$ are natural numbers with $\text{gcd}(p_n, q_n) = 1$. As G has at most N fixed points, whenever $q_n > N$ (this condition is satisfied for any sufficiently large $n \in \mathbb{N}$), we must have $h_n^{q_n} = \text{id}$. By Lemma 4.8, there exists $\delta_n > 0$ such that for every positive circle homeomorphism $g \in \text{Homeo}_+(\mathbb{S}^1)$ which is δ_n -close to the identity, we have that $d_{\mathcal{C}^0}(h_n, gh_n) < \varepsilon_{n+1}$ and $\text{rot}(gh_n) \in \left(\frac{p_n}{q_n}, \frac{p_n}{q_n} + \frac{1}{q_n^3} \right]$. Then, we define h_{n+1} as $f_{m_n} h_n \in G$, where f_{m_n} is the first element of the sequence $(f_n)_{n \in \mathbb{N}}$ such that $d_{\mathcal{C}^0}(f_{m_n}, \text{id}) < \delta_n$. This gives that

$$d_{\mathcal{C}^0}(h_{n+1}, \text{id}) \leq d_{\mathcal{C}^0}(h_{n+1}, h_n) + d_{\mathcal{C}^0}(h_n, \text{id}) < d_{\mathcal{C}^0}(f_{m_n} h_n, h_n) + \sum_{k=0}^n \varepsilon_k < \sum_{k=0}^{n+1} \varepsilon_k.$$

Since $\sum_{k=0}^{n+1} \varepsilon_k < \frac{1}{2}$, $h_{n+1} \in G$ is a positive circle homeomorphism (Remark 2.2). By our assumption on G , the rotation number of h_{n+1} is rational, and we write $\text{rot}(h_{n+1}) = \frac{p_{n+1}}{q_{n+1}}$, which belongs to $\left(\frac{p_n}{q_n}, \frac{p_n}{q_n} + \frac{1}{q_n^3} \right]$ by construction. Hence, the inductive assumptions are satisfied.

We next check that (h_n) is a Cauchy sequence with respect to the d_{C^0} -distance (for which $\text{Homeo}_+(\mathbb{S}^1)$ is complete), so that it admits a limit $h \in \overline{G}$. Indeed, for every $n, m \in \mathbb{N}$ we have

$$d_{C^0}(h_{n+m}, h_n) \leq \sum_{k=n}^{n+m} d_{C^0}(h_{k+1}, h_k) < \sum_{k=n}^{n+m} \varepsilon_k < \sum_{k=n}^{+\infty} \varepsilon_k \xrightarrow{n \rightarrow +\infty} 0.$$

It remains to prove that $\text{rot}(h) \notin \mathbb{Q}$. To see this, for every $n \in \mathbb{N}$, we consider the interval $I_n = \left(\frac{p_n}{q_n}, \frac{p_n}{q_n} + \frac{1}{q_n^2}\right)$. By the classical Dirichlet's approximation theorem, we have that I_n is a nested sequence of intervals, such that $\bigcap I_n$ is an irrational number. Let us detail this for completeness. First, we recall that

$$\frac{p_{n+1}}{q_{n+1}} \in \left(\frac{p_n}{q_n}, \frac{p_n}{q_n} + \frac{1}{q_n^3}\right],$$

so that $\frac{p_{n+1}}{q_{n+1}} > \frac{p_n}{q_n}$. Next, we want to prove that $\frac{p_{n+1}}{q_{n+1}} + \frac{1}{q_{n+1}^2} < \frac{p_n}{q_n} + \frac{1}{q_n^2}$. We will first show that $q_{n+1} > q_n$. Indeed, if $q_{n+1} \leq q_n$ then

$$p_{n+1} \in \left(\frac{p_n q_{n+1}}{q_n}, \frac{p_n q_{n+1}}{q_n} + \frac{q_{n+1}}{q_n^3}\right] \subset \left(\frac{p_n q_{n+1}}{q_n}, \frac{p_n q_{n+1}}{q_n} + \frac{1}{q_n^2}\right],$$

