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Edge-decomposing graphs into coprime forests ^{*†}Tereza Klimošová[‡] Stéphan Thomassé^{§¶}

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Abstract

The Barát-Thomassen conjecture, recently proved in [3], asserts that for every tree T , there is a constant c_T such that every c_T -edge connected graph G with number of edges (size) divisible by the size of T admits an edge partition into copies of T (a T -decomposition). In this paper, we investigate in which case the connectivity requirement can be dropped to a minimum degree condition. For instance, it was shown in [4] that when T is a path with k edges, there is a constant d_k such that every $2d_k$ -edge connected graph G with size divisible by k and minimum degree d_k has a T -decomposition. We show in this paper that when F is a coprime forest (the sizes of its components being a coprime set of integers), any graph G with sufficiently large minimum degree has an F -decomposition provided that the size of F divides the size of G (no connectivity is required). A natural conjecture asked in [4] asserts that for a fixed tree T , any graph G of size divisible by the size of T with sufficiently high minimum degree has a T -decomposition, provided that G is sufficiently highly connected in terms of the maximal degree of T . The case of maximum degree 2 is answered by paths. We provide a counterexample to this conjecture in the case of maximum degree 3.

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1 Introduction

Given a graph G , we denote by $V(G)$ and by $E(G)$ its vertex set and its edge set, respectively. For $X \subseteq V(G)$, $G[X]$ denotes the induced subgraph of G on X . Unless we specify otherwise, we consider graphs to be simple graphs without loops and multigraphs to have multiple edges and loops.

For graphs G and H , we say that G is H -decomposable if there exists a partition $\{E_i\}_{i \in [k]}$ of $E(G)$ such that every E_i forms an isomorphic copy of H . We then call $\{E_i\}_{i \in [k]}$ an H -decomposition of G . Note that if G has an H -decomposition, $|E(H)|$ divides $|E(G)|$. More generally, if there exists a partition $\{E_i\}_{i \in [k]}$ of $E(G)$ such that each E_i forms an isomorphic copy of one of the graphs H_1, \dots, H_j , we call $\{E_i\}_{i \in [k]}$ a H_1, \dots, H_j -decomposition of G .

In [2], Barát and Thomassen conjectured that for a fixed tree T , every sufficiently edge-connected graph with number of edges divisible by $|E(T)|$ has a T -decomposition.

Conjecture 1. *For any tree T on m edges, there exists an integer k_T such that every k_T -edge-connected graph with number of edges divisible by m has a T -decomposition.*

They also observed a relation between T -decompositions and Tutte's conjecture, which states that every 4-edge-connected graph admits a nowhere-zero 3-flow. Until recently it was not even known that any constant edge-connectivity would suffice. Barát and Thomassen have shown that if every 8-edge-connected graph has a $K_{1,3}$ -decomposition, then every 8-edge-connected graph has a nowhere-zero 3-flow and, vice versa, Tutte's 3-flow conjecture would imply that every 10-edge-connected graph with number of edges divisible by 3 has a $K_{1,3}$ -decomposition.

A series of results showing that Conjecture 1 holds for specific trees followed [1, 4–6, 8, 10–14]. Recently, the conjecture was proven by Bensmail, Harutyunyan, Le, Merker and the second author [3].

Theorem 2. *For any tree T , there exists an integer k_T such that every k_T -edge-connected graph with number of edges divisible by $|E(T)|$ has a T -decomposition.*

In [4], the authors posed the following, strengthened version of the conjecture of Barát and Thomassen and they proved it for T being a path.

Conjecture 3. *There is a function f such that, for any fixed tree T with maximum degree Δ_T , every $f(\Delta_T)$ -edge-connected graph with minimum degree at least $f(|E(T)|)$ and number of edges divisible by $|E(T)|$ has a T -decomposition.*

We define the *length* of a path as its number of edges. The following result from [4] answers the previous question for $\Delta_T = 2$.

Theorem 4. *For every integer ℓ , there exists $d = d(\ell)$ such that the edge set of every $2d$ -edge-connected graph G with minimum degree d and number of edges divisible by ℓ has a decomposition into paths of length ℓ .*

However, Conjecture 3 is not true in general. In Section 2, we show that it does not hold even for trees of maximum degree three.

In Section 3, we consider a variation of Conjecture 3 for forests. We call a tree *proper* if it has at least one edge. We call a forest *proper* if each of its connected components is a proper tree. Note that if F is a forest which is not proper, for the purpose of finding an F -decomposition, one can disregard components with a single vertex, provided that the host graph has at least as many vertices as F . Having this in mind, we state our result only for proper forests.

Theorem 2 can be easily extended to proper forests by gluing several copies of a forest together (see 3 for details). Moreover, one can replace edge-connectivity requirement by minimum degree requirement in case of *coprime forests*. We call a forest *coprime* if there is no integer $d > 1$ which divides the number of edges of all of its components.

Theorem 5. *For any proper coprime forest F , there exists an integer $\delta = \delta(F)$ such that every graph with minimum degree at least δ and number of edges divisible by the number of edges of F has an F -decomposition.*

Note that the requirement that the forest is coprime is necessary. For instance, if every component of F has even number of edges, a graph with a connected component that has an odd number of edges does not have any F -decomposition.

