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# One degree of summation's rational solutions to the KPI equation

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## **Abstract**

In this paper, we construct solutions to the Kadomtsev-Petviashvili equation (KPI) by using a Darboux transformation with particular generating functions limited to one degree of summation. With this choice, we get rational solutions expressed in terms of a wronskian of order  $N$  depending on  $N(D + 5)$  real parameters.

In this case, we construct explicitly multi-parametric rational solutions to the KPI equation and we study the patterns of their modulus in the plane  $(x, y)$  and their evolution according to time and parameters.

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33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

## **1 Introduction**

The Kadomtsev-Petviashvili equation (KPI)

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \quad (1)$$

where as usual, subscripts  $x$ ,  $y$  and  $t$  denote partial derivatives was proposed first by Kadomtsev and Petviashvili [1] in 1970. This equation describes weakly non-linear waves in media. It is the subject of countless researches beginning with Petviashvili [2] in 1976, then in 1977 with Manakov, Zakharov, Bordag and Matveev [3], with Krichever [4, 5], Matveev in 1979 [6], Freeman and Nimmo [7, 8] in 1983 among others.

Some recent works related to this study can be cited. Tian, Xu and Zhang successfully proposed an effective and direct approach to study the symmetry-preserving discretization for a class of generalized higher order equations, and

proposed an open problem about symmetries and the multipliers of conservation law [9]. Li and Tian systematically solved the Cauchy problem of the general  $n$ -component nonlinear Schrödinger equations based the Riemann-Hilbert method, and have given the  $N$ -soliton solutions. Moreover, they proposed a conjecture about the law of nonlinear wave propagation [10]. Wang, Tian and Cheng successfully derived the three-component coupled Hirota hierarchy, and firstly obtained the explicit soliton solutions of the equations via dressing method [11]. Yang, Tian and Li successfully solved the soliton solutions of the focusing nonlinear Schrödinger equation with multiple high-order poles under nonzero boundary conditions for the first time [12]. Wu and Tian successfully solved the long-time asymptotic problem of the solution to the non local short pulse equation with the Schwartz-type initial data, without solitons [13]. With respect to soliton resolution conjecture, Li, Tian, Yang and Fan have done some interesting work in deriving the solutions of Wadati-Konno-Ichikawa equation and complex short pulse equation with the help of Dbar-steepest descent method. They solved the long-time asymptotic behavior of the solutions of these equations, and proved the soliton resolution conjecture and the asymptotic stability of solutions of these equations [14, 15, 16].

Here, we use an extended Darboux transformation and some specific generating functions to construct rational solutions to the KPI equation in terms of a second derivative with respect to  $x$  of a logarithm of a wronskian of order  $N$ . These solutions depend on a degree of summation  $S$  and a degree of degree of derivation  $D$ . In this study, we choose  $S = 1$ . So we obtain  $N$  order solutions depending in general on  $N(D + 5)$  real parameters which are rational in  $x$ ,  $y$  and  $t$ . In the following, we restrict ourselves to the case where the order  $N$  is equal to 1.

We construct some explicit rational solutions of order 1, depending on several real parameters, and the representations of their modulus in the plane of the coordinates  $(x, y)$  according to parameters and time  $t$ .

## 2 Multi-parametric rational solutions of order $N$ to the KPI equation

We consider the Kadomtsev-Petviashvili (KPI) equation

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0.$$

We consider the wronskian of order  $N$  of the functions  $f_1, \dots, f_N$  denoted as  $W(f_1, \dots, f_N)$ , and defined by  $\det(\partial_x^{i-1} f_j)_{1 \leq i \leq N, 1 \leq j \leq N}$ ,  $\partial_x^i$  being the partial derivative of order  $i$  with respect to  $x$  and  $\partial_x^0 f_j$  being the function  $f_j$ .

We consider  $a_j, b_{j_s d}, d_{j_s}, e_{j_s}, g_{j_s}, h_{j_s}$ , arbitrary real numbers with  $c_{j_s} = g_{j_s} + ih_{j_s}$ .

We consider the following elementary functions  $f_{j_s}$  defined by

$$f_{j_s}(x, y, t) = e^{ic_{j_s}x - ic_{j_s}^2 y + ic_{j_s}^3 t + d_{j_s} + ie_{j_s}} = e^{i(g_{j_s} + ih_{j_s})x - i(g_{j_s} + ih_{j_s})^2 y + i(g_{j_s} + ih_{j_s})^3 t + d_{j_s} + ie_{j_s}}. \quad (2)$$

Then, we have the following statement :

**Theorem 2.1** *Let  $\psi_j$ , be the functions defined by*

$$\psi_j(x, y, t) = a_j + \sum_{s=1}^S \sum_{d=0}^D b_{j,s,d} \partial_{c_{j,s}}^d e^{ic_{j,s}x - ic_{j,s}^2y + ic_{j,s}^3t + d_{j,s} + ie_{j,s}} = a_j + \sum_{s=1}^S \sum_{d=0}^D b_{j,s,d} \partial_{c_{j,s}}^d f_{j,s}(x, y, t). \quad (3)$$

Then the function  $u$  defined by

$$u(x, y, t) = -2\partial_x^2 \ln(W(\psi_1(x, y, t), \dots, \psi_N(x, y, t))) \quad (4)$$

is a solution to the (KPI) equation (1) depending on real  $N(S(D+5)+1)$  parameters  $a_j, b_{j,s,d}, g_{j,s}, h_{j,s}, d_{j,s}, e_{j,s}, 1 \leq j \leq N, 1 \leq s \leq S, 0 \leq d \leq D$ .

We will call the order  $N$  of the wronskian, the order of the solution. The number  $S$  of terms of the summation will be call the degree of the summation of the solution and  $D$  the degree of derivation of the solution.

In the following, we restrict ourselves to the study with  $S = 1$ .

So we will use some simplified notations.

We consider  $b_{j,d}, d_j, e_j, g_j, h_j$ , arbitrary real numbers with  $c_j = g_j + ih_j$ .

We consider the following elementary functions  $f_j$  defined by

$$f_j(x, y, t) = e^{ic_jx - ic_j^2y + ic_j^3t + d_j + ie_j} = e^{i(g_j + ih_j)x - i(g_j + ih_j)^2y + i(g_j + ih_j)^3t + d_j + ie_j}. \quad (5)$$

Then, we have the following statement :

**Theorem 2.2** *Let  $\phi_j$ , be the functions defined by*

$$\phi_j(x, y, t) = \sum_{d=0}^D b_{j,d} \partial_{c_j}^d e^{ic_jx - ic_j^2y + ic_j^3t + d_j + ie_j} = \sum_{d=0}^D b_{j,d} \partial_{c_j}^d f_j(x, y, t). \quad (6)$$

Then the function  $u$  defined by

$$u(x, y, t) = -2\partial_x^2 \ln(W(\phi_1(x, y, t), \dots, \phi_N(x, y, t))) \quad (7)$$

is a rational solution to the (KPI) equation (1) depending on real  $N(D+5)$  parameters  $b_{j,d}, g_j, h_j, d_j, e_j, 1 \leq j \leq N, 0 \leq d \leq D$ .

### 3 Rational solutions of order 1 with a degree of derivation equal to 1 ( $D = 1$ ) depending on 6 real parameters

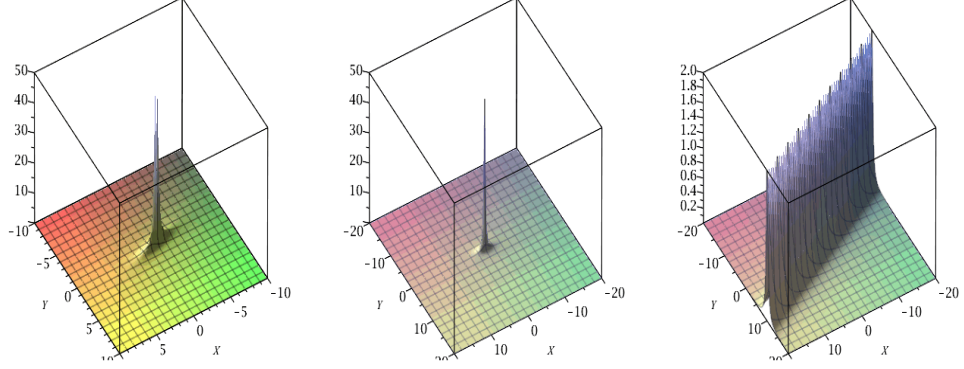
The case  $D = 0$  is not interesting because we get the solution  $u = 0$ .

