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One degree of summation's rational solutions to the KPI equation

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Abstract

In this paper, we construct solutions to the Kadomtsev-Petviashvili equation (KPI) by using a Darboux transformation with particular generating functions limited to one degree of summation. With this choice, we get rational solutions expressed in terms of a wronskian of order N depending on $N(D + 5)$ real parameters.

In this case, we construct explicitly multi-parametric rational solutions to the KPI equation and we study the patterns of their modulus in the plane (x, y) and their evolution according to time and parameters.

PACS numbers :

33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

1 Introduction

The Kadomtsev-Petviashvili equation (KPI)

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \quad (1)$$

where as usual, subscripts x , y and t denote partial derivatives was proposed first by Kadomtsev and Petviashvili [1] in 1970. This equation describes weakly non-linear waves in media. It is the subject of countless researches beginning with Petviashvili [2] in 1976, then in 1977 with Manakov, Zakharov, Bordag and Matveev [3], with Krichever [4, 5], Matveev in 1979 [6], Freeman and Nimmo [7, 8] in 1983 among others.

Some recent works related to this study can be cited. Tian, Xu and Zhang successfully proposed an effective and direct approach to study the symmetry-preserving discretization for a class of generalized higher order equations, and

proposed an open problem about symmetries and the multipliers of conservation law [9]. Li and Tian systematically solved the Cauchy problem of the general n -component nonlinear Schrödinger equations based the Riemann-Hilbert method, and have given the N -soliton solutions. Moreover, they proposed a conjecture about the law of nonlinear wave propagation [10]. Wang, Tian and Cheng successfully derived the three-component coupled Hirota hierarchy, and firstly obtained the explicit soliton solutions of the equations via dressing method [11]. Yang, Tian and Li successfully solved the soliton solutions of the focusing nonlinear Schrödinger equation with multiple high-order poles under nonzero boundary conditions for the first time [12]. Wu and Tian successfully solved the long-time asymptotic problem of the solution to the non local short pulse equation with the Schwartz-type initial data, without solitons [13]. With respect to soliton resolution conjecture, Li, Tian, Yang and Fan have done some interesting work in deriving the solutions of Wadati-Konno-Ichikawa equation and complex short pulse equation with the help of Dbar-steepest descent method. They solved the long-time asymptotic behavior of the solutions of these equations, and proved the soliton resolution conjecture and the asymptotic stability of solutions of these equations [14, 15, 16].

Here, we use an extended Darboux transformation and some specific generating functions to construct rational solutions to the KPI equation in terms of a second derivative with respect to x of a logarithm of a wronskian of order N . These solutions depend on a degree of summation S and a degree of degree of derivation D . In this study, we choose $S = 1$. So we obtain N order solutions depending in general on $N(D + 5)$ real parameters which are rational in x , y and t . In the following, we restrict ourselves to the case where the order N is equal to 1.

We construct some explicit rational solutions of order 1, depending on several real parameters, and the representations of their modulus in the plane of the coordinates (x, y) according to parameters and time t .

2 Multi-parametric rational solutions of order N to the KPI equation

We consider the Kadomtsev-Petviashvili (KPI) equation

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0.$$

We consider the wronskian of order N of the functions f_1, \dots, f_N denoted as $W(f_1, \dots, f_N)$, and defined by $\det(\partial_x^{i-1} f_j)_{1 \leq i \leq N, 1 \leq j \leq N}$, ∂_x^i being the partial derivative of order i with respect to x and $\partial_x^0 f_j$ being the function f_j .

We consider $a_j, b_{j_s d}, d_{j_s}, e_{j_s}, g_{j_s}, h_{j_s}$, arbitrary real numbers with $c_{j_s} = g_{j_s} + ih_{j_s}$.

We consider the following elementary functions f_{j_s} defined by

$$f_{j_s}(x, y, t) = e^{ic_{j_s}x - ic_{j_s}^2 y + ic_{j_s}^3 t + d_{j_s} + ie_{j_s}} = e^{i(g_{j_s} + ih_{j_s})x - i(g_{j_s} + ih_{j_s})^2 y + i(g_{j_s} + ih_{j_s})^3 t + d_{j_s} + ie_{j_s}}. \quad (2)$$

Then, we have the following statement :

Theorem 2.1 *Let ψ_j , be the functions defined by*

$$\psi_j(x, y, t) = a_j + \sum_{s=1}^S \sum_{d=0}^D b_{j,s,d} \partial_{c_{j,s}}^d e^{ic_{j,s}x - ic_{j,s}^2y + ic_{j,s}^3t + d_{j,s} + ie_{j,s}} = a_j + \sum_{s=1}^S \sum_{d=0}^D b_{j,s,d} \partial_{c_{j,s}}^d f_{j,s}(x, y, t). \quad (3)$$

Then the function u defined by

$$u(x, y, t) = -2\partial_x^2 \ln(W(\psi_1(x, y, t), \dots, \psi_N(x, y, t))) \quad (4)$$

is a solution to the (KPI) equation (1) depending on real $N(S(D+5)+1)$ parameters $a_j, b_{j,s,d}, g_{j,s}, h_{j,s}, d_{j,s}, e_{j,s}, 1 \leq j \leq N, 1 \leq s \leq S, 0 \leq d \leq D$.

We will call the order N of the wronskian, the order of the solution. The number S of terms of the summation will be call the degree of the summation of the solution and D the degree of derivation of the solution.

In the following, we restrict ourselves to the study with $S = 1$.

So we will use some simplified notations.

We consider $b_{j,d}, d_j, e_j, g_j, h_j$, arbitrary real numbers with $c_j = g_j + ih_j$.

We consider the following elementary functions f_j defined by

$$f_j(x, y, t) = e^{ic_jx - ic_j^2y + ic_j^3t + d_j + ie_j} = e^{i(g_j + ih_j)x - i(g_j + ih_j)^2y + i(g_j + ih_j)^3t + d_j + ie_j}. \quad (5)$$

Then, we have the following statement :

Theorem 2.2 *Let ϕ_j , be the functions defined by*

$$\phi_j(x, y, t) = \sum_{d=0}^D b_{j,d} \partial_{c_j}^d e^{ic_jx - ic_j^2y + ic_j^3t + d_j + ie_j} = \sum_{d=0}^D b_{j,d} \partial_{c_j}^d f_j(x, y, t). \quad (6)$$

Then the function u defined by

$$u(x, y, t) = -2\partial_x^2 \ln(W(\phi_1(x, y, t), \dots, \phi_N(x, y, t))) \quad (7)$$

is a rational solution to the (KPI) equation (1) depending on real $N(D+5)$ parameters $b_{j,d}, g_j, h_j, d_j, e_j, 1 \leq j \leq N, 0 \leq d \leq D$.

3 Rational solutions of order 1 with a degree of derivation equal to 1 ($D = 1$) depending on 6 real parameters

The case $D = 0$ is not interesting because we get the solution $u = 0$.

In the case $D = 1$, solutions to the KPI equation can be written as

$$u(x, y, t) = -2 \frac{b_{1,1}^2}{(ib_{1,1}x + b_{1,0} + 3itb_{1,1}g_1^2 - 3itb_{1,1}h_1^2 - 2iyb_{1,1}g_1 + 2yb_{1,1}h_1 - 6tb_{1,1}g_1h_1)^2}.$$

In this case, we observe lumps whose intensities depend on 6 real parameters. If $b_{1,1} = 0$, we get the solution $u = 0$.

We give some figures in the (x, y) plane of coordinates.

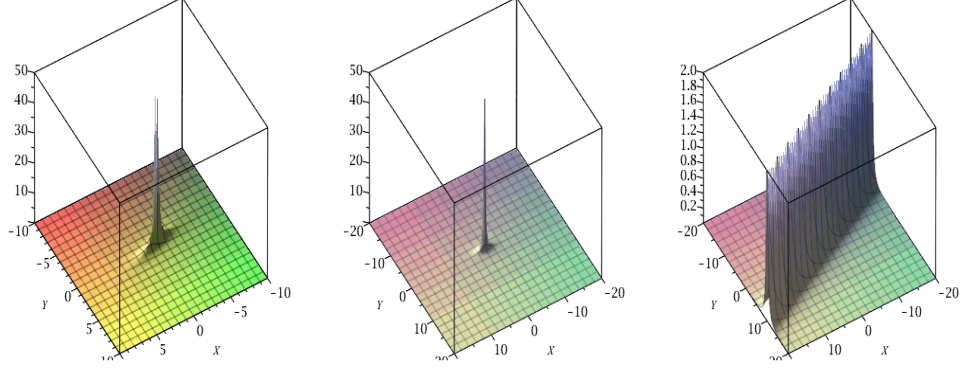


Figure 1. Solutions of order 1 to KPI, on the left for $t = 0$, $b_{1,0} = 1$, $b_{1,1} = 1$, $g_1 = 1$, $h_1 = 1$, $d_1 = 1$, $e_1 = 1$; in the center for $t = 0$, $b_{1,0} = 1$, $b_{1,1} = 1$, $g_1 = 0$, $h_1 = 1$, $d_1 = 1$, $e_1 = 1$; on the right for $t = 0$, $b_{1,0} = 1$, $b_{1,1} = 1$, $g_1 = 1$, $h_1 = 0$, $d_1 = 1$, $e_1 = 1$.