From Theorem 5, we can easily derive, in a sense, a relaxation of Conjecture 3. In particular, for two trees T_1, T_2 with coprime numbers of edges, high minimum degree implies the existence of a T_1, T_2 -decomposition.

Corollary 6. *For any two trees T_1, T_2 with coprime numbers of edges, there exists an integer $\delta = \delta(T_1, T_2)$ such that every graph with minimum degree at least δ has a T_1, T_2 -decomposition.*

The core idea of the proof of Theorem 5 (which is used in Lemma 10) is to partition the host graph along edge-cuts of bounded size into vertex disjoint parts that are highly edge-connected. This is achieved by repeated applications of the following lemma.

Lemma 7. *Let k be an integer and G be a multigraph. Then, there is a cut (A, B) in G of order at most $2k$ such that $G[A]$ is k -edge-connected or has only one vertex.*

We then obtain decompositions of these parts using the result of [3]. Finally, we construct a decomposition of the whole graph by combining decompositions of the parts.

2 Disproving Conjecture 3

In this section, we disprove Conjecture 3. We show that it does not hold even for trees of maximum degree three.

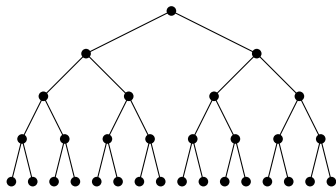


Figure 1: The tree T_4 .

Assume that Conjecture 3 holds for some function f . Let T_k be the complete binary tree of depth k (see Figure 1 for an example). The maximum degree of T_k is three and the number of edges of T_k is $n_k = 2^{k+1} - 2$ for every k . Let \mathbb{T}_k be the set of possible numbers of edges contained in a component of $T_k \setminus e$ for some edge e of T_k . Observe that the components of $T_k \setminus e$ have $2^i - 2$ edges and $n_k - (2^i - 1) = 2^{k+1} - 2^i - 1$ edges for some $i \in [k]$, for every edge e . Thus, $\mathbb{T}_k = \{2^i - 2 \mid i \in [k]\} \cup \{2^{k+1} - 2^i - 1 \mid i \in [k]\}$ and then $|\mathbb{T}_k| \leq 2k$.

It follows that the sum $\sum_{i=1}^{f(3)} t_i$, where $t_i \in \mathbb{T}_k$ for every $i \in [f(3)]$, can attain at most $(2k)^{f(3)}$ different values. Therefore, there exists k_0 such that $(2k)^{f(3)} < n_k$ for every $k > k_0$.

Fix $k > k_0$. Then, there exists $m \in [n_k]$ such that $m \not\equiv \sum_{i=1}^{f(3)} t_i \pmod{n_k}$ for any choice of values of $t_i \in \mathbb{T}_k$.

Let G_1 and G_2 be $f(3)$ -edge-connected graphs with minimum degree at least $f(|E(T_k)|)$ such that the number of edges of G_1 is congruent to m modulo n_k , the number of edges of G_2 is congruent to $n_k - f(3) - m$ modulo n_k and there are sets $S_1 \subseteq V(G_1)$, $S_2 \subseteq V(G_2)$ of size $f(3)$ such that the distance between every two distinct vertices of S_i in G_i , $i = 1, 2$, is greater than $2k$ (where the distance is the length of the shortest path between the two vertices). See Figure 2 for an example of a construction of G_1 with these properties, G_2 can be constructed in an analogous way.

Let G be a graph obtained from the disjoint union of G_1 and G_2 by adding a matching M of size $f(3)$ between the vertices of S_1 and S_2 . Then, G is $f(3)$ -edge-connected, of minimum degree at least $f(|E(T_k)|)$ and with number of edges divisible by $|E(T_k)|$. Assume that there exists a T_k -decomposition \mathcal{T} of G . Since the distance between any two vertices in S_1 and any two vertices in S_2 is greater than the distance between any two vertices in T , every copy of T_k in \mathcal{T} contains at most one vertex of S_1 and S_2 and therefore at most one edge of M . Note that each copy of T with an edge in M contains $t_i \in \mathbb{T}_k$ edges of G_1 . Therefore, the number of edges of G_1 is $cn_k + \sum_{i=1}^{f(3)} t_i$, where c is an integer and $t_i \in \mathbb{T}_k$ for every $i \in [f(3)]$. This yields a contradiction with the choice of the number of edges of G_1 .

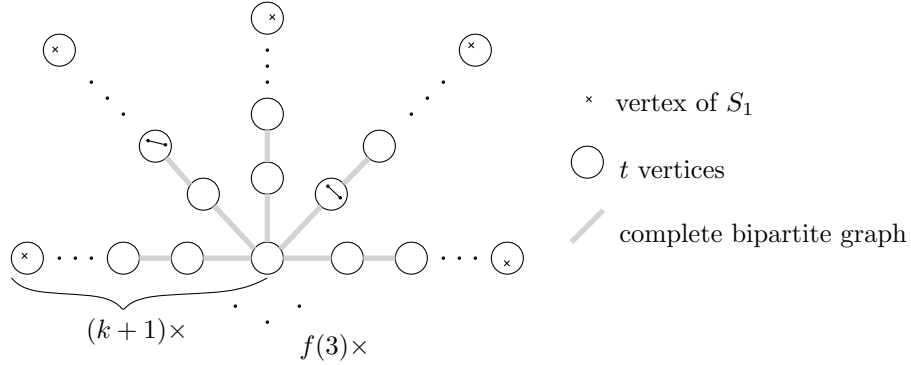


Figure 2: A graph G_1 with the desired properties can be easily constructed by taking a star with $f(3)$ edges, subdividing each edge k times, replacing each vertex by $t = \max(f(|E(T_k)|), n_k)$ vertices and each edge by a complete bipartite graph and finally, adding appropriate number of additional edges between the vertices corresponding to the same vertex of the subdivided star, so that the number of edges of the graph is congruent to m modulo n_k . Taking vertices of S_1 as in the figure yields a set of size $f(3)$ with distance at least $2k + 2$ between every two vertices.

