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# A FABER-KRAHN-TYPE INEQUALITY FOR THE LAPLACIAN WITH DRIFT UNDER ROBIN BOUNDARY CONDITION

FRANÇOIS HAMEL AND EMMANUEL RUSS

ABSTRACT. We prove a Faber-Krahn-type inequality for the Laplacian with drift under Robin boundary condition, provided that the  $\beta$  parameter in the Robin condition is large enough. The proof relies on a compactness argument, on the convergence of Robin eigenvalues to Dirichlet eigenvalues when  $\beta$  goes to infinity, and on a strict Faber-Krahn-type inequality under Dirichlet boundary condition. We also show the existence and uniqueness of drifts  $v$  satisfying some  $L^\infty$  constraints and minimizing or maximizing the principal eigenvalue of  $-\Delta + v \cdot \nabla$  in a fixed domain and with a fixed parameter  $\beta > 0$  in the Robin condition.

*To Nikolai Nadirashvili, with admiration*

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## 1. INTRODUCTION

Throughout this paper,  $d \geq 1$  is an integer. For all  $x \in \mathbb{R}^d$ , denote by  $|x|$  the Euclidean norm of  $x$  and define

$$e_r(x) := \frac{x}{|x|} \text{ for all } x \in \mathbb{R}^d \setminus \{0\}.$$

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Let  $\Omega \subset \mathbb{R}^d$  be a bounded domain (connected open set) of class  $C^2$ , with outward unit normal  $\nu$  on  $\partial\Omega$ . If  $v \in L^\infty(\Omega, \mathbb{R}^d)$  is a bounded measurable vector field, set

$$\|v\|_\infty := \||v|\|_\infty.$$

We are interested in the principal eigenvalue of the operator  $-\Delta + v \cdot \nabla$  in  $\Omega$  under Robin boundary condition on  $\partial\Omega$ . More precisely, let  $\beta > 0$ . By [15, Theorem A.4] and Krein-Rutman theory [2], there exists a principal eigenvalue  $\lambda_1^\beta(\Omega, v)$  of the problem

$$(1.1) \quad \begin{cases} -\Delta \varphi_{\Omega, v}^\beta + v \cdot \nabla \varphi_{\Omega, v}^\beta = \lambda_1^\beta(\Omega, v) \varphi_{\Omega, v}^\beta & \text{in } \Omega, \\ \frac{\partial \varphi_{\Omega, v}^\beta}{\partial \nu} + \beta \varphi_{\Omega, v}^\beta = 0 & \text{on } \partial\Omega. \end{cases}$$

This principal eigenvalue is simple, the corresponding eigenfunction  $\varphi_{\Omega, v}^\beta$  is positive in  $\bar{\Omega}$  by [2, Theorem 4.5], and none of the other eigenvalues corresponds to a positive eigenfunction (see the discussion after [15, Theorem 1.3]). By  $W^{2,p}$  elliptic regularity ([15, Theorem A.29]), the function  $\varphi_{\Omega, v}^\beta$  belongs to  $W^{2,p}(\Omega)$  for all  $p \in [1, \infty)$  and then to  $C^{1,\alpha}(\bar{\Omega})$  for all  $\alpha \in (0, 1)$ . The first line in (1.1) is therefore understood almost everywhere in  $\Omega$ , while the second line is understood in the classical sense. We usually normalize  $\varphi_{\Omega, v}^\beta$  by

$$(1.2) \quad \max_{\bar{\Omega}} \varphi_{\Omega, v}^\beta = 1.$$

Moreover, there holds

$$\lambda_1^\beta(\Omega, v) > 0.$$

The  $C^2$  smoothness of  $\partial\Omega$  is used to derive the regularity of  $\varphi_{\Omega, v}^\beta$ . So is it in some arguments of the proofs of the following main results. We leave as an open question the derivation of Faber-Krahn-type inequalities for weaker formulations of (1.1) under weaker assumptions on  $\partial\Omega$ .

Fix  $\tau \geq 0$  and  $m > 0$ . We are interested in the infimum of  $\lambda_1^\beta(\Omega, v)$  when  $\Omega$  and  $v$  vary under the constraints  $|\Omega| = m$  (throughout the paper,  $|A|$  denotes the  $n$ -dimensional Lebesgue measure of  $A$  for all measurable sets  $A \subset \mathbb{R}^d$ ) and

$$(1.3) \quad \|v\|_\infty \leq \tau.$$

In the sequel,  $\Omega^*$  stands for the Euclidean ball centered at 0 such that  $|\Omega^*| = |\Omega|$ .

Our main result states that, when  $\Omega$  is not a ball,  $\lambda_1^\beta(\Omega, v)$  is (strictly) greater than the corresponding quantity in  $\Omega^*$  when  $v = \tau e_r$ , provided that  $\beta$  is large enough:

**Theorem 1.1.** *Let  $\Omega \subset \mathbb{R}^d$  be a bounded  $C^2$  domain and  $\tau \geq 0$ . Assume that  $\Omega$  is not a ball. Then there exist  $\beta_0 > 0$  and  $\varepsilon > 0$  with the following*

property:

$$(1.4) \quad \forall \beta \geq \beta_0, \forall v \in L^\infty(\Omega, \mathbb{R}^d) \text{ such that } \|v\|_\infty \leq \tau, \\ \lambda_1^\beta(\Omega, v) \geq \lambda_1^\beta(\Omega^*, \tau e_r) + \varepsilon.$$

When  $\tau = 0$ , *i.e.* when  $-\Delta + v \cdot \nabla = -\Delta$  is merely (minus) the Laplacian, it was proved in [9, 10, 13, 14] that, for all  $\beta > 0$ , the following Bossel-Daners inequality is satisfied:

$$(1.5) \quad \lambda_1^\beta(\Omega, 0) \geq \lambda_1^\beta(\Omega^*, 0),$$

and equality holds if and only if  $\Omega = \Omega^*$  up to translation. When  $\tau \neq 0$ , Theorem 1.1 provides on the one hand a quantified strict inequality if  $\Omega$  is not a ball, but the conclusion is only established above some threshold for  $\beta$ , contrary to [10, Theorem 1.1], and it actually can not hold for all  $\beta > 0$ , since

$$\lim_{\beta \rightarrow 0} \lambda_1^\beta(\Omega, v) = \lim_{\beta \rightarrow 0} \lambda_1^\beta(\Omega^*, \tau e_r) = 0$$

for each  $v \in L^\infty(\Omega, \mathbb{R}^d)$  (as follows from Lemma 2.2 below). On the other hand, when  $\Omega = \Omega^*$ , the uniqueness part in Theorem 1.3 below ensures that, for all  $v \in L^\infty(\Omega^*, \mathbb{R}^d)$  with  $\|v\|_\infty \leq \tau$ , if  $v \neq \tau e_r$ , then  $\lambda_1^\beta(\Omega^*, v) > \lambda_1^\beta(\Omega^*, \tau e_r)$  for all  $\beta > 0$ .