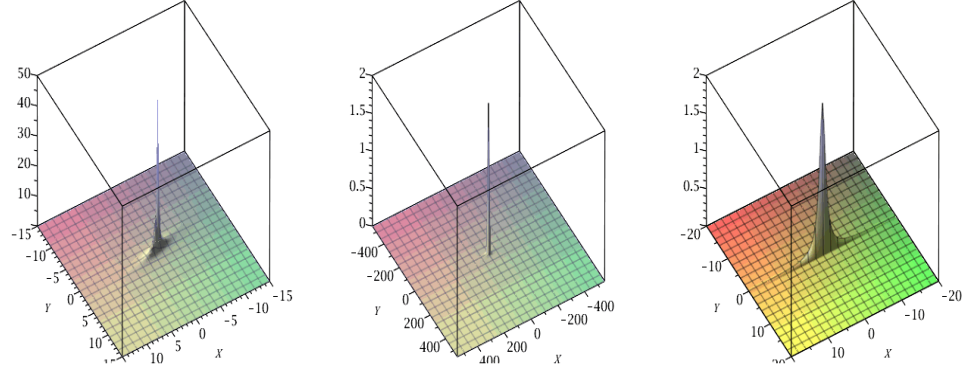


Figure 2. Solutions of order 1 to KPI, on the left for $t = 0$, $b_{1,0} = 1$, $b_{1,1} = 1$, $g_1 = 1$, $h_1 = 1$, $d_1 = 0$, $e_1 = 1$; in the center for $t = 0$, $b_{1,0} = 1$, $b_{1,1} = 1$, $g_1 = 1$, $h_1 = 1$, $d_1 = 1$, $e_1 = 1$; on the right for $t = 10$, $b_{1,0} = 1$, $b_{1,1} = 1$, $g_1 = 0.01$, $h_1 = 10^2$, $d_1 = 1$, $e_1 = 1$.

In figures 1 and 2, we obtain lumps whose modulus depends on the values of parameters and we remark that these structures are more sensitive to g_1 and h_1 parameters than to others.

4 Rational solutions of order 1 with a degree of derivation equal to 2 ($D = 2$) depending on 7 real parameters

In this case, we take $D = 2$; the solutions to the KPI equation can be written as

$$u(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

with

$$\begin{aligned} n(x, y, t) = & -2b_{1,2}^2x^2 + (12tb_{1,2}^2h_1^2 + 8yb_{1,2}^2g_1 - 12tb_{1,2}^2g_1^2 + 8iyb_{1,2}^2h_1 - \\ & 24itb_{1,2}^2g_1h_1 + 2ib_{1,1}b_{1,2})x + 72it^2b_{1,2}^2g_1h_1^3 - 12itb_{1,2}^2g_1 - 6itb_{1,1}b_{1,2}h_1^2 - \\ & 2b_{1,0}b_{1,2} + 108t^2b_{1,2}^2g_1^2h_1^2 + 24tyb_{1,2}^2g_1^3 + 6itb_{1,1}b_{1,2}g_1^2 + 4yb_{1,1}b_{1,2}h_1 - \\ & 16iy^2b_{1,2}^2g_1h_1 - 12tb_{1,1}b_{1,2}g_1h_1 - 18t^2b_{1,2}^2g_1^4 + 12tb_{1,2}^2h_1 + b_{1,1}^2 + 8y^2b_{1,2}^2h_1^2 - \\ & 24ityb_{1,2}^2h_1^3 + 4iyb_{1,2}^2 - 72it^2b_{1,2}^2g_1^3h_1 - 72tyb_{1,2}^2g_1h_1^2 - 4iyb_{1,1}b_{1,2}g_1 - \\ & 18t^2b_{1,2}^2h_1^4 + 72ityb_{1,2}^2g_1^2h_1 - 8y^2b_{1,2}^2g_1^2 \end{aligned}$$

and

$$\begin{aligned} d(x, y, t) = & b_{1,2}x^2 + (6tb_{1,2}g_1^2 + 12itb_{1,2}g_1h_1 - 6tb_{1,2}h_1^2 - 4yb_{1,2}g_1 - 4ib_{1,2}yh_1 - \\ & ib_{1,1})x + 9t^2b_{1,2}g_1^4 + 12ityb_{1,2}h_1^3 - 54t^2b_{1,2}g_1^2h_1^2 - 36ib_{1,2}tg_1^2yh_1 + 9t^2b_{1,2}h_1^4 - \\ & 12tyb_{1,2}g_1^3 - 6ib_{1,2}tg_1 + 36tyb_{1,2}g_1h_1^2 + 2iyb_{1,1}g_1 + 4y^2b_{1,2}g_1^2 + 36it^2b_{1,2}g_1^3h_1 - \\ & 4y^2b_{1,2}h_1^2 + 2iyb_{1,2} + 6tb_{1,1}g_1h_1 - 36ib_{1,2}t^2g_1h_1^3 + 8iy^2b_{1,2}g_1h_1 + 6tb_{1,2}h_1 + \\ & 3itb_{1,1}h_1^2 - 2yb_{1,1}h_1 - 3ib_{1,1}tg_1^2 - b_{1,0} \end{aligned}$$

We give some figures in the (x, y) plane of coordinates.

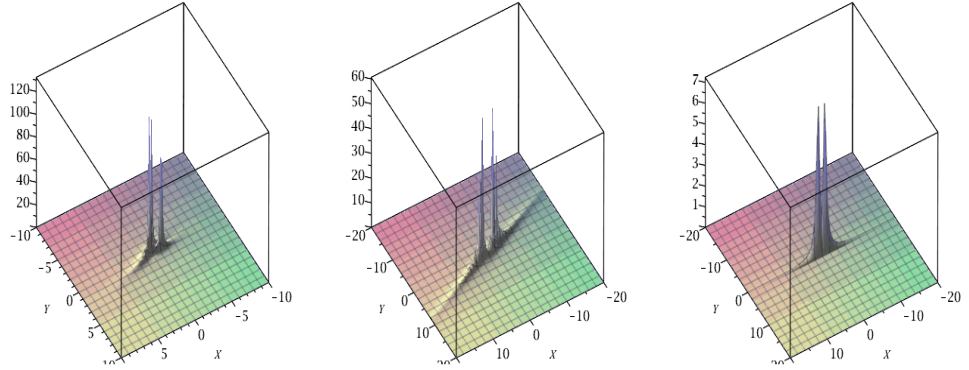


Figure 3. Solutions of order 1 to KPI, on the left for $t = 0$, $b_{1,0} = 1$, $b_{1,1} = 1$, $b_{1,2} = 1$, $g_1 = 1$, $h_1 = 1$, $d_1 = 0$, $e_1 = 1$; in the center for $t = 0$, $b_{1,0} = 1$, $b_{1,1} = 1$, $b_{1,2} = 1$, $g_1 = 1$, $h_1 = 0$, $d_1 = 1$, $e_1 = 1$; on the right for $t = 10$, $b_{1,0} = 1$, $b_{1,1} = 1$, $b_{1,2} = 1$, $g_1 = 0.01$, $h_1 = 10^2$, $d_1 = 1$, $e_1 = 1$.

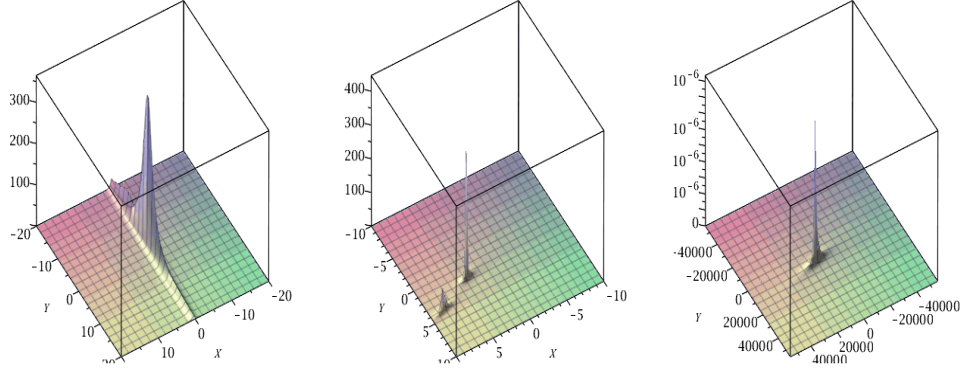


Figure 4. Solutions of order 1 to KPI, on the left for $t = 0$, $b_{1,0} = 1$, $b_{1,1} = 1$, $b_{1,2} = 1$, $g_1 = 0.01$, $h_1 = 0.01$, $d_1 = 0$, $e_1 = 1$; in the center for $t = 1$, $b_{1,0} = 1$, $b_{1,1} = 1$, $b_{1,2} = 1$, $g_1 = 1$, $h_1 = 1$, $d_1 = 1$, $e_1 = 1$; on the right for $t = 10^3$, $b_{1,0} = 1$, $b_{1,1} = 1$, $b_{1,2} = 1$, $g_1 = 0.01$, $h_1 = 10^2$, $d_1 = 1$, $e_1 = 1$.

In figures 3 and 4, we get lumps whose modulus depends on the values of parameters. These structures are more sensitive to g_1 and h_1 parameters than to others. In this case, we observe a maximum of two isolated lumps, as shown for example in the figure 4 in the center.