3 Decomposition into coprime trees

In this section, we prove Theorem 5 after introducing some tools.

Let \mathcal{G} be a set of graphs. We say that G has a \mathcal{G} -decomposition if its edge set can be partitioned into edge disjoint copies of graphs in \mathcal{G} . Decompositions are transitive in the following sense.

Observation 8. *Let G be a graph and let \mathcal{G} and \mathcal{H} be sets of graphs. If the edge set of G can be partitioned in edge-disjoint subgraphs such that each of them has an \mathcal{H} -decomposition, then G has an \mathcal{H} -decomposition. In particular, if every graph in \mathcal{G} has an \mathcal{H} -decomposition and G has a \mathcal{G} -decomposition, G has an \mathcal{H} -decomposition.*

Given a sequence of vertex disjoint proper trees T_1, \dots, T_k , we define a *chain* $T_1 \circ T_2 \circ \dots \circ T_k$ as a graph obtained from T_1, \dots, T_k by choosing two distinct leaves u_i and v_i in each T_i and identifying u_i and v_{i+1} for $i \in [k - 1]$. Note that a chain is a tree.

For a proper tree T and an integer k , we define a *k-chain of T* to be a chain $T_1 \circ T_2 \circ \dots \circ T_k$, where each T_i is isomorphic to T . We denote it by $T^{\circ k}$. Similarly, we define a *k-chain $F^{\circ k}$* of a proper forest F with connected components T_1, \dots, T_m as the tree $T_1^{\circ k} \circ \dots \circ T_m^{\circ k}$.

Observation 9. *Let F be a proper forest and $k \geq 2$. Then $F^{\circ k}$ has an F -decomposition.*

Next, we argue that Theorem 2 can be easily extended to proper forests. In particular, given a fixed proper forest F , if G has number of edges divisible by $|E(F)|$ and is $k_{F^{\circ 2}} + |E(F)|$ -edge-connected, where $k_{F^{\circ 2}}$ is as in Theorem 2, then G has an F -decomposition. Indeed, from Theorem 2 it follows that if $|E(G)|$ is divisible by $|E(F^{\circ 2})| = 2|E(F)|$, then G has an $F^{\circ 2}$ -decomposition. The existence of an F -decomposition follows from Observation 8.

If $|E(G)|$ is divisible by $|E(F)|$ but not by $|E(F^{\circ 2})|$, we can remove any copy of F to make the number of edges of G divisible by $|E(F^{\circ 2})|$, decreasing edge-connectivity by at most $|E(F)|$ (to at least $k_{F^{\circ 2}}$) and then argue as before.

For the proof of Theorem 5, we need the following lemma, which we prove in Section 4.

Lemma 10. *Let T, T_1 and T_2 be trees such that the number of edges of T is coprime with both the number of edges of T_1 and the number of edges of T_2 . Then, there exists an integer $\delta = \delta(T, T_1, T_2)$ such that every graph with minimum degree at least δ and number of edges divisible by the greatest common divisor of $|E(T)|$ and $|E(T_1)| + |E(T_2)|$ has a $\{T, T_1, T_2\}$ -decomposition that contains the same number of copies of T_1 and T_2 .*

Proof of Theorem 5. Let K_1, \dots, K_s be the connected components of the coprime forest F . By Bézout's lemma, there exist integers p_1, \dots, p_s such that $\sum_{i=1}^s p_i |E(K_i)| = 1$. We define $n := 2|V(F)| - \min\{0, p_1, \dots, p_s\}$ and $m := 2|V(F)| + n + \max\{0, p_1, \dots, p_s\}$. Let T_1 be $K_1^{\circ n + p_1} \circ \dots \circ K_s^{\circ n + p_s}$ and T_2 be $K_1^{\circ m - (n + p_1)} \circ \dots \circ K_s^{\circ m - (n + p_s)}$. Then, $|E(T_1)| = n|E(F)| + 1$, i.e. the number of edges of T_1 is coprime with the number of edges of F . Moreover, $|E(T_2)| = m|E(F)| - |E(T_1)| = (m - n)|E(F)| - 1$, thus the number of edges of T_2 is also coprime with number of edges of F . Let r be a prime number greater than $|E(T_1)|$ and $|E(T_2)|$ and define T to be $F^{\circ r}$. Then, $|E(T)|$ is coprime with both $|E(T_1)|$ and $|E(T_2)|$.

Then, by Lemma 10, G has a $\{T, T_1, T_2\}$ -decomposition \mathcal{T} . Let G' be a subgraph of G formed by the copies of T_1 and T_2 in \mathcal{T} . Since the graph $G \setminus E(G')$ has a T -decomposition by Observation 8 and 9, it also has an F -decomposition. Thus, to show that G has an F -decomposition, it suffices to show that G' has an F -decomposition.

