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# Generalization of a Sangaku to the Exinscribed Circles of a Triangle

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# Generalization of a Sangaku to the Exinscribed Circles of a Triangle

David Pigeon

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## Abstract

This text proposes a generalization of a known Sangaku, providing a relation between the radii of circles inscribed in a triangle. By using classical geometric and trigonometric methods, we extend the result to exinscribed circles.

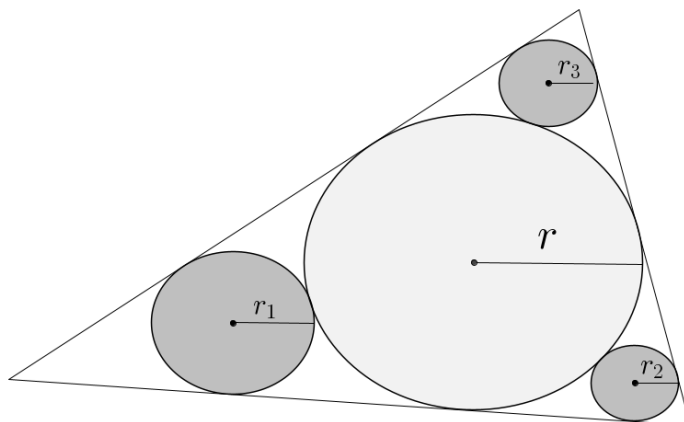
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## 1 Introduction

Sangaku were mathematical puzzles presented on tablets in Japanese temples. In this text, we generalize a classical Sangaku result through the study of inscribed and exinscribed circles of a triangle. These are generally geometric drawings with very little information.

The Sangaku considered here is as follows:



Let  $i \in \{1, 2, 3\}$ <sup>1</sup> :

$$\lambda_i := \sqrt{\frac{r_i}{r}}.$$

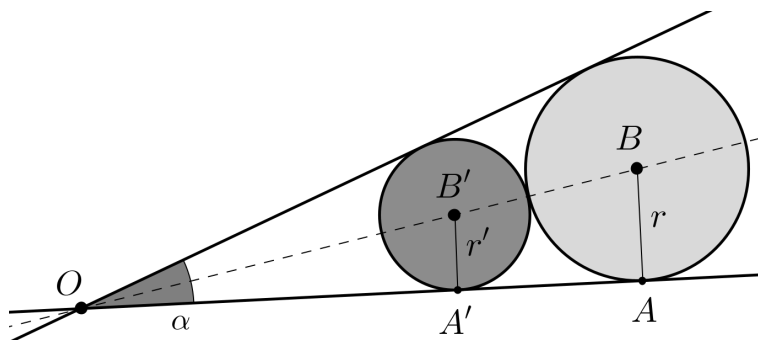
The known result states:

$$\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 = 1$$

In this text, we generalize this result to the exinscribed circle of a triangle (see [1] for a proof in the case of inscribed circles).

## 2 Tangent Circles in an Angle

Let  $\alpha \in ]0, \pi/2[$ . Consider two lines making an angle  $\alpha$  between them, and two circles with centers  $B$  and  $B'$  and radii  $r$  and  $r'$  respectively, tangent to these two lines and tangent to each other such that  $r > r'$ .



**Proposition 1.** *We have the relation:*

$$\frac{r'}{r} = \tan^2\left(\frac{\pi - \alpha}{4}\right) \quad \text{and} \quad \frac{r}{r'} = \tan^2\left(\frac{\pi + \alpha}{4}\right).$$

<sup>1</sup>The triangles considered in this text are not flat, so the radii of the inscribed circles are non-zero.

*Proof.* We have:

$$\begin{aligned} r + r' &= B'B \\ &= BO - B'O \\ &= \frac{r - r'}{\sin(\alpha/2)}. \end{aligned}$$

So, we have:

$$\begin{aligned} \frac{r'}{r} &= -\frac{\sin(\alpha/2) - 1}{\sin(\alpha/2) + 1} & \text{and} & \quad \frac{r}{r'} = -\frac{\sin(\alpha/2) + 1}{\sin(\alpha/2) - 1} \\ &= -\frac{\cos((\pi - \alpha)/2) - 1}{\cos((\pi - \alpha)/2) + 1} & & \quad = \frac{1 - \cos((\pi + \alpha)/2)}{1 + \cos((\pi + \alpha)/2)} \\ &= \frac{\sin^2((\pi - \alpha)/4)}{\cos^2((\pi - \alpha)/4)} & & \quad = \frac{\sin^2((\pi + \alpha)/4)}{\cos^2((\pi + \alpha)/4)} \\ &= \tan^2\left(\frac{\pi - \alpha}{4}\right) & & \quad = \tan^2\left(\frac{\pi + \alpha}{4}\right). \end{aligned}$$

□

### 3 A Sangaku in a Triangle

Let's fix a triangle  $A_1A_2A_3$ . We will use the following notations.

