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# THE SET OF ELEMENTARY TENSORS IS WEAKLY CLOSED IN PROJECTIVE TENSOR PRODUCTS.

COLIN PETITJEAN

ABSTRACT. In this short note, we prove that the set of elementary tensors is weakly closed in the projective tensor product of two Banach spaces. As a result, we are able to answer a question from the literature proving that if  $(x_n) \subset X$  and  $(y_n) \subset Y$  are two weakly null sequences such that  $(x_n \otimes y_n)$  converges weakly in  $X \widehat{\otimes}_\pi Y$ , then  $(x_n \otimes y_n)$  is also weakly null.

## 1. WEAK CONVERGENCE IN PROJECTIVE TENSOR PRODUCT

Let  $X$ ,  $Y$  and  $Z$  be real Banach spaces. We denote  $\mathcal{B}(X \times Y, Z)$  the space of continuous bilinear operators from  $X \times Y$  into  $Z$ . If  $Z = \mathbb{R}$ , we simply write  $\mathcal{B}(X \times Y)$ . For  $x \in X$  and  $y \in Y$ , define the *elementary tensor*  $x \otimes y \in \mathcal{B}(X \times Y)^*$  by:

$$\forall B \in \mathcal{B}(X \times Y), \quad \langle x \otimes y, B \rangle = B(x, y).$$

We then introduce  $X \otimes Y := \text{span}\{x \otimes y : x \in X, y \in Y\}$ . Recall that the norm on  $\mathcal{B}(X \times Y)$  is defined by  $\|B\|_{\mathcal{B}(X \times Y)} = \sup_{x \in B_X, y \in B_Y} |B(x, y)|$ . Let  $\|\cdot\|_\pi$  be the dual norm of  $\|\cdot\|_{\mathcal{B}(X \times Y)}$ . It is well known (see e.g. [1, Proposition VIII. 9. a)]) that if  $u \in X \otimes Y$  then

$$\|u\|_\pi = \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

The *projective tensor product* of  $X$  and  $Y$  is defined as follows:

$$X \widehat{\otimes}_\pi Y = \overline{\text{span}}^{\|\cdot\|_\pi} \{x \otimes y : x \in X, y \in Y\} \subseteq \mathcal{B}(X \times Y)^*.$$

As a consequence of the fundamental linearisation property of tensor products, one easily deduce the following isometric identification  $(X \widehat{\otimes}_\pi Y)^* \equiv \mathcal{B}(X \times Y)$ . Since  $\mathcal{B}(X \times Y) \equiv \mathcal{L}(X, Y^*)$ , where  $\mathcal{L}(X, Y^*)$  stands for the space of bounded linear operators from  $X$  to  $Y^*$ , one also has that  $\mathcal{L}(X, Y^*) \equiv (X \widehat{\otimes}_\pi Y)^*$ .

The aim of this short note is to answer Question 3.9 in [5]:

*Let  $X$  and  $Y$  be Banach spaces. Let  $(x_n)_{n \in \mathbb{N}}$  and  $(y_n)_{n \in \mathbb{N}}$  be weakly null sequences in  $X$  and  $Y$ , respectively, such that  $(x_n \otimes y_n)_{n \in \mathbb{N}}$  is weakly convergent in  $X \widehat{\otimes}_\pi Y$ .*

*Is  $(x_n \otimes y_n)_{n \in \mathbb{N}}$  weakly null in  $X \widehat{\otimes}_\pi Y$ ?*

Let

$$(1) \quad \mathcal{T} = \{x \otimes y : x \in X, y \in Y\}$$

be the set of elementary tensors in  $X \widehat{\otimes}_\pi Y$ . We shall start with a simple but key observation. Recall that a Banach space  $X$  has the approximation property (AP

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in short) if for every  $\varepsilon > 0$ , for every compact subset  $K \subset X$ , there exists a finite rank operator  $T \in \mathcal{L}(X, X)$  such that  $\|Tx - x\| \leq \varepsilon$  for every  $x \in K$ .