In the case  $D = 1$ , solutions to the KPI equation can be written as

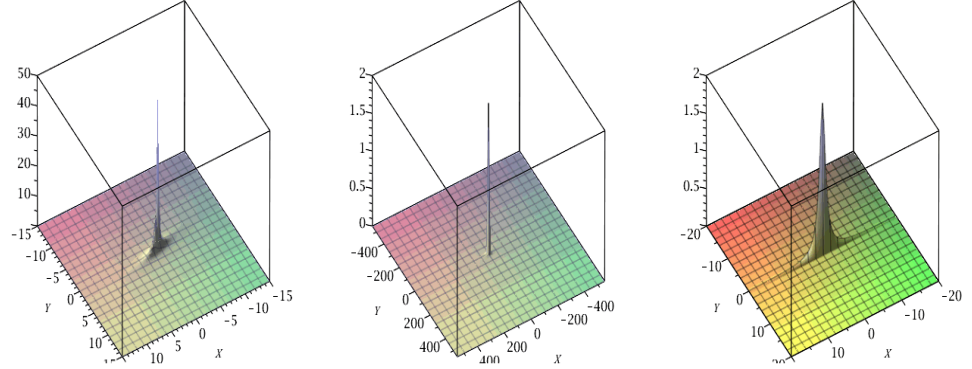
$$u(x, y, t) = -2 \frac{b_{1,1}^2}{(ib_{1,1}x + b_{1,0} + 3itb_{1,1}g_1^2 - 3itb_{1,1}h_1^2 - 2iyb_{1,1}g_1 + 2yb_{1,1}h_1 - 6tb_{1,1}g_1h_1)^2}.$$

In this case, we observe lumps whose intensities depend on 6 real parameters. If  $b_{1,1} = 0$ , we get the solution  $u = 0$ .

We give some figures in the  $(x, y)$  plane of coordinates.



**Figure 1.** Solutions of order 1 to KPI, on the left for  $t = 0$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $g_1 = 1$ ,  $h_1 = 1$ ,  $d_1 = 1$ ,  $e_1 = 1$ ; in the center for  $t = 0$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $g_1 = 0$ ,  $h_1 = 1$ ,  $d_1 = 1$ ,  $e_1 = 1$ ; on the right for  $t = 0$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $g_1 = 1$ ,  $h_1 = 0$ ,  $d_1 = 1$ ,  $e_1 = 1$ .



**Figure 2.** Solutions of order 1 to KPI, on the left for  $t = 0$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $g_1 = 1$ ,  $h_1 = 1$ ,  $d_1 = 0$ ,  $e_1 = 1$ ; in the center for  $t = 0$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $g_1 = 1$ ,  $h_1 = 1$ ,  $d_1 = 1$ ,  $e_1 = 1$ ; on the right for  $t = 10$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $g_1 = 0.01$ ,  $h_1 = 10^2$ ,  $d_1 = 1$ ,  $e_1 = 1$ .

In figures 1 and 2, we obtain lumps whose modulus depends on the values of parameters and we remark that these structures are more sensitive to  $g_1$  and  $h_1$  parameters than to others.

## 4 Rational solutions of order 1 with a degree of derivation equal to 2 ( $D = 2$ ) depending on 7 real parameters

In this case, we take  $D = 2$ ; the solutions to the KPI equation can be written as

$$u(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

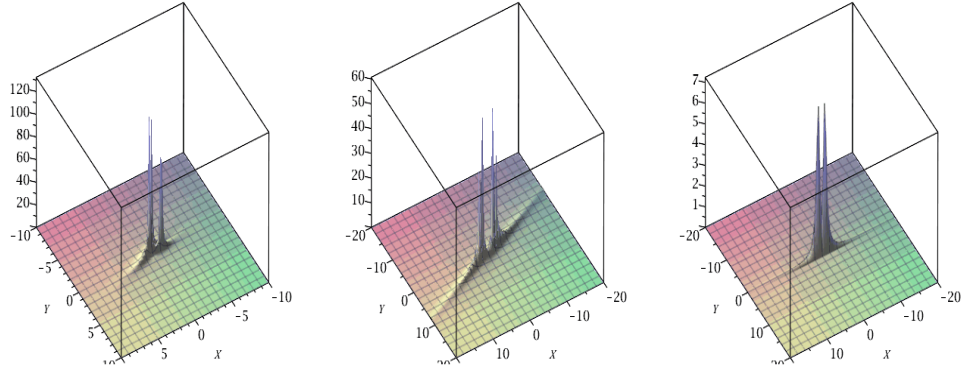
with

$$\begin{aligned} n(x, y, t) = & -2b_{1,2}^2x^2 + (12tb_{1,2}^2h_1^2 + 8yb_{1,2}^2g_1 - 12tb_{1,2}^2g_1^2 + 8iyb_{1,2}^2h_1 - \\ & 24itb_{1,2}^2g_1h_1 + 2ib_{1,1}b_{1,2})x + 72it^2b_{1,2}^2g_1h_1^3 - 12itb_{1,2}^2g_1 - 6itb_{1,1}b_{1,2}h_1^2 - \\ & 2b_{1,0}b_{1,2} + 108t^2b_{1,2}^2g_1^2h_1^2 + 24tyb_{1,2}^2g_1^3 + 6itb_{1,1}b_{1,2}g_1^2 + 4yb_{1,1}b_{1,2}h_1 - \\ & 16iy^2b_{1,2}^2g_1h_1 - 12tb_{1,1}b_{1,2}g_1h_1 - 18t^2b_{1,2}^2g_1^4 + 12tb_{1,2}^2h_1 + b_{1,1}^2 + 8y^2b_{1,2}^2h_1^2 - \\ & 24ityb_{1,2}^2h_1^3 + 4iyb_{1,2}^2 - 72it^2b_{1,2}^2g_1^3h_1 - 72tyb_{1,2}^2g_1h_1^2 - 4iyb_{1,1}b_{1,2}g_1 - \\ & 18t^2b_{1,2}^2h_1^4 + 72ityb_{1,2}^2g_1^2h_1 - 8y^2b_{1,2}^2g_1^2 \end{aligned}$$

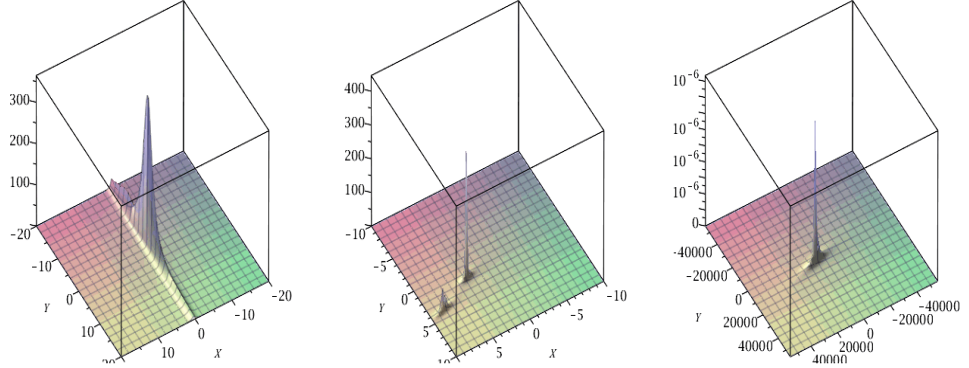
and

$$\begin{aligned} d(x, y, t) = & b_{1,2}x^2 + (6tb_{1,2}g_1^2 + 12itb_{1,2}g_1h_1 - 6tb_{1,2}h_1^2 - 4yb_{1,2}g_1 - 4ib_{1,2}yh_1 - \\ & ib_{1,1})x + 9t^2b_{1,2}g_1^4 + 12ityb_{1,2}h_1^3 - 54t^2b_{1,2}g_1^2h_1^2 - 36ib_{1,2}tg_1^2yh_1 + 9t^2b_{1,2}h_1^4 - \\ & 12tyb_{1,2}g_1^3 - 6ib_{1,2}tg_1 + 36tyb_{1,2}g_1h_1^2 + 2iyb_{1,1}g_1 + 4y^2b_{1,2}g_1^2 + 36it^2b_{1,2}g_1^3h_1 - \\ & 4y^2b_{1,2}h_1^2 + 2iyb_{1,2} + 6tb_{1,1}g_1h_1 - 36ib_{1,2}t^2g_1h_1^3 + 8iy^2b_{1,2}g_1h_1 + 6tb_{1,2}h_1 + \\ & 3itb_{1,1}h_1^2 - 2yb_{1,1}h_1 - 3ib_{1,1}tg_1^2 - b_{1,0} \end{aligned}$$