which is an absurd, because $\left(\frac{p_n q_{n+1}}{q_n}, \frac{p_n q_{n+1}}{q_n} + \frac{1}{q_n^2}\right]$ does not contain any integer. Now, we have the following inequality

$$\begin{aligned} \frac{p_{n+1}}{q_{n+1}} + \frac{1}{q_{n+1}^2} &\leq \frac{p_n}{q_n} + \frac{1}{q_n^3} + \frac{1}{q_{n+1}^2} \leq \frac{p_n}{q_n} + \frac{1}{q_n^3} + \frac{1}{(q_n+1)^2} = \frac{p_n}{q_n} + \frac{q_n^3 + (q_n+1)^2}{q_n^3(q_n+1)^2} \\ &= \frac{p_n}{q_n} + \frac{1}{q_n^2} \frac{q_n^3 + q_n^2 + 2q_n + 1}{q_n^3 + 2q_n^2 + q_n} < \frac{p_n}{q_n} + \frac{1}{q_n^2} \frac{q_n^3 + q_n^2 + 2q_n + 1 + (q_n^2 - q_n - 1)}{q_n^3 + 2q_n^2 + q_n} = \frac{p_n}{q_n} + \frac{1}{q_n^2}. \end{aligned}$$

So, we conclude that $I_{n+1} \subset I_n$ for every $n \in \mathbb{N}$. On the other hand, $(q_n)_{n \in \mathbb{N}}$ is an increasing sequence of integers, and therefore $|I_n| = \frac{1}{q_n^2} \rightarrow 0$. Hence, $\bigcap_{n \in \mathbb{N}} I_n$ converges to a point $\alpha \in \mathbb{R}$. We claim that α is not a rational number, otherwise $\alpha = \frac{p}{q}$ with $p, q \in \mathbb{Z}$, and $q \geq 2$, which implies that $\frac{p}{q} \in I_n = \left(\frac{p_n}{q_n}, \frac{p_n}{q_n} + \frac{1}{q_n^2}\right)$, for every $n \in \mathbb{N}$. Therefore, for n sufficiently large, we have $2q < q_n$ and then

$$p \in \left(\frac{p_n q}{q_n}, \frac{p_n q}{q_n} + \frac{q}{q_n^2}\right) \subset \left(\frac{p_n q}{q_n}, \frac{p_n q}{q_n} + \frac{1}{2q_n}\right),$$

which is an absurd, because $\left(\frac{p_n q}{q_n}, \frac{p_n q}{q_n} + \frac{1}{2q_n}\right)$ does not contain any integer.

Let us go back to our converging sequence (h_n) . By construction we have

$$\text{rot}(h_{n+1}) \in \left(\frac{p_n}{q_n}, \frac{p_n}{q_n} + \frac{1}{q_n^3}\right] \subset \left(\frac{p_n}{q_n}, \frac{p_n}{q_n} + \frac{1}{q_n^2}\right) = I_n,$$

so that $\text{rot}(h_{n+k}) \in I_{n+k} \subset I_n$ for any $k \geq 1$. Hence, by continuity of $\text{rot} : \text{Homeo}_+(\mathbb{S}^1) \rightarrow \mathbb{S}^1$, it follows that $\text{rot}(h) \in I_n$, for every $n \in \mathbb{N}$, and therefore $\text{rot}(h) = \alpha$ is irrational, as desired. \square

Lemma 4.10. *Let $G < \text{Homeo}_+(\mathbb{S}^1)$ be a non-elementary, non-discrete subgroup with at most N fixed points. Then, \overline{G} contains an element which is conjugate to an irrational rotation. Therefore, up to conjugacy, \overline{G} contains $\text{SO}(2)$.*