Note that G' has a $\{T_1, T_2\}$ -decomposition into edge disjoint copies T_1^1, \dots, T_1^ℓ of T_1 and edge disjoint copies T_2^1, \dots, T_2^ℓ of T_2 for some integer ℓ . Let $H_i = T_1^i \cup T_2^i$. We show that H_i has an F -decomposition for every $i \in [\ell]$ and thus, by Observation 8, G' has an F -decomposition. By construction, T_1^i has a (K_1, \dots, K_s) -decomposition \mathcal{T}_1 in which each K_j appears $n + p_j$ -times. Similarly, T_2^i has a (K_1, \dots, K_s) -decomposition \mathcal{T}_2 in which each K_j appears $m - (n + p_j)$ -times. Note that $n + p_j$ and $m - (n + p_j)$ are at least $2|V(F)|$ for every $j \in [s]$. We first construct $m - 4|V(F)|$ copies of F in H_i combining trees from \mathcal{T}_1 and \mathcal{T}_2 in such a way that at the end of the process, there will be exactly $2|V(F)|$ unused copies

of each K_i in each of \mathcal{T}_1 and \mathcal{T}_2 . We construct one copy of F at a time by greedily picking copies of K_1, \dots, K_s one by one, always choosing it from that of \mathcal{T}_1 and \mathcal{T}_2 which contains the greatest number of unused copies of the component (in case of equality we choose arbitrarily). We argue that it is always possible to pick a copy which is vertex disjoint from all the previously picked components in the currently constructed copy of F . Assume that we need to pick a copy of K_j and assume that \mathcal{T}_1 contains greater number of unused copies of K_j - the argument for \mathcal{T}_2 is analogous.

The total number of copies of K_j in \mathcal{T}_1 and \mathcal{T}_2 is m and therefore after using less than $m - 4|V(F)|$ copies of K_j , \mathcal{T}_1 contains more than $2|V(F)|$ unused copies of K_j . By construction, each copy of K_k , $k \neq j$ in \mathcal{T}_1 or \mathcal{T}_2 intersects at most $2|V(K_k)|$ copies of K_j in \mathcal{T}_1 . Therefore, the already chosen components of the currently constructed copy of F intersect at most $2|V(F)|$ copies of K_j in \mathcal{T}_1 , so there exist at least one copy of K_j which is vertex disjoint. Note that after constructing $m - 4|V(F)|$ copies of F in this way, both \mathcal{T}_1 and \mathcal{T}_2 contain $2|V(F)|$ unused copies of each K_j , $j \in [s]$. We denote these collections of unused copies by \mathcal{T}'_1 and \mathcal{T}'_2 respectively and argue that \mathcal{T}'_1 can be partitioned into copies of F . Then, \mathcal{T}'_2 can be partitioned into copies of F by analogous arguments.

We call a copy of K_i , $i \in [s]$, in \mathcal{T}'_1 *bad* if it intersects a copy of K_j in \mathcal{T}'_1 with $j \neq i$. We call all other copies in \mathcal{T}'_1 *good*. Observe that by construction, there are at most $2s$ bad copies in \mathcal{T}'_1 . Any collection of copies in \mathcal{T}'_1 containing one copy of K_i for each $i \in [s]$ such that at most one of the copies is bad forms a copy of the forest F (i.e., the copies in the collection are pairwise disjoint). Thus, it is possible to construct copies of F greedily by picking one bad copy and good copies of the remaining components of F . Since \mathcal{T}'_1 contains $2|V(F)|$ copies of each K_i , $i \in [s]$ and $2s < 2|V(F)|$, there are enough good copies of each component of F for repeatedly creating copies of F in this way, until all the bad copies in \mathcal{T}'_1 are used. Once all the bad copies are used, the remaining good copies in \mathcal{T}'_1 can be arbitrarily partitioned into copies of F . □

4 Proof of Lemma 10

We start by showing that sufficiently highly edge-connected graph can be decomposed into copies of two trees with coprime numbers of edges.

Lemma 11. *Let T_1, T_2 be trees with coprime numbers of edges. Then, there exists an integer $K = K(T_1, T_2)$ such that every K -edge-connected graph has a $\{T_1, T_2\}$ -decomposition with less than $|E(T_1)|$ copies of T_2 .*

Proof. Let m_1, m_2 be numbers of edges of T_1, T_2 , respectively, and let k_{T_1} be as in Theorem 2 and let $K = k_{T_1} + m_1 m_2$. Let G be a K -edge-connected graph and

let n be the smallest non-negative integer such that $m_1 \mid (|E(G)| - nm_2)$. Since m_1 and m_2 are coprime, such n exists and is smaller than m_1 by Bézout's Lemma.

By the greedy algorithm, it is possible to find a collection \mathcal{T} of n edge-disjoint copies of T_2 in G . Then, $G \setminus E(\mathcal{T})$ is a k_{T_1} -edge-connected graph with number of edges divisible by m_1 . The result follows from Theorem 2. □

For the purpose of the proof of Lemma 10, we extend the definition of a graph by allowing hyperedges of size one, which we call *stubs* and we call the resulting object a *stub graph*. Each vertex of a stub graph can be incident with arbitrarily many stubs and the *degree* of a vertex is the number of edges and stubs incident with it. Moreover, we assign a positive integer i_s to each stub s and we call i_s the *index of the stub* s . Intuitively, a stub can be viewed as a remainder of an edge after one of its endvertices has been removed from the graph. The index of the stub then contains some information about the removed endvertex.