We give some figures in the  $(x, y)$  plane of coordinates.



**Figure 3.** Solutions of order 1 to KPI, on the left for  $t = 0$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $b_{1,2} = 1$ ,  $g_1 = 1$ ,  $h_1 = 1$ ,  $d_1 = 0$ ,  $e_1 = 1$ ; in the center for  $t = 0$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $b_{1,2} = 1$ ,  $g_1 = 1$ ,  $h_1 = 0$ ,  $d_1 = 1$ ,  $e_1 = 1$ ; on the right for  $t = 10$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $b_{1,2} = 1$ ,  $g_1 = 0.01$ ,  $h_1 = 10^2$ ,  $d_1 = 1$ ,  $e_1 = 1$ .



**Figure 4.** Solutions of order 1 to KPI, on the left for  $t = 0$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $b_{1,2} = 1$ ,  $g_1 = 0.01$ ,  $h_1 = 0.01$ ,  $d_1 = 0$ ,  $e_1 = 1$ ; in the center for  $t = 1$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $b_{1,2} = 1$ ,  $g_1 = 1$ ,  $h_1 = 1$ ,  $d_1 = 1$ ,  $e_1 = 1$ ; on the right for  $t = 10^3$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $b_{1,2} = 1$ ,  $g_1 = 0.01$ ,  $h_1 = 10^2$ ,  $d_1 = 1$ ,  $e_1 = 1$ .

In figures 3 and 4, we get lumps whose modulus depends on the values of parameters. These structures are more sensitive to  $g_1$  and  $h_1$  parameters than to others. In this case, we observe a maximum of two isolated lumps, as shown for example in the figure 4 in the center.

## 5 Solutions of order 1 with a degree of derivation equal to 3 ( $D = 3$ ) depending on 8 real parameters

In the case  $D = 3$ , we observe multi lumps. As in the previous cases, these structures are more sensitive to  $g_1$  and  $h_1$  parameters than to others. The solutions to the KPI equation can be written as

$$u(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}.$$

with

$$\begin{aligned} n(x, y, t) = & 3b_{1,3}^2x^4 + (-36tb_{1,3}^2h_1^2 + 36tb_{1,3}^2g_1^2 - 24yb_{1,3}^2g_1 + 72itb_{1,3}^2g_1h_1 - \\ & 24iyb_{1,3}^2h_1 - 4ib_{1,2}b_{1,3})x^3 + (216ityb_{1,3}^2h_1^3 + 162t^2b_{1,3}^2g_1^4 - 36itb_{1,2}b_{1,3}g_1^2 - \\ & 648it^2b_{1,3}^2g_1h_1^3 - 972t^2b_{1,3}^2g_1^2h_1^2 + 162t^2b_{1,3}^2h_1^4 - 2b_{1,2}^2 + 648it^2b_{1,3}^2g_1^3h_1 + \\ & 648tyb_{1,3}^2g_1h_1^2 + 36itb_{1,2}b_{1,3}h_1^2 - 24yb_{1,2}b_{1,3}h_1 - 648ityb_{1,3}^2g_1^2h_1 + 144iy^2b_{1,3}^2g_1h_1 - \\ & 216tyb_{1,3}^2g_1^3 + 72tb_{1,2}b_{1,3}g_1h_1 - 72y^2b_{1,3}^2h_1^2 + 72y^2b_{1,3}^2g_1^2 + 24iyb_{1,2}b_{1,3}g_1)x^2 + \\ & (1728ity^2b_{1,3}^2g_1^3h_1 - 1728ity^2b_{1,3}^2g_1h_1^3 + 648it^2b_{1,2}b_{1,3}g_1^2h_1^2 + 144ityb_{1,2}b_{1,3}g_1^3 - \\ & 3240it^2yb_{1,3}^2g_1^4h_1 + 6480it^2yb_{1,3}^2g_1^2h_1^3 - 432tyb_{1,2}b_{1,3}g_1^2h_1 - 4860t^3b_{1,3}^2g_1^4h_1^2 + \\ & 4860t^3b_{1,3}^2g_1^2h_1^4 + 432ty^2b_{1,3}^2h_1^4 + 96iy^3b_{1,3}^2h_1^3 + 8iyb_{1,2}^2h_1 + 432ty^2b_{1,3}^2g_1^4 - \\ & 648t^2yb_{1,3}^2g_1^5 + 288y^3b_{1,3}^2g_1h_1^2 - 432ityb_{1,2}b_{1,3}g_1h_1^2 + 36tb_{1,3}^2 - 48iy^2b_{1,2}b_{1,3}g_1^2 + \\ & 48iy^2b_{1,2}b_{1,3}h_1^2 - 24itb_{1,2}^2g_1h_1 - 108it^2b_{1,2}b_{1,3}g_1^4 - 108it^2b_{1,2}b_{1,3}h_1^4 - 288iy^3b_{1,3}^2g_1^2h_1 - \\ & 648it^2yb_{1,3}^2h_1^5 + 1944it^3b_{1,3}^2g_1^5h_1 - 6480it^3b_{1,3}^2g_1^3h_1^3 + 1944it^3b_{1,3}^2g_1h_1^5 + \end{aligned}$$

