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Sana Benhafsia, Rejeb Hadiji

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Sana Benhafsia *and Rejeb Hadiji†

Abstract

Recently, a great attention has been focused on the study of fractional and non-local operators of elliptic type, both for the pure mathematical research and in view of concrete real-world applications. We consider the following non local problem on $\mathbb{H}_0^s(\Omega) \subset L^{q_s}(\Omega)$, with $q_s := \frac{2n}{n-2s}$, $s \in]0,1[$ and $n \geq 3$

$$\int_{\mathbb{R}^n} p(x)\Big(\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dy\Big)dx - \lambda \int_\Omega |u(x)|^q dx, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n, p : \mathbb{R}^n \to \mathbb{R}$ is a given positive weight presenting a global minimum $p_0 > 0$ at $a \in \Omega$ and λ is a real constant. In this work we show that for $q = 2$ the infimum of (1) over the set $\{u \in \mathbb{H}_0^s(\Omega), ||u||_{L^{q_s}(\Omega)} = 1\}$ does exist for some k, s, λ and n and for $q \geq 2$ we study non ground state solutions using the Mountain Pass Theorem.

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1 Introduction, notations and statement of the results

1.1 The fractional non-linear problem with weight and its relation with the ordinary non-linear problem

Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$. We are interested in the following non-linear problem involving the fractional Laplacian, for $u \in \mathbb{H}_0^s(\Omega)$

$$\begin{aligned}&\int_{\mathbb{R}^n\times\mathbb{R}^n} p(x)\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy - \lambda\int_\Omega |u(x)|^{q-2}u(x)\varphi(x)dx\\ &= \int_\Omega |u(x)|^{q_s-2}u(x)\varphi(x)dx\end{aligned} \tag{2}$$

for any $\varphi \in \mathbb{H}_0^s(\Omega)$, where the space $\mathbb{H}_0^s(\Omega)$ is defined by

$$\mathbb{H}_0^s(\Omega) := \Big\{u \in L^2(\Omega), \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{2}+s}} \in L^2(\mathbb{R}^n\times\mathbb{R}^n \setminus \Omega^c \times \Omega^c), u(x) = 0, \forall x \in \mathbb{R}^n \setminus \Omega\Big\}, \tag{3}$$

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$2 \leq q < q_s$ with q_s is the critical fractional Sobolev exponent $q_s := \frac{2n}{n-2s}$, $\lambda > 0$. Note that it is well known that the embedding $\mathbb{H}_0^s(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for any $1 \leq r \leq \frac{2n}{n-2s}$. Moreover this embedding is compact for $1 \leq r < \frac{2n}{n-2s}$, see ([5], lemma 1.31). We assume that $p : \mathbb{R}^n \to \mathbb{R}$ is a positive bounded weight in $C^1(\mathbb{R}^n)$, we assume also that p represents a global minimum p_0 at $a \in \Omega$ and satisfies in $B(a, 4\eta)$, $\eta > 0$, $\kappa > 0$ and $k > 1$

$$p(x) \leq p_0 + \kappa |x - a|^k. \tag{4}$$

We remind that for $p = 1$, (2) is the weak variational of the following problem

$$\begin{cases} (-\Delta)^s u - \lambda u^{q-2} u = |u|^{q_s - 2} u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{5}$$

which was studied in [5] when $q = 2, n \geq 4s$ and $\lambda \in]0, \lambda_{1,s}[$ where $\lambda_{1,s}$ denotes the first eigenvalue of the non local operator $(-\Delta)^s$ with homogeneous Dirichlet boundary datum. In this paper, we will break the symmetry of the problem by introducing the weight p as described before. we extend the result which was already done in the ordinary Laplacian in [19] to the case of the fractional Laplacian. Other authors gave a basic introduction to the fractional Laplacian operator, see [23] and references therein, see also [2], [3], [5] and [24] where authors dealt with non local fractional problems. Some other authors studied variants of the fractional Laplacian, see [15], [21] and [10].
Let us take p defined as previously in (4), $s \in]0, 1[$ and $n \geq 3$, we define the infimum $S_{s,\lambda}(p)$ by

$$S_{s,\lambda}(p) := \inf_{\substack{u \in \mathbb{H}_0^s(\Omega) \\ ||u||_{L^{q_s}(\mathbb{R}^n)} = 1}} \left\{ \int_{\mathbb{R}^n} p(x) \left(\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy \right) dx - \lambda \int_\Omega |u(x)|^2 dx \right\}. \tag{6}$$

The study of the infimum (6), shows that the existence of minimizers depends, apart from parameter λ, on the behavior of p near its absolute minima and the geometry of the domain Ω.

The well known fractional Sobolev inequalities were first considered in a remarkable paper by Lieb in [20]; see also [13]; [11] or the survey [23].

Let us compare this work to what is known in the literature concerning problems related to the Yamabe problemin the local case

$$S_\lambda(p) = \inf_{\substack{u \in H_0^1(\Omega) \\ ||u||_{L^q(\Omega)} = 1}} \int_\Omega p(x) |\nabla u(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx, \tag{7}$$

where $q = \frac{2n}{n-2}$ is the critical exponent for the Sobolev embedding $H_0^1(\Omega) \subset L^q(\Omega)$. In [19], the authors treated the case where the problem presents a positive weight with the ordinary Laplacian ($s = 1$). They proved in particular, the existence of minimizers of $S_\lambda(p)$ for $0 < \lambda < \lambda_1^{div}$ if $n \geq 4$ and $k > 2$, and for $\lambda^* < \lambda < \lambda_1^{div}$ if $n \geq 3$ and $0 < k < 2$, $k = 2$ is critical for the problem, and in other subcases which are well detailed in ([19],Theorem 1.1) with k is a positive constant that appears in the expression of the weight p, λ_1^{div} is the first eigenvalue of $-div(p(x)\nabla.)$ on Ω with zero Dirichlet boundary condition and λ^* is a positive constant. The method used for the proof of this result is, first to show that $S_\lambda(p) < p_0 S$, with S is the best Sobolev constant defined by

$$S := \inf_{\substack{u \in H^1(\mathbb{R}^n) \\ ||u||_{L^q(\mathbb{R}^n)} = 1}} \int_\Omega |\nabla u(x)|^2 dx, \tag{8}$$

then, they prove that the infimum $S_\lambda(p)$ is achieved. In the same context, similar questions were studied in [16] where the author investigate the problem with a weight and a nonvanishing boundary datum and in [18] where authors dealt with a non-linear eigenvalue problem with a variable weight.

The problem involving the ordinary Laplacian in the case where p is a constant was originally studied by Aubin in [1] and Brezis-Nirenberg in [9], see also [6], [4] we refer to [21] for a complete history of the problem and geometrical motivations.

The authors dealt in [26] with a non ground state solutions in the fractional Laplacian case without weight. The solutions are constructed with a variational method by a min-max procedure on the associated energy functional.

In [15] the authors formulated a fractional s-Yamabe problems that include the boundary Yamabe problem studied by Escobar, see [12]. They highlight a Hopf-type maximum principle together with interplay between analysis of weighted trace Sobolev inequalities and conformal structure of the underlying manifolds and they obtained some properties for the fractional case that are analogous to the original Yamabe problem, see [12].

1.2 Some definitions

One of the aims of this paper is to study non local problems driven by $(-\Delta)^s$ (or its generalization) and with Dirichlet boundary data via variational methods. For this purpose, we need to work in a suitable fractional Sobolev space: for this, we consider a functional analytical setting that is inspired by (but not equivalent to) the fractional Sobolev spaces in order to correctly encode the Dirichlet boundary datum in the variational formulation. This section is devoted to the definition of this space as well as to its properties. Therefore, before setting the main result, we start by defining the fractional Laplacian and the fractional Sobolev spaces.
Let $s \in]0,1[$. Up to normalization factors we define the non local operator $(-\Delta)^s : S(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ by

$$(-\Delta)^s u(x) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy, \tag{9}$$

with $x \in \mathbb{R}^n$, $B(x,\varepsilon)$ is the ball centered in $x \in \mathbb{R}^n$ of radius ε.

We define the Sobolev space $H^s(\mathbb{R}^n)$ the set of functions u such that they are square integrable and their fractional Laplacian $(-\Delta)^{\frac{s}{2}} u$ is also square integrable:

$$H^s(\mathbb{R}^n) := \big\{ u : \mathbb{R}^n \to \mathbb{R}, u \in L^2(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy < +\infty \big\}. \tag{10}$$

$H^s(\mathbb{R}^n)$ is endowed with the norm defined as

$$||u||_{H^s(\mathbb{R}^n)} := ||u||_{L^2(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy \right)^{\frac{1}{2}}.$$

We know that $H^s(\mathbb{R}^n)$ endowed with the norm $\|.\|_{H^s(\mathbb{R}^n)}$ is a Hilbert space.
Thus, $\mathbb{H}_0^s(\Omega)$ is a subspace of $H^s(\mathbb{R}^n)$ and is defined simply as in [5] by

$$\mathbb{H}_0^s(\Omega) = \big\{ u \in H^s(\mathbb{R}^n); u = 0 \text{ a.e. sur } \Omega^c \big\}.$$

The norm in $\mathbb{H}_0^s(\Omega)$ is defined as follows

$$\mathcal{N}(u) := ||u||_{L^2(\Omega)} + \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy \right)^{\frac{1}{2}}.$$

This norm is equivalent to

$$||u||_{\mathbb{H}_0^s(\Omega)} = \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy \right)^{\frac{1}{2}},$$

see ([5], lemma1.28) and ([13], lemma 3.1). We notice that the following identity $||u||_{\mathbb{H}_0^s(\Omega)} = ||(-\Delta)^{\frac{s}{2}} u||_{L^2(\mathbb{R}^n)}$, gives the relation between the fractional Laplacian operator $(-\Delta)^s$ and the fractional Sobolev space $\mathbb{H}_0^s(\Omega)$, see ([5], (4.35)).

In this present paper, $\mathbb{H}_0^s(\Omega)$ will be the functional analytic setting because the classical fractional Sobolev space approach not sufficient for studying the problem, see [3], [5], [26].

We start by recalling some notations and some remarks which will be useful. First, We will start by giving some important definitions afterwards. Let denote by $S_s(p) := S_{s,0}(p)$ and $S_s := S_{s,0}(1)$ the weightless case. Let's denote by $u_{\varepsilon,s,a}$ an extremal function for the weightless Sobolev inequality for the fractional Laplacian operator. Let us fix $\eta > 0$ such that

$$B(a, 4\eta) \subset \Omega \tag{11}$$

For $x \in \mathbb{R}^n$ and $\varepsilon > 0$,

$$U_{\varepsilon,s,a}(x) = \left(\frac{\varepsilon}{\varepsilon^2 + |x-a|^2} \right)^{\frac{n-2s}{2}}, \tag{12}$$

then we set

$$u_{\varepsilon,s,a}(x) = U_{\varepsilon,s,a}(x)\Psi(x), \tag{13}$$

where $\Psi \in \mathbb{C}_0^\infty(\mathbb{R}^n)$ such that $0 \leq \Psi \leq 1$, $\Psi = 1$ in $B(a, \eta)$ and $\Psi = 0$ in $B(a, 2\eta)^c$, with η is a postive real.
We have

$$||(-\Delta)^{\frac{s}{2}} u_{\varepsilon,s,a}(x)||^2_{L^2(\mathbb{R}^n)} = K_{1,s}, \tag{14}$$

the square integration of $u_{\varepsilon,s,a}$ goes as follows

$$||u_{\varepsilon,s,a}||^2_{L^2(\mathbb{R}^n)} = K_{2,s}\varepsilon^{2s} + \mathcal{O}(\varepsilon^{n-2s}), \tag{15}$$

the q_s norm is given by

$$||u_{\varepsilon,s,a}||^{q_s}_{L^{q_s}(\mathbb{R}^n)} = K_2 + \mathcal{O}(\varepsilon^n), \tag{16}$$

and for any $n \geq 4, s \in]0,1[, q \in [2, q_s[$ we have

$$||u_{\varepsilon,s,a}||^q_{L^q(\mathbb{R}^n)} = \tilde{K}_{s,q}\varepsilon^{n - \frac{q(n-2s)}{2}} + \mathcal{O}(\varepsilon^{\frac{q(n-2s)}{2}}) \tag{17}$$

with $K_{2,s}$ is a positive constant, $\tilde{K}_{s,q} = \displaystyle\int_{\mathbb{R}^n} \frac{dy}{(1+|y|^2)^{\frac{q(n-2s)}{2}}}$, $K_2 = \displaystyle\int_{\mathbb{R}^n} \frac{dy}{(1+|y|^2)^n}$ and $\frac{K_{1,s}}{K_2^{\frac{2}{q_s}}} = S_s$.

