The set of elementary tensors is weakly closed in projective tensor products

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THE SET OF ELEMENTARY TENSORS IS WEAKLY CLOSED IN PROJECTIVE TENSOR PRODUCTS.

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ABSTRACT. In this short note, we prove that the set of elementary tensors is weakly closed in the projective tensor product of two Banach spaces. As a result, we are able to answer a question from the literature proving that if $(x_n) \subset X$ and $(y_n) \subset Y$ are two weakly null sequences such that $(x_n \otimes y_n)$ converges weakly in $X \widehat{\otimes}_\pi Y$, then $(x_n \otimes y_n)$ is also weakly null.

1. WEAK CONVERGENCE IN PROJECTIVE TENSOR PRODUCT

Let X, Y and Z be real Banach spaces. We denote $\mathcal{B}(X \times Y, Z)$ the space of continuous bilinear operators from $X \times Y$ into Z. If $Z = \mathbb{R}$, we simply write $\mathcal{B}(X \times Y)$. For $x \in X$ and $y \in Y$, define the *elementary tensor* $x \otimes y \in \mathcal{B}(X \times Y)^*$ by:

$$\forall B \in \mathcal{B}(X \times Y), \quad \langle x \otimes y, B \rangle = B(x, y).$$

We then introduce $X \otimes Y := \operatorname{span}\{x \otimes y \,:\, x \in X,\, y \in Y\}$. Recall that the norm on $\mathcal{B}(X \times Y)$ is defined by $\|B\|_{\mathcal{B}(X\times Y)} = \sup_{x \in B_X, y \in B_Y} |B(x, y)|$. Let $\|\cdot\|_\pi$ be the dual norm of $\|\cdot\|_{\mathcal{B}(X\times Y)}$. It is well known (see e.g. [1, Proposition VIII. 9. a)]) that if $u \in X \otimes Y$ then

$$\|u\|_\pi = \inf\Big\{ \sum_{i=1}^n \|x_i\|\|y_i\| \,:\, u = \sum_{i=1}^n x_i \otimes y_i \Big\}.$$

The *projective tensor product* of X and Y is defined as follows:

$$X \widehat{\otimes}_\pi Y = \overline{\operatorname{span}}^{\|\cdot\|_\pi}\{x \otimes y \,:\, x \in X,\, y \in Y\} \subseteq \mathcal{B}(X \times Y)^*.$$

As a consequence of the fundamental linearisation property of tensor products, one easily deduce the following isometric identification $(X \widehat{\otimes}_\pi Y)^* \equiv \mathcal{B}(X \times Y)$. Since $\mathcal{B}(X \times Y) \equiv \mathcal{L}(X, Y^*)$, where $\mathcal{L}(X, Y^*)$ stands for the space of bounded linear operators from X to Y^*, one also has that $\mathcal{L}(X, Y^*) \equiv (X \widehat{\otimes}_\pi Y)^*$.

The aim of this short note is to answer Question 3.9 in [5]:

Let X and Y be Banach spaces. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be weakly null sequences in X and Y, respectively, such that $(x_n \otimes y_n)_{n\in\mathbb{N}}$ is weakly convergent in $X \widehat{\otimes}_\pi Y$. Is $(x_n \otimes y_n)_{n\in\mathbb{N}}$ weakly null in $X \widehat{\otimes}_\pi Y$?

Let

$$\mathcal{T} = \{x \otimes y \,:\, x \in X,\, y \in Y\} \tag{1}$$

be the set of elementary tensors in $X \widehat{\otimes}_\pi Y$. We shall start with a simple but key observation. Recall that a Banach space X has the approximation property (AP

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in short) if for every $\varepsilon > 0$, for every compact subset $K \subset X$, there exists a finite rank operator $T \in \mathcal{L}(X,X)$ such that $\|Tx - x\| \leq \varepsilon$ for every $x \in K$.