**Lemma 1.1.** *Let  $X, Y$  be Banach spaces such that  $X$  or  $Y$  has the AP. Let  $T \in X \widehat{\otimes}_\pi Y$ . Then  $T \in \mathcal{T}$  if and only if for every linearly independent families  $\{x_1^*, x_2^*\} \subset X^*$  and  $\{y_1^*, y_2^*\} \subset Y^*$  we have:*

$$(\star) \quad \begin{vmatrix} \langle T, x_1^* \otimes y_1^* \rangle & \langle T, x_1^* \otimes y_2^* \rangle \\ \langle T, x_2^* \otimes y_1^* \rangle & \langle T, x_2^* \otimes y_2^* \rangle \end{vmatrix} = 0.$$

*Proof.* Thanks to [6, Proposition 2.8], every  $T \in X \widehat{\otimes}_\pi Y$  can be written as

$$T = \sum_{n=1}^{\infty} x_n \otimes y_n$$

with  $\sum_{n=1}^{\infty} \|x_n\| \|y_n\| \leq 2\|T\|$ . Moreover, the linear map  $\Phi : X \widehat{\otimes}_\pi Y \rightarrow \mathcal{L}(X^*, Y)$  obtained by

$$\forall x^* \in X^*, \quad \Phi\left(\sum_{n=1}^{\infty} x_n \otimes y_n\right)(x^*) = \sum_{n=1}^{\infty} x^*(x_n) y_n$$

defines a bounded operator. Since  $X$  or  $Y$  has the AP,  $\Phi$  is moreover injective; see [6, Proposition 4.6].

If  $T = x \otimes y \in \mathcal{T}$ , then it is straightforward to check that condition  $(\star)$  is verified:

$$\begin{vmatrix} \langle T, x_1^* \otimes y_1^* \rangle & \langle T, x_1^* \otimes y_2^* \rangle \\ \langle T, x_2^* \otimes y_1^* \rangle & \langle T, x_2^* \otimes y_2^* \rangle \end{vmatrix} = \begin{vmatrix} x_1^*(x)y_1^*(y) & x_1^*(x)y_2^*(y) \\ x_2^*(x)y_1^*(y) & x_2^*(x)y_2^*(y) \end{vmatrix} = 0.$$

Assume now that  $T \notin \mathcal{T}$ . Then  $\Phi(T)$  is an operator of rank greater than 2 in  $\mathcal{L}(X^*, Y)$ . Thus, there exists a linearly independent family  $\{x_1^*, x_2^*\} \subset X^*$  such that  $\Phi(T)(x_1^*) \neq 0$ ,  $\Phi(T)(x_2^*) \neq 0$  and  $\{\Phi(T)(x_1^*), \Phi(T)(x_2^*)\} \subset Y$  is a linearly independent family. To finish the proof, simply pick a linearly independent family  $\{y_1^*, y_2^*\} \subset Y^*$  satisfying:

$$\begin{aligned} \langle \Phi(T)(x_1^*), y_1^* \rangle &\neq 0 & \langle \Phi(T)(x_1^*), y_2^* \rangle &= 0 \\ \langle \Phi(T)(x_2^*), y_1^* \rangle &= 0 & \langle \Phi(T)(x_2^*), y_2^* \rangle &\neq 0. \end{aligned}$$

□

**Proposition 1.2.** *Let  $X, Y$  be two Banach spaces such that  $X$  or  $Y$  has the AP. Then the set of elementary tensors  $\mathcal{T}$  is weakly closed in  $X \widehat{\otimes}_\pi Y$ .*