Note that for $n = 3, s \leq \frac{3}{4}, q \in [2, q_s[$ (17) holds true. In order to present our main results, we will need to introduce the first eigenvalue with weight $\lambda_{1,p,s}$ associated to the minimizing problem (6):

$$\lambda_{1,p,s} := \min_{\substack{u \in \mathbb{H}_0^s(\Omega) \\ u \neq 0}} \frac{\displaystyle\int_{\mathbb{R}^n} p(x) \Big(\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dy \Big) dx}{\displaystyle\int_{\mathbb{R}^n} |u(x)|^2 dx}. \tag{18}$$

‘

Let us take

$$E_\lambda(u) := \int_{\mathbb{R}^n} p(x) \Big(\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dy \Big) dx - \lambda \int_{\Omega_\varepsilon} u^q dx.$$

For the sake of clarity, let us analyze a blow up around a minima of p which we suppose 0, we are led to the study $E_\lambda(v_\varepsilon)$ where v_ε is defined by by $v(x) = \varepsilon^{-n/q_s} v_\varepsilon(\frac{x}{\varepsilon})$ and $\Omega_\varepsilon = \varepsilon^{-1}\Omega$. Note that if $||v||_{q_s} = 1$ then $||v_\varepsilon||_{q_s} = 1$. We have

$$\begin{aligned} E_\lambda(v_\varepsilon) = \quad & p_0 \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v_\varepsilon|^2 dx + \varepsilon^k \int_{\mathbb{R}^n} |x|^k \int_{\mathbb{R}^n} \frac{|v_\varepsilon(x) - v_\varepsilon(y)|^2}{|x-y|^{n+2s}} dy dx \\ & - \lambda \varepsilon^{n - \frac{q(n-2s)}{2}} \int_{\Omega_\varepsilon} v_\varepsilon^q dx. \end{aligned} \tag{19}$$

When $q > 2$ and $k > n - \frac{q(n-2s)}{2}$, the term related to $\varepsilon^{n - \frac{q(n-2s)}{2}}$ dominates the one related to ε^k provided that the quantity

$$\int_{\mathbb{R}^n} |x|^k \int_{\mathbb{R}^n} \frac{|v_\varepsilon(x) - v_\varepsilon(y)|^2}{|x-y|^{n+2s}} dy dx \tag{20}$$

is finite, so we find that the energy get strictly below the critical value $p_0^{\frac{n}{2s}} S_s^{\frac{n}{2s}}$, while in the case where $q = 2$ and $k > 2s$ we expect that $E_\lambda(v) < p_0 S_s$, then $S_{s,\lambda}(p)$ is achieved.

It is important to mention that, since the problem is not local, at infinity the term $|x|^k$ has an impact on the integral (20) which is not finite in general for $u \in \mathbb{H}_0^s(\Omega)$, so that we will restrict ourselves to specific values of k.

We expect a competition between the local character of the weight and the non-local character of the operator. It turns out that the local nature of the weight wins. The presence of weight pushes the problem to be non-local. In other words, when k is smaller than $n - 4s$, we find that the local character dominates, on the other hand when k is large we expect that the problem does not admit solutions.
In (6), if we take $\lambda = 0$ we get the following nonexistence result.

Proposition 1.1. *If $\lambda = 0$, $s \in]0, 1[$ and $2 \leq k < n - 4s$ then $S_{s,0}(p) = p_0 S_{s,0}(1)$ and $S_{s,0}(p)$ is never achieved.*

Proof : Let $s \in]0, 1[$ and $2 \leq k < n - 4s$. We recall the notations $S_s(p) := S_{s,0}(p)$ and $S_s := S_{s,0}(1)$. By Theorem2.4, please see section 2, we write

$$p_0 S_s \leq S_s(p) \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} p(x) \frac{|u_{\varepsilon,s,a}(x) - u_{\varepsilon,s,a}(y)|^2}{|x-y|^{n+2s}} dy dx \leq p_0 S_s + o(1), \tag{21}$$

as ε tends to zero. Then we get $S_s(p) = p_0 S_s$.
We suppose that $S_s(p)$ is achieved by some function $u \in \mathbb{H}_0^s(\Omega)$ such that $||u||_{L^{q_s}(\mathbb{R}^n)} = 1$, then

$$p_0 \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dydx \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} p(x)\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dydx = p_0 S_s. \tag{22}$$

Therefore, $S_s = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dydx, ||u||_{L^{q_s}(\mathbb{R}^n)} = 1$ which means that S_s is achieved by a function $u \in \mathbb{H}_0^s(\Omega)$ and this is absurd. See [20], see also [13] , [11] or [23]. □
In what follows, we will concentrate on the case where $\lambda > 0$.

1.3 Statement of the main results

Let us announce the main statements of this paper. We state two Theorems where in the first, we prove existence of minimizers of $S_{s,\lambda}(p)$ in the presence of a linear perturbation.

Theorem 1.2. *Let Ω be an open, bounded subset of $\mathbb{R}^n$ with continuous boundary. Let p be as defined in (4), $n \geq 3$ and $2 \leq k < n - 4s$. The following statements hold true*

1. *If $n = 3$ and $s \in]0, \frac{1}{4}]$ or $n = 4$ and $s \in]0, \frac{1}{2}]$ or $n = 5$ and $s \in]0, \frac{3}{4}]$ there exists a constant $C = C(n, s, k) > 0$ such that for every $\kappa \in]0, C\lambda[$ we have $S_{s,\lambda}(p)$ is achieved.*

2. *If $n \geq 6$ and $s \in]0, 1[$, there exists $\bar{C}_2 := C(n, s, k) > 0$ such that for every $\lambda \in]0, \lambda_{1,p,s}[$ and for every $\kappa \in]0, \bar{C}_2\lambda[$, $S_{s,\lambda}(p)$ is achieved.*

The second Theorem is dealing with problem (2), we prove that it has non ground state solutions with a subcritical pertubation by proceeding with a min-max technique using the Mountain Pass Theorem.

Theorem 1.3. *Let Ω be an open, bounded subset of $\mathbb{R}^n$ with continuous boundary. Let Suppose that $k \in [2, n - 4s[$. If $2 < q < q_s$ then we have*

1. *If $n \geq 4$ then for every $s \in]0, 1[$ and there exists $\lambda_0 > 0$ such that for every $0 < \lambda < \lambda_0$ problem (2) has a nontrivial solution $u \in \mathbb{H}_0^s(\Omega)$.*

2. *If $n = 3$ then for every $s \in]0, \frac{3}{4}[$ and there exists $\lambda_0 > 0$ such that for every $0 < \lambda < \lambda_0$ problem (2) has a nontrivial solution $u \in \mathbb{H}_0^s(\Omega)$.*

In the case where $q = 2$ then we have

1. *If $n = 3$ and $s \in]0, \frac{1}{4}]$ or $n = 4$ and $s \in]0, \frac{1}{2}]$ or $n = 5$ and $s \in]0, \frac{3}{4}]$ there exists a constant $C = C(n, s, k) > 0$ such that for every $\kappa \in]0, C\lambda[$ we have problem (2) has a nontrivial solution $u \in \mathbb{H}_0^s(\Omega)$*

2. *while if $n \geq 6$ and $s \in]0, 1[$, there exists $\bar{C}_2 := C(n, s, k) > 0$ such that for every $\lambda \in]0, \lambda_{1,p,s}[$ and for every $\kappa \in]0, \bar{C}_2\lambda[$, problem (2) has a nontrivial solution $u \in \mathbb{H}_0^s(\Omega)$.*

1.4 Structure of the paper

The paper is structured as follows. The next section §2, proves the a-priori estimate $S_{s,\lambda}(p) < p_0 S_s$. Theorem1.2 which is the first main result of this paper, is proved by mathematical adequate technique using the previous section in order to get existence of minimizing solutions to $S_{s,\lambda}(p)$. In section §3 we carry out non ground state solutions using the minimax technique and proving the Mountain Pass Theorem in the general case. We investigate also a subcritical approximation and we adopt the strategy used from [9] to prove Theorem 1.3 by choosing a suitable test function.

2 Existence of minimizers

First of all, let us prove that the infimum $S_{s,\lambda}(p)$ does exist if $\lambda \leq \lambda_{1,p,s}$. In fact, since by definition of $\lambda_{1,p,s}$ we write

$$\frac{\displaystyle\int_{\mathbb{R}^n} p(x)\Big(\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dy\Big)dx}{\displaystyle\int_{\mathbb{R}^n} |u(x)|^2 dx} \geq \lambda_{1,p,s} = \min_{\substack{u\in \mathbb{H}_0^s(\Omega)\\ u\neq 0}} \frac{\displaystyle\int_{\mathbb{R}^n} p(x)\Big(\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dy\Big)dx}{\displaystyle\int_{\mathbb{R}^n} |u(x)|^2 dx}.$$

Then

$$\begin{aligned}\int_{\mathbb{R}^n} p(x)\Big(\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dy\Big)dx - \lambda\int_\Omega |u(x)|^2dx &\geq \lambda_{1,p,s}\int_\Omega |u(x)|^2dx - \lambda\int_\Omega |u(x)|^2dx \\ &= (\lambda_{1,p,s}-\lambda)\int_\Omega |u(x)|^2dx.\end{aligned} \tag{23}$$

Therefore, we deduce immediately that if $\lambda \leq \lambda_{1,p,s}$, then

$$\inf_{\substack{u\in \mathbb{H}_0^s(\Omega)\\ ||u||_{L^{q_s}(\mathbb{R}^n)}=1}} \int_{\mathbb{R}^n} p(x)\Big(\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dy\Big)dx - \lambda\int_\Omega |u(x)|^2dx \geq 0.$$

So that, the infimum $S_{s,\lambda}(p)$ exists.
The following Theorem plays a crucial role to prove existence of solutions, it is an adaptation of an original argument due to [9] in the context of Yamabe's conjecture.

Proposition 2.1. *Let $s \in]0,1[$ and let a weight p satisfying (4). If $S_{s,\lambda}(p) < p_0 S_s$, then $S_{s,\lambda}(p)$ is achieved.*

Proof : Let us recall the expression of $S_{s,\lambda}(p)$:

$$S_{s,\lambda}(p) = \inf_{\substack{u\in \mathbb{H}_0^s(\Omega)\\ ||u||_{L^{q_s}(\mathbb{R}^n)}=1}} \int_{\mathbb{R}^n} p(x)\Big(\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dy\Big)dx - \lambda\int_\Omega |u(x)|^2dx. \tag{24}$$

Let (u_j) a minimizing sequence of $S_{s,\lambda}(p)$, then $||u_j||_{L^{q_s}(\mathbb{R}^n)} = 1$ and

$$\int_{\mathbb{R}^n} p(x)\int_{\mathbb{R}^n} \frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}dydx - \lambda||u_j||_2^2 = S_{s,\lambda}(p) + o(1), \tag{25}$$

as j tends to $+\infty$.
Since (u_j) is bounded in $\mathbb{H}_0^s(\Omega)$, we extract a subsequence still denoted by (u_j) such that, (u_j) tends weakly to u in $\mathbb{H}_0^s(\Omega)$ (since $\mathbb{H}_0^s(\Omega)$ is reflexive space). Then (u_j) tends strongly to u in $L^2(\Omega)$, and (u_j) tends to u almost everywhere in Ω, with

$$||u||_{L^{q_s}(\mathbb{R}^n)} \leq 1. \tag{26}$$

Let's take $v_j := u_j - u$ then (v_j) tends strongly to 0 in $\mathbb{H}_0^s(\Omega)$ and (v_j) tends to 0 almost everywhere in Ω.
By the definition of $S_{s,\lambda}(p)$ and S_s we have,

$$\int_{\mathbb{R}^n} p(x) \int_{\mathbb{R}^n} \frac{|u_j(x) - u_j(y)|^2}{|x-y|^{n+2s}} dydx \geq p_0 S_s,$$

then

$$\int_{\mathbb{R}^n} p(x) \int_{\mathbb{R}^n} \frac{|u_j(x) - u_j(y)|^2}{|x-y|^{n+2s}} dydx - S_{s,\lambda}(p) + o(1) \geq p_0 S_s - S_{s,\lambda}(p) + o(1),$$

as j tends to $+\infty$, therefore

$$\lambda ||u||^2_{L^2(\mathbb{R}^n)} \geq p_0 S_s - S_{s,\lambda}(p) > 0.$$

Therefore $u \neq 0$. Using the definition of S_s, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} p(x) \bigg(\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dy \bigg) dx + \int_{\mathbb{R}^n} p(x) \int_{\mathbb{R}^n} \frac{|v_j(x) - v_j(y)|^2}{|x-y|^{n+2s}} dydx \\ &+2 \int_{\mathbb{R}^n} p(x) \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v_j(x) - v_j(y))}{|x-y|^{n+2s}} dydx - \lambda ||u||^2_{L^2(\mathbb{R}^n)} = S_{s,\lambda}(p) + o(1), \end{aligned} \tag{27}$$

as j tends to $+\infty$. We define a scalar product on $\mathbb{H}_0^s(\Omega)$ by

$$< u, v >_p = \int_{\mathbb{R}^n} p(x) \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} dydx, \tag{28}$$

for all $u, v \in \mathbb{H}_0^s(\Omega)$. It's obvious that the norm associated with this scalar product is equivalent to the ordinary norm over $\mathbb{H}_0^s(\Omega)$. Then, $\int_{\mathbb{R}^n} p(x) \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v_j(x) - v_j(y))}{|x-y|^{n+2s}} dydx$ tends to zero as j tends to $+\infty$. On the other hand, Since (v_j) is bounded in $L^{q_s}(\Omega)$ and (v_j) tends to 0 almost everywhere in Ω, we deduce from a result of Brezis-Lieb, see [8] that:

$$||u + v_j||^{q_s}_{L^{q_s}(\mathbb{R}^n)} = ||u||^{q_s}_{L^{q_s}(\mathbb{R}^n)} + ||v_j||^{q_s}_{L^{q_s}(\mathbb{R}^n)} + o(1), \tag{29}$$

as j tends to $+\infty$. Therefore

$$1 = \|u\|^{q_s}_{L^{q_s}(\mathbb{R}^n)} + \|v_j\|^{q_s}_{L^{q_s}(\mathbb{R}^n)} + o(1),$$

as j tends to $+\infty$. Since $||u||_{L^{q_s}(\mathbb{R}^n)} \leq 1$ then

$$1 \leq \|u\|^2_{L^{q_s}(\mathbb{R}^n)} + \|v_j\|^2_{L^{q_s}(\mathbb{R}^n)} + o(1),$$

as j tends to $+\infty$. Denoting by $\tilde{v}_j := \frac{v_j}{||v_j||_{L^{q_s}(\mathbb{R}^n)}}$. Since we have

$$p_0 S_s \leq \int_{\mathbb{R}^n} p(x) \int_{\mathbb{R}^n} \frac{|\tilde{v}_j(x) - \tilde{v}_j(y)|^2 |x-y|^{n+2s}}{d} y dx, \tag{30}$$

we obtain

$$1 \leq \|u\|^2_{L^{q_s}(\mathbb{R}^n)} + \frac{1}{p_0 S_s} \int_{\mathbb{R}^n} p(x) \int_{\mathbb{R}^n} \frac{|v_j(x) - v_j(y)|^2}{|x-y|^{n+2s}} dy dx + o(1), \tag{31}$$

as j tends to $+\infty$. Since we have already proved that $S_{s,\lambda}(p) \geq 0$, we deduce from (31) that

$$S_{s,\lambda}(p) \leq S_{s,\lambda}(p) \|u\|^2_{L^{q_s}(\mathbb{R}^n)} + \frac{S_{s,\lambda}(p)}{p_0 S_s} \int_{\mathbb{R}^n} p(x) \int_{\mathbb{R}^n} \frac{|v_j(x) - v_j(y)|^2}{|x-y|^{n+2s}} dy dx + o(1), \tag{32}$$

as j tends to $+\infty$. Combining (27) and (129), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} p(x) \bigg(\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dy \bigg) dx - \lambda \int_\Omega |u(x)|^2 dx + \int_{\mathbb{R}^n} p(x) \int_{\mathbb{R}^n} \frac{|v_j(x) - v_j(y)|^2}{|x-y|^{n+2s}} dy dx \\ &\leq S_{s,\lambda}(p) ||u||^2_{L^{q_s}(\mathbb{R}^n)} + \tfrac{S_{s,\lambda}(p)}{p_0 S_s} \int_{\mathbb{R}^n} p(x) \int_{\mathbb{R}^n} \frac{|v_j(x) - v_j(y)|^2}{|x-y|^{n+2s}} dy dx + o(1), \end{aligned} \tag{33}$$

as j tends to $+\infty$. Thus,

$$\begin{aligned} &\int_{\mathbb{R}^n} p(x) \bigg(\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dy \bigg) dx - \lambda \int_\Omega |u(x)|^2 dx \\ &\leq S_{s,\lambda}(p) ||u||^2_{L^{q_s}(\mathbb{R}^n)} + \bigg[\tfrac{S_{s,\lambda}(p)}{p_0 S_s} - 1 \bigg] \int_{\mathbb{R}^n} p(x) \int_{\mathbb{R}^n} \frac{|v_j(x) - v_j(y)|^2}{|x-y|^{n+2s}} dy dx + o(1), \end{aligned} \tag{34}$$

as j tends to $+\infty$. Since $S_{s,\lambda}(p) < p_0 S_s$, we deduce

$$\int_{\mathbb{R}^n} p(x) \bigg(\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dy \bigg) dx - \lambda \int_\Omega |u(x)|^2 dx \leq S_{s,\lambda}(p) ||u||^2_{L^{q_s}(\mathbb{R}^n)}. \tag{35}$$

This means that u is a minimum of $S_{s,\lambda}(p)$. □

Remark 2.2. *In the case where the weight p and λ are such that $-\infty < S_{s,\lambda}(p) \leq 0$, we prove that the infimum $S_{s,\lambda}(p)$ is achieved. In fact, as in* (26) *we have $||u||_{L^{q_s}(\mathbb{R}^n)} \leq 1$, then*

$$S_{s,\lambda}(p) \leq S_{s,\lambda}(p) ||u||^2_{L^{q_s}(\mathbb{R}^n)}.$$

Again, we deduce (35) *from* (27).

Remark 2.3. *If $S_{s,\lambda}(p)$ is achieved by a function $u \in \mathbb{H}^s_0(\Omega)$, then $u \geq 0$. In fact, thanks to the following inequality $\big| |u_j(x)| - |u_j(y)| \big| \leq |u_j(x) - u_j(y)|$, if (u_j) is a minimising sequence of $S_{s,\lambda}(p)$, $(|u_j|)$ is also a minimising sequence. Therefore, we take a positive minimizing sequence of $S_{s,\lambda}(p)$.*

2.1 A priori estimate on $S_{s,\lambda}(p)$

We need to prove the following proposition and the techniques that it uses since it constitutes an official key in order to apply Proposition2.1.

Theorem 2.4. *Let $s \in]0,1[$, $n \geq 3$ and $2 \leq k < n - 4s$. Then the following estimate holds true*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} p(x)\frac{|u_{\varepsilon,s,a}(x) - u_{\varepsilon,s,a}(y)|^2}{|x-y|^{n+2s}}dxdy \leq p_0 S_s + \kappa C\varepsilon^{2s} + \mathcal{O}(\varepsilon^{n-2s}) + \mathcal{O}(\varepsilon^{k+2s}), \tag{36}$$

as ε tends to zero and C is a positive constant depending on k, s and the dimension n.

Proof : First of all and without loss of generality, we assume that $a = 0$ and we note $U_{\varepsilon,s,0} = U_{\varepsilon,s}$ $u_{\varepsilon,s,0} = u_{\varepsilon,s}$.
The proof makes use of the following estimates and is a bit complicated definetly more difficult than the one for similar results in the case of the fractional Laplacian without weight as it is in the [5] that we will inspire from it to accomplish the proof.
We easily see that for $\rho > 0$ and $x \in B(0,\rho)^c$, then

$$|u_{\varepsilon,s}(x)| \leq |U_{\varepsilon,s}(x)| \leq C\varepsilon^{\frac{n-2s}{2}}, \tag{37}$$

for any $\varepsilon > 0$ and for some positive constant C, possibly depending on η, ρ, s and the dimension n.
We mention that the following assertions hold true
(a) For any $x \in \mathbb{R}^n$ and $y \in B(0,\eta)^c$, with $|x-y| \leq \frac{\eta}{2}$,

$$|u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)| \leq C\varepsilon^{\frac{n-2s}{2}}|x-y|. \tag{38}$$

(b) For any $x, y \in B(0,\eta)^c$,

$$|u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)| \leq C\varepsilon^{\frac{n-2s}{2}} \min\{1, |x-y|\}, \tag{39}$$

for any $\varepsilon > 0$ and for some positive constant C, possibly depending on η, ρ, s and n
Let us begin the proof of the Theorem. We introduce the notations

$$\mathbb{D} := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \in B(0,\eta), y \in B(0,\eta)^c, |x-y| > \frac{\eta}{2}\} \tag{40}$$

and

$$\mathbb{E} := \{(x,y) \in R^n \times \mathbb{R}^n : x \in B(0,\eta), y \in B(0,\eta)^c, |x-y| \leq \frac{\eta}{2}\}, \tag{41}$$

where η is as in (11).

By (13), we have that

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|x|^k\frac{|u_{\varepsilon,s}(x)-u_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}}dxdy &= \int_{B(0,\eta)}\int_{B(0,\eta)}|x|^k\frac{|U_{\varepsilon,s}(x)-U_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}}dxdy\\ &+2\int_{\mathbb{D}}|x|^k\frac{|u_{\varepsilon,s}(x)-u_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}}dxdy\\ &+2\int_{\mathbb{E}}|x|^k\frac{|u_{\varepsilon,s}(x)-u_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}}dxdy\\ &+\int_{B(0,\eta)^c}\int_{B(0,\eta)^c}|x|^k\frac{|u_{\varepsilon,s}(x)-u_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}}dxdy\end{aligned}\tag{42}$$

We start by treating the first term in the right hand side. For $s\in]0,1[$, $2\le k<n-4s$ and $\varepsilon>0$, let us denote by

$$A_{s,k,\varepsilon}:=\int_{|x|\le\eta}|x|^k\bigg(\int_{|y|\le\eta}\frac{|U_{\varepsilon,s}(x)-U_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}}dy\bigg)dx.\tag{43}$$

Proposition 2.5. *For $s\in]0,1[$, $2\le k<n-4s$ and $\varepsilon>0$, we have*

$$A_{s,k,\varepsilon}\le C\varepsilon^{2s},\tag{44}$$

where C is a positive constant depending on s,k and the dimension n.

Proof : Performing a change of variables in (43), we obtain

$$\begin{aligned}A_{s,k,\varepsilon} &= \varepsilon^k\int_{|x|\le\frac{\eta}{\varepsilon}}|x|^k\int_{|y|\le\frac{\eta}{\varepsilon}}\frac{1}{|x-y|^{n+2s}}\bigg|\frac{1}{(1+|x|^2)^{\frac{n-2s}{2}}}-\frac{1}{(1+|y|^2)^{\frac{n-2s}{2}}}\bigg|^2dydx\\ &= \varepsilon^k\int_{|x|\le\frac{\eta}{\varepsilon}}\int_{|y|\le\frac{\eta}{\varepsilon}}\bigg|\frac{|x|^{\frac{k}{2}}}{(1+|x|^2)^{\frac{n-2s}{2}}}-\frac{|x|^{\frac{k}{2}}}{(1+|y|^2)^{\frac{n-2s}{2}}}\bigg|^2\frac{dydx}{|x-y|^{n+2s}},\end{aligned}\tag{45}$$

on the other hand, we have

$$\bigg|\frac{|x|^{\frac{k}{2}}}{(1+|x|^2)^{\frac{n-2s}{2}}}-\frac{|x|^{\frac{k}{2}}}{(1+|y|^2)^{\frac{n-2s}{2}}}\bigg|\le 2\Big(|f_k(x)-f_k(y)|^2+\bigg|\frac{|y|^{\frac{k}{2}}-|x|^{\frac{k}{2}}}{(1+|y|^2)^{\frac{n-2s}{2}}}\bigg|^2\Big)\tag{46}$$

with $f_k(t):=\frac{|t|^{\frac{k}{2}}}{(1+|t|)^{\frac{n-2s}{2}}}, t\in\mathbb{R}^n$.
As a consequence, in order to obtain the inequality (44), we will prove these two following assertions

$$\int_{|x|\le\frac{\eta}{\varepsilon}}\int_{|y|\le\frac{\eta}{\varepsilon}}\frac{|f_k(x)-f_k(y)|^2}{|x-y|^{n+2s}}dydx<+\infty,\tag{47}$$

and

$$\int_{|x|\le\frac{\eta}{\varepsilon}}\int_{|y|\le\frac{\eta}{\varepsilon}}\frac{\big||x|^{\frac{k}{2}}-|y|^{\frac{k}{2}}\big|^2}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}}dydx\le C'+C''\varepsilon^{-k+2s},\tag{48}$$

where C' and C'' are two positive constants depending on s, k and n. We start proving (47). For A a subset domain of $\mathbb{R}^n$, we define the function 1_A by $\forall x \in \mathbb{R}^n$, $1_A(x) = 0$ if $x \notin A$ and $1_A(x) = 1$ if $x \in A$.
Let $\gamma > 0$. at first we take the case where $|x - y| \geq \gamma$.

