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### ► **To cite this version:**

Geoffrey Beck, Charles-Edouard Bréhier, Laurent Chevillard, Isabelle Gallagher, Ricardo Grande, et al.. Transport equation in Fourier and dynamical turbulent fields. Waves 2024, 16th International Conference on Mathematical and Numerical Aspects of Wave Propagation, 2024, Berlin, Germany. <hal-04534268>

**HAL Id: hal-04534268**

**<https://hal.science/hal-04534268v1>**

Submitted on 23 Feb 2025

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## Transport equation in Fourier and dynamical turbulent fields

Geoffrey Beck<sup>1,\*</sup>, Charles-Edouard Bréhier<sup>2</sup>, Laurent Chevillard<sup>3</sup>, Isabelle Gallagher<sup>4</sup>,  
Ricardo Grande<sup>5</sup>, Wandrille Ruffenach<sup>3</sup>

<sup>1</sup>Univ Rennes, IRMAR UMR 6625 & Centre Inria de l'Université de Rennes (MINGuS) & ENS  
Rennes, France

<sup>2</sup>Université de Pau et des Pays de l'Adour, E2S UPPA, CNRS, LMAP, Pau, France

<sup>3</sup>Univ Lyon, Ens de Lyon, Univ Claude Bernard, CNRS, Laboratoire de Physique, 46 allée d'Italie  
F-69342 Lyon, France

<sup>4</sup>DMA - Département de Mathématiques et Applications - ENS Paris & UP Cité - Université Paris  
Cité, France

<sup>5</sup>School for Advanced Studies (SISSA), via Bonomea 265, 34136 Trieste, Italy

\*Email: geoffrey.a.beck@inria.fr

### Abstract

Turbulent cascades characterize the transfer of energy injected by a random force at large scales towards the small scales. We construct a linear equation that mimics the phenomenology of energy cascades when the external force is a statistically homogeneous and stationary stochastic process. In the Fourier variable, this equation can be seen as a wave equation, which corresponds to a wave operator of degree 0 in physical space. Our results give a complete characterization of the solution: it is smooth at any finite time, and, up to smaller order corrections, it converges to a fractional Gaussian field at infinite time. We apply a finite volume method in the Fourier variables formulation in order to reach the invariant measure of the equation.

**Keywords:** Stochastic PDE, Turbulence, Fractional Gaussian fields, Non-self-adjoint Operator, Finite Volume

### 1 Phenomenology of turbulence

We are interested in modelling and studying the behaviour of a fully turbulent fluid which moves freely and where no particular direction is predominant. It is possible to measure the streamwise velocity of a turbulent fluid. This recorded velocity looks very rough. Formulating a precise prediction about these recorded signals is a difficult task, due to the chaotic nature of the signal, and to the many scales that make it up. Nevertheless, a statistical description of such signals is possible, and many predictions regarding its behaviour exist dating back to Kolmogorov. When the Reynolds number is large, a 3D fluid that is stirred by a statistically sta-

tionary force will eventually reach a statistically stationary state in which the velocity variance is finite. In order to dissipate the energy that is constantly injected into the system, the velocity field of the fluid will develop a complex multiscale structure ending up with high values of the velocity gradients. In other words, the fluid has transferred the energy pumped by the forcing at large scales towards small scales (and gets rough). These well-established experimental facts are known as energy cascade.

### 2 A transport equation in Fourier space

This aforementioned cascade phenomenon has no rigorous mathematical explanations in the case of the Navier-Stokes equations. Turbulence is usually associated with a nonlinear equation. For example, in the case of wave turbulence, nonlinearities create wave interactions which allow the transfer of energy to higher and higher wavenumbers. Surprisingly, nonlinearities might not be the only way in which such loss of regularity can occur. Indeed, in [3], the authors have shown that, in the context of internal waves, a loss of regularity can also take place in the case of a linear equation with an operator of degree 0. In [1], we have constructed a stochastic linear partial differential equation that mimics the phenomenology of energy cascades when the random external force  $f$  is a statistically homogeneous, isotropic and stationary real-valued Gaussian field. We moreover assume that  $\mathbb{E}[f(t, x)] = 0$  and