(1) For all  $i \in \{0, 1, 2, 3\}$ , we denote  $\alpha_i$  the angle at  $A_i$ . We assume that the triangle is not flat, i.e., for all  $i \in \{0, 1, 2, 3\}$ ,  $\alpha_i \in ]0, \pi[$ .

(2) We denote:

- (a)  $\mathcal{C}_0$  the circle inscribed in the triangle,  $O_0$  its center, and  $r_0$  its radius;
- (b)  $\mathcal{C}_1$  the exinscribed circle in the triangle on side  $A_2A_3$ ,  $O_1$  its center, and  $r_1$  its radius.

Similarly for the circles  $\mathcal{C}_2$  and  $\mathcal{C}_3$ .

(3) For all  $i \in \{0, 1, 2, 3\}$ , we denote  $\mathcal{C}_{i,1}$  the circle tangent to the circle  $\mathcal{C}_i$  and the two lines  $(A_1A_2)$  and  $(A_1A_3)$ ,  $O_{i,1}$  its center, and  $r_{i,1}$  its radius.

Similarly for the circles  $\mathcal{C}_{i,2}$  and  $\mathcal{C}_{i,3}$ .

(4) For all  $i \in \{0, 1, 2, 3\}$  and all  $j \in \{1, 2, 3\}$ , we define:

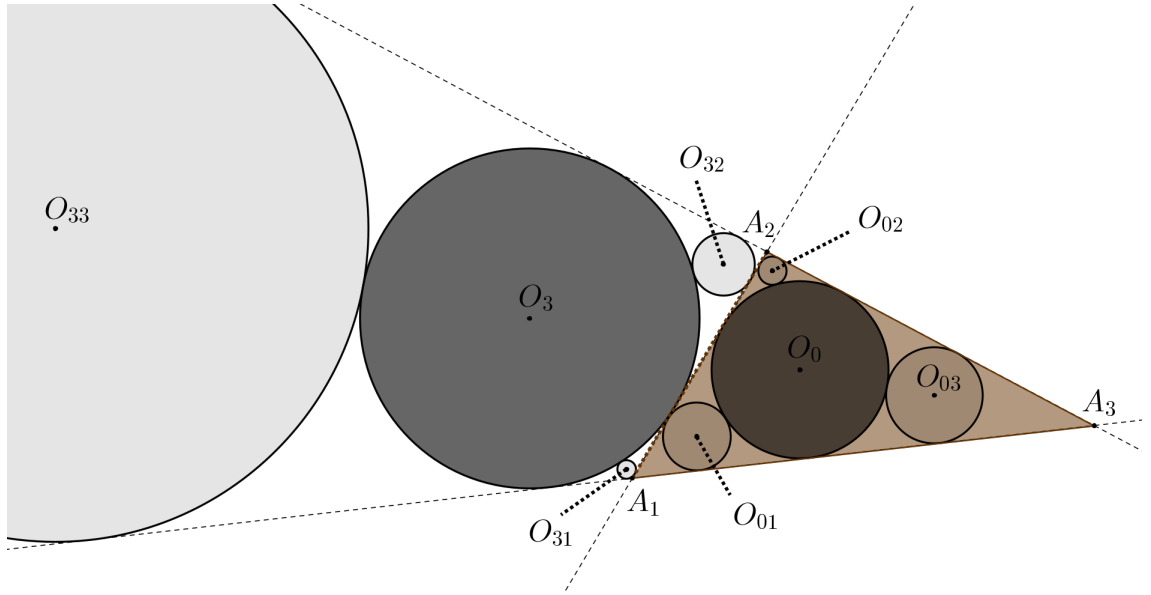
$$\begin{aligned} \lambda_{i,j} &:= \sqrt{\frac{r_{i,j}}{r_i}} \\ \sigma_i &:= \lambda_{i,1}\lambda_{i,2} + \lambda_{i,1}\lambda_{i,3} + \lambda_{i,2}\lambda_{i,3}. \end{aligned}$$

(5) For all  $u, v, w \in ]0, \pi/2[$ :

$$\beta(u, v, w) := \tan(u) \tan(v) + \tan(u) \tan(w) + \tan(w) \tan(v).$$

We note that  $\beta$  is symmetric in the variables  $u, v, w$ .