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{|x-y|\geq\gamma} \frac{|f_k(x) - f_k(y)|^2}{|x-y|^{n+2s}}dydx \\
&\leq 2\left(\int_{\mathbb{R}^n}\int_{|x-y|\geq\gamma} \frac{|x|^k}{(1+|x|^2)^{n-2s}|x-y|^{n+2s}}dydx + \int_{\mathbb{R}^n}\int_{|x-y|\geq\gamma} \frac{|y|^k}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}}\right) \\
&= 2\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|x|^k}{(1+|x|)^{n-2s}}\frac{1_{B(0,\gamma)^c}(x-y)}{|x-y|^{n+2s}}dydx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|y|^k}{(1+|y|)^{n-2s}}\frac{1_{B(0,\gamma)^c}(x-y)}{|x-y|^{n+2s}}dydx\right) \\
&= 4\int_{\mathbb{R}^n}(g_k * h)(y)dy,
\end{aligned} \tag{49}$$

where g_k and h are functions defined by, for all $t \in \mathbb{R}^n$,

$$g_k(t) := f_k^2(t) = \frac{|t|^k}{(1+|t|^2)^{n-2s}}, \text{ and } h(t) := \frac{1_{B(0,\gamma)^c}(t)}{|t|^{n+2s}}.$$

Provided that $k < n - 4s$, we have $g_k \in L^1(\mathbb{R}^n)$. Since $s > 0$, $h \in L^1(\mathbb{R}^n)$. This implies $g_k * h \in L^1(\mathbb{R}^n)$ and we have

$$||g_k * h||_{L^1(\mathbb{R}^n)} \leq ||g_k||_{L^1(\mathbb{R}^n)}||h||_{L^1(\mathbb{R}^n)} < +\infty,$$

thus,

$$\int_{\mathbb{R}^n}\int_{|x-y|\geq\gamma} \frac{|f_k(x) - f_k(y)|^2}{|x-y|^{n+2s}}dydx < +\infty. \tag{50}$$

Now, let's take the case where $|x - y| < \gamma$, we apply Taylor's formula we get

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{|x-y|<\gamma} \frac{|f_k(x) - f_k(y)|^2}{|x-y|^{n+2s}}dydx &\leq C\int_{\mathbb{R}^n}\int_{|x-y|<\gamma} \frac{|y|^k}{(1+|y|^2)^{n-2s}}\frac{1}{|x-y|^{n+2s-2}}dydx \\
&< +\infty
\end{aligned}$$

Since the convolution product of the integrable functions $g_k(x) = \frac{|x|^k}{(1+|x|^2)^{n-2s}}, x \in \mathbb{R}^n$ and $\zeta(z) := \frac{1_{B(0,\gamma)}(z)}{|z|^{n+2s-2}}, z \in \mathbb{R}^n$ is well defined and finite, we have

$$\begin{aligned}\int_{|x|\leq\frac{\eta}{\varepsilon}}\int_{|y|\leq\frac{\eta}{\varepsilon}}\frac{|f_k(x)-f_k(y)|^2}{|x-y|^{n+2s}}dydx &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_k(x)-f_k(y)|^2}{|x-y|^{n+2s}}dydx\\ &= \int_{\mathbb{R}^n}\int_{|x-y|<\gamma}\frac{|f_k(x)-f_k(y)|^2}{|x-y|^{n+2s}}dydx\\ &+ \int_{\mathbb{R}^n}\int_{|x-y|\geq\gamma}\frac{|f_k(x)-f_k(y)|^2}{|x-y|^{n+2s}}dydx\\ &\leq \tilde{C}<+\infty,\end{aligned} \tag{51}$$

where $\tilde{C}$ is a positive constant depending on s, k and the dimension n.
Now, in order to prove (48), we will need to prove some lemmas. Before that, let us explain the sketch of the proof. First we treat the integral in (48) over the set $\{x, y \in \mathbb{R}^n, |x-y| < \gamma < 1\}$, where $\gamma > 0$, and we divide this case into two cases, the first is where $|x| > R$ and $|y| > R$. For the second, we compute (48) for $|x| \leq R, |y| \leq R$. After that, we move to the second part, we treat the integral in (48) over the set $\{x, y \in \mathbb{R}^n, |x-y| > \gamma\}$.

Lemma 2.6. *Let $\gamma, R \in \mathbb{R}_+^*$ and $s \in]0,1[$. We have*
1) Let $\beta \geq 0$, $2 \leq k \leq 2\beta+2$ and let the set $\mathcal{A}$ defined by $\mathcal{A} := \{x, y \in \mathbb{R}^n, |x-y| < \gamma, |x| > R, |y| > R\}$. Then, there exists $\delta_1 := \delta_1(\gamma, R, k, \beta) > 0$ such that for all $x, y \in \mathcal{A}$ we have

$$\big||x|^{\frac{k}{2}} - |y|^{\frac{k}{2}}\big|^2 \leq \delta_1 |x-y|^2 |y|^{2\beta}. \tag{52}$$

2) Let $k \geq 2$ and let the set $\mathcal{B}$ defined by $\mathcal{B} := \{x, y \in \mathbb{R}^n, |x-y| < \gamma, |x| \leq R, |y| \leq R\}$. Then, here exists $\delta_2 := \delta_2 := \delta_2(R, k, s) > 0$, such that for every $x, y \in \mathcal{B}$, we have

$$|y|^{2-2s}\big||x|^{\frac{k}{2}} - |y|^{\frac{k}{2}}\big|^2 \leq \delta_2 |x-y|^2. \tag{53}$$

Proof : We start by proving the first assertion.
1)We introduce the function G defined for $x, y \in \mathcal{A}$ by

$$G(x,y) := \frac{\big||x|^{\frac{k}{2}} - |y|^{\frac{k}{2}}\big|^2}{|x-y|^2|y|^{2\beta}}$$

Proving (52) remains to prove that G is bounded by some $\delta_1 > 0$ which doesn't depend on x and y. If $k = 2$, we have $\big||x| - |y|\big|^2 \leq |x-y|^2$, then

$$\begin{aligned}G(x,y) &\leq \tfrac{1}{|y|^{2\beta}}\\ &\leq \tfrac{1}{R^{2\beta}}\\ &:= \delta_1\end{aligned} \tag{54}$$

Let $2 < k \leq 2\beta + 2$. Let $f(x) := |x|^{\frac{k}{2}}, x \in \mathbb{R}^n$. We have for $x \in \mathbb{R}^n$, $Df(x) = \frac{k}{2}|x|^{\frac{k}{2}-2}x$. By the inequality of finite increments applied to the function f we have for $x, y \in \mathbb{R}^n$

$$\big||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}\big| \leq \frac{k}{2} \sup_{t\in[0,1]} |y + t(x-y)|^{\frac{k}{2}-1}|x-y|, \tag{55}$$

then since $2 < k \leq 2\beta + 2$, we have for $x, y \in \mathcal{A}$

$$\begin{aligned} G(x,y) &\leq \tfrac{k^2}{4} \sup_{t\in[0,1]} \frac{|y + t(x-y)|^{k-2}}{|y|^{2\beta}} \\ &\leq \tfrac{k^2}{4} \sup_{t\in[0,1]} \left(|y|^{1-\frac{2\beta}{k-2}} + t\frac{|x-y|}{|y|^{\frac{2\beta}{k-2}}}\right)^{k-2} \\ &\leq \tfrac{k^2}{4}\left(|y|^{1-\frac{2\beta}{k-2}} + \frac{|x-y|}{|y|^{\frac{2\beta}{k-2}}}\right)^{k-2} \\ &\leq \left(\frac{1}{R^{\frac{-k+2+2\beta}{k-2}}} + \frac{\gamma}{R^{\frac{2\beta}{k-2}}}\right)^{k-2} \\ &:= \delta_1. \end{aligned} \tag{56}$$

Thus, for all $x, y \in \mathcal{A}$, G is bounded on $\mathcal{A}$ and (52) yields.
2) We proceed as previously. If $k = 2$, we have

$$\begin{aligned} |y|^{2-2s}\big||x| - |y|\big|^2 &\leq |y|^{2-2s}|x-y|^2 \\ &\leq R^{2-2s}|x-y|^2 \\ &:= \delta_2|x-y|^2. \end{aligned} \tag{57}$$

Let $k > 2$ and let $H(x,y) := |y|^{2-2s}\frac{\left||x|^{\frac{k}{2}} - |y|^{\frac{k}{2}}\right|^2}{|x-y|^2}$, $x, y \in \mathcal{B}$. Our aim is to prove that H is bounded by some δ_2 which does not depend on x and y.
By 1) we have for $x, y \in \mathbb{R}^n$

$$\big||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}\big|^2 \leq \frac{k^2}{4} \sup_{t\in[0,1]} |y + t(x-y)|^{k-2}|x-y|^2. \tag{58}$$

Since for $t \in [0,1]$, $x, y \in \mathcal{B}$, we have $|y + t(x-y)| = |tx + (1-t)y| \leq |x| + |y| \leq 2R$, then

$$\big||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}\big|^2 \leq \frac{k^2}{4}(2R)^{k-2}|x-y|^2.$$

Using the fact that $k > 2$, we have for $x, y \in \mathcal{B}$,

$$\begin{aligned} H(x,y) &\leq \tfrac{k^2}{4}(2R)^{k-2}|y|^{2-2s} \\ &\leq 2^{k-4}k^2R^{k-2s} \\ &:= \delta_2. \end{aligned} \tag{59}$$

Therefore, H is bounded over $\mathcal{B}$ and the result follows.

□

We move, now, to prove (48) and we take $|x-y|<\gamma<1$.
Let $R>0$, we have two cases
1) If $|x|>R$ and $|y|>R$, we apply the first assertion of the previous lemma for $\beta<\frac{n}{2}-2s$ and $2s<2$

$$\begin{aligned} \int_{\mathcal{A}} \frac{\left||x|^{\frac{k}{2}}-|y|^{\frac{k}{2}}\right|^2}{|x-y|^{n+2s}(1+|y|^2)^{n-2s}}dydx &\leq \delta_1 \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|y|^{2\beta}}{(1+|y|^2)^{n-2s}} \frac{1_{B(0,\gamma)}(x-y)}{|x-y|^{n+2s-2}}dxdy \\ &= \delta_1 \int_{\mathbb{R}^n} h_1 * g_1(z)dz \\ &< \infty, \end{aligned} \tag{60}$$

where h_1 and g_1 are defined by

$$h_1(z) := \frac{|z|^{2\beta}}{(1+|z|^2)^{n-2s}} \text{ and } g_1(z) := \frac{1_{B(0,\gamma)}(z)}{|z|^{n+2s-2}}. \tag{61}$$

Since $\beta<\frac{n}{2}-2s$, $h_1\in L^1(\mathbb{R}^n)$ and since $2s<2$, $g_1\in L^1(\mathbb{R}^n)$. Therefore $h_1*g_1\in L^1(\mathbb{R}^n)$.
2) If $|x|\leq R$ and $|y|\leq R$, we apply the second assertion of the previous lemma for $n\geq 4$ and $\alpha>2s$

$$\begin{aligned} \int_{\mathcal{B}} \frac{\left||x|^{\frac{k}{2}}-|y|^{\frac{k}{2}}\right|^2}{|x-y|^{n+2s}(1+|y|^2)^{n-2s}}dydx &\leq \delta_2 \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|y|^{2s-2}}{(1+|y|^2)^{n-2s}} \frac{1_{B(0,\gamma)}(x-y)}{|x-y|^{n+2s-\alpha}}dxdy \\ &= \delta_2 \int_{\mathbb{R}^n} \tilde{h}_1 * g_1(z)dz \\ &< \infty, \end{aligned} \tag{62}$$

where h_1 and g_1 are the functions defined by

$$\tilde{h}_1(z) := \frac{|z|^{2s-2}}{(1+|z|^2)^{n-2s}}, \text{ and } g_1(z) := \frac{1_{B(0,\gamma)}(z)}{|z|^{n+2s-2}}. \tag{63}$$

Since $n-6s+2>0$, $\tilde{h}_1\in L^1(\mathbb{R}^n)$ and since $2s<2$, $g_1\in L^1(\mathbb{R}^n)$. Therefore $\tilde{h}_1*g_1\in L^1(\mathbb{R}^n)$.
Finally we get

$$\int_{\mathbb{R}^n}\int_{|x-y|<\gamma} \frac{\left||x|^{\frac{k}{2}}-|y|^{\frac{k}{2}}\right|^2}{|x-y|^{n+2s}(1+|y|^2)^{n-2s}}dydx \leq \tilde{C}_0 < +\infty, \tag{64}$$

where $\tilde{C}_0$ is a positive constant depending on s,k,n. Now, to finish the proof of (48), it remains to study the integral in (48) over the set $\Gamma_\varepsilon := \{x,y\in\mathbb{R}^n \text{ such that } |x|<\frac{\eta}{\varepsilon}, |y|<\frac{\eta}{\varepsilon} \text{ and } |x-y|>\gamma\}$. Let

$$X_\varepsilon := \int_{\Gamma_\varepsilon} \frac{\left||x|^{\frac{k}{2}}-|y|^{\frac{k}{2}}\right|^2}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}}dydx. \tag{65}$$

We have

$$X_\varepsilon \leq 2\left(\int_{\Gamma_\varepsilon} \frac{|y|^k}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}} + \int_{\Gamma_\varepsilon} \frac{|x|^k}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}} dydx \right). \tag{66}$$

We will treat both of the terms of the sum (66). The first term will be treated as a convolution product of the functions h_ε and g_k, where
$h_\varepsilon(z) = \frac{1_{B(0,\frac{\eta}{\varepsilon})\setminus B(0,\gamma)}(z)}{|z|^{n+2s}}, z \in \mathbb{R}^n$ and g_k is already defined by $g_k(x) = \frac{|x|^k}{(1+|x|^2)^{n-2s}}, x \in \mathbb{R}^n$. Since $k < n-4s$, the function $g_k \in L^1(\mathbb{R}^n)$ and it's obvious that the function $h_\varepsilon \in L^1(\mathbb{R}^n)$ (as a bounded continuous function in $\mathbb{R}^n$) and we have

$$||h_\varepsilon||_{L^1(\mathbb{R}^n)} = \frac{c_n}{2s}(-\eta^{-2s}\varepsilon^{2s} + \gamma^{-2s}) \leq \frac{\gamma^{-2s}}{2s}. \tag{67}$$