5 Solutions of order 1 with a degree of derivation equal to 3 ($D = 3$) depending on 8 real parameters

In the case $D = 3$, we observe multi lumps. As in the previous cases, these structures are more sensitive to g_1 and h_1 parameters than to others. The solutions to the KPI equation can be written as

$$u(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}.$$

with

$$\begin{aligned} n(x, y, t) = & 3b_{1,3}^2x^4 + (-36tb_{1,3}^2h_1^2 + 36tb_{1,3}^2g_1^2 - 24yb_{1,3}^2g_1 + 72itb_{1,3}^2g_1h_1 - \\ & 24iyb_{1,3}^2h_1 - 4ib_{1,2}b_{1,3})x^3 + (216ityb_{1,3}^2h_1^3 + 162t^2b_{1,3}^2g_1^4 - 36itb_{1,2}b_{1,3}g_1^2 - \\ & 648it^2b_{1,3}^2g_1h_1^3 - 972t^2b_{1,3}^2g_1^2h_1^2 + 162t^2b_{1,3}^2h_1^4 - 2b_{1,2}^2 + 648it^2b_{1,3}^2g_1^3h_1 + \\ & 648tyb_{1,3}^2g_1h_1^2 + 36itb_{1,2}b_{1,3}h_1^2 - 24yb_{1,2}b_{1,3}h_1 - 648ityb_{1,3}^2g_1^2h_1 + 144iy^2b_{1,3}^2g_1h_1 - \\ & 216tyb_{1,3}^2g_1^3 + 72tb_{1,2}b_{1,3}g_1h_1 - 72y^2b_{1,3}^2h_1^2 + 72y^2b_{1,3}^2g_1^2 + 24iyb_{1,2}b_{1,3}g_1)x^2 + \\ & (1728ity^2b_{1,3}^2g_1^3h_1 - 1728ity^2b_{1,3}^2g_1h_1^3 + 648it^2b_{1,2}b_{1,3}g_1^2h_1^2 + 144ityb_{1,2}b_{1,3}g_1^3 - \\ & 3240it^2yb_{1,3}^2g_1^4h_1 + 6480it^2yb_{1,3}^2g_1^2h_1^3 - 432tyb_{1,2}b_{1,3}g_1^2h_1 - 4860t^3b_{1,3}^2g_1^4h_1^2 + \\ & 4860t^3b_{1,3}^2g_1^2h_1^4 + 432ty^2b_{1,3}^2h_1^4 + 96iy^3b_{1,3}^2h_1^3 + 8iyb_{1,2}^2h_1 + 432ty^2b_{1,3}^2g_1^4 - \\ & 648t^2yb_{1,3}^2g_1^5 + 288y^3b_{1,3}^2g_1h_1^2 - 432ityb_{1,2}b_{1,3}g_1h_1^2 + 36tb_{1,3}^2 - 48iy^2b_{1,2}b_{1,3}g_1^2 + \\ & 48iy^2b_{1,2}b_{1,3}h_1^2 - 24itb_{1,2}^2g_1h_1 - 108it^2b_{1,2}b_{1,3}g_1^4 - 108it^2b_{1,2}b_{1,3}h_1^4 - 288iy^3b_{1,3}^2g_1^2h_1 - \\ & 648it^2yb_{1,3}^2h_1^5 + 1944it^3b_{1,3}^2g_1^5h_1 - 6480it^3b_{1,3}^2g_1^3h_1^3 + 1944it^3b_{1,3}^2g_1h_1^5 + \end{aligned}$$

$$\begin{aligned}
& 96 y^2 b_{1,2} b_{1,3} g_1 h_1 + 432 t^2 b_{1,2} b_{1,3} g_1^3 h_1 - 432 t^2 b_{1,2} b_{1,3} g_1 h_1^3 + 144 t y b_{1,2} b_{1,3} h_1^3 - \\
& 2592 t y^2 b_{1,3}^2 g_1^2 h_1^2 + 6480 t^2 y b_{1,3}^2 g_1^3 h_1^2 - 3240 t^2 y b_{1,3}^2 g_1 h_1^4 - 12 t b_{1,2}^2 g_1^2 + \\
& 12 t b_{1,2}^2 h_1^2 + 8 y b_{1,2}^2 g_1 - 6 i b_{1,0} b_{1,3} + 2 i b_{1,1} b_{1,2} + 324 t^3 b_{1,3}^2 g_1^6 - 324 t^3 b_{1,3}^2 h_1^6 - \\
& 96 y^3 b_{1,3}^2 g_1^3) x + 13608 t^3 y b_{1,3}^2 g_1^5 h_1^2 - 22680 t^3 y b_{1,3}^2 g_1^3 h_1^4 - 2160 t^3 b_{1,2} b_{1,3} g_1^3 h_1^3 + \\
& 648 t^3 b_{1,2} b_{1,3} g_1 h_1^5 + 2880 t y^3 b_{1,3}^2 g_1^3 h_1^2 - 1440 t y^3 b_{1,3}^2 g_1 h_1^4 - 216 t^2 y b_{1,2} b_{1,3} h_1^5 - \\
& 96 i y^3 b_{1,2} b_{1,3} g_1 h_1^2 + 72 i t y b_{1,2}^2 g_1^2 h_1 - 144 i t y^2 b_{1,2} b_{1,3} g_1^4 - 144 i t y^2 b_{1,2} b_{1,3} h_1^4 + \\
& 4536 t^3 y b_{1,3}^2 g_1 h_1^6 + 1620 i t^3 b_{1,2} b_{1,3} g_1^4 h_1^2 - 1620 i t^3 b_{1,2} b_{1,3} g_1^2 h_1^4 - 1440 i t y^3 b_{1,3}^2 g_1^4 h_1 + \\
& 2880 i t y^3 b_{1,3}^2 g_1^2 h_1^3 + 3888 i t^2 y^2 b_{1,3}^2 g_1^5 h_1 - 12960 i t^2 y^2 b_{1,3}^2 g_1^3 h_1^3 + 3888 i t^2 y^2 b_{1,3}^2 g_1 h_1^5 - \\
& 4536 i t^3 y b_{1,3}^2 g_1^6 h_1 + 22680 i t^3 y b_{1,3}^2 g_1^4 h_1^3 - 13608 i t^3 y b_{1,3}^2 g_1^2 h_1^5 + 1080 i t^2 y b_{1,2} b_{1,3} g_1 h_1^4 + \\
& 864 i t y^2 b_{1,2} b_{1,3} g_1^2 h_1^2 + 216 i t^2 y b_{1,2} b_{1,3} g_1^5 + 1944 i t^4 b_{1,3}^2 g_1^7 h_1 - 13608 i t^4 b_{1,3}^2 g_1^5 h_1^3 + \\
& 13608 i t^4 b_{1,3}^2 g_1^3 h_1^5 - 1944 i t^4 b_{1,3}^2 g_1 h_1^7 + 648 i t^3 y b_{1,3}^2 h_1^7 - 108 i t^3 b_{1,2} b_{1,3} g_1^6 + \\
& 108 i t^3 b_{1,2} b_{1,3} h_1^6 - 288 i t y^3 b_{1,3}^2 h_1^5 + 192 i y^4 b_{1,3}^2 g_1^3 h_1 - 192 i y^4 b_{1,3}^2 g_1 h_1^3 - 72 i t^2 b_{1,2}^2 g_1^3 h_1 + \\
& 72 i t^2 b_{1,2}^2 g_1 h_1^3 + 32 i y^3 b_{1,2} b_{1,3} g_1^3 + 36 i t b_{1,1} b_{1,3} g_1 - 9720 t^2 y^2 b_{1,3}^2 g_1^4 h_1^2 + 9720 t^2 y^2 b_{1,3}^2 g_1^2 h_1^4 + \\
& 648 t^3 b_{1,2} b_{1,3} g_1^5 h_1 - 432 i t^2 b_{1,3}^2 g_1 h_1 - 16 i y^2 b_{1,2}^2 g_1 h_1 + 144 i t y b_{1,3}^2 h_1 - 18 i t b_{1,0} b_{1,3} g_1^2 + \\
& 18 i t b_{1,0} b_{1,3} h_1^2 + 6 i t b_{1,1} b_{1,2} g_1^2 - 6 i t b_{1,1} b_{1,2} h_1^2 - 648 t^2 y^2 b_{1,3}^2 h_1^6 - 288 y^4 b_{1,3}^2 g_1^2 h_1^2 + \\
& 108 t^2 b_{1,2}^2 g_1^2 h_1^2 + 32 y^3 b_{1,2} b_{1,3} h_1^3 + 24 t y b_{1,2}^2 g_1^3 - 12 y b_{1,0} b_{1,3} h_1 + 4 y b_{1,1} b_{1,2} h_1 - \\
& 6804 t^4 b_{1,3}^2 g_1^6 h_1^2 + 17010 t^4 b_{1,3}^2 g_1^4 h_1^4 - 6804 t^4 b_{1,3}^2 g_1^2 h_1^6 + b_{1,1}^2 + 36 t b_{1,0} b_{1,3} g_1 h_1 - \\
& 12 t b_{1,1} b_{1,2} g_1 h_1 - 96 y^3 b_{1,2} b_{1,3} g_1^2 h_1 - 72 t y b_{1,2}^2 g_1 h_1^2 - 2160 i t^2 y b_{1,2} b_{1,3} g_1^3 h_1^2 - \\
& 2 b_{1,0} b_{1,2} - 36 y^2 b_{1,3}^2 - 12 i t b_{1,2}^2 g_1 - 12 i t b_{1,2} b_{1,3} - 12 i y b_{1,1} b_{1,3} - 36 t b_{1,1} b_{1,3} h_1 + \\
& 144 t y b_{1,3}^2 g_1 - 648 t^3 y b_{1,3}^2 g_1^7 + 648 t^2 y^2 b_{1,3}^2 g_1^6 - 288 t y^3 b_{1,3}^2 g_1^5 + 576 t y^2 b_{1,2} b_{1,3} g_1^3 h_1 - \\
& 576 t y b_{1,2} b_{1,3} g_1 h_1^3 - 1080 t^2 y b_{1,2} b_{1,3} g_1^4 h_1 + 2160 t^2 y b_{1,2} b_{1,3} g_1^2 h_1^3 + 12 i y b_{1,0} b_{1,3} g_1 - \\
& 4 i y b_{1,1} b_{1,2} g_1 - 24 i t y b_{1,2}^2 h_1^3 + 4 i y b_{1,2}^2 + 216 t^2 b_{1,3}^2 h_1^2 + 243 t^4 b_{1,3}^2 g_1^8 + 243 t^4 b_{1,3}^2 h_1^8 + \\
& 48 y^4 b_{1,3}^2 h_1^4 - 18 t^2 b_{1,2}^2 g_1^4 - 18 t^2 b_{1,2}^2 h_1^4 - 8 y^2 b_{1,2}^2 g_1^2 + 8 y^2 b_{1,2}^2 h_1^2 + 12 t b_{1,2}^2 h_1 - \\
& 216 t^2 b_{1,3}^2 g_1^2 + 48 y^4 b_{1,3}^2 g_1^4
\end{aligned}$$