*Proof.* We let  $I$  be the set of all vectors  $(x_1^*, x_2^*, y_1^*, y_2^*)$  such that  $\{x_1^*, x_2^*\} \subset X^*$  and  $\{y_1^*, y_2^*\} \subset Y^*$  are both linearly independent families. Next, for every  $T \in X \widehat{\otimes}_\pi Y$  and  $S = (x_1^*, x_2^*, y_1^*, y_2^*) \in I$ , we define

$$D_S(T) = \begin{vmatrix} \langle T, x_1^* \otimes y_1^* \rangle & \langle T, x_1^* \otimes y_2^* \rangle \\ \langle T, x_2^* \otimes y_1^* \rangle & \langle T, x_2^* \otimes y_2^* \rangle \end{vmatrix}.$$

The result now directly follows from Lemma 1.1 together with the fact that  $D_S$  is continuous with respect to the weak topology. Indeed, one can write  $\mathcal{T}$  as an intersection of weakly closed sets:

$$\mathcal{T} = \bigcap_{S \in I} D_S^{-1}(\{0\}).$$

□

The next corollary answers [5, Question 3.9] positively under rather general assumptions.

**Corollary 1.3.** *Let  $X$  and  $Y$  be Banach spaces such that  $X$  or  $Y$  has the AP. If  $(x_n)_{n \in \mathbb{N}} \subset X$  converges weakly to  $x$ ,  $(y_n)_{n \in \mathbb{N}} \subset Y$  converges weakly to  $y$ , and  $(x_n \otimes y_n)_{n \in \mathbb{N}}$  is weakly convergent in  $X \widehat{\otimes}_\pi Y$ , then  $(x_n \otimes y_n)_{n \in \mathbb{N}}$  converges weakly to  $x \otimes y$ .*

Before proving this corollary, let us point out that the canonical basis  $(e_n)_{n \in \mathbb{N}}$  of  $\ell_2$  shows that if  $(x_n)_{n \in \mathbb{N}} \subset X$  and  $(y_n)_{n \in \mathbb{N}} \subset Y$  are weakly null sequences, the sequence  $(x_n \otimes y_n)_{n \in \mathbb{N}}$  may fail to be weakly null in  $X \widehat{\otimes}_\pi Y$ . Indeed,  $(e_n \otimes e_n)_{n \in \mathbb{N}}$  is isometric to the  $\ell_1$ -canonical basis; see [6, Example 2.10].

*Proof.* Assume first that  $(x_n)_{n \in \mathbb{N}} \subset X$  and  $(y_n)_{n \in \mathbb{N}} \subset Y$  are weakly null sequences such that  $(x_n \otimes y_n)_{n \in \mathbb{N}}$  is weakly convergent in  $X \widehat{\otimes}_\pi Y$ . Since  $\mathcal{T}$  is weakly closed, there exists  $x \in X$  and  $y \in Y$  such that  $x_n \otimes y_n \rightarrow x \otimes y$  in the weak topology. Arguing by contradiction, suppose that  $x \otimes y \neq 0$ . Pick  $x^* \in X^*$  and  $y^* \in Y^*$  such that  $x^*(x) = \|x\| \neq 0$  and  $y^*(y) = \|y\| \neq 0$ . On the one hand,  $x_n \otimes y_n \rightarrow x \otimes y$  weakly, so that

$$\langle x^* \otimes y^*, x_n \otimes y_n \rangle \rightarrow \langle x^* \otimes y^*, x \otimes y \rangle = x^*(x)y^*(y) = \|x\|\|y\| \neq 0.$$

On the other hand, since  $(x_n)_{n \in \mathbb{N}}$  and  $(y_n)_{n \in \mathbb{N}}$  are weakly null, one readily obtains a contradiction:

$$\langle x^* \otimes y^*, x_n \otimes y_n \rangle = x^*(x_n)y^*(y_n) \rightarrow 0.$$

Similarly, if  $(x_n)_{n \in \mathbb{N}} \subset X$  converges weakly to  $x$ ,  $(y_n)_{n \in \mathbb{N}} \subset Y$  converges weakly to  $y$ , and  $(x_n \otimes y_n)_{n \in \mathbb{N}}$  is weakly convergent in  $X \widehat{\otimes}_\pi Y$ , then we write:

$$(x - x_n) \otimes (y - y_n) = x \otimes y - x \otimes y_n - x_n \otimes y + x_n \otimes y_n.$$