So, their convolution product is integrable and by (67) we deduce

$$\int_{\Gamma_\varepsilon} \frac{|y|^k}{(1+|y|^2)^{n-2s}} \frac{1}{|x-y|^{n+2s}} dydx \leq ||g_k * h_\varepsilon||_{L^1(\mathbb{R}^n)} \leq ||g_k||_{L^1(\mathbb{R}^n)} ||h_\varepsilon||_{L^1(\mathbb{R}^n)} \leq \tilde{C}_2 < +\infty, \tag{68}$$

where $\tilde{C}_2$ is a positive constant depending on s, k and n.
Now, we treat the second term of the sum in (66).

$$\begin{aligned}
&\int_{\Gamma_\varepsilon} \frac{|x|^k}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}} 1_{B(0,\gamma)^c}(x-y) dydx \\
&\leq \int_{|x|<\frac{\eta}{\varepsilon}} \int_{\mathbb{R}^n} \frac{|x|^k}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}} 1_{B(0,\gamma)^c}(x-y) dydx \\
&\leq \int_{B(0,\frac{\eta}{\varepsilon})\setminus B(0,2)} \int_{B(0,1)} \frac{|x|^k}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}} 1_{B(0,\gamma)^c}(x-y) dydx \\
&+ \int_{B(0,2)} \int_{\mathbb{R}^n \setminus B(0,1)} \frac{|x|^k}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}} 1_{B(0,\gamma)^c}(x-y) dydx \\
&+ \int_{B(0,2)} \int_{B(0,1)} \frac{|x|^k}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}} 1_{B(0,\gamma)^c}(x-y) dydx \\
&+ \int_{B(0,\frac{\eta}{\varepsilon})\setminus B(0,2)} \int_{\mathbb{R}^n \setminus B(0,1)} \frac{|x|^k}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}} 1_{B(0,\gamma)^c}(x-y) dydx
\end{aligned} \tag{69}$$

Let's start by the first term in the sum (69). Since for $|y| \leq 1$, $\frac{1}{(1+|y|^2)^{n-2s}} \leq 1$ and $|x-y| \geq |x| - |y| \geq$

$|x|-1$. Then we write

$$\begin{aligned}
&\int_{B(0,\frac{\eta}{\varepsilon})\setminus B(0,2)}\int_{B(0,1)}\frac{|x|^k}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}}1_{B(0,\gamma)^c}(x-y)dydx\\
&\leq \int_{B(0,\frac{\eta}{\varepsilon})\setminus B(0,2)}\int_{B(0,1)}\frac{|x|^k}{|x-y|^{n+2s}}dydx\\
&\leq \int_{B(0,\frac{\eta}{\varepsilon})\setminus B(0,2)}\int_{B(0,1)}\frac{|x|^k}{(|x|-1)^{n+2s}}dydx \qquad (70)\\
&= c_n\int_2^{\frac{\eta}{\varepsilon}}\frac{t^{k+n-1}}{|t-1|^{n+2s}}dt\\
&\leq \tilde{c}\varepsilon^{-k+2s},
\end{aligned}$$

where c_n is a positive constant corresponding to the measure of the unit sphere of $\mathbb{R}^n$ and $\tilde{c}$ is a positive constant depending on s, k and n.
For the second and third term of the sum in (69), since $|x-y|\geq\gamma$ then we get

$$\begin{aligned}
&\int_{B(0,2)}\int_{\mathbb{R}^n\setminus B(0,1)}\frac{|x|^k}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}}1_{B(0,\gamma)^c}(x-y)dydx\\
&+\int_{B(0,2)}\int_{B(0,1)}\frac{|x|^k}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}}1_{B(0,\gamma)^c}(x-y)dydx \qquad (71)\\
&\leq \tilde{c}_1,
\end{aligned}$$

where $\tilde{c}_1$ is a positive constant depending on s, k and n.
For the fourth and last term of (69), we have

$$\begin{aligned}
&\int_{B(0,\frac{\eta}{\varepsilon})\setminus B(0,2)}\int_{\mathbb{R}^n\setminus B(0,1)}\frac{|x|^k}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}}1_{B(0,\gamma)^c}(x-y)dydx\\
&\leq \int_{B(0,\frac{\eta}{\varepsilon})\setminus B(0,2)}\int_{B(0,2)\setminus B(0,1)}\frac{|x|^k}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}}1_{B(0,\gamma)^c}(x-y)dydx \qquad (72)\\
&+\int_{B(0,\frac{\eta}{\varepsilon})\setminus B(0,2)}\int_{B(0,2)^c}\frac{|x|^k}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}}1_{B(0,\gamma)^c}(x-y)dydx
\end{aligned}$$

To estimate the first term of the sum on the right hand side in (72), we use the fact that if x and y are such that $|x-y|>\gamma$ and $1<|y|<2$ we have

$$\frac{|x|}{|x-y|}\leq\frac{|x-y|+|y|}{|x-y|}\leq 1+\frac{|y|}{\gamma}\leq\frac{2+\gamma}{\gamma} \qquad (73)$$

Consequently we have

$$\frac{|x|^k}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}} \leq \big(\frac{2+\gamma}{\gamma}\big)^{n+2s}|x|^{k-n-2s} \tag{74}$$

Then we get

$$\begin{aligned}&\int_{B(0,\frac{\eta}{\varepsilon})\setminus B(0,2)}\int_{B(0,2)\setminus B(0,1)}\frac{|x|^k}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}}1_{B(0,\gamma)^c}(x-y)dydx\\ &\leq C\varepsilon^{-k+2s}\end{aligned} \tag{75}$$

where C is a positive constant depending on γ, s, k and n.
For the second term of the sum in (72) and similarly to (68) we have

$$\begin{aligned}&\int_{B(0,\frac{\eta}{\varepsilon})\setminus B(0,2)}\int_{B(0,2)^c}\frac{|x|^k}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}}1_{B(0,\gamma)^c}(x-y)dydx\\ &\leq \int_{B(0,\frac{\eta}{\varepsilon})\setminus B(0,2)}\int_{\mathbb{R}^n}\frac{|x|^k}{(1+|x|^2)^{n-2s}}\frac{1_{B(0,\gamma)^c}(x-y)}{|x-y|^{n+2s}}dydx\\ &= ||f_\varepsilon * \tilde{h}||_{L^1(\mathbb{R}^n)}\\ &\leq ||f_\varepsilon||_{L^1(\mathbb{R}^n)}||\tilde{h}||_{L^1(\mathbb{R}^n)}\\ &\leq C_3,\end{aligned} \tag{76}$$

where $\forall x, y \in \mathbb{R}^n$, $f_\varepsilon(x) := \dfrac{|x|^k 1_{B(0,\frac{\eta}{\varepsilon})\setminus B(0,2)}(x)}{(1+|x|^2)^{n-2s}}$, $\tilde{h}(x) := \dfrac{1_{B(0,\gamma)^c}}{|x|^{n+2s}}$ and C_3 is a positive constant depending only on s, k and n.
Thus, (72) gives

$$\int_{B(0,\frac{\eta}{\varepsilon})\setminus B(0,2)}\int_{\mathbb{R}^n\setminus B(0,1)}\frac{|x|^k}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}}1_{B(0,\gamma)^c}(x-y)dydx \leq C_2\varepsilon^{-k+2s}+C_3. \tag{77}$$

Therefore, by (70), (71) and (77), (69) becomes

$$\int_{\Gamma_\varepsilon}\frac{|x|^k}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}}1_{B(0,\gamma)^c}(x-y)dydx \quad \leq \tilde{c}\varepsilon^{-k+2s}+\tilde{c}_1+C_2\varepsilon^{-k+2s}+C_3. \tag{78}$$

Then, by (78) and (68) we get

$$\begin{aligned}X_\varepsilon &= \int_{\Gamma_\varepsilon}\frac{\big||x|^{\frac{k}{2}}-|y|^{\frac{k}{2}}\big|^2}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}}dydx\\ &\leq 2(\tilde{C}_2+\tilde{c}\varepsilon^{-k+2s}+\tilde{c}_1+C_2\varepsilon^{-k+2s}+C_3)\\ &\leq \tilde{C}_3\varepsilon^{-k+2s}+\tilde{C}_4,\end{aligned} \tag{79}$$

with $\tilde{C}_3$ and $\tilde{C}_4$ are two positive constants depending on s, k and n.
Thus, by (64) and (79)

$$\begin{aligned}
&\int_{|x|\leq\frac{\eta}{\varepsilon}}\int_{|y|\leq\frac{\eta}{\varepsilon}}\frac{\left||x|^{\frac{k}{2}}-|y|^{\frac{k}{2}}\right|^2}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}}dydx\\
&=\int_{|x|\leq\frac{\eta}{\varepsilon}}\int_{|y|\leq\frac{\eta}{\varepsilon}}\frac{\left||x|^{\frac{k}{2}}-|y|^{\frac{k}{2}}\right|^2}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}}1_{B(0,\gamma)}(x-y)dydx\\
&+\int_{|x|\leq\frac{\eta}{\varepsilon}}\int_{|y|\leq\frac{\eta}{\varepsilon}}\frac{\left||x|^{\frac{k}{2}}-|y|^{\frac{k}{2}}\right|^2}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}}1_{B(0,\gamma)^c}(x-y)dydx\\
&\leq\int_{\mathbb{R}^n}\int_{|x-y|<\gamma}\frac{\left||x|^{\frac{k}{2}}-|y|^{\frac{k}{2}}\right|^2}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}}dydx\\
&+\int_{\Gamma_\varepsilon}\frac{\left||x|^{\frac{k}{2}}-|y|^{\frac{k}{2}}\right|^2}{(1+|y|^2)^{n-2s}|x-y|^{n+2s}}dydx\\
&\leq\tilde{C}_0+\tilde{C}_3\varepsilon^{-k+2s}+\tilde{C}_4,
\end{aligned}\tag{80}$$

thus, by taking $C'=\tilde{C}_0+\tilde{C}_4$ and $C''=\tilde{C}_3$, (48) yields.
We finally conclude the following assertion
For all $s\in]0,1[, 2\leq k<n-4s, n\in\mathbb{N}$ such that $n\geq 3$,

$$\begin{aligned}
A_{s,k,\varepsilon}&=\int_{|x|\leq\eta}|x|^k\left(\int_{|y|\leq\eta}\frac{|u_{\varepsilon,s}(x)-u_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}}dy\right)dx\\
&\leq(2\tilde{C}+C')\varepsilon^k+C''\varepsilon^{2s}\\
&=\varepsilon^k\tilde{\tilde{C}}+C''\varepsilon^{2s}\\
&\leq C\varepsilon^{2s},
\end{aligned}\tag{81}$$

where C is a positive constant depending on s, k and n. □

We move, now, to the fourth term in the right hand side of (42). By using the assertion (39), we

have

$$\begin{aligned}
&\int_{B(0,\eta)^c}\int_{B(0,\eta)^c}|x|^k\frac{|u_{\varepsilon,s}(x)-u_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}}dxdy\\
&\leq C\varepsilon^{n-2s}\int_{B(0,\eta)^c\times B(0,\eta)^c}|x|^k\frac{\min\{1,|x-y|\}}{|x-y|^{n+2s}}dydx\\
&= C\varepsilon^{n-2s}\bigg(\int_{B(0,\eta)^c\times B(0,\eta)^c}|x|^k\frac{1_{B(0,1)}(x-y)}{|x-y|^{n+2s-2}}\\
&+\int_{B(0,\eta)^c\times B(0,\eta)^c}|x|^k\frac{1_{B(0,1)^c}(x-y)}{|x-y|^{n+2s-2}}\bigg)\\
&= C\varepsilon^{n-2s}\int_{R^n}f_{1,k}*f_2(x)dx+C\varepsilon^{n-2s}\int_{R^n}f_{1,k}*f_3(x)dx,
\end{aligned}\tag{82}$$

where for all $x\in\mathbb{R}^n$,

$$f_{1,k}(x):=|x|^k1_{B(0,\eta)^c}(x),\ f_2(x):=\frac{1_{B(0,1)\cap B(0,\eta)^c}(x)}{|x|^{n+2s}}\text{ and }f_3(x):=\frac{1_{B(0,1)^c\cap B(0,\eta)^c}(x)}{|x|^{n+2s}}.\tag{83}$$

Since $f_{1,k}, f_2$ and f_3 are integrable in $\mathbb{R}^n$, then $f_{1,k}*f_2$ and $f_{1,k}*f_3$ are integrable in $\mathbb{R}^n$. Therefore

$$\int_{B(0,\eta)^c}\int_{B(0,\eta)^c}|x|^k\frac{|u_{\varepsilon,s}(x)-u_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}}dxdy=\mathcal{O}(\varepsilon^{n-2s}).\tag{84}$$

We move, now, to study the integral over the set $\mathbb{E}$. By using the assertion (38), we have

$$\begin{aligned}
&\int_{\mathbb{E}}|x|^k\frac{|u_{\varepsilon,s}(x)-u_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}}dxdy\\
&\leq C\varepsilon^{n-2s}\int_{x\in B(0,\eta),y\in B(0,\eta)^c,|x-y|\leq\frac{\eta}{2}}|x|^k\frac{|x-y|^2}{|x-y|^{n+2s}}dxdy\\
&\leq C\varepsilon^{n-2s}\eta^k\int_{x\in B(0,\eta),y\in B(0,\eta)^c,|x-y|\leq\frac{\eta}{2}}\frac{1}{|x-y|^{n+2s-2}}dxdy\\
&\leq C\varepsilon^{n-2s}\eta^k\int_{|x|\leq\eta}dx\int_{|\xi|\leq\frac{\eta}{2}}\frac{1}{|\xi|^{n+2s-2}}d\xi\\
&=\mathcal{O}(\varepsilon^{n-2s}),
\end{aligned}\tag{85}$$

as ε tends to zero. In both these estimates, we use that $s\in]0,1[$.
Now, in (42), it remains to estimate the integral on $\mathbb{D}$, that is,

$$\int_{\mathbb{D}}|x|^k\frac{|u_{\varepsilon,s}(x)-u_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}}dxdy.\tag{86}$$