Lemma 1.1. *Let X, Y be Banach spaces such that X or Y has the AP. Let $T \in X\widehat{\otimes}_\pi Y$. Then $T \in \mathcal{T}$ if and only if for every linearly independent families $\{x_1^*, x_2^*\} \subset X^*$ and $\{y_1^*, y_2^*\} \subset Y^*$ we have:*

$$(\star) \qquad \begin{vmatrix} \langle T, x_1^* \otimes y_1^* \rangle & \langle T, x_1^* \otimes y_2^* \rangle \\ \langle T, x_2^* \otimes y_1^* \rangle & \langle T, x_2^* \otimes y_2^* \rangle \end{vmatrix} = 0.$$

Proof. Thanks to [6, Proposition 2.8], every $T \in X\widehat{\otimes}_\pi Y$ can be written as

$$T = \sum_{n=1}^{\infty} x_n \otimes y_n$$

with $\sum_{n=1}^{\infty} \|x_n\|\|y_n\| \leq 2\|T\|$. Moreover, the linear map $\Phi : X\widehat{\otimes}_\pi Y \to \mathcal{L}(X^*, Y)$ obtained by

$$\forall x^* \in X^*, \quad \Phi\Big(\sum_{n=1}^{\infty} x_n \otimes y_n\Big)(x^*) = \sum_{n=1}^{\infty} x^*(x_n) y_n$$

defines a bounded operator. Since X or Y has the AP, Φ is moreover injective; see [6, Proposition 4.6].

If $T = x \otimes y \in \mathcal{T}$, then it is straightforward to check that condition $(\star)$ is verified:

$$\begin{vmatrix} \langle T, x_1^* \otimes y_1^* \rangle & \langle T, x_1^* \otimes y_2^* \rangle \\ \langle T, x_2^* \otimes y_1^* \rangle & \langle T, x_2^* \otimes y_2^* \rangle \end{vmatrix} = \begin{vmatrix} x_1^*(x) y_1^*(y) & x_1^*(x) y_2^*(y) \\ x_2^*(x) y_1^*(y) & x_2^*(x) y_2^*(y) \end{vmatrix} = 0.$$

Assume now that $T \notin \mathcal{T}$. Then $\Phi(T)$ is an operator of rank greater than 2 in $\mathcal{L}(X^*, Y)$. Thus, there exists a linearly independent family $\{x_1^*, x_2^*\} \subset X^*$ such that $\Phi(T)(x_1^*) \neq 0$, $\Phi(T)(x_2^*) \neq 0$ and $\{\Phi(T)(x_1^*), \Phi(T)(x_2^*)\} \subset Y$ is a linearly independent family. To finish the proof, simply pick a linearly independent family $\{y_1^*, y_2^*\} \subset Y^*$ satisfying:

$$\begin{matrix} \langle \Phi(T)(x_1^*), y_1^* \rangle \neq 0 & \langle \Phi(T)(x_1^*), y_2^* \rangle = 0 \\ \langle \Phi(T)(x_2^*), y_1^* \rangle = 0 & \langle \Phi(T)(x_2^*), y_2^* \rangle \neq 0. \end{matrix}$$

□

Proposition 1.2. *Let X, Y be two Banach spaces such that X or Y has the AP. Then the set of elementary tensors $\mathcal{T}$ is weakly closed in $X\widehat{\otimes}_\pi Y$.*

Proof. We let I be the set of all vectors $(x_1^*, x_2^*, y_1^*, y_2^*)$ such that $\{x_1^*, x_2^*\} \subset X^*$ and $\{y_1^*, y_2^*\} \subset Y^*$ are both linearly independent families. Next, for every $T \in X\widehat{\otimes}_\pi Y$ and $S = (x_1^*, x_2^*, y_1^*, y_2^*) \in I$, we define

$$D_S(T) = \begin{vmatrix} \langle T, x_1^* \otimes y_1^* \rangle & \langle T, x_1^* \otimes y_2^* \rangle \\ \langle T, x_2^* \otimes y_1^* \rangle & \langle T, x_2^* \otimes y_2^* \rangle \end{vmatrix}.$$

The result now directly follows from Lemma 1.1 together with the fact that D_S is continuous with respect to the weak topology. Indeed, one can write $\mathcal{T}$ as an intersection of weakly closed sets:

$$\mathcal{T} = \bigcap_{S \in I} D_S^{-1}(\{0\}).$$

□

The next corollary answers [5, Question 3.9] positively under rather general assumptions.