But,  $x \otimes y_n \xrightarrow[n \rightarrow +\infty]{w} x \otimes y$  and  $x_n \otimes y \xrightarrow[n \rightarrow +\infty]{w} x \otimes y$ . Therefore  $((x - x_n) \otimes (y - y_n))_{n \in \mathbb{N}}$  converges weakly and moreover the weak limit must be 0 thanks to the first part of the proof. This implies that  $x_n \otimes y_n \xrightarrow[n \rightarrow +\infty]{w} x \otimes y$ .  $\square$

In connection with Proposition 1.2, we also wish to mention [2, Theorem 2.3] which we describe now. If  $C$  and  $D$  are subsets of  $X$  and  $Y$  respectively, then let

$$C \otimes D := \{x \otimes y : x \in C, y \in D\} \subset \mathcal{T}.$$

Theorem 2.3 in [2] states that if  $C$  and  $D$  are bounded then  $\overline{C}^w \otimes \overline{D}^w = \overline{C \otimes D}^w$  in  $X \widehat{\otimes}_\pi Y$ . The technique which we introduced in the present note permits to remove the boundedness assumption in the particular case when  $C$  and  $D$  are subspaces. It also allows us to slightly simplify the original proof of [2, Theorem 2.3]. The next lemma is the main ingredient.

**Lemma 1.4.** *Let  $X$  and  $Y$  be Banach spaces such that  $X$  or  $Y$  has the AP. Let  $(x_s)_s \subset X$  and  $(y_s)_s \subset Y$  be two nets such that  $x_s \rightarrow x^{**}$  in the weak\*-topology of  $X^{**}$ ,  $y_s \rightarrow y^{**}$  in the weak\*-topology of  $Y^{**}$ , and  $(x_s \otimes y_s)_s$  converges in the weak\*-topology of  $(X \widehat{\otimes}_\pi Y)^{**}$ . Then  $(x_s \otimes y_s)_s$  converges weakly\* to  $x^{**} \otimes y^{**}$ .*

The proof is essentially the same as that of Corollary 1.3, so we leave the details to the reader.

**Corollary 1.5.** *Let  $X$  and  $Y$  be Banach spaces such that  $X$  or  $Y$  has the AP. If  $C$  and  $D$  are subsets of  $X$  and  $Y$  respectively, then  $\overline{C}^w \otimes \overline{D}^w = \overline{C \otimes D}^w$  if one of the following additional assumptions are satisfied:*

- (i) *If  $C$  and  $D$  are subspaces.*
- (ii) *If  $C$  and  $D$  are bounded.*

*Proof.* First of all, it is readily seen that one has  $\overline{C}^w \otimes \overline{D}^w \subset \overline{C \otimes D}^w$  without any additional assumption on  $C$  and  $D$  (see the first part of the proof of [2, Theorem 2.3]). Therefore we only have to prove the reverse inclusion in both cases.

To prove (i), it suffices to apply Proposition 1.2:

$$C \otimes D \subset \overline{C} \otimes \overline{D} \implies \overline{C \otimes D}^w \subset \overline{\overline{C} \otimes \overline{D}}^w = \overline{C} \otimes \overline{D}.$$

To prove (ii), let  $z \in \overline{C \otimes D}^w$ . We fix a net  $(x_s \otimes y_s)_s \subset C \otimes D$  which converges weakly to  $z$ . Thanks to Proposition 1.2, there exist  $x \in X$  and  $y \in Y$  such that  $z = x \otimes y$ . Since  $C$  and  $D$  are bounded, up to taking a suitable subnet, we may assume that both  $x_s \rightarrow x^{**}$  in the weak\*-topology of  $X^{**}$  and  $y_s \rightarrow y^{**}$  in the weak\*-topology of  $Y^{**}$ . Thanks to Lemma 1.4,  $x_s \otimes y_s \rightarrow x^{**} \otimes y^{**}$  in the weak\*-topology of  $(X \widehat{\otimes}_\pi Y)^{**}$ . By uniqueness of the limit,  $x^{**} \otimes y^{**} = z = x \otimes y$ . We distinguish two cases.