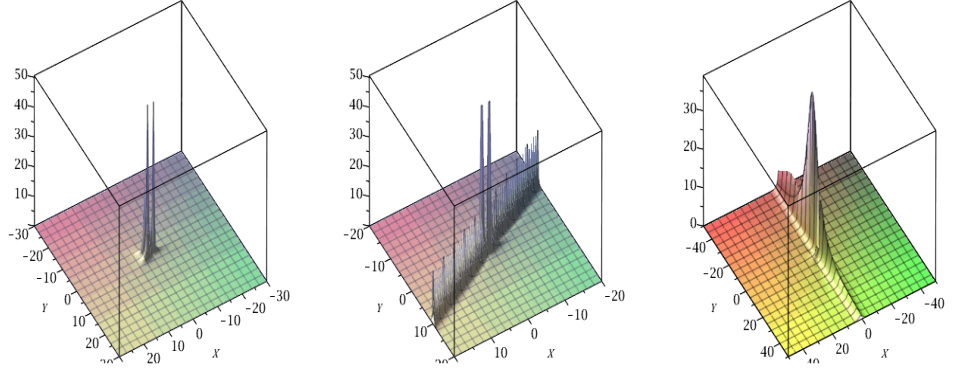
$$\begin{aligned}
& 96 y^2 b_{1,2} b_{1,3} g_1 h_1 + 432 t^2 b_{1,2} b_{1,3} g_1^3 h_1 - 432 t^2 b_{1,2} b_{1,3} g_1 h_1^3 + 144 t y b_{1,2} b_{1,3} h_1^3 - \\
& 2592 t y^2 b_{1,3}^2 g_1^2 h_1^2 + 6480 t^2 y b_{1,3}^2 g_1^3 h_1^2 - 3240 t^2 y b_{1,3}^2 g_1 h_1^4 - 12 t b_{1,2}^2 g_1^2 + \\
& 12 t b_{1,2}^2 h_1^2 + 8 y b_{1,2}^2 g_1 - 6 i b_{1,0} b_{1,3} + 2 i b_{1,1} b_{1,2} + 324 t^3 b_{1,3}^2 g_1^6 - 324 t^3 b_{1,3}^2 h_1^6 - \\
& 96 y^3 b_{1,3}^2 g_1^3) x + 13608 t^3 y b_{1,3}^2 g_1^5 h_1^2 - 22680 t^3 y b_{1,3}^2 g_1^3 h_1^4 - 2160 t^3 b_{1,2} b_{1,3} g_1^3 h_1^3 + \\
& 648 t^3 b_{1,2} b_{1,3} g_1 h_1^5 + 2880 t y^3 b_{1,3}^2 g_1^3 h_1^2 - 1440 t y^3 b_{1,3}^2 g_1 h_1^4 - 216 t^2 y b_{1,2} b_{1,3} h_1^5 - \\
& 96 i y^3 b_{1,2} b_{1,3} g_1 h_1^2 + 72 i t y b_{1,2}^2 g_1^2 h_1 - 144 i t y^2 b_{1,2} b_{1,3} g_1^4 - 144 i t y^2 b_{1,2} b_{1,3} h_1^4 + \\
& 4536 t^3 y b_{1,3}^2 g_1 h_1^6 + 1620 i t^3 b_{1,2} b_{1,3} g_1^4 h_1^2 - 1620 i t^3 b_{1,2} b_{1,3} g_1^2 h_1^4 - 1440 i t y^3 b_{1,3}^2 g_1^4 h_1 + \\
& 2880 i t y^3 b_{1,3}^2 g_1^2 h_1^3 + 3888 i t^2 y^2 b_{1,3}^2 g_1^5 h_1 - 12960 i t^2 y^2 b_{1,3}^2 g_1^3 h_1^3 + 3888 i t^2 y^2 b_{1,3}^2 g_1 h_1^5 - \\
& 4536 i t^3 y b_{1,3}^2 g_1^6 h_1 + 22680 i t^3 y b_{1,3}^2 g_1^4 h_1^3 - 13608 i t^3 y b_{1,3}^2 g_1^2 h_1^5 + 1080 i t^2 y b_{1,2} b_{1,3} g_1 h_1^4 + \\
& 864 i t y^2 b_{1,2} b_{1,3} g_1^2 h_1^2 + 216 i t^2 y b_{1,2} b_{1,3} g_1^5 + 1944 i t^4 b_{1,3}^2 g_1^7 h_1 - 13608 i t^4 b_{1,3}^2 g_1^5 h_1^3 + \\
& 13608 i t^4 b_{1,3}^2 g_1^3 h_1^5 - 1944 i t^4 b_{1,3}^2 g_1 h_1^7 + 648 i t^3 y b_{1,3}^2 h_1^7 - 108 i t^3 b_{1,2} b_{1,3} g_1^6 + \\
& 108 i t^3 b_{1,2} b_{1,3} h_1^6 - 288 i t y^3 b_{1,3}^2 h_1^5 + 192 i y^4 b_{1,3}^2 g_1^3 h_1 - 192 i y^4 b_{1,3}^2 g_1 h_1^3 - 72 i t^2 b_{1,2}^2 g_1^3 h_1 + \\
& 72 i t^2 b_{1,2}^2 g_1 h_1^3 + 32 i y^3 b_{1,2} b_{1,3} g_1^3 + 36 i t b_{1,1} b_{1,3} g_1 - 9720 t^2 y^2 b_{1,3}^2 g_1^4 h_1^2 + 9720 t^2 y^2 b_{1,3}^2 g_1^2 h_1^4 + \\
& 648 t^3 b_{1,2} b_{1,3} g_1^5 h_1 - 432 i t^2 b_{1,3}^2 g_1 h_1 - 16 i y^2 b_{1,2}^2 g_1 h_1 + 144 i t y b_{1,3}^2 h_1 - 18 i t b_{1,0} b_{1,3} g_1^2 + \\
& 18 i t b_{1,0} b_{1,3} h_1^2 + 6 i t b_{1,1} b_{1,2} g_1^2 - 6 i t b_{1,1} b_{1,2} h_1^2 - 648 t^2 y^2 b_{1,3}^2 h_1^6 - 288 y^4 b_{1,3}^2 g_1^2 h_1^2 + \\
& 108 t^2 b_{1,2}^2 g_1^2 h_1^2 + 32 y^3 b_{1,2} b_{1,3} h_1^3 + 24 t y b_{1,2}^2 g_1^3 - 12 y b_{1,0} b_{1,3} h_1 + 4 y b_{1,1} b_{1,2} h_1 - \\
& 6804 t^4 b_{1,3}^2 g_1^6 h_1^2 + 17010 t^4 b_{1,3}^2 g_1^4 h_1^4 - 6804 t^4 b_{1,3}^2 g_1^2 h_1^6 + b_{1,1}^2 + 36 t b_{1,0} b_{1,3} g_1 h_1 - \\
& 12 t b_{1,1} b_{1,2} g_1 h_1 - 96 y^3 b_{1,2} b_{1,3} g_1^2 h_1 - 72 t y b_{1,2}^2 g_1 h_1^2 - 2160 i t^2 y b_{1,2} b_{1,3} g_1^3 h_1^2 - \\
& 2 b_{1,0} b_{1,2} - 36 y^2 b_{1,3}^2 - 12 i t b_{1,2}^2 g_1 - 12 i t b_{1,2} b_{1,3} - 12 i y b_{1,1} b_{1,3} - 36 t b_{1,1} b_{1,3} h_1 + \\
& 144 t y b_{1,3}^2 g_1 - 648 t^3 y b_{1,3}^2 g_1^7 + 648 t^2 y^2 b_{1,3}^2 g_1^6 - 288 t y^3 b_{1,3}^2 g_1^5 + 576 t y^2 b_{1,2} b_{1,3} g_1^3 h_1 - \\
& 576 t y b_{1,2} b_{1,3} g_1 h_1^3 - 1080 t^2 y b_{1,2} b_{1,3} g_1^4 h_1 + 2160 t^2 y b_{1,2} b_{1,3} g_1^2 h_1^3 + 12 i y b_{1,0} b_{1,3} g_1 - \\
& 4 i y b_{1,1} b_{1,2} g_1 - 24 i t y b_{1,2}^2 h_1^3 + 4 i y b_{1,2}^2 + 216 t^2 b_{1,3}^2 h_1^2 + 243 t^4 b_{1,3}^2 g_1^8 + 243 t^4 b_{1,3}^2 h_1^8 + \\
& 48 y^4 b_{1,3}^2 h_1^4 - 18 t^2 b_{1,2}^2 g_1^4 - 18 t^2 b_{1,2}^2 h_1^4 - 8 y^2 b_{1,2}^2 g_1^2 + 8 y^2 b_{1,2}^2 h_1^2 + 12 t b_{1,2}^2 h_1 - \\
& 216 t^2 b_{1,3}^2 g_1^2 + 48 y^4 b_{1,3}^2 g_1^4
\end{aligned}$$