Corollary 1.3. *Let X and Y be Banach spaces such that X or Y has the AP. If $(x_n)_{n\in\mathbb{N}} \subset X$ converges weakly to x, $(y_n)_{n\in\mathbb{N}} \subset Y$ converges weakly to y, and $(x_n \otimes y_n)_{n\in\mathbb{N}}$ is weakly convergent in $X\widehat{\otimes}_\pi Y$, then $(x_n \otimes y_n)_{n\in\mathbb{N}}$ converges weakly to $x \otimes y$.*

Before proving this corollary, let us point out that the canonical basis $(e_n)_{n\in\mathbb{N}}$ of ℓ_2 shows that if $(x_n)_{n\in\mathbb{N}} \subset X$ and $(y_n)_{n\in\mathbb{N}} \subset Y$ are weakly null sequences, the sequence $(x_n \otimes y_n)_{n\in\mathbb{N}}$ may fail to be weakly null in $X\widehat{\otimes}_\pi Y$. Indeed, $(e_n \otimes e_n)_{n\in\mathbb{N}}$ is isometric to the ℓ_1-canonical basis; see [6, Example 2.10].

Proof. Assume first that $(x_n)_{n\in\mathbb{N}} \subset X$ and $(y_n)_{n\in\mathbb{N}} \subset Y$ are weakly null sequences such that $(x_n \otimes y_n)_{n\in\mathbb{N}}$ is weakly convergent in $X\widehat{\otimes}_\pi Y$. Since $\mathcal{T}$ is weakly closed, there exists $x \in X$ and $y \in Y$ such that $x_n \otimes y_n \to x \otimes y$ in the weak topology. Arguing by contradiction, suppose that $x \otimes y \neq 0$. Pick $x^* \in X^*$ and $y^* \in Y^*$ such that $x^*(x) = \|x\| \neq 0$ and $y^*(y) = \|y\| \neq 0$. On the one hand, $x_n \otimes y_n \to x \otimes y$ weakly, so that

$$\langle x^* \otimes y^*, x_n \otimes y_n\rangle \to \langle x^* \otimes y^*, x \otimes y\rangle = x^*(x)y^*(y) = \|x\|\|y\| \neq 0.$$

On the other hand, since $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are weakly null, one readily obtains a contradiction:

$$\langle x^* \otimes y^*, x_n \otimes y_n\rangle = x^*(x_n)y^*(y_n) \to 0.$$

Similarly, if $(x_n)_{n\in\mathbb{N}} \subset X$ converges weakly to x, $(y_n)_{n\in\mathbb{N}} \subset Y$ converges weakly to y, and $(x_n \otimes y_n)_{n\in\mathbb{N}}$ is weakly convergent in $X\widehat{\otimes}_\pi Y$, then we write:

$$(x - x_n) \otimes (y - y_n) = x \otimes y - x \otimes y_n - x_n \otimes y + x_n \otimes y_n.$$

But, $x \otimes y_n \xrightarrow[n\to+\infty]{w} x \otimes y$ and $x_n \otimes y \xrightarrow[n\to+\infty]{w} x \otimes y$. Therefore $\big((x-x_n)\otimes(y-y_n)\big)_{n\in\mathbb{N}}$ converges weakly and moreover the weak limit must be 0 thanks to the first part of the proof. This implies that $x_n \otimes y_n \xrightarrow[n\to+\infty]{w} x \otimes y$. □

In connection with Proposition 1.2, we also wish to mention [2, Theorem 2.3] which we describe now. If C and D are subsets of X and Y respectively, then let

$$C \otimes D := \{x \otimes y \ : \ x \in C,\ y \in D\} \subset \mathcal{T}.$$

Theorem 2.3 in [2] states that if C and D are bounded then $\overline{C}^w \otimes \overline{D}^w = \overline{C \otimes D}^w$ in $X\widehat{\otimes}_\pi Y$. The technique which we introduced in the present note permits to remove the boundedness assumption in the particular case when C and D are subspaces. It also allows us to slighly simplify the original proof of [2, Theorem 2.3]. The next lemma is the main ingredient.