For this, recalling that: $u_{\varepsilon,s}(x) = U_{\varepsilon,s}(x)$, for any $x \in B(0,\eta)$.
We note that for any $(x,y) \in \mathbb{D}$,

$$\begin{aligned} |u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)|^2 &= |U_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)|^2 \\ &= |(U_{\varepsilon,s}(x) - U_{\varepsilon,s}(y)) + (U_{\varepsilon,s}(y) - u_{\varepsilon,s}(y))|^2 \\ &\le |(U_{\varepsilon,s}(x) - U_{\varepsilon,s}(y))|^2 + |U_{\varepsilon,s}(y) - u_{\varepsilon,s}(y)|^2 \\ &\quad +2|(U_{\varepsilon,s}(x) - U_{\varepsilon,s}(y))||U_{\varepsilon,s}(y) - u_{\varepsilon,s}(y)|, \end{aligned} \tag{87}$$

so

$$\begin{aligned} &\int_{\mathbb{D}} |x|^k \frac{|u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}} dxdy \\ &\le \int_{\mathbb{D}} |x|^k \frac{|(U_{\varepsilon,s}(x) - U_{\varepsilon,s}(y))|^2}{|x-y|^{n+2s}} dxdy \\ &+ \int_{\mathbb{D}} |x|^k \frac{|U_{\varepsilon,s}(y) - u_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}} dxdy \\ &+2 \int_{\mathbb{D}} |x|^k \frac{|U_{\varepsilon,s}(x) - U_{\varepsilon,s}(y)||U_{\varepsilon,s}(y) - u_{\varepsilon,s}(y)|}{|x-y|^{n+2s}} dxdy. \end{aligned} \tag{88}$$

Hence, to estimate (86), we bound the three terms on the right hand side of (88). By exploiting (37) (here used with $\rho = \eta$), we obtain

$$\begin{aligned} \int_{\mathbb{D}} |x|^k \frac{|U_{\varepsilon,s}(y) - u_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}} dxdy &\le \int_{\mathbb{D}} |x|^k \frac{(|U_{\varepsilon,s}(y)| + |u_{\varepsilon,s}(y)|)^2}{|x-y|^{n+2s}} dxdy \\ &\le 4 \int_{\mathbb{D}} |x|^k \frac{|U_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}} dxdy \\ &\le C\varepsilon^{n-2s} \int_{x\in B(0,\eta), y\in B(0,\eta)^c, |x-y|>\frac{\eta}{2}} \frac{|x|^k}{|x-y|^{n+2s}} dxdy \\ &\le C\eta^k \varepsilon^{n-2s} \int_{x\in B(0,\eta), y\in B(0,\eta)^c, |x-y|>\frac{\eta}{2}} \frac{1}{|x-y|^{n+2s}} dxdy \\ &\le C\eta^k \varepsilon^{n-2s} \int_{|\zeta-a|\le\eta} d\zeta \int_{|\xi|>\frac{\eta}{2}} \frac{1}{|\xi|^{n+2s}} d\xi \\ &= \mathcal{O}(\varepsilon^{n-2s}), \end{aligned} \tag{89}$$

as ε tends to zero.

We estimate now the last term on the right-hand side of (88).

$$\begin{aligned} &\int_{\mathbb{D}} |x|^k \frac{|U_{\varepsilon,s}(x) - U_{\varepsilon,s}(y)||U_{\varepsilon,s}(y) - u_{\varepsilon,s}(y)|}{|x-y|^{n+2s}} dxdy \\ &\le \int_{\mathbb{D}} |x|^k \frac{|U_{\varepsilon,s}(x)||U_{\varepsilon,s}(y) - u_{\varepsilon,s}(y)|}{|x-y|^{n+2s}} dxdy + \int_{\mathbb{D}} |x|^k \frac{|U_{\varepsilon,s}(y)||U_{\varepsilon,s}(y) - u_{\varepsilon,s}(y)|}{|x-y|^{n+2s}} dxdy. \end{aligned} \tag{90}$$

We increase the two terms of the sum separately each one

$$\begin{aligned}\int_{\mathbb{D}} |x|^k \frac{|U_{\varepsilon,s}(y)||U_{\varepsilon,s}(y)-u_{\varepsilon,s}(y)|}{|x-y|^{n+2s}}dxdy &\leq 2\int_{\mathbb{D}} |x|^k \frac{|U_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}}dxdy \\ &\leq C\varepsilon^{n-2s}\int_{x\in B(0,\eta),y\in B(0,\eta)^c,|x-y|>\frac{\eta}{2}} \frac{|x|^k}{|x-y|^{n+2s}} \\ &\leq C\varepsilon^{n-2s}\eta^k \int_{x\in B(0,\eta),y\in B(0,\eta)^c,|x-y|>\frac{\eta}{2}} \frac{1}{|x-y|^{n+2s}}dxdy \\ &= \mathcal{O}(\varepsilon^{n-2s}),\end{aligned} \tag{91}$$

as ε tends to zero, and we have by [5]

$$\begin{aligned}\int_{\mathbb{D}} |x|^k \frac{|U_{\varepsilon,s}(x)||U_{\varepsilon,s}(y)-u_{\varepsilon,s}(y)|}{|x-y|^{n+2s}}dxdy &\leq \int_{\mathbb{D}} |x|^k \frac{|U_{\varepsilon,s}(x)|(|U_{\varepsilon,s}(y)|)+(|u_{\varepsilon,s}(y)|)}{|x-y|^{n+2s}}dxdy \\ &\leq 2\int_{\mathbb{D}} |x|^k \frac{|U_{\varepsilon,s}(x)||U_{\varepsilon,s}(y)|}{|x-y|^{n+2s}}dxdy \\ &= \mathcal{O}(\varepsilon^{n-2s}).\end{aligned} \tag{92}$$

Finally we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x|^k \frac{|u_{\varepsilon,s}(x)-u_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}}dxdy \leq \kappa C\varepsilon^{2s} + 2\int_{\mathbb{D}} |x|^k \frac{|U_{\varepsilon,s}(x)-U_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}}dxdy + \mathcal{O}(\varepsilon^{n-2s}). \tag{93}$$

Therefore, we write

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} p(x)\frac{|u_{\varepsilon,s}(x)-u_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}}dxdy \leq p_0 S_s + \kappa C\varepsilon^{2s} + \mathcal{O}(\varepsilon^{n-2s}) + 2\int_{\mathbb{D}} |x|^k \frac{|U_{\varepsilon,s}(x)-U_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}}dxdy.$$

Now, we prove that the last term on the right hand-side of the previous inequality is $\mathcal{O}(\varepsilon^{k+2s}) + \mathcal{O}(\varepsilon^{n-2s})$.
Note that for $(x,y)\in\mathbb{D}$ we have $x\in B(0,\eta)$ and $y\in B(0,\eta)^c$

$$\begin{aligned}\int_{\mathbb{D}} |x|^k \frac{|U_{\varepsilon,s}(x)-U_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}}dxdy &\leq \int_{B(0,\eta)} |x|^k \int_{B(0,\eta)^c} \frac{|U_{\varepsilon,s}(x)|^2+|U_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}} 1_{B(0,\frac{\eta}{2})^c(x-y)}dxdy \\ &\leq \int_{B(0,\eta)} |x|^k \int_{B(0,\eta)^c} \frac{|U_{\varepsilon,s}(x)|^2}{|x-y|^{n+2s}} 1_{B(0,\frac{\eta}{2})^c(x-y)}dxdy \\ &+C\int_{B(0,\eta)} |x|^k \int_{B(0,\eta)^c} |U_{\varepsilon,s}(y)|^2 1_{B(0,\frac{\eta}{2})^c(x-y)}dxdy.\end{aligned}$$

Thanks to (37), the second term on the right hand-side is bounded by $C\varepsilon^{n-2s}$.

So, it remains to prove that

$$\int_{B(0,\eta)} |x|^k \int_{B(0,\eta)^c} \frac{|U_{\varepsilon,s}(x)|^2}{|x-y|^{n+2s}} 1_{B(0,\frac{\eta}{2})^c}(x-y)dxdy = \mathcal{O}(\varepsilon^{k+2s}).$$

In fact, let $g_{1,k,\varepsilon}$ and h be two functions such that, for all $x \in \mathbb{R}^n$, $g_{1,k,\varepsilon}(x) := |x|^k |U_{\varepsilon,s}(x)|^2$ and for all $z \in \mathbb{R}^n$, $h(z) := \frac{1_{B(0,\frac{\eta}{2})^c}(z)}{|z|^{n+2s}}$. We have $h \in L^1(\mathbb{R}^n)$ since $2s > 0$.
Let $x \in \mathbb{R}^n$,

$$\begin{aligned} ||g_{1,k,\varepsilon}||_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |x|^k |U_{\varepsilon,s}(x)|^2 dx \\ &= \varepsilon^{n-2s} \int_{\mathbb{R}^n} |x|^k \frac{\Psi(x)}{(\varepsilon^2+|x|^2)^{n-2s}} dx \\ &\leq \varepsilon^{n-2s} \int_{\mathbb{R}^n} \frac{|x|^k}{(\varepsilon^2+|x|^2)^{n-2s}} dx \\ &= \varepsilon^{-n+2s} \int_{\mathbb{R}^n} \frac{|x|^k}{\big(1+\frac{|x|^2}{\varepsilon^2}\big)^{n-2s}} dx \\ &= \varepsilon^{k+2s} \int_{\mathbb{R}^n} \frac{|z|^k}{(1+|z|^2)^{n-2s}} dz. \end{aligned} \tag{94}$$

Since $k < n-4s$, the integral $\int_{\mathbb{R}^n} \frac{|z|^k}{(1+|z|^2)^{n-2s}} dz$ is finite. Therefore, $||g_{1,k,\varepsilon}||_1 = \mathcal{O}(\varepsilon^{k+2s})$ and we have

$$\begin{aligned} \int_{B(0,\eta)} |x|^k \int_{B(0,\eta)^c} \frac{|U_{\varepsilon,s}(x)|^2}{|x-y|^{n+2s}} 1_{B(0,\frac{\eta}{2})^c}(x-y)dxdy &= ||g*h||_{L^1(\mathbb{R}^n)} \\ &\leq ||g_{1,k,\varepsilon}||_{L^1} ||h||_{L^1} \\ &= \mathcal{O}(\varepsilon^{k+2s}), \end{aligned} \tag{95}$$

for $2 \leq k < n-4s$, which completes the proof of Theorem2.4. □

Finally, we conclude the proof of Theorem1.2. Thanksby to Theorem2.4 and (15) we write the inequality satisfied by the energy

$$\begin{aligned} S_{s,\lambda}(p) &\leq E_\lambda(u_{\varepsilon,s,a}) \\ &\leq p_0 S_s + \kappa C \varepsilon^{2s} - \lambda \varepsilon^{2s} + \mathcal{O}(\varepsilon^{n-2s}) + \mathcal{O}(\varepsilon^{k+2s}) \\ &= p_0 S_s + \varepsilon^{2s}\big(-\lambda + \kappa C + \mathcal{O}(\varepsilon^{n-4s}) + \mathcal{O}(\varepsilon^k)\big). \end{aligned} \tag{96}$$

Since $k \in [2, n-4s[$, let us distinguish two subcases:

If $n = 3$ and $s \in]0, \frac{1}{4}]$ or $n = 4$ and $s \in]0, \frac{1}{2}]$ or $n = 5$ and $s \in]0, \frac{3}{4}]$ or if $n \geq 6$ and $s \in]0, 1[$ there exists a constant $C = C(n,s,k) > 0$ such that for every $\kappa \in]0, C\lambda[$ we have $S_{s,\lambda}(p) \leq E_\lambda(u_{\varepsilon,s,a}) < p_0 S_s$.

In all the cases above $S_{s,\lambda}(p)$ is achieved for $0 < \lambda < \lambda_{1,p,s}$ and for every $\kappa \in]0, \frac{1}{C}\lambda[$ thus the Theorem 1.2 follows at once.

We end this section by stating two remarks, the first concerns the limiting case of k:

Remark 2.7. *We notice that when $k = 2s$, in (96), we obtain $E_\lambda(u_{\varepsilon,s,a}) \leq p_0 S_s + \varepsilon^{2s}(-\lambda + C')$. So that, if $C' < \lambda_{1,p,s}$ and $\lambda \in]C', \lambda_{1,p,s}[$, then $E_\lambda(u_{\varepsilon,s,a}) < p_0 S_s$ and $S_{s,\lambda}(p)$ is achieved.*

The second ensure the existence of non negative solution for our minimization problem:

Remark 2.8. *As a consequence of (2.8) and the proof of Theorem 1.2 we obtain actually that $S_{s,\lambda}(p)$ possesses a positive solution.*

In the next section we will deal with a non minimization problem.