In the following drawing, we have represented what happens on the side  $A_1A_2$ .



**Remark 2.** *The Sangaku stated in the introduction says:*

$$\sigma_0 = 1.$$

*It corresponds to the circle inscribed in the triangle. We generalize this result to exinscribed circles.*

**Theorem 3.** *For all  $i \in \{0, 1, 2, 3\}$ , we have:*

$$\sigma_i = 1.$$

**Lemma 4.** *Let  $v, w \in ]0, \pi/2[$  such that  $v + w < \pi/2$ . Then*

$$\lim_{t \rightarrow (\pi/2 - v - w)^-} \beta(t, v, w) = 1.$$

*Proof.* (Lemma 4) Let  $t \in ]0, \pi/2[$  such that  $t + v + w < \pi/2$ . Using addition

formulas for tangents, we have:

$$\begin{aligned}
\beta(t, v, w) &= \tan(t) \tan(v) + \tan(t) \tan(w) + \tan(w) \tan(v) \\
&= \tan(t)(\tan(v) + \tan(w)) + \tan(w) \tan(v) \\
&= \tan(t) \tan(v+w)(1 - \tan(v) \tan(w)) + \tan(w) \tan(v) \\
&= \tan(t) \tan(v+w) + \tan(w) \tan(v)(1 - \tan(t) \tan(v+w)) \\
&= 1 - \frac{\tan(t) + \tan(v+w)}{\tan(t+v+w)} + \tan(w) \tan(v) \frac{\tan(t) + \tan(v+w)}{\tan(t+v+w)}
\end{aligned}$$

All the tangents appearing in the last relation are strictly positive. Taking the limit as  $t \rightarrow (\pi/2 - v - w)^-$  in the previous relation, we have as  $\pi/2 - v - w < \pi/2$ :

$$\lim_{t \rightarrow (\pi/2 - v - w)^-} \beta(t, v, w) = 1.$$

□

We extend the function  $\beta$  by continuity, setting for all  $u, v, w \in ]0, \pi/2[$  such that  $u + v + w = \pi/2$ :

$$\beta(u, v, w) := 1.$$

*Proof.* (Theorem 3) Let  $i \in \{0, 1, 2, 3\}$ . We have two cases.

(1) Suppose  $i := 0$ . By Proposition 1, we have:

$$\forall j \in \{1, 2, 3\}, \lambda_{0,j} = \tan\left(\frac{\pi - \alpha_j}{4}\right).$$

Also:

$$\sum_{j=1}^3 \frac{\pi - \alpha_j}{4} = \frac{1}{4} \left( 3\pi - \sum_{j=1}^3 \alpha_j \right) = \frac{1}{4} (3\pi - \pi) = \frac{\pi}{2}.$$

Therefore, by Lemma 4, we directly have:

$$\sigma_0 = \beta\left(\frac{\pi - \alpha_1}{4}, \frac{\pi - \alpha_2}{4}, \frac{\pi - \alpha_3}{4}\right) = 1.$$

(2) Suppose  $i \in \{1, 2, 3\}$ . By symmetry, it suffices to consider one case. Let's take for example  $i := 1$ . By Proposition 1, we have:

$$\begin{aligned}
\lambda_{1,1} &= \tan\left(\frac{\pi + \alpha_1}{4}\right) \\
\lambda_{1,2} &= \tan\left(\frac{\pi - (\pi - \alpha_2)}{4}\right) = \tan\left(\frac{\alpha_2}{4}\right) \\
\lambda_{1,3} &= \tan\left(\frac{\pi - (\pi - \alpha_2)}{4}\right) = \tan\left(\frac{\alpha_2}{4}\right)
\end{aligned}$$

Also:

$$\frac{\pi + \alpha_1}{4} + \frac{\alpha_2}{4} + \frac{\alpha_3}{4} = \frac{1}{4} \left( \pi + \sum_{j=1}^3 \alpha_j \right) = \frac{\pi}{2}.$$

Therefore, by Lemma 4, we directly have:

$$\sigma_1 := \beta \left( \frac{\pi + \alpha_1}{4}, \frac{\alpha_2}{4}, \frac{\alpha_3}{4} \right) = 1.$$

□

## References

- [1] Géry Huvent, *Sangaku : Le mystère des énigmes géométriques japonaises*, Dunod, (2008).