Lemma 1.4. *Let X and Y be Banach spaces such that X or Y has the AP. Let $(x_s)_s \subset X$ and $(y_s)_s \subset Y$ be two nets such that $x_s \to x^{**}$ in the weak*-topology of X^{**}, $y_s \to y^{**}$ in the weak*-topology of Y^{**}, and $(x_s \otimes y_s)_s$ converges in the weak*-topology of $(X\widehat{\otimes}_\pi Y)^{**}$. Then $(x_s \otimes y_s)_s$ converges weakly* to $x^{**} \otimes y^{**}$.*

The proof is essentially the same as that of Corollary 1.3, so we leave the details to the reader.

Corollary 1.5. *Let X and Y be Banach spaces such that X or Y has the AP. If C and D are subsets of X and Y respectively, then $\overline{C}^w \otimes \overline{D}^w = \overline{C \otimes D}^w$ if one of the following additional assumptions are satisfied:*

(*i*) *If C and D are subspaces.*
(*ii*) *If C and D are bounded.*

Proof. First of all, it is readily seen that one has $\overline{C}^w \otimes \overline{D}^w \subset \overline{C \otimes D}^w$ without any additional assumption on C and D (see the first part of the proof of [2, Theorem 2.3]). Therefore we only have to prove the reverse inclusion in both cases.

To prove (i), it suffices to apply Proposition 1.2:

$$C \otimes D \subset \overline{C} \otimes \overline{D} \implies \overline{C \otimes D}^w \subset \overline{\overline{C} \otimes \overline{D}}^w = \overline{C} \otimes \overline{D}.$$

To prove (ii), let $z \in \overline{C \otimes D}^w$. We fix a net $(x_s \otimes y_s)_s \subset C \otimes D$ which converges weakly to z. Thanks to Proposition 1.2, there exist $x \in X$ and $y \in Y$ such that $z = x \otimes y$. Since C and D are bounded, up to taking a suitable subnet, we may assume that both $x_s \to x^{**}$ in the weak*-topology of X^{**} and $y_s \to y^{**}$ in the weak*-topology of Y^{**}. Thanks to Lemma 1.4, $x_s \otimes y_s \to x^{**} \otimes y^{**}$ in the weak*-topology of $(X \widehat{\otimes}_\pi Y)^{**}$. By uniqueness of the limit, $x^{**} \otimes y^{**} = z = x \otimes y$. We distinguish two cases.

If $z = 0$ then $x^{**} = 0$ or $y^{**} = 0$. Say $x^{**} = 0$ for instance. This means that $0 \in \overline{C}^w$. Now pick any $y \in C$ and observe that $z = 0 \otimes y$, which was to be shown.

If $z \neq 0$, then it is readily seen that $x^{**} \in \operatorname{span}\{x\}$ and $y^{**} \in \operatorname{span}\{y\}$. Therefore $x^{**} \in \overline{C}^{w^*} \cap X = \overline{C}^w$ and $y^{**} \in \overline{D}^{w^*} \cap Y = \overline{D}^w$, which concludes the proof. □

2. Applications to vector-valued Lipschitz free spaces

If M is a pointed metric space, with base point $0 \in M$, and if X is a real Banach space, then $\operatorname{Lip}_0(M, X)$ stands for the vector space of all Lipschitz maps from M to X which satisfy $f(0) = 0$. Equipped with the Lipschitz norm:

$$\forall f \in \operatorname{Lip}_0(M, X), \quad \|f\|_L = \sup_{x \neq y \in M} \frac{\|f(x) - f(y)\|_X}{d(x, y)},$$

$\operatorname{Lip}_0(M, X)$ naturally becomes a Banach space. When $X = \mathbb{R}$, it is customary to omit the reference to X, that is $\operatorname{Lip}_0(M) := \operatorname{Lip}_0(M, \mathbb{R})$. Next, for $x \in M$, we let $\delta(x) \in \operatorname{Lip}_0(M)^*$ be the evaluation functional defined by $\langle \delta(x), f \rangle = f(x)$, $\forall f \in \operatorname{Lip}_0(M)$. The Lipschitz free space over M is the Banach space

$$\mathcal{F}(M) := \overline{\operatorname{span}}^{\|\cdot\|} \{\delta(x) \,:\, x \in M\} \subset \operatorname{Lip}_0(M)^*.$$

The universal extension property of Lipschitz free spaces states that for every $f \in \operatorname{Lip}_0(M, X)$, there exists a unique continuous linear operator $\overline{f} \in \mathcal{L}(\mathcal{F}(M), X)$ such that:

(i) $f = \overline{f} \circ \delta$, and
(ii) $\|\overline{f}\|_{\mathcal{L}(\mathcal{F}(M), X)} = \|f\|_L$.