and

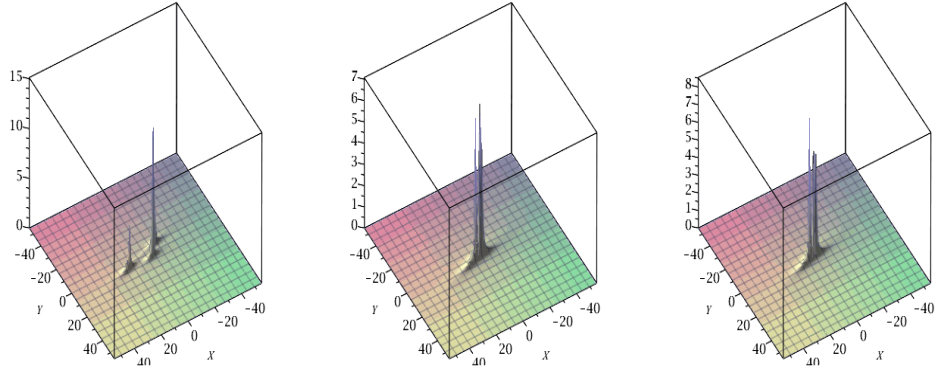
$$\begin{aligned}
d(x, y, t) &= i b_{1,3} x^3 + (9 i t b_{1,3} g_1^2 - 18 t b_{1,3} g_1 h_1 - 9 i t b_{1,3} h_1^2 - 6 i y b_{1,3} g_1 + 6 y b_{1,3} h_1 + \\
& b_{1,2}) x^2 + (27 i t^2 b_{1,3} g_1^4 - 108 t^2 b_{1,3} g_1^3 h_1 - 162 i t^2 b_{1,3} g_1^2 h_1^2 + 108 t^2 b_{1,3} g_1 h_1^3 + \\
& 27 i t^2 b_{1,3} h_1^4 - 36 i t y b_{1,3} g_1^3 \\
& + 108 t y b_{1,3} g_1^2 h_1 + 108 i t y b_{1,3} g_1 h_1^2 - 36 t y b_{1,3} h_1^3 + 12 i y^2 b_{1,3} g_1^2 - 24 y^2 b_{1,3} g_1 h_1 - \\
& 12 i y^2 b_{1,3} h_1^2 + 6 t b_{1,2} g_1^2 + 12 i t b_{1,2} g_1 h_1 - 6 t b_{1,2} h_1^2 + 18 t b_{1,3} g_1 + 18 i t b_{1,3} h_1 - \\
& 4 y b_{1,2} g_1 - 4 i y b_{1,2} h_1 - 6 y b_{1,3} - i b_{1,1}) x - 54 t^2 b_{1,2} g_1^2 h_1^2 - b_{1,0} - 162 t^3 b_{1,3} g_1 h_1^5 + \\
& 54 t^2 y b_{1,3} h_1^5 - 4 y^2 b_{1,2} h_1^2 - 162 t^3 b_{1,3} g_1^5 h_1 + 9 t^2 b_{1,2} h_1^4 + 4 y^2 b_{1,2} g_1^2 - 8 y^3 b_{1,3} h_1^3 - \\
& 2 y b_{1,1} h_1 + 9 t^2 b_{1,2} g_1^4 + 54 t^2 b_{1,3} g_1^3 + 12 y^2 b_{1,3} g_1 - 54 i t^2 b_{1,3} h_1^3 + 12 i y^2 b_{1,3} h_1 + \\
& 27 i t^3 b_{1,3} g_1^6 - 27 i t^3 b_{1,3} h_1^6 - 8 i y^3 b_{1,3} g_1^3 + 3 i t b_{1,1} h_1^2 - 6 i t b_{1,2} g_1 - 3 i t b_{1,1} g_1^2 + \\
& 2 i y b_{1,1} g_1 + 6 t b_{1,2} h_1 + 2 i y b_{1,2} - 6 i t b_{1,3} + 36 i t^2 b_{1,2} g_1^3 h_1 - 36 i t^2 b_{1,2} g_1 h_1^3 + \\
& 12 i t y b_{1,2} h_1^3 + 8 i y^2 b_{1,2} g_1 h_1 - 405 i t^3 b_{1,3} g_1^4 h_1^2 - 54 i t^2 y b_{1,3} g_1^5 + 405 i t^3 b_{1,3} g_1^2 h_1^4 + \\
& 36 i t y^2 b_{1,3} g_1^4 + 36 i t y^2 b_{1,3} h_1^4 + 24 i y^3 b_{1,3} g_1 h_1^2 + 162 i t^2 b_{1,3} g_1^2 h_1 + 24 y^3 b_{1,3} g_1^2 h_1 - \\
& 162 t^2 b_{1,3} g_1 h_1^2 + 54 t y b_{1,3} h_1^2 + 6 t b_{1,1} g_1 h_1 - 12 t y b_{1,2} g_1^3 + 540 t^3 b_{1,3} g_1^3 h_1^3 - \\
& 54 t y b_{1,3} g_1^2 - 108 i t y b_{1,3} g_1 h_1 - 270 i t^2 y b_{1,3} g_1 h_1^4 - 216 i t y^2 b_{1,3} g_1^2 h_1^2 + 540 i t^2 y b_{1,3} g_1^3 h_1^2 - \\
& 36 i t y b_{1,2} g_1^2 h_1 + 36 t y b_{1,2} g_1 h_1^2 - 144 t y^2 b_{1,3} g_1^3 h_1 + 270 t^2 y b_{1,3} g_1^4 h_1 + 144 t y^2 b_{1,3} g_1 h_1^3 - \\
& 540 t^2 y b_{1,3} g_1^2 h_1^3.
\end{aligned}$$

We give some figures in the  $(x, y)$  plane of coordinates.





**Figure 5.** Solutions of order 1 to KPI, on the left for  $t = 0$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $b_{1,2} = 1$ ,  $b_{1,3} = 1$ ,  $g_1 = 0$ ,  $h_1 = 1$ ,  $d_1 = 1$ ,  $e_1 = 1$ ; in the center for  $t = 0$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $b_{1,2} = 1$ ,  $b_{1,3} = 1$ ,  $g_1 = 1$ ,  $h_1 = 0$ ,  $d_1 = 1$ ,  $e_1 = 1$ ; on the right for  $t = 0$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $b_{1,2} = 1$ ,  $b_{1,3} = 1$ ,  $g_1 = 0.01$ ,  $h_1 = 0.01$ ,  $d_1 = 1$ ,  $e_1 = 1$ .



**Figure 6.** Solutions of order 1 to KPI, on the left for  $t = 0$ ,  $b_{1,0} = 10^3$ ,  $b_{1,1} = 1$ ,  $b_{1,2} = 1$ ,  $b_{1,3} = 1$ ,  $g_1 = 1$ ,  $h_1 = 1$ ,  $d_1 = 1$ ,  $e_1 = 1$ ; in the center for  $t = 0$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $b_{1,2} = 1$ ,  $b_{1,3} = 10^3$ ,  $g_1 = 1$ ,  $h_1 = 0$ ,  $d_1 = 1$ ,  $e_1 = 1$ ; on the right for  $t = 0$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 10^3$ ,  $b_{1,2} = 1$ ,  $b_{1,3} = 10^3$ ,  $g_1 = 0.01$ ,  $h_1 = 0.01$ ,  $d_1 = 1$ ,  $e_1 = 1$ .

In figures 5 and 6, we get lumps depending on 8 real parameters. These structures are more sensitive to  $g_1$  and  $h_1$  parameters than to others. In this case, we observe a maximum of three distinct isolated lumps, as shown for example in the figure 6 to the left.