We write $E(G)$ to denote the set consisting of the edges and the stubs of a stub graph G and we call it the *edge set* of G , but we do not refer to stubs as edges otherwise. We denote the set of stubs of index i in a stub graph G by $S_i(G)$. A *subgraph* of a stub graph is also a stub graph and we say that two subgraphs are edge-disjoint if their edge sets are disjoint, i.e., they do not share any edge or stub.

Let \mathcal{T} be a set of trees (without stubs). We extend the definition of \mathcal{T} -decomposition for graphs to stub graphs. Informally, in a \mathcal{T} -decomposition of a stub graph, a stub incident with a vertex v plays the role of a subtree of $T \in \mathcal{T}$, such that the vertex v is a leaf of this subtree.

A *twig* is a pair (T, r) , where T is a proper tree and r is a leaf of T . We say that r is the *root* of the twig. Let G be a stub graph, s a stub incident with a vertex v in G and (T, r) a twig disjoint from G . A stub graph obtained from G by *expanding* s by (T, r) is the stub graph obtained from $G \setminus s$ and T by identifying v and r .

An *embedding* of a tree T in a stub graph G is a subgraph T' of G such that each stub in T' has different index and there exists a mapping \mathcal{S} assigning a twig (T_s, r_s) to each stub s in T' such that expanding every stub s in T' by $\mathcal{S}(s)$ yields a copy of T . See Figure 3 for an example.

We say that a stub graph G has a \mathcal{T} -decomposition if its edge set can be decomposed into disjoint sets $\{E_i\}_{i \in [k]}$ such that each E_i forms an embedding of some $T \in \mathcal{T}$. Note that this definition coincides with the definition of a \mathcal{T} -decomposition in the usual sense if G has no stubs.

We denote the graph obtained from G by removing all the stubs by G^- and we say that G is *k-edge-connected* if G^- is k -edge-connected. The next observation asserts that a \mathcal{T} -decomposition of G^- can be easily extended to a \mathcal{T} -decomposition of G .

Observation 12. *If G^- has a \mathcal{T} -decomposition \mathcal{T}_1 , there exists a \mathcal{T} -decomposition \mathcal{T}_2 of G such that $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Moreover, given $T \in \mathcal{T}$, there exists \mathcal{T}_2 such that $\mathcal{T}_2 \setminus \mathcal{T}_1$ contains only embeddings of T .*

Proof. It follows from the fact that a stub s (with the vertex incident to it) forms an embedding of any proper tree T . It is enough to let $\mathcal{S}(s) = (T, r)$, where r is some leaf of T . \square

In particular, a stub graph G with no edges has a \mathcal{T} -decomposition for any set of proper trees \mathcal{T} , since G^- has a trivial (empty) \mathcal{T} -decomposition. It follows that Lemma 11 holds for stub graphs as well.

Next, we introduce some more tools and terminology. Let G be a multigraph. We call a partition (A, B) of $V(G)$ into two parts a *cut* in G . We denote $E(A, B)$ the set of edges of G incident with a vertex in both A and B and call $|E(A, B)|$ the *order of the cut* (A, B) . We now prove Lemma 7. Note that the definition of a cut and the statement of the lemma trivially extend to stub graphs (by considering the multigraph G^- instead of the stub graph G).

Proof of Lemma 7. Let (A, B) be a cut in G of order at most $2k$ such that A is inclusion-wise minimal. Assume that A has more than one vertex and let (A_1, A_2) be a cut in $G[A]$. By minimality of A , we have that the cuts $(A_1, V(G) \setminus A_1)$ and $(A_2, V(G) \setminus A_2)$ have order greater than $2k$. Since $|E(A_1, V(G) \setminus A_1)| + |E(A_2, V(G) \setminus A_2)| = |E(A, B)| + 2|E(A_1, A_2)|$, (A_1, A_2) has order greater than k . \square

The following results of Czumaj and Strothmann were originally proven only for simple graphs, however, they easily extend to multigraphs.

Theorem 13 (Czumaj, Strothmann [7], extended). *Every 2-edge-connected multigraph G contains a spanning tree T such that $\deg_T v \leq (\deg_G v + 3)/2$ for every vertex v of G .*

To find such a tree, it is enough to take an out-branching in a strongly connected balanced orientation of G .

Theorem 14 (Czumaj, Strothmann [7], extended). *Let p be a positive integer. If a multigraph G contains 2^p edge-disjoint spanning trees, then G has a spanning tree T such that $\deg_T v \leq \deg_G v / 2^p + 3p/2$ for every vertex v of G .*

The following Corollary 16 of Theorem 14 was essentially proven in [14]. Our version differs in some details. Among other things, we use the following theorem of Nash-Williams and Tutte to replace the requirement of having a collection of edge-disjoint spanning trees by edge-connectivity.