In particular, the next isometric identification holds:

$$\operatorname{Lip}_0(M, X) \equiv \mathcal{L}(\mathcal{F}(M), X).$$

A direct application (in the case $X = \mathbb{R}$) provides another basic yet important information:

$$\operatorname{Lip}_0(M) \equiv \mathcal{F}(M)^*.$$

It also follows from basic tensor product theory that $\mathrm{Lip}_0(M, X^*) \equiv (\mathcal{F}(M)\widehat{\otimes}_\pi X)^*$, which leads to the next definition (see [4] for more details):

Definition 2.1 (Vector-valued Lispschitz free spaces)**.** Let M be a pointed metric space and let X be a Banach space. We define the X-valued Lipschitz free space over M to be: $\mathcal{F}(M, X) := \mathcal{F}(M)\widehat{\otimes}_\pi X$.

2.1. **Weak closure of $\delta(M, X)$.** It is proved in [3, Proposition 2.9] that $\delta(M) = \{\delta(x) : x \in M\}$ is weakly closed in $\mathcal{F}(M)$ provided that M is complete. Our first aim is to prove the vector-valued counterpart. For this purpose, we need to identify a set that corresponds to $\delta(M)$ in the vector-valued case. A legitimate set to look at is the following:

$$\delta(M, X) := \{\delta(y) \otimes x \ : \ y \in M,\, x \in X\} \subset \mathcal{F}(M, X).$$

Notice that this does not exactly correspond to $\delta(M)$ in the case $X = \mathbb{R}$ since we have $\delta(M, \mathbb{R}) = \mathbb{R} \cdot \delta(M)$. This discrepancy is not a major issue since $\mathbb{R} \cdot \delta(M)$ is also a weakly closed set when M is complete. The next result is thus a natural extension to the vector valued setting of [3, Proposition 2.9].

Proposition 2.2. *Let M be a complete pointed metric space and X be a Banach space such that $\mathcal{F}(M)$ or X have the approximation property. Then $\delta(M, X)$ is weakly closed in $\mathcal{F}(M, X)$.*

Proof. In what follows, $\mathcal{T}$ denotes the elementary tensors in $\mathcal{F}(M)\widehat{\otimes}_\pi X$. Consider a net $(\delta(m_\alpha) \otimes x_\alpha)_\alpha \subset \delta(M, X)$ which is weakly convergent. Since $\delta(M, X) \subset \mathcal{T}$ and $\mathcal{T}$ is weakly closed (Proposition 1.2), there exist $\gamma \in \mathcal{F}(M)$ and $x \in X$ such that the net goes to $\gamma \otimes x$ in the weak topology. We may assume that $x \neq 0$, otherwise there is nothing to do. Pick $x^* \in X^*$ such that $x^*(x) \neq 0$. Then, for every $f \in \mathrm{Lip}_0(M)$, we have that $f(m_\alpha)x^*(x_\alpha) \to f(\gamma)x^*(x)$. So the net $\left(\frac{x^*(x_\alpha)}{x^*(x)}\delta(m_\alpha)\right)_\alpha \subset \mathbb{R} \cdot \delta(M)$ weakly converges to γ. Since $\mathbb{R} \cdot \delta(M)$ is weakly closed, there is $\lambda \in \mathbb{R}$ and $m \in M$ such that $\gamma = \lambda\delta(m)$. Consequently $\gamma \otimes x = \delta(m) \otimes \lambda x \in \delta(M, X)$. □

2.2. **Natural preduals.** Next, following [3, Section 3], $S \subset \mathrm{Lip}_0(M)$ is a natural predual of $\mathcal{F}(M)$ if $S^* \equiv \mathcal{F}(M)$ and $\delta(B(0, r))$ is $\sigma(\mathcal{F}(M), S)$-closed for every $r \geq 0$. A reasonable extension of this notion in the vector-valued setting is the following.