and

$$\begin{aligned}
d(x, y, t) &= i b_{1,3} x^3 + (9 i t b_{1,3} g_1^2 - 18 t b_{1,3} g_1 h_1 - 9 i t b_{1,3} h_1^2 - 6 i y b_{1,3} g_1 + 6 y b_{1,3} h_1 + \\
& b_{1,2}) x^2 + (27 i t^2 b_{1,3} g_1^4 - 108 t^2 b_{1,3} g_1^3 h_1 - 162 i t^2 b_{1,3} g_1^2 h_1^2 + 108 t^2 b_{1,3} g_1 h_1^3 + \\
& 27 i t^2 b_{1,3} h_1^4 - 36 i t y b_{1,3} g_1^3 \\
& + 108 t y b_{1,3} g_1^2 h_1 + 108 i t y b_{1,3} g_1 h_1^2 - 36 t y b_{1,3} h_1^3 + 12 i y^2 b_{1,3} g_1^2 - 24 y^2 b_{1,3} g_1 h_1 - \\
& 12 i y^2 b_{1,3} h_1^2 + 6 t b_{1,2} g_1^2 + 12 i t b_{1,2} g_1 h_1 - 6 t b_{1,2} h_1^2 + 18 t b_{1,3} g_1 + 18 i t b_{1,3} h_1 - \\
& 4 y b_{1,2} g_1 - 4 i y b_{1,2} h_1 - 6 y b_{1,3} - i b_{1,1}) x - 54 t^2 b_{1,2} g_1^2 h_1^2 - b_{1,0} - 162 t^3 b_{1,3} g_1 h_1^5 + \\
& 54 t^2 y b_{1,3} h_1^5 - 4 y^2 b_{1,2} h_1^2 - 162 t^3 b_{1,3} g_1^5 h_1 + 9 t^2 b_{1,2} h_1^4 + 4 y^2 b_{1,2} g_1^2 - 8 y^3 b_{1,3} h_1^3 - \\
& 2 y b_{1,1} h_1 + 9 t^2 b_{1,2} g_1^4 + 54 t^2 b_{1,3} g_1^3 + 12 y^2 b_{1,3} g_1 - 54 i t^2 b_{1,3} h_1^3 + 12 i y^2 b_{1,3} h_1 + \\
& 27 i t^3 b_{1,3} g_1^6 - 27 i t^3 b_{1,3} h_1^6 - 8 i y^3 b_{1,3} g_1^3 + 3 i t b_{1,1} h_1^2 - 6 i t b_{1,2} g_1 - 3 i t b_{1,1} g_1^2 + \\
& 2 i y b_{1,1} g_1 + 6 t b_{1,2} h_1 + 2 i y b_{1,2} - 6 i t b_{1,3} + 36 i t^2 b_{1,2} g_1^3 h_1 - 36 i t^2 b_{1,2} g_1 h_1^3 + \\
& 12 i t y b_{1,2} h_1^3 + 8 i y^2 b_{1,2} g_1 h_1 - 405 i t^3 b_{1,3} g_1^4 h_1^2 - 54 i t^2 y b_{1,3} g_1^5 + 405 i t^3 b_{1,3} g_1^2 h_1^4 + \\
& 36 i t y^2 b_{1,3} g_1^4 + 36 i t y^2 b_{1,3} h_1^4 + 24 i y^3 b_{1,3} g_1 h_1^2 + 162 i t^2 b_{1,3} g_1^2 h_1 + 24 y^3 b_{1,3} g_1^2 h_1 - \\
& 162 t^2 b_{1,3} g_1 h_1^2 + 54 t y b_{1,3} h_1^2 + 6 t b_{1,1} g_1 h_1 - 12 t y b_{1,2} g_1^3 + 540 t^3 b_{1,3} g_1^3 h_1^3 - \\
& 54 t y b_{1,3} g_1^2 - 108 i t y b_{1,3} g_1 h_1 - 270 i t^2 y b_{1,3} g_1 h_1^4 - 216 i t y^2 b_{1,3} g_1^2 h_1^2 + 540 i t^2 y b_{1,3} g_1^3 h_1^2 - \\
& 36 i t y b_{1,2} g_1^2 h_1 + 36 t y b_{1,2} g_1 h_1^2 - 144 t y^2 b_{1,3} g_1^3 h_1 + 270 t^2 y b_{1,3} g_1^4 h_1 + 144 t y^2 b_{1,3} g_1 h_1^3 - \\
& 540 t^2 y b_{1,3} g_1^2 h_1^3.
\end{aligned}$$

We give some figures in the (x, y) plane of coordinates.

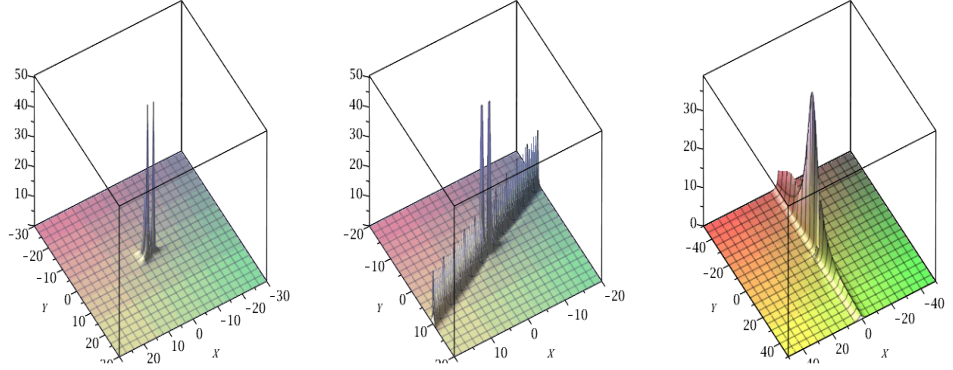


Figure 5. Solutions of order 1 to KPI, on the left for $t = 0$, $b_{1,0} = 1$, $b_{1,1} = 1$, $b_{1,2} = 1$, $b_{1,3} = 1$, $g_1 = 0$, $h_1 = 1$, $d_1 = 1$, $e_1 = 1$; in the center for $t = 0$, $b_{1,0} = 1$, $b_{1,1} = 1$, $b_{1,2} = 1$, $b_{1,3} = 1$, $g_1 = 1$, $h_1 = 0$, $d_1 = 1$, $e_1 = 1$; on the right for $t = 0$, $b_{1,0} = 1$, $b_{1,1} = 1$, $b_{1,2} = 1$, $b_{1,3} = 1$, $g_1 = 0.01$, $h_1 = 0.01$, $d_1 = 1$, $e_1 = 1$.

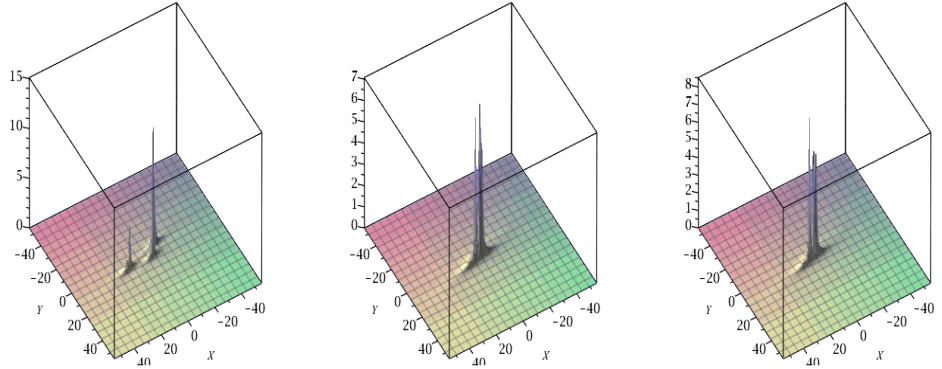


Figure 6. Solutions of order 1 to KPI, on the left for $t = 0$, $b_{1,0} = 10^3$, $b_{1,1} = 1$, $b_{1,2} = 1$, $b_{1,3} = 1$, $g_1 = 1$, $h_1 = 1$, $d_1 = 1$, $e_1 = 1$; in the center for $t = 0$, $b_{1,0} = 1$, $b_{1,1} = 1$, $b_{1,2} = 1$, $b_{1,3} = 10^3$, $g_1 = 1$, $h_1 = 0$, $d_1 = 1$, $e_1 = 1$; on the right for $t = 0$, $b_{1,0} = 1$, $b_{1,1} = 10^3$, $b_{1,2} = 1$, $b_{1,3} = 10^3$, $g_1 = 0.01$, $h_1 = 0.01$, $d_1 = 1$, $e_1 = 1$.

In figures 5 and 6, we get lumps depending on 8 real parameters. These structures are more sensitive to g_1 and h_1 parameters than to others. In this case, we observe a maximum of three distinct isolated lumps, as shown for example in the figure 6 to the left.