The following question nevertheless remains open:

**Open problem 1.2.** *Let  $\Omega \subset \mathbb{R}^d$  be a bounded  $C^2$  domain,  $\tau \geq 0$  and  $v \in L^\infty(\Omega, \mathbb{R}^d)$  with  $\|v\|_\infty \leq \tau$ . Does the inequality*

$$\lambda_1^\beta(\Omega, v) \geq \lambda_1^\beta(\Omega^*, \tau e_r)$$

*hold for all  $\beta > 0$ ?*

Recall that, under Dirichlet boundary condition, it was proved in [17, Theorem 1.1] and [18, Remark 6.9] that, whenever (1.3) holds,

$$(1.6) \quad \lambda_1^D(\Omega, v) \geq \lambda_1^D(\Omega^*, \tau e_r),$$

where  $\lambda_1^D(\Omega, v)$  stands for the principal eigenvalue of  $-\Delta + v \cdot \nabla$  under Dirichlet boundary condition. Moreover, equality holds in (1.6) if and only if, up to translation,  $\Omega = \Omega^*$  and  $v = \tau e_r$ . The inequalities (1.4)-(1.6) are called Faber-Krahn-type inequalities. This terminology originates from the results of Faber [16] and Krahn [22, 23], who proved that

$$\lambda_1^D(\Omega, 0) \geq \lambda_1^D(\Omega^*, 0),$$

with equality if and only if, up to translation,  $\Omega = \Omega^*$ . The latter inequality means that a radially symmetric membrane which is fixed at its boundary has the lowest fundamental tone among all equimeasurable membranes, answering a conjecture of Rayleigh [29] set in dimension  $d = 2$ . Since these pioneering papers, much work has been done on various related optimization eigenvalue problems for elliptic operators, for instance on higher eigenvalues or functions of the eigenvalues of  $-\Delta$  under Dirichlet boundary

condition [3, 4, 6, 11, 12, 26, 27, 28, 32], under Neumann boundary condition [28, 30, 31], or for the first eigenvalue of  $\Delta^2$  under boundary conditions  $\varphi = \frac{\partial \varphi}{\partial \nu} = 0$  on  $\partial\Omega$  [5, 25]. We refer to the surveys [7, 20, 21] for many more references on these topics.

The second main result deals with an optimization problem when the domain  $\Omega$  is fixed and  $v$  varies under the constraint (1.3). Define, for all  $\beta > 0$  and  $\tau \geq 0$  given:

$$(1.7) \quad \underline{\lambda}^\beta(\Omega, \tau) := \inf \left\{ \lambda_1^\beta(\Omega, v) : \|v\|_\infty \leq \tau \right\}$$

and

$$(1.8) \quad \bar{\lambda}^\beta(\Omega, \tau) := \sup \left\{ \lambda_1^\beta(\Omega, v) : \|v\|_\infty \leq \tau \right\}.$$

We claim that these lower and upper bounds are positive real numbers, are uniquely reached and provide an identity relating the optimizing vector fields and the corresponding eigenfunctions:

**Theorem 1.3.** *[Optimization in fixed domains] Let  $\Omega \subset \mathbb{R}^d$  be a bounded  $C^2$  domain,  $\tau \geq 0$  and  $\beta > 0$ .*

- (1) *There exists a unique  $\underline{v} \in L^\infty(\Omega, \mathbb{R}^d)$  meeting  $\|\underline{v}\|_\infty \leq \tau$  such that  $\underline{\lambda}^\beta(\Omega, \tau) = \lambda_1^\beta(\Omega, \underline{v})$ . One has  $|\underline{v}(x)| = \tau$  for almost every  $x \in \Omega$ . Moreover, if  $\underline{\varphi} := \varphi_{\Omega, \underline{v}}^\beta$  is the corresponding eigenfunction, then*

$$(1.9) \quad \underline{v}(x) \cdot \nabla \underline{\varphi}(x) = -\tau |\nabla \underline{\varphi}(x)| \text{ for almost every } x \in \Omega.$$

*Lastly, if  $\lambda \in \mathbb{R}$  and  $\phi \in \bigcap_{1 \leq p < \infty} W^{2,p}(\Omega)$  satisfy*

$$(1.10) \quad \begin{cases} -\Delta \phi - \tau |\nabla \phi| = \lambda \phi & \text{and } \phi \geq 0 \text{ in } \Omega, \\ \frac{\partial \phi}{\partial \nu} + \beta \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

*and  $\max_{\bar{\Omega}} \phi = 1$ , then  $\lambda = \underline{\lambda}^\beta(\Omega, \tau)$  and  $\phi = \underline{\varphi}$  in  $\bar{\Omega}$ .*

- (2) *Similarly, there exists a unique  $\bar{v} \in L^\infty(\Omega, \mathbb{R}^d)$  meeting  $\|\bar{v}\|_\infty \leq \tau$  such that  $\bar{\lambda}^\beta(\Omega, \tau) = \lambda_1^\beta(\Omega, \bar{v})$ . One has  $|\bar{v}(x)| = \tau$  for almost every  $x \in \Omega$ . Moreover, if  $\bar{\varphi} := \varphi_{\Omega, \bar{v}}^\beta$  is the corresponding eigenfunction, then*

$$(1.11) \quad \bar{v}(x) \cdot \nabla \bar{\varphi}(x) = \tau |\nabla \bar{\varphi}(x)| \text{ for almost every } x \in \Omega.$$

*Lastly, if  $\lambda \in \mathbb{R}$  and  $\phi \in \bigcap_{1 \leq p < \infty} W^{2,p}(\Omega)$  satisfy*

$$(1.12) \quad \begin{cases} -\Delta \phi + \tau |\nabla \phi| = \lambda \phi & \text{and } \phi \geq 0 \text{ in } \Omega, \\ \frac{\partial \phi}{\partial \nu} + \beta \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

*and  $\max_{\bar{\Omega}} \phi = 1$ , then  $\lambda = \bar{\lambda}^\beta(\Omega, \tau)$  and  $\phi = \bar{\varphi}$  in  $\bar{\Omega}$ .*