If  $z = 0$  then  $x^{**} = 0$  or  $y^{**} = 0$ . Say  $x^{**} = 0$  for instance. This means that  $0 \in \overline{C}^w$ . Now pick any  $y \in C$  and observe that  $z = 0 \otimes y$ , which was to be shown.

If  $z \neq 0$ , then it is readily seen that  $x^{**} \in \text{span}\{x\}$  and  $y^{**} \in \text{span}\{y\}$ . Therefore  $x^{**} \in \overline{C}^{w*} \cap X = \overline{C}^w$  and  $y^{**} \in \overline{D}^{w*} \cap Y = \overline{D}^w$ , which concludes the proof.  $\square$

## 2. APPLICATIONS TO VECTOR-VALUED LIPSCHITZ FREE SPACES

If  $M$  is a pointed metric space, with base point  $0 \in M$ , and if  $X$  is a real Banach space, then  $\text{Lip}_0(M, X)$  stands for the vector space of all Lipschitz maps from  $M$  to  $X$  which satisfy  $f(0) = 0$ . Equipped with the Lipschitz norm:

$$\forall f \in \text{Lip}_0(M, X), \quad \|f\|_L = \sup_{x \neq y \in M} \frac{\|f(x) - f(y)\|_X}{d(x, y)},$$

$\text{Lip}_0(M, X)$  naturally becomes a Banach space. When  $X = \mathbb{R}$ , it is customary to omit the reference to  $X$ , that is  $\text{Lip}_0(M) := \text{Lip}_0(M, \mathbb{R})$ . Next, for  $x \in M$ , we let  $\delta(x) \in \text{Lip}_0(M)^*$  be the evaluation functional defined by  $\langle \delta(x), f \rangle = f(x)$ ,  $\forall f \in \text{Lip}_0(M)$ . The Lipschitz free space over  $M$  is the Banach space

$$\mathcal{F}(M) := \overline{\text{span}}^{\|\cdot\|} \{\delta(x) : x \in M\} \subset \text{Lip}_0(M)^*.$$

The universal extension property of Lipschitz free spaces states that for every  $f \in \text{Lip}_0(M, X)$ , there exists a unique continuous linear operator  $\overline{f} \in \mathcal{L}(\mathcal{F}(M), X)$  such that:

- (i)  $f = \overline{f} \circ \delta$ , and
- (ii)  $\|\overline{f}\|_{\mathcal{L}(\mathcal{F}(M), X)} = \|f\|_L$ .

In particular, the next isometric identification holds:

$$\text{Lip}_0(M, X) \cong \mathcal{L}(\mathcal{F}(M), X).$$

A direct application (in the case  $X = \mathbb{R}$ ) provides another basic yet important information:

$$\text{Lip}_0(M) \cong \mathcal{F}(M)^*.$$

It also follows from basic tensor product theory that  $\text{Lip}_0(M, X^*) \equiv (\mathcal{F}(M) \widehat{\otimes}_\pi X)^*$ , which leads to the next definition (see [4] for more details):

**Definition 2.1** (Vector-valued Lipschitz free spaces). Let  $M$  be a pointed metric space and let  $X$  be a Banach space. We define the  $X$ -valued Lipschitz free space over  $M$  to be:  $\mathcal{F}(M, X) := \mathcal{F}(M) \widehat{\otimes}_\pi X$ .