Theorem 15 (Nash-Williams [9], Tutte [15]). *If a multigraph G is $2k$ -edge-connected, then G contains k edge-disjoint spanning trees.*

Corollary 16. *For every $\varepsilon > 0$ and integer m , there exists L such that every 2^n -edge-connected stub graph G with minimum degree at least L , where $n = 2 + m + \lceil \log(1/\varepsilon) \rceil$, has 2^m edge-disjoint spanning trees T_1, \dots, T_{2^m} such that*

$$\sum_{i \in [2^m]} \deg_{T_i} v \leq \varepsilon \deg_G v$$

for every $v \in V(G)$.

Proof. By Theorem 15, the graph G^- contains $2^{n-1} = 2^m \cdot 2^{\lceil \log(1/\varepsilon) \rceil + 1}$ edge-disjoint spanning trees. Thus, G^- contains 2^m edge-disjoint $2^{\lceil \log(1/\varepsilon) \rceil + 1}$ -edge-connected spanning subgraphs (formed by unions of the spanning trees). From Theorem 14 applied to these $2^{\lceil \log(1/\varepsilon) \rceil + 1}$ -edge-connected graphs, it follows that G^- contains 2^m edge-disjoint spanning trees T_1, \dots, T_{2^m} such that

$$\sum_{i \in [2^m]} \deg_{T_i} v \leq \frac{\deg_{G^-} v}{2^{\lceil \log(1/\varepsilon) \rceil + 1}} + 3 \cdot 2^m (\lceil \log(1/\varepsilon) \rceil + 1) \leq \frac{\varepsilon \deg_G v}{2} + 3 \cdot 2^m (\lceil \log(1/\varepsilon) \rceil + 1)$$

for every $v \in V(G)$. Moreover $3 \cdot 2^m (\lceil \log(1/\varepsilon) \rceil + 1) \leq \varepsilon L/2$ for L sufficiently large. The result follows. \square

Corollary 16 implies the following.

Corollary 17. *For any positive integers k_0 and δ_0 , there exists d_0 such that the edge set of every $16k_0$ -edge-connected stub graph G with minimum degree at least d_0 can be decomposed into a k_0 -edge-connected graph and a stub graph of minimum degree at least δ_0 .*

Proof. Let $d_0 = \max(2\delta_0, L)$, where L is as in Corollary 16 for $\varepsilon = 1/2$ and $m = \lceil \log k_0 \rceil$. Observe that 2^m in Corollary 16 is then less than $16k_0$ (and equal to $8k_0$ if k_0 is a power of two). Thus, by Corollary 16, there exist k_0 edge-disjoint spanning trees T_1, \dots, T_{k_0} such that $\sum_{i \in [k_0]} \deg_{T_i} v \leq 1/2 \deg_G v$ for every v . By the choice of d_0 , $1/2 \deg_G v \geq \delta_0$ for every v of G . Thus, $G \setminus (\bigcup_{i \in [k_0]} E(T_i))$ has minimum degree at least δ_0 . \square

We need the following easy consequence of Bézout identity for the proof of Lemma 10.

Observation 18. *Let a, b be positive integers and let c be an integer such that $c > ab$ and c is divisible by the greatest common divisor of a and b . Then there exist non-negative integers k_a and k_b such that $k_b < a$ satisfying $k_a a + k_b b = c$.*

Proof. Let d be the greatest common divisor of a and b and let k_c be an integer such that $k_c d = c$. By Bézout identity there exist integers x and y satisfying $xa + yb = d$, thus $xk_c a + yk_c b = c$. Note that then also $(xk_c + ib)a + (yk_c - ia)b = c$ for any integer i . Let i be such that $0 \leq xk_c - ia < a$. Then $k_b = xk_c - ia$ and $k_a = xk_c + ib$ satisfy the claim of the observation, in particular, since $c > ab$ and $k_b b < ab$, k_a must be positive. \square

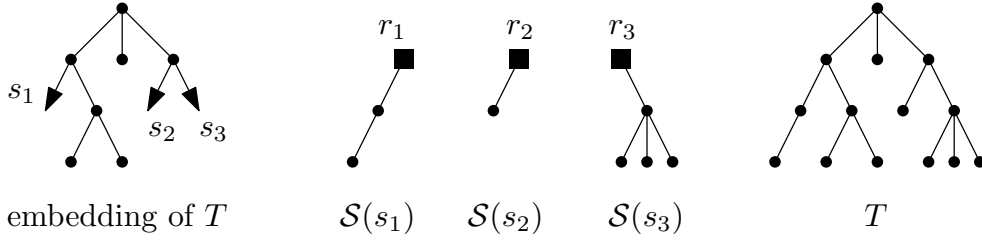


Figure 3: Embedding of a tree. Stubs are depicted as arrows, roots are depicted as squares.

We now prove Lemma 10.

Proof of Lemma 10. Let $k_0 = \max(K(T, T_1), K(T, T_2), k_T)$, where K is as in Lemma 11 and k_T is as in Theorem 2. Let $m = \max(|E(T)|, |E(T_1)|, |E(T_2)|)$, d_0 be as in Corollary 17 for k_0 and $\delta_0 = 64k_0m + 2|E(T)|(|E(T_1)| + |E(T_2)|)$. Let $k = 16k_0$ and $\delta = d_0 + 2k$.

Let $(H_0, G_0) := (\emptyset, G)$ and we repeat the following recursive procedure, obtaining pairs of stub graphs (H_i, G_i) until G_i is empty. Let n be the number of steps before G_n is empty.