- (3) *If  $\Omega = \Omega^*$ , then  $\underline{v} = \tau e_r$ ,  $\bar{v} = -\tau e_r$  in  $\Omega^*$  and the functions  $\underline{\varphi}$  and  $\bar{\varphi}$  are radially decreasing in  $\Omega^*$ .*

**Remark 1.4.** Notice that, in part (1) of Theorem 1.3, from elliptic regularity theory applied to (1.10), since  $\phi$  and  $|\nabla\phi|$  belong to  $C^{0,\alpha}(\overline{\Omega})$  for all  $\alpha \in (0, 1)$ , the function  $\phi$  belongs to  $C_{loc}^{2,\alpha}(\Omega)$  for all  $\alpha \in (0, 1)$ , and the first line of (1.10) holds in the classical sense in  $\Omega$ . Similarly, in part (2), the solution  $\phi$  of (1.12) belongs to  $C_{loc}^{2,\alpha}(\Omega)$  for all  $\alpha \in (0, 1)$ , and the first line of (1.12) holds in the classical sense in  $\Omega$ .

We point out that similar properties had been derived in [17, 18] for the extremal quantities  $\underline{\lambda}(\Omega, \tau)$  and  $\overline{\lambda}(\Omega, \tau)$  defined like  $\underline{\lambda}^\beta(\Omega, \tau)$  and  $\overline{\lambda}^\beta(\Omega, \tau)$  in (1.7)-(1.8) with the Dirichlet eigenvalues  $\lambda_1(\Omega, v)$  instead of the Robin ones  $\lambda_1^\beta(\Omega, v)$ . The asymptotic behavior as  $\tau \rightarrow +\infty$  of the eigenfunctions associated with  $\underline{\lambda}(\Omega, \tau)$  was analyzed in [19].

The paper is organized as follows. In Section 2, we provide comparisons results between Robin, Dirichlet and Neumann eigenvalues in a fixed domain and for a given drift, and prove convergence of the Robin eigenvalues when  $\beta \rightarrow +\infty$  (resp. when  $\beta \rightarrow 0$ ) to the corresponding Dirichlet (resp. Neumann) eigenvalues. Section 3 is devoted to the proof of Theorem 1.3. Finally, we establish Theorem 1.1 in Section 4.

## 2. COMPARISONS AND CONVERGENCE RESULTS BETWEEN ROBIN, DIRICHLET AND NEUMANN PRINCIPAL EIGENVALUES

This section is concerned with some comparisons and convergence results for Robin and Dirichlet principal eigenvalues in a given domain  $\Omega$ . The results will be used in the proofs of the main Theorems 1.1 and 1.3.

We first start with an auxiliary comparison lemma between sub- and super-solutions.

**Lemma 2.1.** *Let  $\mu \in \mathbb{R}$ ,  $\beta \geq 0$ , and  $v \in L^\infty(\Omega, \mathbb{R}^d)$ . Let  $\psi, \varphi \in W^{2,p}(\Omega)$  for all  $1 \leq p < \infty$ , such that  $\psi \geq 0$  and  $\varphi \geq 0$  in  $\Omega$ ,  $\|\psi\|_\infty = \|\varphi\|_\infty = 1$ , and*

$$\begin{cases} -\Delta\psi + v \cdot \nabla\psi \geq \mu\psi & \text{a.e. in } \Omega, \\ -\Delta\varphi + v \cdot \nabla\varphi \leq \mu\varphi & \text{a.e. in } \Omega. \end{cases}$$

Assume also that

$$\frac{\partial\psi}{\partial\nu} + \beta\psi \geq 0 \geq \frac{\partial\varphi}{\partial\nu} + \beta\varphi \text{ on } \partial\Omega.$$

Then  $\psi = \varphi$  in  $\overline{\Omega}$ .

*Proof.* The argument is reminiscent of the proof of [17, Lemma 2.1]. Remember first that  $\psi$  and  $\varphi$  belong to  $C^{1,\alpha}(\overline{\Omega})$  for all  $\alpha \in (0, 1)$ . Furthermore,  $\psi > 0$  in  $\Omega$  from the interior strong maximum principle (otherwise,  $\psi$  would be identically 0 in  $\Omega$ , contradicting  $\|\psi\|_\infty = 1$ ). Observe now that  $\psi > 0$  on  $\partial\Omega$ . Indeed, if there exists  $x_0 \in \partial\Omega$  such that  $\psi(x_0) = 0$ , then the Hopf lemma shows that  $\frac{\partial\psi}{\partial\nu}(x_0) < 0$ , which is impossible by the boundary condition satisfied by  $\psi$ . Thus, being continuous in  $\overline{\Omega}$ ,  $\psi$  is bounded below by a

positive constant, so that there exists  $\gamma > 0$  such that  $\gamma\psi > \varphi$  in  $\Omega$ . Define

$$\gamma^* := \inf \{ \gamma > 0 : \gamma\psi > \varphi \text{ in } \Omega \}$$

and  $w := \gamma^*\psi - \varphi$ . Note that, since  $\varphi \geq 0$  in  $\Omega$  and  $\|\varphi\|_\infty = 1$ ,  $\gamma^* > 0$ . The function  $w$  is nonnegative in  $\bar{\Omega}$ ,

$$\frac{\partial w}{\partial \nu} + \beta w \geq 0 \text{ on } \partial\Omega$$

and

$$-\Delta w + v \cdot \nabla w - \mu w \geq 0 \text{ a.e. in } \Omega.$$

If  $w > 0$  in  $\Omega$ , then, as before,  $w$  is bounded below by a positive constant in  $\Omega$ , so there exists  $\delta > 0$  such that  $w > \delta\varphi$  in  $\Omega$ , which entails in turn

$$\frac{\gamma^*}{1 + \delta} \psi > \varphi \text{ in } \Omega,$$

contradicting the definition of  $\gamma^*$ , since  $\gamma^* > 0$ . Therefore, there exists  $x_0 \in \Omega$  such that  $w(x_0) = 0$ , and since  $w \geq 0$  in  $\Omega$ , the strong maximum principle indicates that  $w(x) = 0$  everywhere in  $\Omega$  and then in  $\bar{\Omega}$  by continuity, meaning that  $\gamma^*\psi = \varphi$  in  $\bar{\Omega}$ . The condition  $\|\varphi\|_\infty = \|\psi\|_\infty = 1$  finally yields  $\varphi = \psi$  in  $\bar{\Omega}$ .  $\square$