3 Existence of non ground state solutions

It's obvious to see that equation (2) is the Euler-Lagrange equation of the functional $\Phi_{p,s} : \mathbb{H}_0^s(\Omega) \to \mathbb{R}$ defined as

$$\Phi_{p,s}(u) := \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} p(x)\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy - \frac{\lambda}{q}\int_\Omega |u(x)|^q dx - \frac{1}{q_s}\int_\Omega |u(x)|^{q_s}dx. \tag{97}$$

Notice that the functional $\Phi_{p,s}$ does not satisfy the Palais-Smale condition globally and this is due to the lack of compactness of the embedding $\mathbb{H}_0^s(\Omega)$ into $L^{q_s}(\Omega)$. As a consequence, an estimate of the critical level of $\Phi_{p,s}$ is necessary and as we will see in what follows, we cannot apply the classical Mountain Pass Theorem, that's why we will apply a variant of the Mountain Pass Theorem without the Palais-Smale condition, as given in ([9],Theorem 2.2).
Thanks to the fact that the embedding $\mathbb{H}_0^s(\Omega)$ into $L^r(\Omega)$ is compact for $r \in [1, q_s[$ and it is continuous for $r = q_s$ with Ω a domain with continuous boundary, the energy functional $\Phi_{p,s}$ is well defined.
Moreover, $\Phi_{p,s}$ is Fréchet differentiable in $u \in \mathbb{H}_0^s(\Omega)$, and, for any $\varphi \in \mathbb{H}_0^s(\Omega)$,

$$\begin{aligned} <\Phi'_{p,s}(u), \varphi> &= \int_{\mathbb{R}^n\times\mathbb{R}^n} p(x)\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy \\ &\quad -\lambda\int_\Omega |u(x)|^{q-2}u(x)\varphi(x)dx - \int_\Omega |u(x)|^{q_s-2}u(x)\varphi(x)dx. \end{aligned} \tag{98}$$

Notice that critical points of $\Phi_{p,s}$ are solutions to problem (2). In order to find these critical points, we proceed as follows; first we start proving that $\Phi_{p,s}$ has a suitable geometric sturcture as stated in conditions (2.9) and (2.10) of ([9],Theorem 2.2).

Proposition 3.1. *There exist $\lambda_0 > 0$, $\rho > 0$ and $\beta > 0$ such that, for all $\lambda \in]0, \lambda_0[$ $u \in \mathbb{H}_0^s(\Omega)$ with $\|u\|_{\mathbb{H}_0^s(\Omega)} = \rho$, we have $\Phi_{p,s}(u) \geq \beta$.*

Proof : Let $u \in \mathbb{H}_0^s(\Omega)$ such that $\|u\|_{\mathbb{H}_0^s(\Omega)} \leq 1$. Thanks to the continuous Sobolev embeddings $\mathbb{H}_0^s(\Omega)$ into $L^r(\Omega)$, where $1 \leq r \leq q_s$, we have

$$\begin{aligned} \Phi_{p,s}(u) &= \tfrac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} p(x)\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy - \frac{\lambda}{q}\int_\Omega |u(x)|^q dx - \frac{1}{q_s}\int_\Omega |u(x)|^{q_s}dx \\ &\geq \tfrac{1}{2}p_0\|u\|^2_{\mathbb{H}_0^s(\Omega)} - \tfrac{\lambda}{q}\alpha_q^{-\frac{q}{2}}\|u\|^q_{\mathbb{H}_0^s(\Omega)} - \tfrac{1}{q_s}S_s^{-\frac{q_s}{2}}\|u\|^{q_s}_{\mathbb{H}_0^s(\Omega)} \\ &\geq (\tfrac{1}{2}p_0 - \tfrac{\lambda}{q}\alpha_q^{-\frac{q}{2}})\|u\|^2_{\mathbb{H}_0^s(\Omega)} - \tfrac{1}{q_s}S_s^{-\frac{q_s}{2}}\|u\|^{q_s}_{\mathbb{H}_0^s(\Omega)}, \end{aligned} \tag{99}$$

where α_q is the best Sobolev constant of the embedding $\mathbb{H}_0^s(\Omega)$ into $L^q(\Omega)$.
Let $\lambda_0 := \frac{1}{2}qp_0\alpha_q^{\frac{q}{2}}$. Hence, it easily follows that for $\lambda \in]0, \lambda_0[$,

$$\Phi_{p,s}(u) \geq \tilde{\alpha}_1 \|u\|^2_{\mathbb{H}_0^s(\Omega)} \big(1 - \tilde{\alpha}_2 \|u\|^{q_s-2}_{\mathbb{H}_0^s(\Omega)}\big), \tag{100}$$

for suitable positive constants $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$. Now let $u \in \|u\|_{\mathbb{H}_0^s(\Omega)}$ be such that $\|u\|_{\mathbb{H}_0^s(\Omega)} = \rho > 0$. Since $q_s > 2$, we choose ρ sufficiently small in order to get $1 - \tilde{\alpha}_2\rho^{q_s-2} > 0$ and by defining $\beta := \tilde{\alpha}_1\rho^2(1 - \tilde{\alpha}_2\rho^{q_s-2}) > 0$, we obtain $\Phi_{p,s}(u) \geq \beta > 0$ which finishes the proof of Proposition 3.1. □

Proposition 3.2. *Let $\lambda \in]0, \lambda_0[$. There exists $e \in \mathbb{H}_0^s(\Omega)$ such that $e \geq 0$ a.e in $\mathbb{R}^n$, $\|e\|_{\mathbb{H}_0^s(\Omega)} > \rho$, and $\Phi_{p,s}(e) < \beta$, where ρ and λ_0 are given in Proposition 3.1.*

Proof : Let u be a fixed function in $\mathbb{H}_0^s(\Omega)$, such that $\|u\|_{\mathbb{H}_0^s(\Omega)} = 1$ and $u \geq 0$ a.e in $\mathbb{R}^n$; we remark that this choice is possible. In fact, we replace any $u \in \mathbb{H}_0^s(\Omega)$ by its positive part since, if $u \in \mathbb{H}_0^s(\Omega)$, $u^+ \in \mathbb{H}_0^s(\Omega)$ where $u^+(x) := max\{u(x), 0\}$.
Let $\zeta > 0$, since p is bounded, there exists a positive constant M such that for all $x \in \mathbb{R}^n$, $p(x) \leq M$, and we have

$$\begin{aligned}
\Phi_{p,s}(\zeta u) &= \frac{\zeta^2}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} p(x) \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dxdy \\
&\quad - \frac{\lambda}{q}\zeta^q \int_\Omega |u(x)|^q dx - \frac{\zeta^{q_s}}{q_s} \int_\Omega |u(x)|^{q_s} dx \\
&\leq \frac{\zeta^2}{2} M \|u\|^2_{\mathbb{H}_0^s(\Omega)} - \frac{\lambda}{q}\zeta^q \int_\Omega |u(x)|^q dx - \frac{\zeta^{q_s}}{q_s} \int_\Omega |u(x)|^{q_s} dx \\
&= \frac{M\zeta^2}{2} - \frac{\lambda}{q}\zeta^q \int_\Omega |u(x)|^q dx - \frac{\zeta^{q_s}}{q_s} \int_\Omega |u(x)|^{q_s} dx.
\end{aligned} \tag{101}$$

Since $\frac{\lambda}{q}\zeta^q \int_\Omega |u(x)|^q dx \geq 0$ and $1 < 2 < q_s$, by passing to the limit $\zeta \to +\infty$ and for u fixed, we get

$$\Phi_{p,s}(\zeta u) \to -\infty,$$

so we take $e = \zeta u$, with ζ large enough so that we get $\Phi_{p,s}(\zeta u) < 0$ and the assertion yields since $\beta > 0$. □

Proposition 3.3. *Let $n \geq 3$ and $s \in]0, 1[$. Let*

$$c := \inf_{P \in \mathcal{P}} \sup_{v \in P([0,1])} \Phi_{p,s}(v), \tag{102}$$

where

$$\mathcal{P} := \{P \in C([0, 1]; \mathbb{H}_0^s(\Omega)) : P(0) = 0, P(1) = e\}, \tag{103}$$

with e given in Proposition 3.2. Then $\beta \leq c < \frac{s}{n}(p_0 S_s)^{\frac{n}{2s}}$, where β given in Proposition 3.1.

Proof : For all $P \in \mathcal{P}$, the function $t \mapsto \|P(t)\|_{\mathbb{H}^s_0(\Omega)}$ is continuous in $[0,1]$, $\|P(0)\|_{\mathbb{H}^s_0(\Omega)} = \|0\|_{\mathbb{H}^s_0(\Omega)} = 0 < \rho$ and $\|P(1)\|_{\mathbb{H}^s_0(\Omega)} = \|e\|_{\mathbb{H}^s_0(\Omega)} > \rho$, therefore, there must be a real $\bar{t} \in]0,1[$ such that $\|P(\bar{t})\| = \rho$, with ρ given in Proposition 3.1. Then, by (99) and taking $P(\bar{t})$ as a function test

$$\max_{v \in P([0,1])} \Phi_{p,s}(v) \geq \Phi_{p,s}(P(\bar{t})) \geq \inf_{\substack{v \in \mathbb{H}^s_0(\Omega) \\ \|v\|_{\mathbb{H}^s_0(\Omega)} = \rho}} \Phi_{p,s}(v) \geq \beta \tag{104}$$

Finally, we obtain $c \geq \beta$. Now Let us prove that $c < \frac{s}{n}(p_0 S_s)^{\frac{n}{2s}}$.
Let $v_{\varepsilon,s,a} := \frac{u_{\varepsilon,s,a}}{||u_{\varepsilon,s,a}||_{L^{q_s}(\mathbb{R}^n)}}$ where $u_{\varepsilon,s,a}$ is defined in (13).

Let us consider $\tilde{X}_\varepsilon := \displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} p(x)\frac{|v_{\varepsilon,s,a}(x) - v_{\varepsilon,s,a}(y)|^2}{|x-y|^{n+2s}}dydx$, then we write

$$\Phi_{p,s}(tv_{\varepsilon,s,a}) = \frac{1}{2}t^2\tilde{X}_\varepsilon - \frac{t^{q_s}}{q_s} - \frac{t^q}{q}\lambda\int_\Omega |v_{\varepsilon,s,a}|^q dx \leq \frac{1}{2}t^2\tilde{X}_\varepsilon - \frac{t^{q_s}}{q_s}$$

,
therefore $\lim\limits_{t\to+\infty} \Phi_{p,s}(tv_{\varepsilon,s,a}) = -\infty$ and $\sup\limits_{t\geq 0} \Phi_{p,s}(tv_{\varepsilon,s,a})$ is achieved at some $t_\varepsilon \geq 0$. If $t_\varepsilon = 0$, then $\sup\limits_{t\geq 0} \Phi_{p,s}(tv_{\varepsilon,s,a}) = 0$ and there is nothing to prove. So we assume that $t_\varepsilon > 0$. Since $\frac{\partial \Phi_{p,s}}{\partial t}(t_\varepsilon v_{\varepsilon,s,a}) = 0$, we have

$$t_\varepsilon\tilde{X}_\varepsilon - t_\varepsilon^{q_s-1} - \lambda t_\varepsilon^{q-1}\int_\Omega |v_{\varepsilon,s}(x)|^q dx = 0, \tag{105}$$

and so

$$t_\varepsilon \leq \tilde{X}_\varepsilon^{\frac{1}{q_s-2}}. \tag{106}$$

We set

$$Y_\varepsilon := \sup_{t\geq 0} \Phi_{p,s}(tv_{\varepsilon,s,a}) = \Phi_{p,s}(t_\varepsilon v_{\varepsilon,s,a}). \tag{107}$$

The function $t \mapsto (\frac{1}{2}t^2\tilde{X}_\varepsilon - \frac{t^{q_s}}{q_s})$ is increasing on the interval $[0, \tilde{X}_\varepsilon^{\frac{1}{q_s-2}}]$, we have, by (106),

$$\begin{aligned}
Y_\varepsilon &= \frac{1}{2}t_\varepsilon^2\tilde{X}_\varepsilon - \frac{t_\varepsilon^{q_s}}{q_s} - \frac{\lambda}{q}t_\varepsilon^q\int_\Omega |v_{\varepsilon,s,a}(x)|^q dx \\
&\leq \frac{1}{2}\tilde{X}_\varepsilon^{\frac{2}{q_s-2}}\tilde{X}_\varepsilon - \frac{\tilde{X}_\varepsilon^{\frac{q_s}{q_s-2}}}{q_s} - \frac{\lambda}{q}t_\varepsilon^q\int_\Omega |v_{\varepsilon,s,a}(x)|^q dx \\
&= \frac{1}{2}\tilde{X}_\varepsilon^{\frac{n}{2s}} - \frac{\tilde{X}_\varepsilon^{\frac{n}{2s}}}{q_s} - \frac{\lambda}{q}t_\varepsilon^q\int_\Omega |v_{\varepsilon,s,a}(x)|^q dx \\
&= \frac{s}{n}\tilde{X}_\varepsilon^{\frac{n}{2s}} - \frac{\lambda}{q}t_\varepsilon^q\int_\Omega |v_{\varepsilon,s,a}(x)|^q dx
\end{aligned} \tag{108}$$

We claim that

$$t_\varepsilon \to (p_0 S_s)^{\frac{1}{q_s-2}}, \tag{109}$$

as $\varepsilon \to 0$.
In fact, by (105) we have

$$\tilde{X}_\varepsilon - t_\varepsilon^{q_s-2} - \lambda t_\varepsilon^{q-2} \int_\Omega |v_{\varepsilon,s,a}(x)|^q dx = 0. \tag{110}$$

Since $q < q_s$ using (110) we get

$$\tilde{X}_\varepsilon - t_\varepsilon^{q_s-2} + o(t_\varepsilon^{q-2}) = 0.$$

which implies the claim (109).