If G_i is k -edge-connected or has only one vertex, we let $H_{i+1} = G_i$ and $G_{i+1} = \emptyset$. Assume that G_i is not k -edge-connected and has more than one vertex. Then, by Lemma 7, there exists a cut (A, B) in G_i of order at most $2k$ such that $G[A]$ is k -edge-connected or has only one vertex.

Let $H_{i+1} = G_i[A]$ and let C_{i+1} be the set of edges $uv \in E(G_i)$ with $u \in A$ and $v \in B$. Let G_{i+1} be the stub graph obtained from $G_i[B]$ by adding a stub with index $i + 1$ incident with v for every edge in C_{i+1} incident with v .

Observe that $V(G) = \bigcup_{i=0}^n V(H_i)$. Moreover, the graphs G_i and H_i have the following properties.

Claim 1. *For every $i \in [n]$, the following holds:*

- *There are at most $2k$ stubs with index i created during the procedure (i.e., in total, there are at most $2k$ stubs with index i in the stub graphs H_1, \dots, H_n),*
- *G_i has minimum degree at least δ ,*
- *H_i is k -edge-connected or has only one vertex,*
- *H_i has minimum degree at least $\delta - 2k$ (minimum of the empty set is ∞).*

Proof. Since there is a one-to-one correspondence between the stubs of index i and the edges in C_i and $|C_i| \leq 2k$, there are at most $2k$ stubs with index i . Moreover, $\deg_{G_{i-1}}(v) = \deg_{G_i}(v)$ for every $v \in V(G_i)$ and therefore G_i has minimum degree at least δ by induction.

By construction, H_i is k -edge-connected or has only one vertex and since G_{i-1} has minimum degree at least δ , H_i has minimum degree at least $\delta - 2k$ as required. \lrcorner

We call a $\{T, T_1, T_2\}$ -decomposition *balanced* if the numbers of copies of T_1 and T_2 differ by at most $|E(T)|$. Since G_n is empty, it has a balanced $\{T, T_1, T_2\}$ -decomposition \mathcal{T}_n (the trivial one). Next, we proceed inductively, constructing a balanced $\{T, T_1, T_2\}$ -decomposition \mathcal{T}_{i-1} of G_{i-1} from a balanced decomposition \mathcal{T}_i of G_i for $i \geq 2$. Moreover, in each step we increase the number of copies of T_1 in the decomposition by at most $|E(T)|$ and keep the number of copies of T_2 the same, or the other way round, we increase the number of copies of T_2 in the decomposition by at most $|E(T)|$ and keep the number of copies of T_1 the same.

In the last step, we construct a $\{T, T_1, T_2\}$ -decomposition of G from a balanced $\{T, T_1, T_2\}$ -decomposition of G_1 in a similar way, ensuring that the numbers of copies of T_1 and T_2 in the constructed decomposition are the same.

Roughly speaking, each step of construction has two phases: first, we replace every stub s in G_i of index i by a subtree $\mathcal{S}(s)$ in G_{i-1} . Then we decompose the remaining part of G_{i-1} using Lemma 11. In the last step, when $i=1$, we will proceed in a slightly different way to ensure that the resulting decomposition will contain the same number of copies of T_1 and T_2 .

More formally, given a balanced $\{T, T_1, T_2\}$ -decomposition \mathcal{T}_i of G_i , let $j = 1$ if the number of copies of T_1 in \mathcal{T}_i is smaller than the number of copies of T_2 and $j = 2$ otherwise. Recall that $V(G_{i-1}) = V(H_i) \dot{\cup} V(G_i)$ and $E(G_{i-1}) = E(H_i) \dot{\cup} C_i \dot{\cup} E(G_i) \setminus S_i(G_i)$.

If H_i has more than one vertex, it is k -edge-connected and thus, by Corollary 17, H_i contains a spanning subgraph R_i with minimum degree at least $4km$ such that $H'_i = H_i \setminus E(R_i)$ is $K(T, T_j)$ -edge-connected. If H_i has only one vertex, let $R_i = H_i$ and H'_i is an isolated vertex.

We replace every embedding of T, T_1 or T_2 in \mathcal{T}_i which contains a stub $s \in S_i$ by an embedding of T, T_1 or T_2 in $G_{i-1} \setminus E(H'_i)$. This yields a partial $\{T, T_1, T_2\}$ -decomposition \mathcal{T}' of G_{i-1} .

Moreover, \mathcal{T}' is such that $E(H'_i) \subseteq E(G_{i-1}) \setminus E(\mathcal{T}') \subseteq E(H_i)$. Thus, the stub graph $H''_i = (V(H_i), E(G_{i-1}) \setminus E(\mathcal{T}'))$ has a $\{T, T_j\}$ -decomposition \mathcal{T}'' that contains at most $|E(T)|$ copies of T_j by Lemma 11 and by Observation 12, because H''_i is either $K(T, T_j)$ -edge-connected or has only one vertex (and therefore no edges). Then, $\mathcal{T}_{i+1} = \mathcal{T}' \cup \mathcal{T}''$ forms a $\{T, T_1, T_2\}$ -decomposition of G_{i-1} . Since there are at most $|E(T)|$ copies of T_j in \mathcal{T}' , from the choice of j it follows that if the difference between the number of copies of T_1 and T_2 in \mathcal{T}_i was at most $|E(T)|$, the difference in \mathcal{T}_{i-1} is also at most $|E(T)|$.