**2.1. Weak closure of  $\delta(M, X)$ .** It is proved in [3, Proposition 2.9] that  $\delta(M) = \{\delta(x) : x \in M\}$  is weakly closed in  $\mathcal{F}(M)$  provided that  $M$  is complete. Our first aim is to prove the vector-valued counterpart. For this purpose, we need to identify a set that corresponds to  $\delta(M)$  in the vector-valued case. A legitimate set to look at is the following:

$$\delta(M, X) := \{\delta(y) \otimes x : y \in M, x \in X\} \subset \mathcal{F}(M, X).$$

Notice that this does not exactly correspond to  $\delta(M)$  in the case  $X = \mathbb{R}$  since we have  $\delta(M, \mathbb{R}) = \mathbb{R} \cdot \delta(M)$ . This discrepancy is not a major issue since  $\mathbb{R} \cdot \delta(M)$  is also a weakly closed set when  $M$  is complete. The next result is thus a natural extension to the vector valued setting of [3, Proposition 2.9].

**Proposition 2.2.** *Let  $M$  be a complete pointed metric space and  $X$  be a Banach space such that  $\mathcal{F}(M)$  or  $X$  have the approximation property. Then  $\delta(M, X)$  is weakly closed in  $\mathcal{F}(M, X)$ .*

*Proof.* In what follows,  $\mathcal{T}$  denotes the elementary tensors in  $\mathcal{F}(M) \widehat{\otimes}_\pi X$ . Consider a net  $(\delta(m_\alpha) \otimes x_\alpha)_\alpha \subset \delta(M, X)$  which is weakly convergent. Since  $\delta(M, X) \subset \mathcal{T}$  and  $\mathcal{T}$  is weakly closed (Proposition 1.2), there exist  $\gamma \in \mathcal{F}(M)$  and  $x \in X$  such that the net goes to  $\gamma \otimes x$  in the weak topology. We may assume that  $x \neq 0$ , otherwise there is nothing to do. Pick  $x^* \in X^*$  such that  $x^*(x) \neq 0$ . Then, for every  $f \in \text{Lip}_0(M)$ , we have that  $f(m_\alpha)x^*(x_\alpha) \rightarrow f(\gamma)x^*(x)$ . So the net  $(\frac{x^*(x_\alpha)}{x^*(x)}\delta(m_\alpha))_\alpha \subset \mathbb{R} \cdot \delta(M)$  weakly converges to  $\gamma$ . Since  $\mathbb{R} \cdot \delta(M)$  is weakly closed, there is  $\lambda \in \mathbb{R}$  and  $m \in M$  such that  $\gamma = \lambda\delta(m)$ . Consequently  $\gamma \otimes x = \delta(m) \otimes \lambda x \in \delta(M, X)$ .  $\square$

**2.2. Natural preduals.** Next, following [3, Section 3],  $S \subset \text{Lip}_0(M)$  is a natural predual of  $\mathcal{F}(M)$  if  $S^* \equiv \mathcal{F}(M)$  and  $\delta(B(0, r))$  is  $\sigma(\mathcal{F}(M), S)$ -closed for every  $r \geq 0$ . A reasonable extension of this notion in the vector-valued setting is the following.

**Definition 2.3.** Let  $M$  be a pointed metric space and  $X$  be a Banach space with  $\dim(X) \geq 2$ . We say that a Banach space  $S$  is a natural predual of  $\mathcal{F}(M, X^*)$  if  $Y^* \equiv \mathcal{F}(M, X^*)$  and

$$\delta(B(0, r), X^*) = \{\delta(m) \otimes x^* : m \in B(0, r), x^* \in X^*\} \subset \mathcal{F}(M, X^*)$$

is  $\sigma(\mathcal{F}(M, X^*), S)$ -closed for every  $r \geq 0$ .