Proof. We will prove that any $f \in \overline{G}$ with irrational rotation number $\alpha = \text{rot}(f)$ (by Lemma 4.9, we can find such an element in G) is conjugate to the rotation R_α . We argue by way of contradiction. Let us denote by Λ the minimal f -invariant Cantor set. Recall that a gap of Λ is a connected component of the complement $\mathbb{S}^1 \setminus \Lambda$. Up to conjugating G by some circle homeomorphism, we can assume that the size of any gap does not exceed $\frac{1}{4}$. After the discussion in Section 2, f is semi-conjugate to the rotation R_α . Denoting by (q_n) the sequence of denominators of rational approximations of α , we have $R_\alpha^{q_n} \rightarrow \text{id}$, which implies that for any $x \in \Lambda$ which is not in the

closure of any wandering interval, we have $f^{q_n}(x) \rightarrow x$. (This is because any such x satisfies that $\{x\} = h^{-1}(h(x))$, where $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is a continuous monotone map such that $hf = R_\alpha h$ giving the semi-conjugacy.) After the choice of the size of the gaps, we can take $n_0 \in \mathbb{N}$ sufficiently large so that $d_\infty(f^{q_n}, \text{id}) < 1/2$ for any $n \geq n_0$, and then take a subsequence $(q_{n_j})_{j \in \mathbb{N}} \subset (q_n)_{n \geq n_0}$ such that $f^{q_{n_j}}$ is positive for any $j \in \mathbb{N}$.

Take $\varepsilon > 0$ smaller than the size of the N largest gaps. By Lemma 4.6, we can choose a positive element $g \in G$ which is ε -close to the identity. Note that for any $j \in \mathbb{N}$ sufficiently large, g crosses $f^{q_{n_j}}$ twice on a small neighborhood of any of the N largest gaps, hence at least $2N$ times. This contradicts the assumption that $g^{-1}f^{q_{n_j}} \in \overline{G}$ have at most N fixed points. So we conclude that $f \in \overline{G}$ with $\text{rot}(f) \notin \mathbb{Q}$ is conjugate to an irrational rotation and therefore \overline{G} contains a conjugate copy of $\text{SO}(2)$. \square

We finally use a result by Glibin and Markovic [10, Theorem 1.2]. (A self-contained argument for the conclusion in the case $N = 2$ is given in the second author's PhD thesis [4].)

Theorem 4.11 (Glibin–Markovic). *Let $G \leq \text{Homeo}_+(\mathbb{S}^1)$ be a closed transitive subgroup which contains a non-constant continuous path. Then we have the following alternative:*

- (1) *either G is conjugate to $\text{SO}(2)$, or*
- (2) *G is conjugate to $\text{PSL}^{(k)}(2, \mathbb{R})$, for some $k \geq 1$, or*
- (3) *G is conjugate to $\text{Homeo}_+^{(k)}(\mathbb{S}^1)$, for some $k \geq 1$, where $\text{Homeo}_+^{(k)}(\mathbb{S}^1)$ is the group of all homeomorphisms commuting with the group of order k rotations.*

We can now put everything together and prove the main result of our work.

Proof of Theorem C. Let G be a non-elementary subgroup with at most N fixed points. If G is non-locally discrete, then after Lemma 4.5, G is non-discrete, so that by Lemma 4.10 its closure \overline{G} contains a conjugate copy of the subgroup of rotations $\text{SO}(2)$. In particular, \overline{G} is closed, transitive and contains a non-constant continuous path. As G is non-elementary, the first possibility in Theorem 4.11 cannot occur. On the other hand, after Lemma 4.7, \overline{G} has at most N fixed points, so that the third possibility in Theorem 4.11 cannot occur either. We conclude that \overline{G} is conjugate to $\text{PSL}^{(k)}(2, \mathbb{R})$, for some $k \geq 1$, as desired. \square

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Institut de Mathématiques de Bourgogne (IMB, UMR CNRS 5584)

Université de Bourgogne

9 av. Alain Savary, 21000 Dijon, France

christian.bonatti@u-bourgogne.fr

joao.carnevale@hotmail.com

michele.triestino@u-bourgogne.fr