Let $K \in \mathcal{T}_i$ be an embedding of T, T_1 or T_2 containing a stub s with index i (i.e., $s \in S_i(G_i)$). Note that K contains at most one stub in $S_i(G_i)$. For each such K , we construct $K' \in \mathcal{T}'$ such that $K \setminus S_i(G_i) \subseteq K'$ in the following way. We assume that K is an embedding of T , the construction for T_1 and T_2 is analogous.

Let \mathcal{I} be the set of the indices of the stubs in K . Let v be the vertex incident with s and let uv be the edge in C_i corresponding to the stub s . We find an embedding S_s of $\mathcal{S}(s)$ such that

- $v \in V(S_s)$ and it corresponds to the root of $\mathcal{S}(s)$,
- $uv \in E(S_s)$ and $S_s \setminus v \subseteq R_i$, and
- no stub in S_s has its index in \mathcal{I} .

Then, $K' := S_s \cup (K \setminus s)$ is an embedding of T in G_{i-1} .

Moreover, we ensure that $E(S_{s_1})$ and $E(S_{s_2})$ are disjoint for every two distinct stubs $s_1, s_2 \in S_i(G_i)$. Thus, the edge sets of the embeddings in \mathcal{T}' will be mutually disjoint.

We construct an embedding S_s of $\mathcal{S}(s)$ greedily. Starting from S_s which consists of the root in v and the edge uv , we add edges and stubs one by one. At the same time, we remove the used edges and stubs from R_i , making sure that no edge and no stub is used in more than one embedding. Let \mathcal{I}' be the set of the indices of the stubs in S_s . Assume that S_s is not yet an embedding of $\mathcal{S}(s)$ and let $w \in R_i$ be a vertex of S_s to which we need to add an edge or a stub. We argue that either w has a neighbor w' in $R_i \setminus S_s$ and therefore we can extend S_s by the edge ww' (removing ww' from R_i) or w is incident with a stub in R_i such that its index is not in $\mathcal{I} \cup \mathcal{I}'$. This is indeed the case; at the beginning, w had degree at least $4km$ in R_i , at most $(2k-1)m$ edges and stubs from R_i were removed by embedding trees corresponding to stubs in $S_i(G_i) \setminus \{s\}$ and at most m edges and stubs incident with v were removed or cannot be used for extending S_s because the other endpoint of the edge is already in S_s . This leaves at least $2km$ available edges and stubs incident with v . Since $|\mathcal{I} \cup \mathcal{I}'| < m$, R_i contains a stub incident with w such that its index is not in $\mathcal{I} \cup \mathcal{I}'$, or an edge ww' with $w' \notin S_s$.

At the last step, it remains to construct a $\{T, T_1, T_2\}$ -decomposition of G from a balanced $\{T, T_1, T_2\}$ -decomposition of G_1 . Note that H_1 is not a single vertex and has no stubs. Let R_1, H'_1 be subgraphs of H_1 defined as before, in particular, H'_1 is k_T -edge-connected (since $k_0 \geq k_T$).

Let t, t_1 and t_2 be the numbers of embeddings of T, T_1 and T_2 in \mathcal{T}_1 respectively. Without loss of generality, we assume that $t_1 \leq t_2$.

Since $|E(G)|$ is divisible by the greatest common divisor of $|E(T)|$ and $|E(T_1)| + |E(T_2)|$, $|E(G)| - t|E(T)| - t_2(|E(T_1)| + |E(T_2)|)$ is also divisible by the greatest common divisor of $|E(T)|$ and $|E(T_1)| + |E(T_2)|$. By Observation 18, there exists an integer $0 \leq t' < |E(T)|$ such that $|E(G)| - t|E(T)| - t_2(|E(T_1)| + |E(T_2)|) - t'(|E(T_1)| + |E(T_2)|)$ is divisible by $|E(T)|$.

We greedily construct $t' + (t_2 - t_1)$ copies of T_1 and t' copies of T_2 in R_1 , denote the resulting partial $\{T_1, T_2\}$ -decomposition by \mathcal{T}^* and remove its edges from R_1 . Note that the minimum degree of R_1 decreases by at most $2|E(T)|(|E(T_1)| +$

$|E(T_2)|$) and thus it is still at least $4km$. Thus, we can proceed in the same way as above, i.e., we construct a partial $\{T, T_1, T_2\}$ -decomposition \mathcal{T}' from \mathcal{T}_1 by expanding the stubs, using the remaining edges of R_1 .

Then, $\mathcal{T}^* \cup \mathcal{T}'$ is a partial $\{T, T_1, T_2\}$ -decomposition that contains the same number of copies of T_1 and T_2 . As before, let H_1'' be a graph obtained from H_1' by adding unused edges of R_1 . By our choice of t' , the number of edges of H_1'' is divisible by $|E(T)|$ and thus by Theorem 2 it has a T -decomposition \mathcal{T}'' . Then, $\mathcal{T}^* \cup \mathcal{T}' \cup \mathcal{T}''$ is a $\{T, T_1, T_2\}$ -decomposition that contains the same number of copies of T_1 and T_2 . \square

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