Let now  $\Omega \subset \mathbb{R}^d$  be a bounded  $C^2$  domain and  $v \in L^\infty(\Omega, \mathbb{R}^d)$ . Denote by  $\lambda_1^D(\Omega, v)$  the principal eigenvalue of  $-\Delta + v \cdot \nabla$  in  $\Omega$  under Dirichlet boundary condition and by  $\varphi_{\Omega, v}^D \in \bigcap_{1 \leq p < \infty} W^{2,p}(\Omega)$  the corresponding principal eigenfunction (which is positive in  $\Omega$ ) normalized by

$$\|\varphi_{\Omega, v}^D\|_\infty = 1.$$

We will show that the map  $\beta \mapsto \lambda_1^\beta(\Omega, v)$  is increasing in  $(0, \infty)$  and converges to  $\lambda_1^D(\Omega, v)$  at infinity, and to 0 (that is, the principal eigenvalue of  $-\Delta + v \cdot \nabla$  in  $\Omega$  under Neumann boundary condition) as  $\beta \rightarrow 0$ :

**Lemma 2.2.** *Let  $\Omega \subset \mathbb{R}^d$  be a bounded  $C^2$  domain and  $v \in L^\infty(\Omega, \mathbb{R}^d)$ . Then the map  $\beta \mapsto \lambda_1^\beta(\Omega, v)$  is increasing in  $(0, +\infty)$ . Furthermore,*

$$(2.1) \quad \lim_{\beta \rightarrow +\infty} \lambda_1^\beta(\Omega, v) = \lambda_1^D(\Omega, v).$$

and

$$\lim_{\beta \rightarrow 0} \lambda_1^\beta(\Omega, v) = 0.$$

*Proof.* Let  $0 < \beta_1 < \beta_2$  and assume by way of contradiction that  $\lambda_1^{\beta_2}(\Omega, v) \leq \lambda_1^{\beta_1}(\Omega, v)$ . Set  $\varphi_1 := \varphi_{\Omega, v}^{\beta_1}$  and  $\varphi_2 := \varphi_{\Omega, v}^{\beta_2}$ . Both functions  $\varphi_1$  and  $\varphi_2$  are positive in  $\bar{\Omega}$  and they satisfy

$$\begin{cases} -\Delta \varphi_1 + v \cdot \nabla \varphi_1 = \lambda_1^{\beta_1}(\Omega, v) \varphi_1 & \text{a.e. in } \Omega, \\ -\Delta \varphi_2 + v \cdot \nabla \varphi_2 = \lambda_1^{\beta_2}(\Omega, v) \varphi_2 \leq \lambda_1^{\beta_1}(\Omega, v) \varphi_2 & \text{a.e. in } \Omega, \end{cases}$$

together with

$$(2.2) \quad \frac{\partial \varphi_1}{\partial \nu} + \beta_1 \varphi_1 = 0 = \frac{\partial \varphi_2}{\partial \nu} + \beta_2 \varphi_2 > \frac{\partial \varphi_2}{\partial \nu} + \beta_1 \varphi_2 \text{ on } \partial \Omega.$$

Lemma 2.1 applied with  $(\mu, \beta, \psi, \varphi) := (\lambda_1^{\beta_1}(\Omega, v), \beta_1, \varphi_1, \varphi_2)$  then entails  $\varphi_1 = \varphi_2$  in  $\bar{\Omega}$ , contradicting the strict inequality in (2.2). Finally,

$$\lambda_1^{\beta_1}(\Omega, v) < \lambda_1^{\beta_2}(\Omega, v),$$

and the map  $\beta \mapsto \lambda_1^\beta(\Omega, v)$  is increasing in  $(0, +\infty)$ .

Let now  $\beta > 0$  and assume by way of contradiction that  $\lambda_1^\beta(\Omega, v) \geq \lambda_1^D(\Omega, v)$ . Both functions  $\varphi_{\Omega, v}^\beta$  and  $\varphi_{\Omega, v}^D$  are positive in  $\Omega$  and they satisfy

$$\begin{cases} -\Delta \varphi_{\Omega, v}^\beta + v \cdot \nabla \varphi_{\Omega, v}^\beta = \lambda_1^\beta(\Omega, v) \varphi_{\Omega, v}^\beta & \text{a.e. in } \Omega, \\ -\Delta \varphi_{\Omega, v}^D + v \cdot \nabla \varphi_{\Omega, v}^D = \lambda_1^D(\Omega, v) \varphi_{\Omega, v}^D \leq \lambda_1^\beta(\Omega, v) \varphi_{\Omega, v}^D & \text{a.e. in } \Omega. \end{cases}$$

Furthermore, the Hopf lemma implies that  $\frac{\partial \varphi_{\Omega, v}^D}{\partial \nu} < 0$  on  $\partial \Omega$ , whence

$$(2.3) \quad \frac{\partial \varphi_{\Omega, v}^\beta}{\partial \nu} + \beta \varphi_{\Omega, v}^\beta = 0 > \frac{\partial \varphi_{\Omega, v}^D}{\partial \nu} + \beta \varphi_{\Omega, v}^D \text{ on } \partial \Omega.$$

Lemma 2.1 applied with  $(\mu, \beta, \psi, \varphi) := (\lambda_1^\beta(\Omega, v), \beta, \varphi_{\Omega, v}^\beta, \varphi_{\Omega, v}^D)$  then entails  $\varphi_{\Omega, v}^\beta = \varphi_{\Omega, v}^D$  in  $\bar{\Omega}$ , contradicting the strict inequality in (2.3) (or the fact that  $\varphi_{\Omega, v}^\beta > 0 = \varphi_{\Omega, v}^D$  on  $\partial \Omega$ ). Finally,

$$\lambda_1^\beta(\Omega, v) < \lambda_1^D(\Omega, v)$$

for all  $\beta > 0$ .