6 Solutions of order 1 with a degree of derivation equal to 4 ($D = 4$) depending on 9 real parameters

In the case $D = 4$, the expression of the solutions with all parameters being too long, we present only one of them with particular values of parameters. For example we choose :

$$b_{1,0} = 1, b_{1,1} = 1, b_{1,2} = 1, b_{1,3} = 1, g_1 = 1, h_1 = 1, d_1 = 1, e_1 = 1.$$

So, the solution to the KPI equation, with these choices of parameters, can be written as

$$u(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

with

$$\begin{aligned} n(x, y, t) = & -4x^6 + (48y - 144it + 6i + 48iy)x^5 + (72it + 5 - 84iy - 480iy^2 + \\ & 2160t^2 + 1440iyt - 252t + 60y - 1440yt)x^4 + (-17280iyt^2 + 3168yt - 3888it^2 + \\ & 192it - 40iy + 1440iyt + 17280it^3 - 24t - 64y - 1280y^3 + 192iy^2 - 672y^2 - \\ & 17280t^2y - 1728t^2 + 11520ty^2 + 1280iy^3)x^3 + (-77760t^4 - 23040ty^3 + 103680t^3y + \\ & 24iy + 384iy^2 - 6912ty^2 + 288y^2 + 103680it^2y^2 + 28512t^3 + 2112y^3 + 960iy^3 + \\ & 432yt - 15552iy^2t + 38880iyt^2 + 2160it^2 + 10 + 3840y^4 - 15552it^3 - 2304iyt - \\ & 103680it^3y + 72t - 23040ity^3 - 12960t^2y - 2376t^2)x^2 + (-16y - 84t - 256y^3 + \\ & 96y^2 + 3024t^2 - 4i - 28512t^3 + 138240t^2y^3 - 1536y^4 + 62208t^4 - 3072y^5 - \\ & 40iy - 1472iy^3 + 101088it^4 - 3456iy^4 - 864it^2 - 186624it^5 - 3072iy^5 - 168iy^2 - \\ & 12096it^3 + 34560ity^3 + 311040it^4y - 138240it^2y^3 + 9792iy^2t + 720iyt - 9504iyt^2 + \\ & 46080ity^4 - 62208it^2y^2 - 51840it^3y - 196992t^3y + 114048t^2y^2 - 11520ty^3 + \\ & 48it - 1008yt + 31104t^2y - 6336ty^2 + 311040t^4y - 414720t^3y^2)x - 936t^2 - \\ & 7776t^3 - 131328t^2y^3 - 704y^4 + 22032t^4 - 768y^5 - 12y^2 - 1 - 8y + 72t - 336y^3 - \\ & 622080it^4y^2 - 7776iyt^2 + 269568it^3y^2 - 13824ity^4 + 373248it^5y + 276480it^3y^3 - \\ & 18432ity^5 - 24192it^2y^2 + 117504it^3y - 6912ity^3 - 357696it^4y - 34560it^2y^3 + \\ & 2592iy^2t - 139968t^5 + 186624t^6 + 2048iy^6 - 88128it^4 + 1536iy^4 - 288it^2 + \\ & 93312it^5 + 2304iy^5 + 32iy^2 + 2592it^3 + 24iy - 12it + 63936t^3y - 69120t^2y^2 + \\ & 16512ty^3 - 373248t^5y + 276480t^3y^3 - 138240t^2y^4 + 18432ty^5 - 144iy^3 + 192yt + \\ & 1728t^2y + 1296ty^2 + 77760t^4y + 165888t^3y^2 + 25344ty^4 \end{aligned}$$

and

$$\begin{aligned} d(x, y, t) = & -x^4 + (-24it + 8y + 8iy + i)x^3 + (216t^2 + 144ity - 144ty - 48iy^2 - \\ & 54t + 36it - 18iy + 6y + 1)x^2 + (864it^3 - 864t^2y - 864it^2y + 576ty^2 + 64iy^3 - \\ & 64y^3 - 432t^2 - 540it^2 + 504ty + 72ity - 72y^2 + 48iy^2 + 30it + 42t - 10y - \\ & 4iy - i)x - 1 - 2y + 12t + 112y^3 + 24y^2 - 144t^2 + 4iy + 16iy^3 + 468it^2 + \\ & 20iy^2 - 1296it^3 - 384iy^3t - 720iy^2t - 252iyt + 2376it^2y + 1728it^2y^2 - 1728iyt^3 + \\ & 1512t^3 + 64y^4 - 1296t^4 - 12it + 1728t^3y - 384ty^3 - 96ty - 216t^2y - 576ty^2. \end{aligned}$$

We get multi lumps. We present some figures in the (x, y) plane :

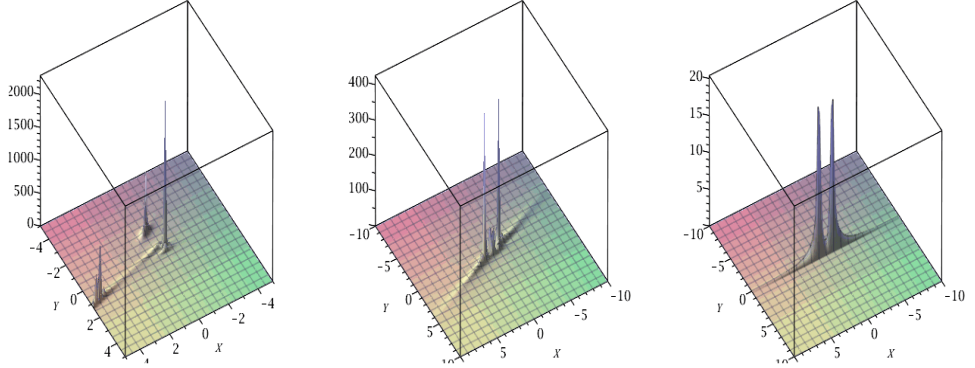


Figure 7. Solutions of order 1 to KPI, on the left for $t = 0$, $b_{1,0} = 1$, $b_{1,1} = 1$, $b_{1,2} = 1$, $b_{1,3} = 1$, $b_{1,4} = 1$, $g_1 = 1$, $h_1 = 1$, $d_1 = 1$, $e_1 = 1$; in the center for $t = 0$, $b_{1,0} = 1$, $b_{1,1} = 1$, $b_{1,2} = 1$, $b_{1,3} = 1$, $b_{1,4} = 1$, $g_1 = 1$, $h_1 = 0$, $d_1 = 1$, $e_1 = 1$; on the right for $t = 0$, $b_{1,0} = 1$, $b_{1,1} = 1$, $b_{1,2} = 1$, $b_{1,3} = 1$, $b_{1,4} = 1$, $g_1 = 10^3$, $h_1 = 0,01$, $d_1 = 1$, $e_1 = 1$.

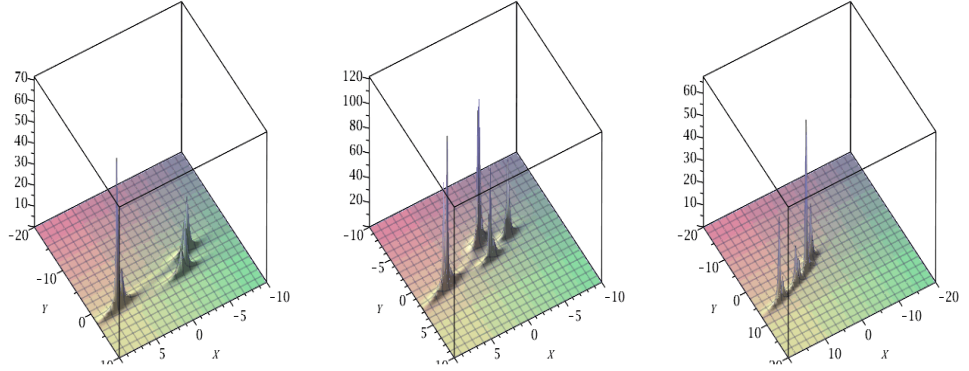


Figure 8. Solutions of order 1 to KPI, on the left for $t = 0$, $b_{1,0} = 10^3$, $b_{1,1} = 1$, $b_{1,2} = 1$, $b_{1,3} = 1$, $b_{1,4} = 1$, $g_1 = 1$, $h_1 = 1$, $d_1 = 1$, $e_1 = 1$; in the center for $t = 0$, $b_{1,0} = 10^2$, $b_{1,1} = 1$, $b_{1,2} = 1$, $b_{1,3} = 1$, $b_{1,4} = 1$, $g_1 = 1$, $h_1 = 0$, $d_1 = 0,01$, $e_1 = 0,01$; on the right for $t = 1$, $b_{1,0} = 1$, $b_{1,1} = 1$, $b_{1,2} = 1$, $b_{1,3} = 1$, $b_{1,4} = 1$, $g_1 = 10^3$, $h_1 = 0,01$, $d_1 = 1$, $e_1 = 1$.

In figures 7 and 8, we get lumps depending on 9 real parameters. These structures are more sensitive to g_1 and h_1 parameters than to others. In this case, we observe a maximum of four distinct isolated lumps, as shown for example in the figure 8 in the center. In Figure 8, on the right, there are four isolated lumps aligned on a non rectilinear curve.

7 Solutions of order 1 with a degree of derivation equal to 5 ($D = 5$) depending on 10 real parameters

In the case $D = 5$, the expression of the solutions with all parameters being too long, we present only one of them with particular values of parameters. For example we choose :

$$b_{1,0} = 1, b_{1,1} = 1, b_{1,2} = 1, b_{1,3} = 1, b_{1,4} = 1, b_{1,5} = 1, g_1 = 1, h_1 = 1, d_1 = 1, e_1 = 1.$$