## 6 Solutions of order 1 with a degree of derivation equal to 4 ( $D = 4$ ) depending on 9 real parameters

In the case  $D = 4$ , the expression of the solutions with all parameters being too long, we present only one of them with particular values of parameters. For example we choose :

$$b_{1,0} = 1, b_{1,1} = 1, b_{1,2} = 1, b_{1,3} = 1, g_1 = 1, h_1 = 1, d_1 = 1, e_1 = 1.$$

So, the solution to the KPI equation, with these choices of parameters, can be written as

$$u(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

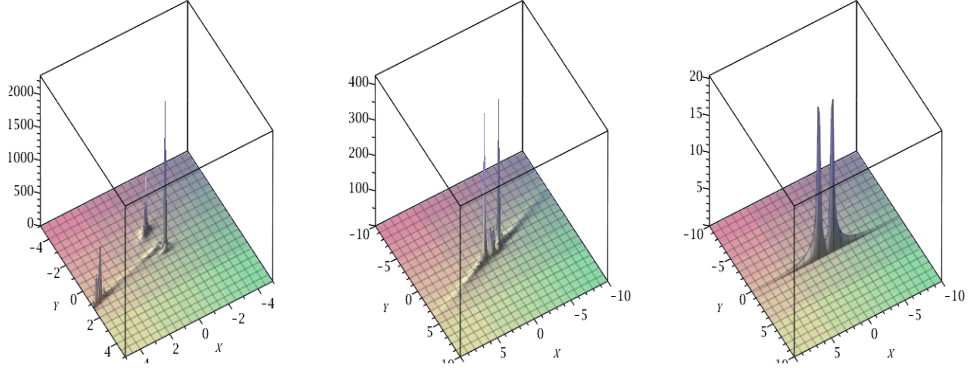
with

$$\begin{aligned} n(x, y, t) = & -4x^6 + (48y - 144it + 6i + 48iy)x^5 + (72it + 5 - 84iy - 480iy^2 + \\ & 2160t^2 + 1440iyt - 252t + 60y - 1440yt)x^4 + (-17280iyt^2 + 3168yt - 3888it^2 + \\ & 192it - 40iy + 1440iyt + 17280it^3 - 24t - 64y - 1280y^3 + 192iy^2 - 672y^2 - \\ & 17280t^2y - 1728t^2 + 11520ty^2 + 1280iy^3)x^3 + (-77760t^4 - 23040ty^3 + 103680t^3y + \\ & 24iy + 384iy^2 - 6912ty^2 + 288y^2 + 103680it^2y^2 + 28512t^3 + 2112y^3 + 960iy^3 + \\ & 432yt - 15552iy^2t + 38880iyt^2 + 2160it^2 + 10 + 3840y^4 - 15552it^3 - 2304iyt - \\ & 103680it^3y + 72t - 23040ity^3 - 12960t^2y - 2376t^2)x^2 + (-16y - 84t - 256y^3 + \\ & 96y^2 + 3024t^2 - 4i - 28512t^3 + 138240t^2y^3 - 1536y^4 + 62208t^4 - 3072y^5 - \\ & 40iy - 1472iy^3 + 101088it^4 - 3456iy^4 - 864it^2 - 186624it^5 - 3072iy^5 - 168iy^2 - \\ & 12096it^3 + 34560ity^3 + 311040it^4y - 138240it^2y^3 + 9792iy^2t + 720iyt - 9504iyt^2 + \\ & 46080ity^4 - 62208it^2y^2 - 51840it^3y - 196992t^3y + 114048t^2y^2 - 11520ty^3 + \\ & 48it - 1008yt + 31104t^2y - 6336ty^2 + 311040t^4y - 414720t^3y^2)x - 936t^2 - \\ & 7776t^3 - 131328t^2y^3 - 704y^4 + 22032t^4 - 768y^5 - 12y^2 - 1 - 8y + 72t - 336y^3 - \\ & 622080it^4y^2 - 7776iyt^2 + 269568it^3y^2 - 13824ity^4 + 373248it^5y + 276480it^3y^3 - \\ & 18432ity^5 - 24192it^2y^2 + 117504it^3y - 6912ity^3 - 357696it^4y - 34560it^2y^3 + \\ & 2592iy^2t - 139968t^5 + 186624t^6 + 2048iy^6 - 88128it^4 + 1536iy^4 - 288it^2 + \\ & 93312it^5 + 2304iy^5 + 32iy^2 + 2592it^3 + 24iy - 12it + 63936t^3y - 69120t^2y^2 + \\ & 16512ty^3 - 373248t^5y + 276480t^3y^3 - 138240t^2y^4 + 18432ty^5 - 144iy^3 + 192yt + \\ & 1728t^2y + 1296ty^2 + 77760t^4y + 165888t^3y^2 + 25344ty^4 \end{aligned}$$

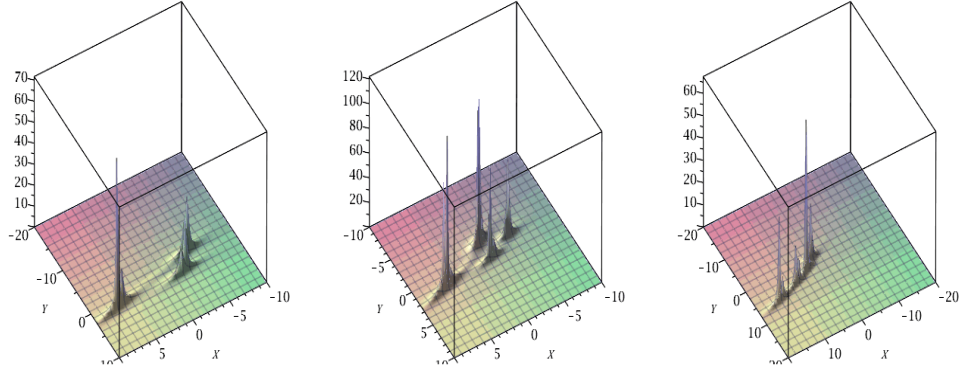
and

$$\begin{aligned} d(x, y, t) = & -x^4 + (-24it + 8y + 8iy + i)x^3 + (216t^2 + 144ity - 144ty - 48iy^2 - \\ & 54t + 36it - 18iy + 6y + 1)x^2 + (864it^3 - 864t^2y - 864it^2y + 576ty^2 + 64iy^3 - \\ & 64y^3 - 432t^2 - 540it^2 + 504ty + 72ity - 72y^2 + 48iy^2 + 30it + 42t - 10y - \\ & 4iy - i)x - 1 - 2y + 12t + 112y^3 + 24y^2 - 144t^2 + 4iy + 16iy^3 + 468it^2 + \\ & 20iy^2 - 1296it^3 - 384iy^3t - 720iy^2t - 252iyt + 2376it^2y + 1728it^2y^2 - 1728iyt^3 + \\ & 1512t^3 + 64y^4 - 1296t^4 - 12it + 1728t^3y - 384ty^3 - 96ty - 216t^2y - 576ty^2. \end{aligned}$$

We get multi lumps. We present some figures in the  $(x, y)$  plane :



**Figure 7.** Solutions of order 1 to KPI, on the left for  $t = 0$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $b_{1,2} = 1$ ,  $b_{1,3} = 1$ ,  $b_{1,4} = 1$ ,  $g_1 = 1$ ,  $h_1 = 1$ ,  $d_1 = 1$ ,  $e_1 = 1$ ; in the center for  $t = 0$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $b_{1,2} = 1$ ,  $b_{1,3} = 1$ ,  $b_{1,4} = 1$ ,  $g_1 = 1$ ,  $h_1 = 0$ ,  $d_1 = 1$ ,  $e_1 = 1$ ; on the right for  $t = 0$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $b_{1,2} = 1$ ,  $b_{1,3} = 1$ ,  $b_{1,4} = 1$ ,  $g_1 = 10^3$ ,  $h_1 = 0,01$ ,  $d_1 = 1$ ,  $e_1 = 1$ .



**Figure 8.** Solutions of order 1 to KPI, on the left for  $t = 0$ ,  $b_{1,0} = 10^3$ ,  $b_{1,1} = 1$ ,  $b_{1,2} = 1$ ,  $b_{1,3} = 1$ ,  $b_{1,4} = 1$ ,  $g_1 = 1$ ,  $h_1 = 1$ ,  $d_1 = 1$ ,  $e_1 = 1$ ; in the center for  $t = 0$ ,  $b_{1,0} = 10^2$ ,  $b_{1,1} = 1$ ,  $b_{1,2} = 1$ ,  $b_{1,3} = 1$ ,  $b_{1,4} = 1$ ,  $g_1 = 1$ ,  $h_1 = 0$ ,  $d_1 = 0,01$ ,  $e_1 = 0,01$ ; on the right for  $t = 1$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $b_{1,2} = 1$ ,  $b_{1,3} = 1$ ,  $b_{1,4} = 1$ ,  $g_1 = 10^3$ ,  $h_1 = 0,01$ ,  $d_1 = 1$ ,  $e_1 = 1$ .