Let us now turn to the proof of (2.1). Pick up any increasing sequence  $(\beta_k)_{k \in \mathbb{N}}$  of positive real numbers with  $\lim_{k \rightarrow +\infty} \beta_k = +\infty$  and set  $\lambda_k := \lambda_1^{\beta_k}(\Omega, v)$  for all  $k \in \mathbb{N}$ . The sequence  $(\lambda_k)_{k \in \mathbb{N}}$  is increasing and bounded above by  $\lambda_1^D(\Omega, v)$  and therefore converges to some  $\mu \leq \lambda_1^D(\Omega, v)$ . For all  $k$ , if  $\varphi_k$  is defined as  $\varphi_k := \theta_k \varphi_{\Omega, v}^{\beta_k}$  with  $\theta_k > 0$  such that  $\|\varphi_k\|_{L^2(\Omega)} = 1$ , then

$$(2.4) \quad \begin{cases} -\Delta \varphi_k + v \cdot \nabla \varphi_k = \lambda_k \varphi_k & \text{a.e. in } \Omega, \\ \frac{\partial \varphi_k}{\partial \nu} + \beta_k \varphi_k = 0 & \text{on } \partial \Omega. \end{cases}$$

We claim that the sequence  $(\varphi_k)_{k \in \mathbb{N}}$  is bounded in  $H^1(\Omega)$ . Indeed, for all  $k \in \mathbb{N}$ ,

$$\begin{aligned} \lambda_k \int_{\Omega} \varphi_k^2 &= - \int_{\Omega} \varphi_k \Delta \varphi_k + \int_{\Omega} (v \cdot \nabla \varphi_k) \varphi_k \\ &= \int_{\Omega} |\nabla \varphi_k|^2 - \int_{\partial \Omega} \varphi_k \frac{\partial \varphi_k}{\partial \nu} + \int_{\Omega} (v \cdot \nabla \varphi_k) \varphi_k \\ &= \int_{\Omega} |\nabla \varphi_k|^2 + \beta_k \int_{\partial \Omega} \varphi_k^2 + \int_{\Omega} (v \cdot \nabla \varphi_k) \varphi_k. \end{aligned}$$



From this, we derive, for all  $\varepsilon > 0$ ,

$$(2.5) \quad \begin{aligned} \int_{\Omega} |\nabla \varphi_k|^2 + \beta_k \int_{\partial\Omega} \varphi_k^2 &\leq \lambda_k \int_{\Omega} \varphi_k^2 + \|v\|_{\infty} \int_{\Omega} \varphi_k |\nabla \varphi_k| \\ &\leq \left( \lambda_k + \frac{1}{2\varepsilon} \|v\|_{\infty} \right) \int_{\Omega} \varphi_k^2 + \frac{\varepsilon}{2} \|v\|_{\infty} \int_{\Omega} |\nabla \varphi_k|^2. \end{aligned}$$

Provided  $\varepsilon \|v\|_{\infty} < 2$ , recalling that the sequences  $(\lambda_k)_{k \in \mathbb{N}}$  and  $(\|\varphi_k\|_{L^2(\Omega)})_{k \in \mathbb{N}}$  are bounded, one obtains that the sequence  $(\varphi_k)_{k \in \mathbb{N}}$  is bounded in  $H^1(\Omega)$ , that is, there is  $M \in \mathbb{R}_+$  such that

$$\|\varphi_k\|_{H^1(\Omega)} = \sqrt{\|\varphi_k\|_{L^2(\Omega)}^2 + \|\nabla \varphi_k\|_{L^2(\Omega)}^2} \leq M$$

for all  $k \in \mathbb{N}$ . Therefore, there exists  $\varphi \in H^1(\Omega)$  such that, up to a subsequence,

$$(2.6) \quad \begin{aligned} \varphi_k \rightharpoonup \varphi \text{ weakly in } H^1(\Omega), \quad \varphi_k \rightarrow \varphi \text{ strongly in } L^2(\Omega), \\ \varphi_k \rightarrow \varphi \text{ a.e. in } \Omega, \end{aligned}$$

as  $k \rightarrow +\infty$ , whence

$$(2.7) \quad \|\varphi\|_{L^2(\Omega)} = 1 \quad \text{and} \quad \varphi \geq 0 \text{ a.e. in } \Omega.$$

Moreover, since  $\lim_{k \rightarrow +\infty} \beta_k = +\infty$ , (2.5) shows that

$$(2.8) \quad \lim_{k \rightarrow +\infty} \text{tr}(\varphi_k) = 0 \text{ strongly in } L^2(\partial\Omega),$$

where  $\text{tr} : H^1(\Omega) \rightarrow L^2(\partial\Omega)$  denotes the trace operator. Since this trace operator is compact from  $H^1(\Omega)$  to  $L^2(\partial\Omega)$  with the topologies induced by the norms (see [24, Corollary 18.4]), and since  $\varphi_k \rightharpoonup \varphi$  weakly in  $H^1(\Omega)$  as  $k \rightarrow +\infty$ , it follows that  $\text{tr}(\varphi_k) \rightarrow \text{tr}(\varphi)$  strongly in  $L^2(\partial\Omega)$  as  $k \rightarrow +\infty$ , whence  $\text{tr}(\varphi) = 0$  by (2.8), meaning that  $\varphi \in H_0^1(\Omega)$ . Consider now  $\psi \in C_c^\infty(\Omega)$ . One has

$$\begin{aligned} \int_{\Omega} \nabla \varphi \cdot \nabla \psi + \int_{\Omega} (v \cdot \nabla \varphi) \psi &= \lim_{k \rightarrow +\infty} \int_{\Omega} \nabla \varphi_k \cdot \nabla \psi + \int_{\Omega} (v \cdot \nabla \varphi_k) \psi \\ &= \lim_{k \rightarrow +\infty} \lambda_k \int_{\Omega} \varphi_k \psi \\ &= \mu \int_{\Omega} \varphi \psi, \end{aligned}$$

which means that  $\varphi$  is an  $H_0^1(\Omega)$  weak solution of

$$\begin{cases} -\Delta \varphi + v \cdot \nabla \varphi = \mu \varphi & \text{in } \Omega, \\ \text{tr}(\varphi) = 0 & \text{on } \partial\Omega. \end{cases}$$

Elliptic  $H^2$  and  $W^{2,p}$  estimates show that  $\varphi \in W^{2,p}(\Omega)$  for all  $1 \leq p < \infty$ , and, since  $\varphi \geq 0$  in  $\Omega$  and  $\|\varphi\|_{L^2(\Omega)} = 1$ , the strong maximum principle entails that  $\varphi > 0$  in  $\Omega$ . Thus, by uniqueness of the principal eigenvalue of  $-\Delta + v \cdot \nabla$  under Dirichlet boundary condition, one gets that

$$\mu = \lambda_1^D(\Omega, v),$$

which ends the proof.