The solution to the KPI equation can be written as

$$u(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

with

$$\begin{aligned} n(x, y, t) = & 5x^8 + (240it - 80y - 80iy - 8i)x^7 + (3360ty - 5040t^2 - 3360ity - \\ & 112y + 576t + 1120iy^2 - 8 + 192iy - 240it)x^6 + (-4032ity + 8640t^2 + 96iy - \\ & 60480it^3 + 4i - 168t + 60480t^2y + 60480it^2y - 960iy^2 - 4480iy^3 - 40320ty^2 - \\ & 12672ty + 192y + 2304y^2 + 4480y^3 + 14688it^2 - 576it)x^5 + (-5 + 40y - \\ & 372t - 14080y^3 - 1560y^2 + 12960t^2 + 97920ity^2 - 604800it^2y^2 - 276480it^2y + \\ & 134400ity^3 + 604800it^3y + 13920ity + 60480t^2y + 57600ty^2 - 4480iy^3 + 132it + \\ & 129600it^3 - 15840it^2 - 124iy - 1920iy^2 + 134400ty^3 - 604800t^3y - 480ty - \\ & 190080t^3 - 22400y^4 + 453600t^4)x^4 + (-24y + 216t + 2240y^3 - 992y^2 - 19008t^2 - \\ & 3360ity - 108480ity^2 + 1036800it^2y^2 - 537600ity^4 + 1612800it^2y^3 + 109440it^2y - \\ & 3628800it^4y - 491520ity^3 + 483840it^3y + 20i + 4838400t^3y^2 - 3628800t^4y - \\ & 1612800t^2y^3 - 351360t^2y + 69120ty^2 - 1399680it^4 + 2177280it^5 + 72it + 138240it^3 + \\ & 15040iy^3 + 35840iy^5 - 288it^2 + 43520iy^4 + 40iy + 1152iy^2 + 107520ty^3 - \\ & 1520640t^2y^2 + 2903040t^3y + 8928ty + 319680t^3 + 25600y^4 - 1036800t^4 + 35840y^5)x^3 + \\ & (10 + 120y - 288t + 7552y^3 + 528y^2 + 9144t^2 - 15552ity^2 + 829440it^2y^2 + \\ & 691200ity^4 + 967680it^2y^3 - 11197440it^3y^2 + 185760it^2y + 16174080it^4y - 9676800it^3y^3 - \\ & 13063680it^5y + 645120ity^5 + 193920ity^3 - 3732480it^3y - 3744ity + 21772800it^4y^2 - \\ & 8294400t^3y^2 - 1013760ty^4 - 2177280t^4y + 5806080t^2y^3 + 207360t^2y - 76032ty^2 - \\ & 6531840t^6 - 240it + 2695680it^4 - 71680iy^6 - 4665600it^5 - 313632it^3 - 2240iy^3 - \\ & 98304iy^5 + 10800it^2 - 44160iy^4 - 96iy + 144iy^2 - 516480ty^3 + 2134080t^2y^2 - \\ & 1866240t^3y - 9676800t^3y^3 + 4838400t^2y^4 - 645120ty^5 + 13063680t^5y - 3168ty - \\ & 66528t^3 + 5971968t^5 + 23040y^4 - 777600t^4 + 21504y^5)x^2 + (-136y + 156t - \\ & 896y^3 + 24192ity^2 - 1168128it^2y^2 + 368640ity^4 - 5264640it^2y^3 + 15240960it^3y^2 - \\ & 29030400it^4y^3 + 26127360it^6y - 64224it^2y - 3870720it^2y^5 - 10782720it^4y + \\ & 19353600it^3y^4 + 28753920it^3y^3 - 5225472it^5y + 258048ity^5 + 184320ity^3 + \\ & 2090880it^3y + 1680ity - 31104000it^4y^2 - 7464960it^2y^4 - 384y^2 + 1584t^2 - 4i - \\ & 4147200t^3y^2 + 1044480ty^4 + 29030400t^4y^3 - 52254720t^5y^2 + 26127360t^6y + \\ & 860160ty^6 - 3870720t^2y^5 + 17625600t^4y - 2776320t^2y^3 + 67104t^2y - 7776ty^2 - \\ & 40960y^7 + 11197440t^6 + 13810176it^6 - 533952it^4 + 61440iy^6 - 2239488it^5 + \\ & 408it - 11197440it^7 - 7776it^3 - 1792iy^3 + 32256iy^5 - 3744it^2 - 6656iy^4 + \\ & 40960iy^7 - 40iy - 48iy^2 + 145920ty^3 - 93312t^2y^2 - 1372032t^3y + 39813120t^4y^2 - \end{aligned}$$

$3870720 t^3 y^3 - 5529600 t^2 y^4 + 1548288 t y^5 - 46282752 t^5 y + 192 t y - 106272 t^3 - 10419840 t^5 - 90112 y^6 - 16896 y^4 + 1824768 t^4 - 75264 y^5)x + 224 y^3 - 936 t^2 + 8398080 t^8 + 20480 y^8 + 108 y^2 - 15303168 t^6 + 11197440 t^7 + 67392 t^3 + 656 i y^3 + 13568 i y^5 + 432 i t^2 + 2176 i y^4 + 8192 i y^7 + 4520448 t^3 y^2 + 17664 t y^4 - 295488 t^4 + 16384 i y^6 + 3592512 i t^5 - 132 i t - 491520 t y^6 - 774144 t^2 y^5 - 5857920 t^4 y - 972288 t^2 y^3 - 22752 t^2 y + 4656 t y^2 + 49152 y^7 - 13436928 t^7 + 2612736 t^6 + 17694720 t^3 y^4 - 51425280 t^4 y^3 + 44789760 t^5 y^2 + 5225472 t^6 y - 1 - 8 y + 72 t - 4608 t y^3 - 181440 t^2 y^2 + 798336 t^3 y - 36028800 t^4 y^2 + 16243200 t^3 y^3 - 1382400 t^2 y^4 - 408576 t y^5 + 21026304 t^5 y - 245760 t y^7 + 7741440 t^3 y^5 - 29030400 t^4 y^4 + 34836480 t^5 y^3 - 22394880 t^7 y + 672 t y + 24 i y + 22394880 i t^7 y - 3888 i t y^2 + 266688 i t^2 y^2 - 240384 i t y^4 + 1163520 i t^2 y^3 - 905472 i t^3 y^2 - 5806080 i t^4 y^3 - 53747712 i t^6 y - 17856 i t^2 y + 5750784 i t^2 y^5 - 3312576 i t^4 y - 13824000 i t^3 y^4 + 55240704 i t^5 y^2 - 663552 i t y^6 - 245760 i t y^7 - 7741440 i t^3 y^5 + 34836480 i t^5 y^3 + 2580480 i t^2 y^6 - 10160640 i t^3 y^3 + 30730752 i t^5 y - 740352 i t y^5 - 52992 i t y^3 - 245376 i t^3 y - 1200 i t y - 52254720 i t^6 y^2 + 272 i y^2 - 7464960 i t^4 y^2 + 5368320 i t^2 y^4 - 29376 t^3 + 1197504 t^5 + 50688 y^6 + 2896 y^4 - 768528 t^4 + 7168 y^5$

and

$d(x, y, t) = i x^5 + (-30 t - 10 i y + 10 y + 1)x^4 + (-360 i t^2 + 240 i t y + 240 t y - 80 y^2 + 60 t + 84 i t - 28 y - 8 i y - i)x^3 + (2160 t^3 - 2160 t^2 y + 2160 i t^2 y - 1440 i t y^2 + 160 y^3 + 160 i y^3 - 1296 t^2 + 1080 i t^2 - 1224 i t y + 144 t y + 168 i y^2 + 120 y^2 + 54 t - 96 i t + 18 i y - 6 y - 1)x^2 + (6480 i t^4 - 8640 t^3 y - 8640 i t^3 y + 8640 t^2 y^2 - 1920 t y^3 + 1920 i t y^3 - 320 i y^4 - 7344 i t^3 - 6480 t^3 + 864 i t^2 y + 11664 t^2 y + 2880 i t y^2 - 3456 t y^2 - 544 i y^3 + 64 y^3 + 2232 t^2 + 540 i t^2 + 528 i t y - 1104 t y - 108 i y^2 + 72 y^2 - 30 i t - 42 t + 10 y + 4 i y + i)x + 2 y - 12 t + 720 i t y^2 - 232 y^3 - 24 y^2 - 216 t^2 - 12960 i t^4 - 828 i t^2 + 1 + 1920 t y^4 + 12960 t^4 y - 5760 t^2 y^3 - 5544 t^2 y + 2856 t y^2 + 5184 t y^3 - 12960 t^2 y^2 - 1728 t^3 y + 128 i y^5 + 104 i y^3 + 9936 i t^3 + 320 i y^4 - 4 i y + 216 t y - 14688 i t^2 y^2 + 252 i t y - 5760 i t^2 y^3 - 12960 i t^4 y - 8136 i t^2 y + 384 i t y^3 + 31968 i t^3 y + 17280 i t^3 y^2 - 20 i y^2 + 12 i t - 1512 t^3 - 7776 t^5 - 384 y^4 + 14256 t^4 - 128 y^5.$

We present some figures in the (x, y) plane :

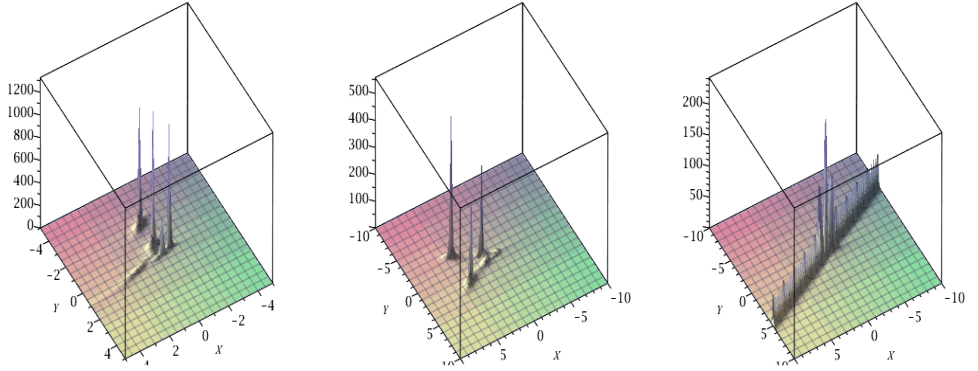


Figure 9. Solutions of order 1 to KPI, on the left for $t = 0$, $b_{1,0} = 1$, $b_{1,1} = 1$, $b_{1,2} = 1$, $b_{1,3} = 1$, $b_{1,4} = 1$, $b_{1,5} = 1$, $g_1 = 1$, $h_1 = 1$, $d_1 = 1$, $e_1 = 1$; in the center for $t = 0$, $b_{1,0} = 10^3$, $b_{1,1} = 1$, $b_{1,2} = 1$, $b_{1,3} = 1$, $b_{1,4} = 1$, $b_{1,5} = 1$, $g_1 = 0$, $h_1 = 1$, $d_1 = 1$, $e_1 = 1$; on the right for $t = 0$, $b_{1,0} = 10^3$, $b_{1,1} = 1$,

$$b_{1,2} = 1, b_{1,3} = 1, b_{1,4} = 1, b_{1,5} = 1, g_1 = 1, h_1 = 0, d_1 = 1, e_1 = 1.$$