Using Theorem 2.4 together with (17) and (109) we have

$$\begin{aligned}
Y_\varepsilon &\leq \frac{s}{n}\tilde{X}_\varepsilon^{\frac{n}{2s}} - \frac{\lambda}{q} t_\varepsilon^q \int_\Omega |v_{\varepsilon,s,a}(x)|^q dx \\
&\leq \frac{s}{n}\left((p_0 S_s)^{\frac{n}{2s}} + \frac{n}{2s}(p_0 S_s)^{\frac{n}{2s}-1}\kappa C \varepsilon^{2s} + \mathcal{O}(\varepsilon^{n-2s}) + \mathcal{O}(\varepsilon^{k+2s}) \right) \\
&\quad -\frac{\lambda}{q}\left((p_0 S_s)^{\frac{1}{q_s-2}} + o(1) \right)^q \left(\tilde{K}_{s,q}\varepsilon^{n-\frac{q(n-2s)}{2}} + \mathcal{O}(\varepsilon^{\frac{q(n-2s)}{2}}) \right) \\
&\leq \frac{s}{n}(p_0 S_s)^{\frac{n}{2s}} + \frac{1}{2}\kappa C (p_0 S_s)^{\frac{n}{2s}-1}\varepsilon^{2s} + \mathcal{O}(\varepsilon^{n-2s}) + o(\varepsilon^{2s}) \\
&\quad -\tfrac{\lambda}{q}(p_0 S_s)^{\frac{q}{q_s-2}}(1+o(1))\left(\tilde{K}_{s,q}\varepsilon^{n-\frac{q(n-2s)}{2}} + \mathcal{O}(\varepsilon^{\frac{q(n-2s)}{2}}) \right) \\
&\leq \frac{s}{n}(p_0 S_s)^{\frac{n}{2s}} + \frac{1}{2}\kappa C (p_0 S_s)^{\frac{n}{2s}-1}\varepsilon^{2s} + \mathcal{O}(\varepsilon^{n-2s}) + o(\varepsilon^{2s}) \\
&\quad -\tfrac{\lambda}{q}(p_0 S_s)^{\frac{q}{q_s-2}}\tilde{K}_{s,q}\varepsilon^{n-\frac{q(n-2s)}{2}} + o(\varepsilon^{n-\frac{q(n-2s)}{2}}).
\end{aligned} \tag{111}$$

therefore, if $n \geq 4$ we obtain for $\varepsilon > 0$ sufficiently small

$$Y_\varepsilon < \frac{s}{n} p_0^{\frac{n}{2s}} S_s^{\frac{n}{2s}}, \tag{112}$$

When $n = 3$, using the calculation in 111 and 17 it results that for $Y_\varepsilon < \frac{s}{3} p_0^{\frac{3}{2s}} S_s^{\frac{3}{2s}}$ for $s \in]0, \frac{3}{4}[$. hence the proof is complete. □

3.1 Proof of Theorem 1.3

Proposition 3.1 and 3.2 give that the geometry of the variant of the Mountain Pass Theorem given in ([9], Theorem 2.2) is fulfilled by $\Phi_{p,s}$. Furthermore, we easily have $\Phi_{p,s}(0) = 0 < \beta$, with β given in Proposition 3.1 and by Proposition 3.3, we deduce that all hypotheses of ([9], Theorem 2.2) are satisfied, then, there is a sequence $(u_j)_{j\in\mathbb{N}}$ in $\mathbb{H}_0^s(\Omega)$ such that

$$\Phi_{p,s}(u_j) \to c \tag{113}$$

and

$$sup\big\{| < \Phi'_{p,s}(u_j), \varphi > | : \varphi \in \mathbb{H}^s_0(\Omega), \|\varphi\|_{\mathbb{H}^s_0(\Omega)} = 1\big\} \to 0 \tag{114}$$

as j tends to $+\infty$.
In order to finish the proof of Theorem 1.3 we proceed as in [9] and ([5], chapter 14) and we need to establish the following result

Lemma 3.4. *The sequence $(u_j)_{j\in\mathbb{N}}$ is bounded in $\mathbb{H}^s_0(\Omega)$.*

Proof : For any $j \in \mathbb{N}$, by (113) and (114), it is easily follows that there exists $k > 0$ such that

$$|\Phi_{p,s}(u_j)| \leq k \tag{115}$$

and

$$| < \Phi'_{p,s}(u_j), \frac{u_j}{\|u_j\|_{\mathbb{H}^s_0(\Omega)}} > | \leq k \tag{116}$$

By (115) and (116), we have

$$\Phi_{p,s}(u_j) - \frac{1}{2} < \Phi'_{p,s}(u_j), u_j > \leq k(1 + \|u_j\|_{\mathbb{H}^s_0(\Omega)}) \tag{117}$$

Moreover, we have

$$\begin{aligned} \Phi_{p,s}(u_j) - \tfrac{1}{2} < \Phi'_{p,s}(u_j), u_j > &= (-\tfrac{1}{q_s} + \tfrac{1}{2})\|u_j\|^{q_s}_{L^{q_s}(\Omega)} + \lambda(-\tfrac{1}{q} + \tfrac{1}{2})\|u_j\|^q_{L^q(\Omega)} \\ &= \tfrac{s}{n}\|u_j\|^{q_s}_{L^{q_s}(\Omega)} + \lambda(-\tfrac{1}{q} + \tfrac{1}{2})\|u_j\|^q_{L^q(\Omega)}, \end{aligned} \tag{118}$$

then, by this and (117), we get that, for any $j \in \mathbb{N}$,

$$\|u_j\|^{q_s}_{L^{q_s}(\Omega)} \leq k_*(1 + \|u_j\|_{\mathbb{H}^s_0(\Omega)}), \tag{119}$$

for a convenient positive constant k_*.
Therefore, as a consequence of (115) and as in the proof of Proposition 3.1, we have

$$k \geq \phi_{p,s}(u_j) \geq \frac{1}{2}p_0\|u\|^2_{\mathbb{H}^s_0(\Omega)} - \frac{\lambda}{2}\|u\|^q_{L^q(\Omega)} - \frac{1}{q_s}\|u\|^{q_s}_{L^{q_s}(\Omega)}, \tag{120}$$

then combining this with (119), we get that, for any $j \in \mathbb{N}$,

$$\|u\|^2_{\mathbb{H}^s_0(\Omega)} \leq \bar{k}(1 + \|u\|^2_{\mathbb{H}^s_0(\Omega)}), \tag{121}$$

where $\bar{k}$ is a convenient positive constant. So that $(u_j)_{j\in\mathbb{N}}$ is bounded in $\mathbb{H}^s_0(\Omega)$. □

Lemma 3.5. *Problem* (2) *admits a non trivial solution $u_\infty \in \mathbb{H}^s_0(\Omega)$.*

Proof : Since the sequence $(u_j)_{j\in\mathbb{N}}$ is bounded, and recalling that the fractional Sobolev space $\mathbb{H}^s_0(\Omega)$ is a reflexive space, extract a subsequence, still denoted by u_j, so that

$$\begin{aligned} &u_j \rightharpoonup u \text{ weakly in } \mathbb{H}^s_0(\Omega), \\ &u_j \rightharpoonup u \text{ weakly in } L^{q_s}(\Omega), \\ &u_j \to u \text{ strongly in } L^q(\Omega) \text{ for any } q \in [1, q_s[, \\ &u_j \to u \text{ a.e. on } \Omega. \end{aligned}$$

By taking into account the scalar product defined in (28), for any $\varphi \in \mathbb{H}_0^s(\Omega)$ we have

$$\int_{\mathbb{R}^{2n}} p(x)\frac{(u_j(x)-u_j(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy = \int_{\mathbb{R}^{2n}} p(x)\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy + o(1),$$

as j tends to $+\infty$. We have also
$|u_j|^{q_s-2}u_j \to |u|^{q_s-2}u$ weakly in $L^{\frac{q_s}{q_s-1}}(\Omega)$ as j tends to $+\infty$, and $|u_j|^{q-2}u_j \to |u|^{q-2}u$ strongly in $L^{\frac{q}{q-1}}(\Omega)$ as j tends to $+\infty$.
Moreover, for any $\varphi \in \mathbb{H}_0^s(\Omega)$, we have

$$\begin{aligned} < \Phi'_{p,s}(u_j), \varphi > \; &= \int_{\mathbb{R}^n\times\mathbb{R}^n} p(x)\frac{(u_j(x)-u_j(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy \\ &-\lambda\int_\Omega |u_j(x)|^{q-2}u(x)\varphi(x)dx - \int_\Omega |u_j(x)|^{q_s-2}u(x)\varphi(x)dx \end{aligned} \tag{122}$$

By (114), $< \Phi'_{p,s}(u_j), \varphi >$ tends to zero as j tends to $+\infty$. Thus, passing to the limit as j tends to $+\infty$ in (122), u satisfies

$$\begin{aligned} &\int_{\mathbb{R}^n\times\mathbb{R}^n} p(x)\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy \\ &-\lambda\int_\Omega |u(x)|^{q-2}u(x)\varphi(x)dx - \int_\Omega |u(x)|^{q_s-2}u(x)\varphi(x)dx = 0, \end{aligned} \tag{123}$$

for any $\varphi \in \mathbb{H}_0^s(\Omega)$; that is, u is a solution of problem 2, and the second point follows.
Now let us prove that the solution u_∞ is not zero. Let us suppose by contradiction that $u \equiv 0$ in Ω. Since $u \in \mathbb{H}_0^s(\Omega)$ $u \equiv 0$ in $\mathbb{R}^n$. We know that $(u_j)_{j\in\mathbb{N}}$ is bounded in $\mathbb{H}_0^s(\Omega)$, and

$$\begin{aligned} < \Phi'_{p,s}(u_j), u_j > \; &= \int_{\mathbb{R}^n\times\mathbb{R}^n} p(x)\frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}dxdy \\ &-\lambda\int_\Omega |u_j(x)|^q dx - \int_\Omega |u_j(x)|^{q_s} dx, \end{aligned} \tag{124}$$

Since (114) holds true for any $\varphi \in \mathbb{H}_0^s(\Omega)$, $< \Phi'_{p,s}(u_j), u_j >$ tends to zero as j tends to $+\infty$. We have also $\int_\Omega |u_j(x)|^q dx \to 0$ as j tends to $+\infty$. Thus, we write

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} p(x)\frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}dxdy - \int_\Omega |u_j(x)|^{q_s}dx \to 0, \text{ as } j \to +\infty. \tag{125}$$

We know that the sequence $(\|u_j\|_{\mathbb{H}_0^s(\Omega)})_{j\in\mathbb{N}}$ is bounded in $\mathbb{R}$.
Hence, the sequence $\left(\int_{\mathbb{R}^n\times\mathbb{R}^n} p(x)\frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}dxdy\right)_{j\in\mathbb{N}}$ is also bounded in $\mathbb{R}$ since the weight p is bounded. Extract a subsequence, we assume that

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} p(x)\frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}dxdy \to l, \tag{126}$$

as j tends to $+\infty$, and by (125) we easily deduce that

$$\int_\Omega |u_j(x)|^{q_s}dx \to l, \tag{127}$$

as j tends to $+\infty$.
Moreover, by (113), we have

$$\tfrac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} p(x)\frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}dxdy - \frac{\lambda}{q}\int_\Omega |u_j(x)|^q dx - \frac{1}{q_s}\int_\Omega |u_j(x)|^{q_s}dx \to c, \tag{128}$$

as j tends to $+\infty$. Since $\displaystyle\int_\Omega |u_j(x)|^q dx \to 0$ as j tends to $+\infty$, using (126) and (127), we deduce that

$$c = (\frac{1}{2} - \frac{1}{q_s})l = \frac{s}{n}l \tag{129}$$

Using Proposition 3.3 $c \geq \beta > 0$, it is obvious that $l > 0$. Furthermore, by definition of S_s, we have

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} p(x)\frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}dxdy \geq p_0 S_s \|u_j\|^2_{L^{q_s}(\Omega)}. \tag{130}$$

Passing to the limit as j tends to $+\infty$ and combining with (126) and (127), we get

$$l \geq p_0 S_s l^{\frac{2}{q_s}}, \tag{131}$$

and taking into account (129), we have

$$c \geq \frac{s}{n} p_0^{\frac{n}{2s}} S_s^{\frac{n}{2s}}, \tag{132}$$

a contradiction to the fact that $c < \frac{s}{n} p_0^{\frac{n}{2s}} S_s^{\frac{n}{2s}}$. Thus u is not trivial and the proof of Theorem 1.3 is complete.

□

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