Lastly, let us investigate the limit of  $\lambda_1^\beta(\Omega, v)$  as  $\beta \rightarrow 0$ . Pick up any decreasing sequence  $(\beta_k)_{k \in \mathbb{N}}$  of positive real numbers with  $\lim_{k \rightarrow +\infty} \beta_k = 0$  and set  $\lambda_k := \lambda_1^{\beta_k}(\Omega, v)$  for all  $k \in \mathbb{N}$ . The sequence  $(\lambda_k)_{k \in \mathbb{N}}$  is decreasing and bounded below by 0, and therefore converges to some  $\lambda \geq 0$ . For all  $k$ , if  $\varphi_k$  is defined as  $\varphi_k := \theta_k \varphi_{\Omega, v}^{\beta_k}$  with  $\theta_k > 0$  such that  $\|\varphi_k\|_{L^2(\Omega)} = 1$ , then as above (2.4)-(2.5) still hold and there exists  $\varphi \in H^1(\Omega)$  satisfying (2.6)-(2.7), up to a subsequence. Pick now any  $\psi \in H^1(\Omega)$ . For all  $k \in \mathbb{N}$ , one has

$$\lambda_k \int_{\Omega} \varphi_k \psi = \int_{\Omega} \nabla \varphi_k \cdot \nabla \psi + \beta_k \int_{\partial\Omega} \varphi_k \psi + \int_{\Omega} (v \cdot \nabla \varphi_k) \psi.$$

But  $\beta_k \rightarrow 0$  as  $k \rightarrow +\infty$  and the sequence  $(\text{tr}(\varphi_k))_{k \in \mathbb{N}}$  is bounded in  $L^2(\partial\Omega)$  (since so is  $(\varphi_k)_{k \in \mathbb{N}}$  in  $H^1(\Omega)$ ). Hence, by (2.6), the passage to the limit as  $k \rightarrow +\infty$  in the above formula leads to

$$\lambda \int_{\Omega} \varphi \psi = \int_{\Omega} \nabla \varphi \cdot \nabla \psi + \int_{\Omega} (v \cdot \nabla \varphi) \psi.$$

In other words,  $\varphi$  is an  $H^1(\Omega)$  weak solution of

$$\begin{cases} -\Delta \varphi + v \cdot \nabla \varphi = \lambda \varphi & \text{in } \Omega, \\ \frac{\partial \varphi}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Elliptic  $H^2$  and  $W^{2,p}$  estimates show that  $\varphi \in W^{2,p}(\Omega)$  for all  $1 \leq p < \infty$ , and by (2.7) the strong maximum principle and Hopf lemma entail that  $\varphi > 0$  in  $\bar{\Omega}$ . Thus, by uniqueness of the principal eigenvalue of  $-\Delta + v \cdot \nabla$  under Neumann boundary condition, one gets that  $\lambda = 0$ . The proof of Lemma 2.2 is thereby complete.  $\square$

### 3. OPTIMIZATION OF THE PRINCIPAL EIGENVALUE IN A FIXED DOMAIN

This section is devoted to the proof of Theorem 1.3.

*Proof of Theorem 1.3. Part 1.* We first focus on the infimum problem and begin with the existence part. Let  $(v_k)_{k \in \mathbb{N}}$  be a sequence of vector fields in  $L^\infty(\Omega, \mathbb{R}^d)$  such that  $\|v_k\|_\infty \leq \tau$  for all  $k$  and

$$\lim_{k \rightarrow +\infty} \lambda_1^\beta(\Omega, v_k) = \underline{\lambda}^\beta(\Omega, \tau).$$

For all  $k \in \mathbb{N}$ , define  $\lambda_k := \lambda_1^\beta(\Omega, v_k)$  and let  $\varphi_k := \varphi_{\Omega, v_k}^\beta$  be the corresponding eigenfunction, normalized with  $\max_{\bar{\Omega}} \varphi_k = 1$ . Since  $(v_k)_{k \in \mathbb{N}}$  is bounded in  $L^\infty(\Omega, \mathbb{R}^d)$  and the sequence  $(\lambda_k)_{k \in \mathbb{N}}$  is bounded,  $W^{2,p}$  elliptic estimates ([15, Theorem A.29]) show that the sequence  $(\varphi_k)_{k \in \mathbb{N}}$  is bounded in  $W^{2,p}(\Omega)$  for all  $1 \leq p < \infty$ . Up to a subsequence, there exist  $\varphi \in \bigcap_{1 \leq p < \infty} W^{2,p}(\Omega)$  and  $f \in L^\infty(\Omega)$  such that, as  $k \rightarrow +\infty$ ,

$$\varphi_k \rightharpoonup \varphi \text{ weakly in } W^{2,p}(\Omega)$$

for all  $1 \leq p < \infty$ ,

$$\varphi_k \rightarrow \varphi \text{ strongly in } C^{1,\alpha}(\overline{\Omega})$$

for all  $\alpha \in (0, 1)$ , and

$$v_k \cdot \nabla \varphi_k \xrightarrow{*} f \text{ weak-* in } L^\infty(\Omega).$$

As a consequence,

$$-\Delta \varphi + f = \underline{\lambda}^\beta(\Omega, \tau) \varphi \text{ a.e. in } \Omega$$

and

$$-\Delta \varphi - \tau |\nabla \varphi| \leq \underline{\lambda}^\beta(\Omega, \tau) \varphi \text{ a.e. in } \Omega.$$

Moreover,  $\varphi \geq 0$  in  $\Omega$ ,  $\|\varphi\|_\infty = 1$  and

$$\frac{\partial \varphi}{\partial \nu} + \beta \varphi = 0 \text{ on } \partial \Omega.$$

Define now  $v \in L^\infty(\Omega, \mathbb{R}^d)$  by

$$(3.1) \quad v(x) := \begin{cases} -\tau \frac{\nabla \varphi(x)}{|\nabla \varphi(x)|} & \text{if } \nabla \varphi(x) \neq 0, \\ 0 & \text{if } \nabla \varphi(x) = 0. \end{cases}$$

Notice that  $\|v\|_\infty \leq \tau$ , which entails that  $\underline{\lambda}^\beta(\Omega, \tau) \leq \lambda_1^\beta(\Omega, v)$ . On the one hand,

$$(3.2) \quad -\Delta \varphi + v \cdot \nabla \varphi = -\Delta \varphi - \tau |\nabla \varphi| \leq \underline{\lambda}^\beta(\Omega, \tau) \varphi \leq \lambda_1^\beta(\Omega, v) \varphi \text{ a.e. in } \Omega.$$