Definition 2.3. Let M be a pointed metric space and X be a Banach space with $\dim(X) \geq 2$. We say that a Banach space S is a natural predual of $\mathcal{F}(M, X^*)$ if $Y^* \equiv \mathcal{F}(M, X^*)$ and

$$\delta(B(0, r), X^*) = \{\delta(m) \otimes x^* \ : \ m \in B(0, r),\, x^* \in X^*\} \subset \mathcal{F}(M, X^*)$$

is $\sigma(\mathcal{F}(M, X^*), S)$-closed for every $r \geq 0$.

Notice again that $\delta(B(0, r), \mathbb{R}) = \mathbb{R} \cdot \delta(B(0, r))$. In the next statement, $\mathrm{lip}_0(M)$ denotes the subspace of $\mathrm{Lip}_0(M)$ of all uniformly locally flat functions. Recall that $f \in \mathrm{Lip}_0(M)$ is uniformly locally flat if

$$\lim_{d(x,y)\to 0} \frac{|f(x) - f(y)|}{d(x, y)} = 0.$$

Lemma 2.4. *Let M be a separable pointed metric space. Suppose that $S \subset \mathrm{lip}_0(M)$ is a natural predual of $\mathcal{F}(M)$. Then, for every $r \geq 0$, $\mathbb{R} \cdot \delta(B(0, r))$ is weak* closed in $\mathcal{F}(M)$.*

Proof. Let us fix $r \geq 0$. Let $(\lambda_n \delta(x_n))_n \subset \mathbb{R} \cdot \delta(B(0,r))$ be a sequence converging to some $\gamma \in \mathcal{F}(M)$ in the weak* topology. We assume that $\gamma \neq 0$, otherwise there is nothing to do. Since a weak* convergent sequence is bounded, and by weak* lower-semi-continuity of the norm, we may assume that there exists $C > 0$ such that for every n:

$$0 < \frac{\|\gamma\|}{2} \leq |\lambda_n| \|\delta(x_n)\| = |\lambda_n| d(x_n, 0) \leq C.$$

Thus, $d(x_n, 0) \neq 0$ and $\lambda_n \neq 0$ for every n. Up to extracting a further subsequence, we may assume that the sequence $(\lambda_n d(x_n, 0))_n$ converges to some $\ell \neq 0$. Since $(x_n)_n \subset B(0,r)$, we also assume that $(d(x_n, 0))_n$ converges to some d. We will distinguish two cases.

If $d \neq 0$, then $(\lambda_n)_n$ converges to $\lambda := \frac{\ell}{d}$ and so $(\delta(x_n))_n$ weak* converges to $\frac{\gamma}{\lambda}$. Since S is a natural predual of $\mathcal{F}(M)$, $\delta(B(0,r))$ is weak* closed in $\mathcal{F}(M)$. So there exists $x \in M$ such that $\gamma = \lambda \delta(x)$.

If $d = 0$, then $(\delta(x_n))_n$ converges to 0 in the norm topology (and $(\lambda_n)_n$ tends to infinity). Note that we may write:

$$\lambda_n \delta(x_n) = \lambda_n d(x_n, 0) \frac{\delta(x_n) - \delta(0)}{d(x_n, 0)}.$$

Since $S \subset \mathrm{lip}_0(M)$, the sequence $(\frac{\delta(x_n) - \delta(0)}{d(x_n, 0)})_n$ weak* converges to 0. Moreover the sequence $(\lambda_n d(x_n, 0))_n$ converges to $\ell \neq 0$. Consequently $(\lambda_n \delta(x_n))_n$ weak* converges to 0 and so $\gamma = 0$, which is a contradiction. □

Before going further, we need to introduce the injective tensor product of two Banach spaces. Recall that, to define the projective tensor product, we introduced $x \otimes y$ as an element of $\mathcal{B}(X \times Y)^*$. For the injective tensor product, we change the point of view since we now consider $x \otimes y$ as an element of $\mathcal{B}(X^* \times Y^*)$ defined as follows:

$$\forall (x^*, y^*) \in X^* \times Y^*, \quad \langle x \otimes y, (x^*, y^*) \rangle = x^*(x) y^*(y).$$