Notice again that  $\delta(B(0, r), \mathbb{R}) = \mathbb{R} \cdot \delta(B(0, r))$ . In the next statement,  $\text{lip}_0(M)$  denotes the subspace of  $\text{Lip}_0(M)$  of all uniformly locally flat functions. Recall that  $f \in \text{Lip}_0(M)$  is uniformly locally flat if

$$\lim_{d(x,y) \rightarrow 0} \frac{|f(x) - f(y)|}{d(x,y)} = 0.$$

**Lemma 2.4.** *Let  $M$  be a separable pointed metric space. Suppose that  $S \subset \text{lip}_0(M)$  is a natural predual of  $\mathcal{F}(M)$ . Then, for every  $r \geq 0$ ,  $\mathbb{R} \cdot \delta(B(0, r))$  is weak\* closed in  $\mathcal{F}(M)$ .*

*Proof.* Let us fix  $r \geq 0$ . Let  $(\lambda_n \delta(x_n))_n \subset \mathbb{R} \cdot \delta(B(0, r))$  be a sequence converging to some  $\gamma \in \mathcal{F}(M)$  in the weak\* topology. We assume that  $\gamma \neq 0$ , otherwise there is nothing to do. Since a weak\* convergent sequence is bounded, and by weak\* lower-semi-continuity of the norm, we may assume that there exists  $C > 0$  such that for every  $n$ :

$$0 < \frac{\|\gamma\|}{2} \leq |\lambda_n| \|\delta(x_n)\| = |\lambda_n| d(x_n, 0) \leq C.$$

Thus,  $d(x_n, 0) \neq 0$  and  $\lambda_n \neq 0$  for every  $n$ . Up to extracting a further subsequence, we may assume that the sequence  $(\lambda_n d(x_n, 0))_n$  converges to some  $\ell \neq 0$ . Since  $(x_n)_n \subset B(0, r)$ , we also assume that  $(d(x_n, 0))_n$  converges to some  $d$ . We will distinguish two cases.

If  $d \neq 0$ , then  $(\lambda_n)_n$  converges to  $\lambda := \frac{\ell}{d}$  and so  $(\delta(x_n))_n$  weak\* converges to  $\frac{\gamma}{\lambda}$ . Since  $S$  is a natural predual of  $\mathcal{F}(M)$ ,  $\delta(B(0, r))$  is weak\* closed in  $\mathcal{F}(M)$ . So there exists  $x \in M$  such that  $\gamma = \lambda \delta(x)$ .

If  $d = 0$ , then  $(\delta(x_n))_n$  converges to 0 in the norm topology (and  $(\lambda_n)_n$  tends to infinity). Note that we may write:

$$\lambda_n \delta(x_n) = \lambda_n d(x_n, 0) \frac{\delta(x_n) - \delta(0)}{d(x_n, 0)}.$$

Since  $S \subset \text{lip}_0(M)$ , the sequence  $(\frac{\delta(x_n) - \delta(0)}{d(x_n, 0)})_n$  weak\* converges to 0. Moreover the sequence  $(\lambda_n d(x_n, 0))_n$  converges to  $\ell \neq 0$ . Consequently  $(\lambda_n \delta(x_n))_n$  weak\* converges to 0 and so  $\gamma = 0$ , which is a contradiction.  $\square$

Before going further, we need to introduce the injective tensor product of two Banach spaces. Recall that, to define the projective tensor product, we introduced  $x \otimes y$  as an element of  $\mathcal{B}(X \times Y)^*$ . For the injective tensor product, we change the point of view since we now consider  $x \otimes y$  as an element of  $\mathcal{B}(X^* \times Y^*)$  defined as follows:

$$\forall (x^*, y^*) \in X^* \times Y^*, \quad \langle x \otimes y, (x^*, y^*) \rangle = x^*(x) y^*(y).$$

In this case, we denote  $\|\cdot\|_\varepsilon$  the canonical norm on  $\mathcal{B}(X^* \times Y^*)$ . Thus, if  $u = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$  then

$$\|u\|_\varepsilon = \sup \left\{ \left| \sum_{i=1}^n x^*(x_i) y^*(y_i) \right| : x^* \in B_{X^*}, y^* \in B_{Y^*} \right\}.$$

The *injective tensor product* of  $X$  and  $Y$  is defined by:

$$X \widehat{\otimes}_\varepsilon Y = \overline{\text{span}}^{\|\cdot\|_\varepsilon} \{x \otimes y : x \in X, y \in Y\} \subseteq \mathcal{B}(X^* \times Y^*).$$