On the other hand,  $\varphi_{\Omega, v}^\beta > 0$  in  $\overline{\Omega}$ ,

$$\frac{\partial \varphi_{\Omega, v}^\beta}{\partial \nu} + \beta \varphi_{\Omega, v}^\beta = 0 = \frac{\partial \varphi}{\partial \nu} + \beta \varphi \text{ on } \partial \Omega$$

and  $\|\varphi_{\Omega, v}^\beta\|_\infty = 1$ . Lemma 2.1 applied with  $(\mu, \beta, \psi, \varphi) := (\lambda_1^\beta(\Omega, v), \beta, \varphi_{\Omega, \tau}^\beta, \varphi)$  yields

$$\varphi_{\Omega, v}^\beta = \varphi \text{ in } \overline{\Omega}.$$

As a consequence, all inequalities in (3.2) are equalities and

$$\underline{\lambda}^\beta(\Omega, \tau) = \lambda_1^\beta(\Omega, v).$$

Furthermore, since  $\varphi \in W^{2,p}(\Omega)$  for each  $1 \leq p < \infty$ , it follows that

$$|\nabla(\partial_{x_i} \varphi)| \times \mathbf{1}_{\{\partial_{x_i} \varphi = 0\}} = 0 \text{ a.e. in } \Omega$$

for each  $1 \leq i \leq d$ , where  $\partial_{x_i} \varphi := \frac{\partial \varphi}{\partial x_i}$ , whence

$$\Delta \varphi \times \mathbf{1}_{\{\nabla \varphi = 0\}} = 0 \text{ a.e. in } \Omega.$$

Since  $-\Delta \varphi + v \cdot \nabla \varphi = \lambda_1^\beta(\Omega, v) \varphi > 0$  a.e. in  $\Omega$ , one gets that the set  $\{x \in \Omega : \nabla \varphi(x) = 0\}$  is negligible. Therefore, in addition to  $v \cdot \nabla \varphi = -\tau |\nabla \varphi|$  a.e. in  $\Omega$ , (3.1) also entails that  $|v(x)| = \tau$  for almost every  $x \in \Omega$ . The vector field  $\underline{v} := v$  and the function

$$\underline{\varphi} := \varphi = \varphi_{\Omega, v}^\beta$$

then fulfill the required conclusions of part 1 of Theorem 1.3.

Let us now turn to the uniqueness result in part 1 of Theorem 1.3. Assume that  $w \in L^\infty(\Omega, \mathbb{R}^d)$  is such that  $\|w\|_\infty \leq \tau$  and  $\lambda_1^\beta(\Omega, w) = \underline{\lambda}^\beta(\Omega, \tau)$ . One has

$$(3.3) \quad \begin{cases} -\Delta \varphi_{\Omega, v}^\beta + w \cdot \nabla \varphi_{\Omega, v}^\beta \geq -\Delta \varphi_{\Omega, v}^\beta - \tau |\nabla \varphi_{\Omega, v}^\beta| = \underline{\lambda}^\beta(\Omega, \tau) \varphi_{\Omega, v}^\beta, \\ -\Delta \varphi_{\Omega, w}^\beta + w \cdot \nabla \varphi_{\Omega, w}^\beta = \underline{\lambda}^\beta(\Omega, \tau) \varphi_{\Omega, w}^\beta, \end{cases}$$

a.e. in  $\Omega$ , together with

$$\frac{\partial \varphi_{\Omega, v}^\beta}{\partial \nu} + \beta \varphi_{\Omega, v}^\beta = 0 = \frac{\partial \varphi_{\Omega, w}^\beta}{\partial \nu} + \beta \varphi_{\Omega, w}^\beta \quad \text{on } \partial\Omega.$$

Furthermore, both functions  $\varphi_{\Omega, v}^\beta$  and  $\varphi_{\Omega, w}^\beta$  are positive (in  $\bar{\Omega}$ ), with  $L^\infty$  norms equal to 1. Lemma 2.1 applied with

$$(\mu, \beta, \psi, \varphi) := (\underline{\lambda}^\beta(\Omega, \tau), \beta, \varphi_{\Omega, v}^\beta, \varphi_{\Omega, w}^\beta),$$

and the vector field  $w$  instead of  $v$ , then entails

$$\varphi_{\Omega, v}^\beta = \varphi_{\Omega, w}^\beta \quad \text{in } \bar{\Omega}.$$

Consequently, the first line in (3.3) then yields

$$w \cdot \nabla \varphi_{\Omega, v}^\beta = -\tau |\nabla \varphi_{\Omega, v}^\beta| \quad \text{a.e. in } \Omega,$$

that is,  $w \cdot \nabla \varphi = -\tau |\nabla \varphi|$  a.e. in  $\Omega$ . Since  $\nabla \varphi \neq 0$  a.e. in  $\Omega$  and  $\|w\|_\infty \leq \tau$ , one concludes that

$$w = -\tau \frac{\nabla \varphi}{|\nabla \varphi|} \quad \text{a.e. in } \Omega,$$

that is,  $w = v$  a.e. in  $\Omega$ .

Lastly, let  $\lambda \in \mathbb{R}$  and  $\phi \in \bigcap_{1 \leq p < \infty} W^{2,p}(\Omega)$  satisfy

$$\begin{cases} -\Delta \phi - \tau |\nabla \phi| = \lambda \phi \quad \text{and } \phi \geq 0 & \text{in } \Omega, \\ \frac{\partial \phi}{\partial \nu} + \beta \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

and  $\max_{\bar{\Omega}} \phi = 1$ . Define  $q \in L^\infty(\Omega, \mathbb{R}^d)$  by

$$q(x) := \begin{cases} -\tau \frac{\nabla \phi(x)}{|\nabla \phi(x)|} & \text{if } \nabla \phi(x) \neq 0, \\ 0 & \text{if } \nabla \phi(x) = 0. \end{cases}$$

Notice that  $\|q\|_\infty \leq \tau$ . Since  $-\tau |\nabla \phi| = q \cdot \nabla \phi$  a.e. in  $\Omega$ , the nonnegativity of  $\phi$  and the uniqueness of the pair of principal eigenvalue and principal normalized eigenfunction imply that

$$\lambda = \lambda_1^\beta(\Omega, q) \geq \underline{\lambda}^\beta(\Omega, \tau), \quad \text{and } \phi = \varphi_{\Omega, q}^\beta \text{ in } \bar{\Omega}.$$