In figures 7 and 8, we get lumps depending on 9 real parameters. These structures are more sensitive to  $g_1$  and  $h_1$  parameters than to others. In this case, we observe a maximum of four distinct isolated lumps, as shown for example in the figure 8 in the center. In Figure 8, on the right, there are four isolated lumps aligned on a non rectilinear curve.

## 7 Solutions of order 1 with a degree of derivation equal to 5 ( $D = 5$ ) depending on 10 real parameters

In the case  $D = 5$ , the expression of the solutions with all parameters being too long, we present only one of them with particular values of parameters. For example we choose :

$b_{1,0} = 1, b_{1,1} = 1, b_{1,2} = 1, b_{1,3} = 1, b_{1,4} = 1, b_{1,5} = 1, g_1 = 1, h_1 = 1, d_1 = 1, e_1 = 1.$

The solution to the KPI equation can be written as

$$u(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

with

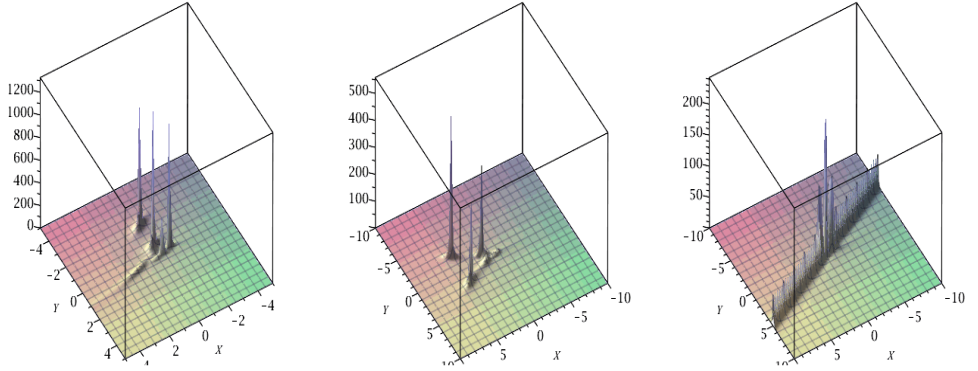
$$\begin{aligned} n(x, y, t) = & 5x^8 + (240it - 80y - 80iy - 8i)x^7 + (3360ty - 5040t^2 - 3360ity - \\ & 112y + 576t + 1120iy^2 - 8 + 192iy - 240it)x^6 + (-4032ity + 8640t^2 + 96iy - \\ & 60480it^3 + 4i - 168t + 60480t^2y + 60480it^2y - 960iy^2 - 4480iy^3 - 40320ty^2 - \\ & 12672ty + 192y + 2304y^2 + 4480y^3 + 14688it^2 - 576it)x^5 + (-5 + 40y - \\ & 372t - 14080y^3 - 1560y^2 + 12960t^2 + 97920ity^2 - 604800it^2y^2 - 276480it^2y + \\ & 134400ity^3 + 604800it^3y + 13920ity + 60480t^2y + 57600ty^2 - 4480iy^3 + 132it + \\ & 129600it^3 - 15840it^2 - 124iy - 1920iy^2 + 134400ty^3 - 604800t^3y - 480ty - \\ & 190080t^3 - 22400y^4 + 453600t^4)x^4 + (-24y + 216t + 2240y^3 - 992y^2 - 19008t^2 - \\ & 3360ity - 108480ity^2 + 1036800it^2y^2 - 537600ity^4 + 1612800it^2y^3 + 109440it^2y - \\ & 3628800it^4y - 491520ity^3 + 483840it^3y + 20i + 4838400t^3y^2 - 3628800t^4y - \\ & 1612800t^2y^3 - 351360t^2y + 69120ty^2 - 1399680it^4 + 2177280it^5 + 72it + 138240it^3 + \\ & 15040iy^3 + 35840iy^5 - 288it^2 + 43520iy^4 + 40iy + 1152iy^2 + 107520ty^3 - \\ & 1520640t^2y^2 + 2903040t^3y + 8928ty + 319680t^3 + 25600y^4 - 1036800t^4 + 35840y^5)x^3 + \\ & (10 + 120y - 288t + 7552y^3 + 528y^2 + 9144t^2 - 15552ity^2 + 829440it^2y^2 + \\ & 691200ity^4 + 967680it^2y^3 - 11197440it^3y^2 + 185760it^2y + 16174080it^4y - 9676800it^3y^3 - \\ & 13063680it^5y + 645120ity^5 + 193920ity^3 - 3732480it^3y - 3744ity + 21772800it^4y^2 - \\ & 8294400t^3y^2 - 1013760ty^4 - 2177280t^4y + 5806080t^2y^3 + 207360t^2y - 76032ty^2 - \\ & 6531840t^6 - 240it + 2695680it^4 - 71680iy^6 - 4665600it^5 - 313632it^3 - 2240iy^3 - \\ & 98304iy^5 + 10800it^2 - 44160iy^4 - 96iy + 144iy^2 - 516480ty^3 + 2134080t^2y^2 - \\ & 1866240t^3y - 9676800t^3y^3 + 4838400t^2y^4 - 645120ty^5 + 13063680t^5y - 3168ty - \\ & 66528t^3 + 5971968t^5 + 23040y^4 - 777600t^4 + 21504y^5)x^2 + (-136y + 156t - \\ & 896y^3 + 24192ity^2 - 1168128it^2y^2 + 368640ity^4 - 5264640it^2y^3 + 15240960it^3y^2 - \\ & 29030400it^4y^3 + 26127360it^6y - 64224it^2y - 3870720it^2y^5 - 10782720it^4y + \\ & 19353600it^3y^4 + 28753920it^3y^3 - 5225472it^5y + 258048ity^5 + 184320ity^3 + \\ & 2090880it^3y + 1680ity - 31104000it^4y^2 - 7464960it^2y^4 - 384y^2 + 1584t^2 - 4i - \\ & 4147200t^3y^2 + 1044480ty^4 + 29030400t^4y^3 - 52254720t^5y^2 + 26127360t^6y + \\ & 860160ty^6 - 3870720t^2y^5 + 17625600t^4y - 2776320t^2y^3 + 67104t^2y - 7776ty^2 - \\ & 40960y^7 + 11197440t^6 + 13810176it^6 - 533952it^4 + 61440iy^6 - 2239488it^5 + \\ & 408it - 11197440it^7 - 7776it^3 - 1792iy^3 + 32256iy^5 - 3744it^2 - 6656iy^4 + \\ & 40960iy^7 - 40iy - 48iy^2 + 145920ty^3 - 93312t^2y^2 - 1372032t^3y + 39813120t^4y^2 - \end{aligned}$$

$3870720 t^3 y^3 - 5529600 t^2 y^4 + 1548288 t y^5 - 46282752 t^5 y + 192 t y - 106272 t^3 - 10419840 t^5 - 90112 y^6 - 16896 y^4 + 1824768 t^4 - 75264 y^5)x + 224 y^3 - 936 t^2 + 8398080 t^8 + 20480 y^8 + 108 y^2 - 15303168 t^6 + 11197440 t^7 + 67392 t^3 + 656 i y^3 + 13568 i y^5 + 432 i t^2 + 2176 i y^4 + 8192 i y^7 + 4520448 t^3 y^2 + 17664 t y^4 - 295488 t^4 + 16384 i y^6 + 3592512 i t^5 - 132 i t - 491520 t y^6 - 774144 t^2 y^5 - 5857920 t^4 y - 972288 t^2 y^3 - 22752 t^2 y + 4656 t y^2 + 49152 y^7 - 13436928 t^7 + 2612736 t^6 + 17694720 t^3 y^4 - 51425280 t^4 y^3 + 44789760 t^5 y^2 + 5225472 t^6 y - 1 - 8 y + 72 t - 4608 t y^3 - 181440 t^2 y^2 + 798336 t^3 y - 36028800 t^4 y^2 + 16243200 t^3 y^3 - 1382400 t^2 y^4 - 408576 t y^5 + 21026304 t^5 y - 245760 t y^7 + 7741440 t^3 y^5 - 29030400 t^4 y^4 + 34836480 t^5 y^3 - 22394880 t^7 y + 672 t y + 24 i y + 22394880 i t^7 y - 3888 i t y^2 + 266688 i t^2 y^2 - 240384 i t y^4 + 1163520 i t^2 y^3 - 905472 i t^3 y^2 - 5806080 i t^4 y^3 - 53747712 i t^6 y - 17856 i t^2 y + 5750784 i t^2 y^5 - 3312576 i t^4 y - 13824000 i t^3 y^4 + 55240704 i t^5 y^2 - 663552 i t y^6 - 245760 i t y^7 - 7741440 i t^3 y^5 + 34836480 i t^5 y^3 + 2580480 i t^2 y^6 - 10160640 i t^3 y^3 + 30730752 i t^5 y - 740352 i t y^5 - 52992 i t y^3 - 245376 i t^3 y - 1200 i t y - 52254720 i t^6 y^2 + 272 i y^2 - 7464960 i t^4 y^2 + 5368320 i t^2 y^4 - 29376 t^3 + 1197504 t^5 + 50688 y^6 + 2896 y^4 - 768528 t^4 + 7168 y^5$