$$\mathbb{E}[f(t, x)f(t', x')] = \delta(t - t') C_f(x - x'),$$

where the notation  $\delta(t)$  stands for the Dirac distribution, and the correlation function  $C_f$  is

a smooth real-valued function. In the Fourier variable  $k$ , the equation of [1] can be seen as a linear transport equation:

$$\begin{cases} \partial_t \hat{u} + \operatorname{div}_k \left( \frac{k}{|k|} \hat{u} \right) + \frac{(H+\frac{1}{2})}{|k|} \hat{u} = \hat{f}, & |k| > \kappa, \\ \hat{u}(t, k) = 0 & |k| \leq \kappa, \\ \hat{u}(t = 0, k) = 0. \end{cases} \quad (1)$$

where  $\hat{\cdot}$  stands for the spatial Fourier-transform,  $\kappa > 0$  corresponds to the inverse of the integral length scale and  $H \in (0, 1)$  is the Hurst-index.

### 3 Dynamic in the physical space governed by an operator of degree 0

The solution of the previous equation is the Fourier transform of the solution of  $\partial_t u + Au = f$ , where the unbounded operator  $A : D(A) \rightarrow X$  is defined by

$$A : u \mapsto \mathcal{F}^{-1} \left[ \operatorname{div}_k \left( \frac{k}{|k|} \hat{u} \right) + \frac{H + \frac{1}{2}}{|k|} \hat{u} \right],$$

with  $\mathcal{F}^{-1}$  denoting the inverse Fourier transform, with the domain

$$D(A) := \{u \in X \mid \partial_{|k|} \hat{u} \in L^2(\mathbb{R}_k^d \setminus B(0, \kappa)) \text{ and } \hat{u}|_{|k|=\kappa} = 0\},$$

and  $X$  stands for the following Hilbert space:

$$X := \{u \in L^2(\mathbb{R}_x^d) \mid \hat{u}(k) = 0 \text{ for all } |k| < \kappa\}.$$

The operator  $A$  is the sum of a differential operator of degree 0 plus a regularizing operator. One can show that the operator  $A$  is not self-adjoint and have no discrete spectrum. Our results give a complete characterization of the solution: it is smooth at any finite time, and, up to smaller order corrections, it converges in law to a fractional Gaussian field at infinite time. Moreover, this limiting field is Holder-continuous. This loss of regularity at infinite time is what is expected in linear turbulence.

### 4 Numerical scheme

It is worth mentioning that we are mostly interested in the statistical behavior of the model, in the large time regime: it is therefore sufficient to propose numerical algorithms which are efficient for the approximation in distribution of the stochastic processes (instead of their trajectories). The numerical simulation of the stochastic evolution equation (1) requires to solve several non-trivial issues. Indeed, our objective

is to study the long-time behavior of the system, in order to observe the power-law behavior of the generated fractional random fields. We thus need to ensure first that the numerical approximation reaches a stationary state, which depends on the chosen numerical scheme and on the numerical discretization parameters, and second, that this stationary state is an accurate approximation of the stationary state of the system (1). Even for deterministic dynamical systems, crude methods may fail and the design of effective methods is a non trivial task. The addition of stochastic external forcing naturally introduces additional difficulties, see for instance [2]. In this work, the fact that the external forcing is  $\delta$ -correlated in time  $t$  and in the Fourier variable  $k$  is an important challenge. We have first discretized in the Fourier domain using the finite volume method and we then describe the temporal discretization using a splitting method and exponential integrators for Ornstein-Uhlenbeck dynamics. We conduct numerical simulations in one, two and three spatial dimensions.

### Acknowledgments

G. Beck is partly supported by the BOURGEONS project, grant ANR-23-CE40-0014-01 of the French National Research Agency (ANR). C-E. Bréhier is partially supported by the project SIMALIN (ANR-19-CE40-0016) operated by the French National Research Agency. L. Chevillard is partly supported by the Simons Foundation Award number 1151711.

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