In the sequel, we will use a classical result from tensor product theory (see e.g. [6, Theorem 5.33]): If  $X^*$  or  $Y^*$  has the Radon-Nikodým property (RNP in short) and that  $X^*$  or  $Y^*$  has the AP, then  $(X \widehat{\otimes}_\varepsilon Y)^* \equiv X^* \widehat{\otimes}_\pi Y^*$ . The RNP has many characterizations, we refer to reader to Section VII.6 in [1] for a nice overview.

Assume now that there exists a subspace  $S$  of  $\text{Lip}_0(M)$  such that  $S^* \equiv \mathcal{F}(M)$ . Then one has

$$\mathcal{F}(M, X^*) = \mathcal{F}(M) \widehat{\otimes}_\pi X^* \equiv (S \widehat{\otimes}_\varepsilon X)^*$$

whenever either  $\mathcal{F}(M)$  or  $X^*$  has the AP and either  $\mathcal{F}(M)$  or  $X^*$  has the RNP. It is quite natural to wonder whether there are conditions which ensure that  $S \widehat{\otimes}_\varepsilon X$  is a natural predual of  $\mathcal{F}(M, X^*)$ . The next result asserts that this sometimes relies on the scalar case.

**Proposition 2.5.** *Let  $M$  be a separable pointed metric space,  $S \subset \text{lip}_0(M)$  be a natural predual of  $\mathcal{F}(M)$  and  $X$  be a Banach space (with  $\dim(X) \geq 2$ ). Assume moreover that either  $\mathcal{F}(M)$  or  $X^*$  has the AP and either  $\mathcal{F}(M)$  or  $X^*$  has the RNP. Then  $S\widehat{\otimes}_\varepsilon X$  is a natural predual of  $\mathcal{F}(M, X^*)$ .*

*Proof.* To show that  $S\widehat{\otimes}_\varepsilon X$  is a natural predual, we essentially follow the proof of Proposition 2.2. First of all, we show that  $\mathcal{T} := \{\gamma \otimes x^* : \gamma \in \mathcal{F}(M), x \in X^*\}$  is weak\* closed in  $\mathcal{F}(M, X^*)$ . Indeed, it is not hard to show that if  $T \in \mathcal{F}(M, X^*)$ , then  $T \in \mathcal{T}$  if and only if for every linearly independent families  $\{f_1, f_2\} \subset S$  and  $\{x_1, x_2\} \subset X$  we have:

$$\begin{vmatrix} \langle T, f_1 \otimes x_1 \rangle & \langle T, f_1 \otimes x_2 \rangle \\ \langle T, f_2 \otimes x_1 \rangle & \langle T, f_2 \otimes x_2 \rangle \end{vmatrix} = 0.$$

Accordingly,  $\mathcal{T}$  is weak\* closed. Now we fix  $r > 0$ . Let us consider a net  $(\delta(m_\alpha) \otimes x_\alpha^*)_\alpha \subset \delta(B(0, r), X^*)$  which weak\* converges to some  $\gamma \otimes x^* \in \mathcal{T}$ . We may assume that  $x^* \neq 0$  otherwise there is nothing to do. Consider  $x \in X$  such that  $x^*(x) \neq 0$ . Then, for every  $f \in S$  we have that  $f(m_\alpha)x^*(x_\alpha) \rightarrow f(\gamma)x^*(x)$ . So the net  $(\frac{x^*(x_\alpha)}{x^*(x)}\delta(m_\alpha))_\alpha \subset \mathbb{R} \cdot \delta(M)$  weak\* converges to  $\gamma$ . Since  $\mathbb{R} \cdot \delta(M)$  is weak\* closed (Lemma 2.4), there is  $\lambda \in \mathbb{R}$  and  $m \in M$  such that  $\gamma = \lambda\delta(m)$ .  $\square$

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