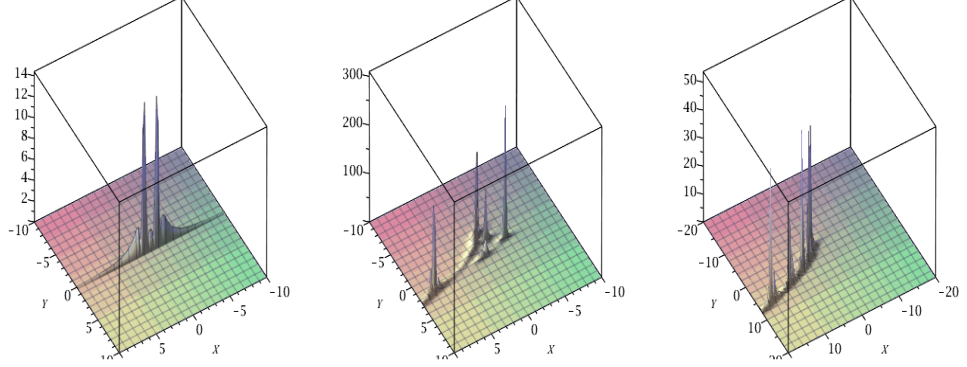


Figure 10. Solutions of order 1 to KPI, on the left for $t = 0$, $b_{1,0} = 10^3$, $b_{1,1} = 1, b_{1,2} = 1, b_{1,3} = 1, b_{1,4} = 1, b_{1,5} = 1, g_1 = 10^3, h_1 = 0,01, d_1 = 1, e_1 = 1$; in the center for $t = 0$, $b_{1,0} = 10^2, b_{1,1} = 1, b_{1,2} = 1, b_{1,3} = 1, b_{1,4} = 1, b_{1,5} = 1, g_1 = 1, h_1 = 1, d_1 = 1, e_1 = 1$; on the right for $t = 1, b_{1,0} = 1, b_{1,1} = 1, b_{1,2} = 1, b_{1,3} = 1, b_{1,4} = 1, b_{1,5} = 1, g_1 = 1, h_1 = 1, d_1 = 1, e_1 = 1$.

In figures 9 and 10, we get lumps depending on 10 real parameters. In this case, we observe a maximum of five distinct isolated lumps, as shown for example in the figure 10 in the center. In Figure 10, on the right, there are five lumps aligned on a non rectilinear curve.

8 Conclusion

By means of a Darboux transformation with particular generating functions, we construct multi-parametric solutions to the KPI equation expressed in terms of a wronskian of order N . These solutions depend on the order of the determinant N , the degree of summation S and the degree of derivation D . In this study, to get rational solutions, we restrict ourselves to the case where $S = 1$. In the general case, these rational solutions can be written as a second derivative with respect to x of a logarithm of a wronskian of order N , depending on $N(D + 5)$ real parameters. For a rational solution of order N , the numerator is a polynomial in x, y and t of degree $2N - 2$ and the denominator a polynomial of degree $2N$ in x, y and t .

We only give the explicit expressions of the solutions in the simple case of order $N = 1$ and degrees of summation between $S = 1$ and $S = 5$.

All these solutions are different from these constructed with another approach given by the present author [50, 51, 52, 54, 56].

It would be relevant to study the cases of order greater or equal to 2 and to realize an exhaustive classification of the solutions to the KPI equation.

We postpone this study to another publication.

References

- [1] B.B. Kadomtsev, V.I. Petviashvili, On the stability of solitary waves in weakly dispersing media, *Sov. Phys. Dokl.*, V. 15, N. 6, 539-541, 1970
- [2] V.I. Petviashvili, Equation of an extraordinary soliton, *Plasma Phys.*, V. 2, 469-472, 1976
- [3] S.V. Manakov, V.E. Zakharov, L.A. Bordag, V.B. Matveev, Two-dimensional solitons of the Kadomtsev-Petviashvili equation and their interaction, *Phys. Letters*, V. 63A, N. 3, 205-206, 1977
- [4] I. Krichever, Rational solutions of the Kadomtsev-Petviashvili equation and integrable systems of n particles on a line, *Funct. Anal. and Appl.*, V. 12, N. 1 : 76-78, 1978
- [5] I. Krichever, S.P. Novikov Holomorphic bundles over Riemann surfaces and the KPI equation, *Funkt. Ana. E Pril.*, V. 12, 41-52, 1979
- [6] V.B. Matveev, Darboux transformation and explicit solutions of the Kadomtsev-Petviashvili equation depending on functional parameters, *Letters in Mathematical Physics*, V. 3, 213-216, 1979
- [7] N. C Freeman, J.J.C. Nimmo Rational solutions of the KdV equation in wronskian form, *Phys. Letters*, V. 96 A, N. 9, 443-446, 1983
- [8] N. C Freeman, J.J.C. Nimmo The use of Bäcklund transformations in obtaining N -soliton solutions in wronskian form, *J. Phys. A : Math. Gen.*, V. 17 A, 1415-1424, 1984
- [9] S.F. Tian, M.J. Xu, T.T. Zhang, A symmetry-preserving difference scheme and analytical solutions of a generalized higher-order beam equation, *Proceedings of the Royal Society London A*, N. 477, 2021
- [10] Y. Li, S.F. Tian, J.J. Yang, Riemann-Hilbert problem and interactions of solitons in the n -component nonlinear Schrödinger equations, *Stud. Appl. Math.*, N. 148, 577-605, 2022
- [11] Z.Y. Wang, S.F. Tian, J. Cheng, The Dbar-dressing method and soliton solutions for the three-component coupled Hirota equations, *J. Math. Phys.*, V. 62, 093510, (2021)
- [12] J.J. Yang, S.F. Tian, ZQ. Li, Riemann-Hilbert problem for the focusing nonlinear Schrödinger equation with multiple high-order poles under nonzero boundary conditions, *Phys. D*, V. 432, 133162, 2022
- [13] X. Wu, S.F. Tian, On long-time asymptotics to the nonlocal short pulse equation with the Schwartz-type initial data: Without solitons, *Phys. D*, V. 448, 133733, 2023

- [14] Z.Q. Li, S.F. Tian, J.J. Yang, On the soliton resolution and the asymptotic stability of N-soliton solution for the Wadati-Konno-Ichikawa equation with finite density initial data in space-time solitonic regions, *Adv. Math.*, V. 409, 108639, 2022
- [15] Z.Q. Li, S.F. Tian, J.J. Yang, Soliton Resolution for the Wadati-Konno-Ichikawa Equation with Weighted Sobolev Initial Data, *Ann. Henri Poincaré*, V. 23, 2611-2655, 2022
- [16] Z.Q. Li, S.F. Tian, J.J. Yang, F. Fan, Soliton resolution for the complex short pulse equation with weighted Sobolev initial data in space-time solitonic regions, *J. Differ. Equ.*, V. 329, 31-88, 2022
- [17] V.B. Matveev, M.A. Salle, Darboux transformation and solitons, *Series in Nonlinear Dynamics*, Springer Verlag, Berlin, 1991
- [18] P. Gaillard, V.B. Matveev, Wronskian addition formula and its applications, *Max-Planck-Institut für Mathematik, MPI 02-31*, V. 161, 2002
- [19] P. Gaillard, A new family of deformations of Darboux-Pöschl-Teller potentials, *Lett. Math. Phys.*, V. 68, 77-90, 2004
- [20] P. Gaillard, V.B. Matveev, New formulas for the eigenfunctions of the two-particle Calogero-Moser system, *Lett. Math. Phys.*, V. 89, 1-12, 2009
- [21] P. Gaillard, V.B. Matveev, Wronskian and Casorati determinant representations for Darboux-Pöschl-Teller potentials and their difference extensions, *J. Phys A : Math. Theor.*, V. 42, 404409-1-16, 2009
- [22] P. Dubard, P. Gaillard, C. Klein, V. B. Matveev, On multi-rogue wave solutions of the NLS equation and positon solutions of the KdV equation, *Eur. Phys. J. Spe. Top.*, V. 185, 247-258, 2010
- [23] P. Gaillard, Families of quasi-rational solutions of the NLS equation and multi-rogue waves, *J. Phys. A : Meth. Theor.*, V. 44, 435204-1-15, 2011
- [24] P. Gaillard, Wronskian representation of solutions of the NLS equation and higher Peregrine breathers, *J. Math. Sciences : Adv. Appl.*, V. 13, N. 2, 71-153, 2012
- [25] P. Gaillard, Degenerate determinant representation of solution of the NLS equation, higher Peregrine breathers and multi-rogue waves, *J. Math. Phys.*, V. 54, 013504-1-32, 2013
- [26] P. Gaillard, Wronskian representation of solutions of NLS equation and seventh order rogue waves, *J. Mod. Phys.*, V. 4, N. 4, 246-266, 2013
- [27] P. Gaillard, V.B. Matveev, Wronskian addition formula and Darboux-Pöschl-Teller potentials, *J. Math.*, V. 2013, ID 645752, 1-10, 2013