In this case, we denote $\|\cdot\|_\varepsilon$ the canonical norm on $\mathcal{B}(X^* \times Y^*)$. Thus, if $u = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$ then

$$\|u\|_\varepsilon = \sup \Big\{ \Big| \sum_{i=1}^n x^*(x_i) y^*(y_i) \Big| \, : \, x^* \in B_{X^*}, y^* \in B_{Y^*} \Big\}.$$

The *injective tensor product* of X and Y is defined by:

$$X \widehat{\otimes}_\varepsilon Y = \overline{\mathrm{span}}^{\|\cdot\|_\varepsilon} \{ x \otimes y \, : \, x \in X, \, y \in Y \} \subseteq \mathcal{B}(X^* \times Y^*).$$

In the sequel, we will use a classical result from tensor product theory (see e.g. [6, Theorem 5.33]): If X^* or Y^* has the Radon-Nikodým property (RNP in short) and that X^* or Y^* has the AP, then $(X \widehat{\otimes}_\varepsilon Y)^* \equiv X^* \widehat{\otimes}_\pi Y^*$. The RNP has many characterizations, we refer to reader to Section VII.6 in [1] for a nice overview.

Assume now that there exists a subspace S of $\mathrm{Lip}_0(M)$ such that $S^* \equiv \mathcal{F}(M)$. Then one has

$$\mathcal{F}(M, X^*) = \mathcal{F}(M) \widehat{\otimes}_\pi X^* \equiv (S \widehat{\otimes}_\varepsilon X)^*$$

whenever either $\mathcal{F}(M)$ or X^* has the AP and either $\mathcal{F}(M)$ or X^* has the RNP. It is quite natural to wonder whether there are conditions which ensure that $S \widehat{\otimes}_\varepsilon X$ is a natural predual of $\mathcal{F}(M, X^*)$. The next result asserts that this sometimes relies on the scalar case.

Proposition 2.5. *Let M be a separable pointed metric space, $S \subset \mathrm{lip}_0(M)$ be a natural predual of $\mathcal{F}(M)$ and X be a Banach space (with $\dim(X) \geq 2$). Assume moreover that either $\mathcal{F}(M)$ or X^* has the AP and either $\mathcal{F}(M)$ or X^* has the RNP. Then $S\widehat{\otimes}_\varepsilon X$ is a natural predual of $\mathcal{F}(M, X^*)$.*

Proof. To show that $S\widehat{\otimes}_\varepsilon X$ is a natural predual, we essentially follow the proof of Proposition 2.2. First of all, we show that $\mathcal{T} := \{\gamma \otimes x^* : \gamma \in \mathcal{F}(M),\, x \in X^*\}$ is weak* closed in $\mathcal{F}(M, X^*)$. Indeed, it is not hard to show that if $T \in \mathcal{F}(M, X^*)$, then $T \in \mathcal{T}$ if and only if for every linearly independent families $\{f_1, f_2\} \subset S$ and $\{x_1, x_2\} \subset X$ we have:

$$\begin{vmatrix} \langle T, f_1 \otimes x_1 \rangle & \langle T, f_1 \otimes x_2 \rangle \\ \langle T, f_2 \otimes x_1 \rangle & \langle T, f_2 \otimes x_2 \rangle \end{vmatrix} = 0.$$

Accordingly, $\mathcal{T}$ is weak* closed. Now we fix $r > 0$. Let us consider a net $(\delta(m_\alpha) \otimes x^*_\alpha)_\alpha \subset \delta(B(0,r), X^*)$ which weak* converges to some $\gamma \otimes x^* \in \mathcal{T}$. We may assume that $x^* \neq 0$ otherwise there is nothing to do. Consider $x \in X$ such that $x^*(x) \neq 0$. Then, for every $f \in S$ we have that $f(m_\alpha)x^*(x_\alpha) \to f(\gamma)x^*(x)$. So the net $\left(\frac{x^*(x_\alpha)}{x^*(x)}\delta(m_\alpha)\right)_\alpha \subset \mathbb{R} \cdot \delta(M)$ weak* converges to γ. Since $\mathbb{R} \cdot \delta(M)$ is weak* closed (Lemma 2.4), there is $\lambda \in \mathbb{R}$ and $m \in M$ such that $\gamma = \lambda\delta(m)$. □

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