and

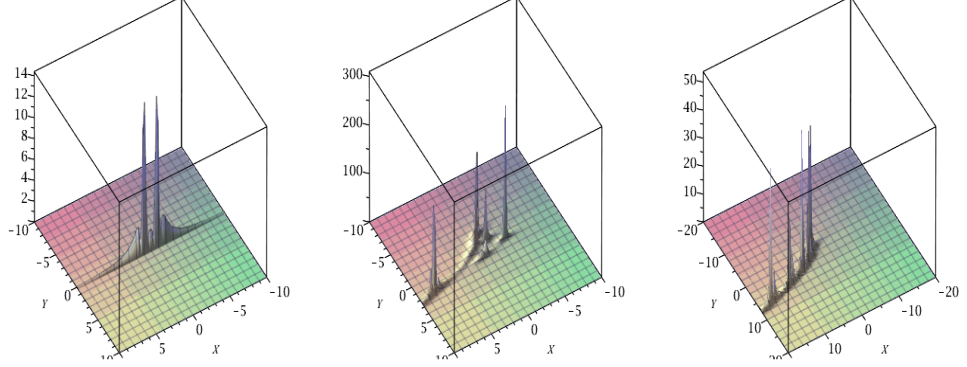
$d(x, y, t) = i x^5 + (-30 t - 10 i y + 10 y + 1)x^4 + (-360 i t^2 + 240 i t y + 240 t y - 80 y^2 + 60 t + 84 i t - 28 y - 8 i y - i)x^3 + (2160 t^3 - 2160 t^2 y + 2160 i t^2 y - 1440 i t y^2 + 160 y^3 + 160 i y^3 - 1296 t^2 + 1080 i t^2 - 1224 i t y + 144 t y + 168 i y^2 + 120 y^2 + 54 t - 96 i t + 18 i y - 6 y - 1)x^2 + (6480 i t^4 - 8640 t^3 y - 8640 i t^3 y + 8640 t^2 y^2 - 1920 t y^3 + 1920 i t y^3 - 320 i y^4 - 7344 i t^3 - 6480 t^3 + 864 i t^2 y + 11664 t^2 y + 2880 i t y^2 - 3456 t y^2 - 544 i y^3 + 64 y^3 + 2232 t^2 + 540 i t^2 + 528 i t y - 1104 t y - 108 i y^2 + 72 y^2 - 30 i t - 42 t + 10 y + 4 i y + i)x + 2 y - 12 t + 720 i t y^2 - 232 y^3 - 24 y^2 - 216 t^2 - 12960 i t^4 - 828 i t^2 + 1 + 1920 t y^4 + 12960 t^4 y - 5760 t^2 y^3 - 5544 t^2 y + 2856 t y^2 + 5184 t y^3 - 12960 t^2 y^2 - 1728 t^3 y + 128 i y^5 + 104 i y^3 + 9936 i t^3 + 320 i y^4 - 4 i y + 216 t y - 14688 i t^2 y^2 + 252 i t y - 5760 i t^2 y^3 - 12960 i t^4 y - 8136 i t^2 y + 384 i t y^3 + 31968 i t^3 y + 17280 i t^3 y^2 - 20 i y^2 + 12 i t - 1512 t^3 - 7776 t^5 - 384 y^4 + 14256 t^4 - 128 y^5.$

We present some figures in the  $(x, y)$  plane :



**Figure 9.** Solutions of order 1 to KPI, on the left for  $t = 0$ ,  $b_{1,0} = 1$ ,  $b_{1,1} = 1$ ,  $b_{1,2} = 1$ ,  $b_{1,3} = 1$ ,  $b_{1,4} = 1$ ,  $b_{1,5} = 1$ ,  $g_1 = 1$ ,  $h_1 = 1$ ,  $d_1 = 1$ ,  $e_1 = 1$ ; in the center for  $t = 0$ ,  $b_{1,0} = 10^3$ ,  $b_{1,1} = 1$ ,  $b_{1,2} = 1$ ,  $b_{1,3} = 1$ ,  $b_{1,4} = 1$ ,  $b_{1,5} = 1$ ,  $g_1 = 0$ ,  $h_1 = 1$ ,  $d_1 = 1$ ,  $e_1 = 1$ ; on the right for  $t = 0$ ,  $b_{1,0} = 10^3$ ,  $b_{1,1} = 1$ ,

$$b_{1,2} = 1, b_{1,3} = 1, b_{1,4} = 1, b_{1,5} = 1, g_1 = 1, h_1 = 0, d_1 = 1, e_1 = 1.$$



**Figure 10.** Solutions of order 1 to KPI, on the left for  $t = 0$ ,  $b_{1,0} = 10^3$ ,  $b_{1,1} = 1, b_{1,2} = 1, b_{1,3} = 1, b_{1,4} = 1, b_{1,5} = 1, g_1 = 10^3, h_1 = 0,01, d_1 = 1, e_1 = 1$ ; in the center for  $t = 0$ ,  $b_{1,0} = 10^2, b_{1,1} = 1, b_{1,2} = 1, b_{1,3} = 1, b_{1,4} = 1, b_{1,5} = 1, g_1 = 1, h_1 = 1, d_1 = 1, e_1 = 1$ ; on the right for  $t = 1, b_{1,0} = 1, b_{1,1} = 1, b_{1,2} = 1, b_{1,3} = 1, b_{1,4} = 1, b_{1,5} = 1, g_1 = 1, h_1 = 1, d_1 = 1, e_1 = 1$ .

In figures 9 and 10, we get lumps depending on 10 real parameters. In this case, we observe a maximum of five distinct isolated lumps, as shown for example in the figure 10 in the center. In Figure 10, on the right, there are five lumps aligned on a non rectilinear curve.

## 8 Conclusion

By means of a Darboux transformation with particular generating functions, we construct multi-parametric solutions to the KPI equation expressed in terms of a wronskian of order  $N$ . These solutions depend on the order of the determinant  $N$ , the degree of summation  $S$  and the degree of derivation  $D$ . In this study, to get rational solutions, we restrict ourselves to the case where  $S = 1$ . In the general case, these rational solutions can be written as a second derivative with respect to  $x$  of a logarithm of a wronskian of order  $N$ , depending on  $N(D + 5)$  real parameters. For a rational solution of order  $N$ , the numerator is a polynomial in  $x, y$  and  $t$  of degree  $2N - 2$  and the denominator a polynomial of degree  $2N$  in  $x, y$  and  $t$ .

We only give the explicit expressions of the solutions in the simple case of order  $N = 1$  and degrees of summation between  $S = 1$  and  $S = 5$ .

All these solutions are different from these constructed with another approach given by the present author [50, 51, 52, 54, 56].

It would be relevant to study the cases of order greater or equal to 2 and to realize an exhaustive classification of the solutions to the KPI equation.

We postpone this study to another publication.

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