- [28] P. Gaillard, Two-parameters determinant representation of seventh order rogue waves solutions of the NLS equation, *J. Theor. Appl. Phys.*, V. 7, N. 45, 1-6, 2013
- [29] P. Gaillard, Deformations of third order Peregrine breather solutions of the NLS equation with four parameters, *Phys. Rev. E*, V. 88, 042903-1-9, 2013
- [30] P. Gaillard, Two parameters deformations of ninth Peregrine breather solution of the NLS equation and multi rogue waves, *J. Math.*, V. 2013, 1-111, 2013
- [31] P. Gaillard, Six-parameters deformations of fourth order Peregrine breather solutions of the NLS equation, *J. Math. Phys.*, V. 54, 073519-1-22, 2013
- [32] P. Gaillard, The fifth order Peregrine breather and its eight-parameters deformations solutions of the NLS equation, *Commun. Theor. Phys.*, V. 61, 365-369, 2014
- [33] P. Gaillard, Ten parameters deformations of the sixth order Peregrine breather solutions of the NLS equation, *Phys. Scripta*, V. 89, 015004-1-7, 2014
- [34] P. Gaillard, Higher order Peregrine breathers, their deformations and multi-rogue waves, *J. Of Phys. : Conf. Ser.*, V. 482, 012016-1-7, 2014
- [35] P. Gaillard, M. Gastineau, Eighteen parameter deformations of the Peregrine breather of order ten solutions of the NLS equation, *Int. J. Mod. Phys. C*, V. 26, N. 2, 1550016-1-14, 2014
- [36] P. Gaillard, Two parameters wronskian representation of solutions of nonlinear Schrödinger equation, eight Peregrine breather and multi-rogue waves, *J. Math. Phys.*, V. 5, 093506-1-12, 2014
- [37] P. Gaillard, Hierarchy of solutions to the NLS equation and multi-rogue waves, *J. Phys. : Conf. Ser.*, V. 574, 012031-1-5, 2015
- [38] P. Gaillard, Tenth Peregrine breather solution of the NLS, *Ann. Phys.*, V. 355, 293-298, 2015
- [39] P. Gaillard, M. Gastineau, The Peregrine breather of order nine and its deformations with sixteen parameters solutions of the NLS equation *Phys. Lett. A*, V. 379, 1309-1313, 2015
- [40] P. Gaillard, Other $2N-2$ parameters solutions to the NLS equation and $2N+1$ highest amplitude of the modulus of the N -th order AP breather, *J. Phys. A: Math. Theor.*, V. 48, 145203-1-23, 2015
- [41] P. Gaillard, Multi-parametric deformations of the Peregrine breather of order N solutions to the NLS equation and multi-rogue waves, *Adv. Res.*, V. 4, N. 5, 346-364, 2015

- [42] P. Gaillard, Higher order Peregrine breathers solutions to the NLS equation, *Jour. Phys. : Conf. Ser.*, V. 633, 012106-1-6, 2016
- [43] P. Gaillard, M. Gastineau Patterns of deformations of Peregrine breather of order 3 and 4, solutions to the NLS equation with multi-parameters, *Journal of Theoretical and Applied Physics*, V. 10,1-7, 2016
- [44] P. Gaillard, M. Gastineau Twenty parameters families of solutions to the NLS equation and the eleventh Peregrine breather, *Commun. Theor. Phys*, V. **65**, N. 2, 136-144, 2016
- [45] P. Gaillard, Rational solutions to the KPI equation and multi rogue waves, *Annals Of Physics*, V. 367, 1-5, 2016
- [46] P. Gaillard, M. Gastineau Twenty two parameters deformations of the twelfth Peregrine breather solutions to the NLS equation, *Adv. Res.*, V. 10, 83-89, 2016
- [47] P. Gaillard, Towards a classification of the quasi rational solutions to the NLS equation, *Theor. And Math. Phys.*, V. 189, N. 1, 1440-1449, 2016
- [48] P. Gaillard, Fredholm and Wronskian representations of solutions to the KPI equation and multi-rogue waves, *Jour. of Math. Phys.*, V. 57, 063505-1-13, doi: 10.1063/1.4953383, 2016
- [49] P. Gaillard, M. Gastineau Families of deformations of the thirteenth Peregrine breather solutions to the NLS equation depending on twenty four parameters, *Jour. Of Bas. And Appl. Res. Int.*, V. 21, N. 3, 130-139, 2017
- [50] P. Gaillard, From Fredholm and Wronskian representations to rational solutions to the KPI equation depending on $2N - 2$ parameters, *Int. Jour. of Appl. Sci. And Math.*, V. 4, N. 3, 60-70, 2017
- [51] P. Gaillard, Families of rational solutions of order 5 to the KPI equation depending on 8 parameters, *New Hor. in Math. Phys.*, V. 1, N. 1, 26-31, 2017
- [52] P. Gaillard, 6-th order rational solutions to the KPI Equation depending on 10 parameters, *Jour. Of Bas. And Appl. Res. Int.*, V. 21, N. 2, 92-98, 2017
- [53] P. Gaillard, N -Order rational solutions to the Johnson equation depending on $2N - 2$ parameters, *Int. Jour. of Adv. Res. in Phys. Sci.*, V. 4, N. 9, 19-37, 2017
- [54] P. Gaillard, Families of rational solutions to the KPI equation of order 7 depending on 12 parameters, *Int. Jour. of Adv. Res. in Phys. Sci.*, V. 4, N. 11, 24-30, 2017
- [55] P. Gaillard, Rational solutions to the Johnson equation and rogue waves, *Int. Jour. of Inn. In Sci. and Math.*, V. 6, N. 1, 14-19, 2018

- [56] P. Gaillard, Multiparametric families of solutions of the KPI equation, the structure of their rational representations and multi-rogue waves, *Theo. And Mat. Phys.*, V. 196, N. 2, 1174-1199, 2018
- [57] P. Gaillard, The Johnson equation, Fredholm and wronskian representations of solutions and the case of order three, *Adv. In Math. Phys.*, V. 2018, 1-18, 2018
- [58] P. Gaillard, Families of solutions of order 5 to the Johnson equation depending on 8 parameters, *NHIMP*, V. 2, N. 4, 53-61, 2018
- [59] P. Gaillard, Multiparametric families of solutions to the Johnson equation, *J. Phys. : Conf. Series*, V. 1141, 012102-1-10, 2018
- [60] P. Gaillard, Rational solutions to the Boussinesq equation, *Fund. Jour. Of Math. And Appl.*, V. 2, 109-112, 2019
- [61] P. Gaillard, Differential relations for the solutions to the NLS equation and their different representations, *Comm. In Adv. Math. Sci.*, V. 2, N. 4, 1-4, 2019
- [62] P. Gaillard, Multi-parametric families of solutions of order N to the Boussinesq and KP equations and the degenerate rational case, *UJMA*, V. 3, N. 2, 44-52, 2020
- [63] P. Gaillard, The mKdV equation and multi-parameters rational solutions, *Wave Motion*, V. 100, 102667-1-9, 2021
- [64] P. Gaillard, Degenerate Riemann theta functions, Fredholm and wronskian representations of the solutions to the KdV equation and the degenerate rational case, *Jour. Of Geom. And Phys.*, V. 161, 104059-1-12, 2021
- [65] P. Gaillard, $2N$ parameter solutions to the Burgers equation, *Jour. Of Appl. Nonlin. Dyn.*, V. 1, N. 1, 1-6, 2021
- [66] P. Gaillard, Multi-Parametric rational Solutions to the KdV equation, *Asi. Jour. Of Res. and Rev. In Phys.*, V. 3, N. 3, 14-21, 2021
- [67] P. Gaillard, Multiparametric solutions to the Gardner equation and the degenerate rational case, *Jour. Of Appl. Ana. And Appl.*, V. 11, N.4, 2102-2113, 2021
- [68] P. Gaillard, Other families of rational solutions to the KPI equation, *Asi. Res. Jour Of Math.*, V. 17, N. 6, 27-34, 2021
- [69] P. Gaillard, Rational solutions to the mKdV equation associated to particular polynomials, *Wave Motion*, V. 107, 102824-1-11, 2021
- [70] P. Gaillard, Multiparametric rational solutions of order N to the KPI equation and the explicit case of order 3, *Arch. Of cur. Res. Int.*, V. 21, N . 6, 58-71, 2021

- [71] P. Gaillard, Rational solutions to the KPI equation from particular polynomials, *Wave Motion*, V. 108, 102828-1-9, 2022
- [72] P. Gaillard, Rogue Waves of the Lakshmanan Porsezian Daniel Equation Depending on Multi-parameters, *As. Jour. Of Adv. Res. And Rep.*, V 16, N. 3, 32-40, 2022
- [73] P. Gaillard, Quasi-rational and rational solutions to the defocusing NLS equation, *Jour. Of Math. Sci. And Mod. Res. And Rep.*, V 5, N. 1, 24-34, 2022
- [74] P. Gaillard, Rational solutions to the KdV equation in terms of particular polynomials, *Jour. Of Nonlin. Mod. and Ana.*, V. 4, N. 4, 615-626, 2022
- [75] P. Gaillard, Rational solutions to the Painlevé II equation from particular polynomials, *Int. J. Of Appl. Nonlin. Sci.*, V. 3, N. 3, 189-196, 2022
- [76] P. Gaillard, Multi-lump solutions to the KPI equation with a zero degree of derivation, *Jour. Of Theor. And Appl. Phys.*, V. 16, N. 4, 1-8, 2022
- [77] P. Gaillard, Families of solutions to the KPI equation given by an extended Darboux transformation, *Part. Diff. Equ. And Appl.*, V. 3, N. 80, 1-14, 2022
- [78] P. Gaillard, Rogue waves of the Hirota equation in terms of quasi-rational solutions depending on multi-parameters, *WSEAS Trans. On Math.*, DOI: 10.37394/23206.2023.22.24, V. 22, 190-203, 2023
- [79] P. Gaillard, Different representations of the solutions to the cylindrical nonlinear Schrödinger equation, *Dis. Nonlin. And Compl.*, V. 12, N. 3, 539-553, 2023
- [80] P. Gaillard, Rational solutions to the KdV equation from Riemann theta functions, *Jour. Of Theor. And Appl. Phys.*, V. 17, N. 3, 1-5, 2023
- [81] P. Gaillard, On Hadamard's coefficients for two dimensional Schrödinger operators, *Int. Jour. Of Theor. Phys.*, V. 62, N. 205, 1-16, 2023
- [82] P. Gaillard, Rational solutions to the Burgers' equation from particular polynomials, *WSEAS Trans. On Math*, DOI: 10.37394/23206.2023.22.94, 867-873, 2023
- [83] P. Gaillard, Rational solutions to the Gardner equation from particular polynomials, *WSEAS Equ.*, DOI: 10.37394/232021.2023.3.2, V. 3, 1-24, 2023