Both functions  $\phi = \varphi_{\Omega,q}^\beta$  and  $\varphi = \varphi_{\Omega,v}^\beta$  are positive in  $\overline{\Omega}$  with  $L^\infty$  norms equal to 1, and they satisfy

$$(3.4) \quad \begin{cases} -\Delta \varphi_{\Omega,q}^\beta + v \cdot \nabla \varphi_{\Omega,q}^\beta \geq -\Delta \varphi_{\Omega,q}^\beta - \tau |\nabla \varphi_{\Omega,q}^\beta| = \lambda_1^\beta(\Omega, q) \varphi_{\Omega,q}^\beta, \\ -\Delta \varphi_{\Omega,v}^\beta + v \cdot \nabla \varphi_{\Omega,v}^\beta = \underline{\lambda}^\beta(\Omega, \tau) \varphi_{\Omega,v}^\beta \leq \lambda_1^\beta(\Omega, q) \varphi_{\Omega,v}^\beta, \end{cases}$$

a.e. in  $\Omega$ , together with

$$\frac{\partial \varphi_{\Omega,q}^\beta}{\partial \nu} + \beta \varphi_{\Omega,q}^\beta = 0 = \frac{\partial \varphi_{\Omega,v}^\beta}{\partial \nu} + \beta \varphi_{\Omega,v}^\beta \quad \text{on } \partial\Omega.$$

Lemma 2.1 applied with  $(\mu, \beta, \psi, \varphi) = (\lambda_1^\beta(\Omega, q), \beta, \varphi_{\Omega,q}^\beta, \varphi_{\Omega,v}^\beta)$  then entails

$$\varphi_{\Omega,q}^\beta = \varphi_{\Omega,v}^\beta \quad \text{in } \overline{\Omega},$$

that is,  $\phi = \varphi = \underline{\varphi}$  in  $\overline{\Omega}$ . Furthermore, all inequalities in (3.4) are equalities and

$$\underline{\lambda}^\beta(\Omega, \tau) = \lambda_1^\beta(\Omega, q),$$

whence  $\lambda = \underline{\lambda}^\beta(\Omega, \tau)$ . All properties in part 1 of Theorem 1.3 have now been proved.

*Part 2.* Notice that, for all  $v \in L^\infty(\Omega)$ ,  $\lambda_1^\beta(\Omega, v) \leq \lambda_1^D(\Omega, v)$  by Lemma 2.2. Since

$$\sup_{v \in L^\infty(\Omega, \mathbb{R}^d), \|v\|_\infty \leq \tau} \lambda_1^D(\Omega, v) < +\infty$$

by [8, Proposition 5.1], [17, Theorem 1.5] or [18, Theorem 6.6], it follows that the quantity  $\overline{\lambda}^\beta(\Omega, \tau)$  defined in (1.8) is a real number. Then, arguments similar to those in part 1 above yield the conclusions of part 2.

*Part 3.* Consider now the case  $\Omega = \Omega^*$  and denote

$$\phi := \varphi_{\Omega^*, \tau e_r}^\beta.$$

This function  $\phi$  is positive in  $\overline{\Omega^*}$ , it is of class  $W^{2,p}(\Omega^*)$  for all  $1 \leq p < \infty$ , and  $\max_{\overline{\Omega^*}} \phi = 1$ . For any  $\mathcal{R} \in O(d)$  (the group of orthogonal transformations in  $\mathbb{R}^d$ ), the function  $\phi \circ \mathcal{R}$  satisfies the same equation as  $\phi$  in  $\Omega^*$  and the same boundary condition on  $\partial\Omega^*$ . The uniqueness of the pair of eigenvalue and principal normalized eigenfunction then entails that  $\phi \circ \mathcal{R} = \phi$  in  $\overline{\Omega^*}$  for any  $\mathcal{R} \in O(d)$ , that is,  $\phi$  is radially symmetric in  $\overline{\Omega^*}$ . Let  $R$  denote the radius of  $\Omega^*$ . For any  $\sigma \in (0, R]$ , there holds

$$-\Delta \phi + \tau e_r \cdot \nabla \phi = \lambda_1^\beta(\Omega^*, \tau e_r) \phi > 0$$

almost everywhere in  $\{x : |x| \leq \sigma\}$  and  $\phi$  is constant on the sphere  $\{y : |y| = \sigma\}$ . The weak maximum principle then implies that  $\phi(x) \geq \phi(y)$  for all  $|x| \leq |y| = \sigma$ , and the Hopf lemma even yields  $e_r \cdot \nabla \phi(y) < 0$  for all  $|y| = \sigma$ . As a conclusion,  $\phi$  is radially decreasing and

$$\tau e_r \cdot \nabla \phi = -\tau |\nabla \phi|$$

everywhere in  $\overline{\Omega^*} \setminus \{0\}$  and the function  $\phi$  then fulfills (1.10) in  $\Omega^*$  with  $\lambda := \lambda_1^\beta(\Omega^*, \tau e_r)$ . It then follows from the last result of part 1 of the present theorem that

$$\underline{\lambda}^\beta(\Omega^*, \tau) = \lambda_1^\beta(\Omega^*, \tau e_r),$$

and the uniqueness of the vector field minimizing  $\lambda_1^\beta(\Omega^*, v)$  implies that  $\underline{v} = \tau e_r$ .

By denoting  $\psi := \varphi_{\Omega^*, -\tau e_r}^\beta$ , one proves similarly that  $\psi$  is radially decreasing and one still has  $\tau e_r \cdot \nabla \psi = -\tau |\nabla \psi|$ , that is,  $-\tau e_r \cdot \nabla \psi = \tau |\nabla \psi|$ , everywhere in  $\overline{\Omega^*} \setminus \{0\}$ . Part 2 of the present theorem then implies that

$$\overline{\lambda}^\beta(\Omega^*, \tau) = \lambda_1^\beta(\Omega^*, -\tau e_r)$$

and  $\overline{v} = -\tau e_r$ . The proof of Theorem 1.3 is thereby complete.  $\square$

#### 4. PROOF OF THE MINIMIZATION RESULT

Let us now prove Theorem 1.1. Arguing by contradiction, assume that the conclusion does not hold. There exist then a sequence  $(\beta_k)_{k \in \mathbb{N}}$  of positive numbers such that  $\lim_{k \rightarrow +\infty} \beta_k = +\infty$  and a sequence of vector fields  $(v_k)_{k \in \mathbb{N}}$  such that, for all  $k \in \mathbb{N}$ ,  $\|v_k\|_\infty \leq \tau$  and

$$(4.1) \quad \lambda_1^{\beta_k}(\Omega, v_k) < \lambda_1^{\beta_k}(\Omega^*, \tau e_r) + \frac{1}{k+1}.$$

For all  $k \in \mathbb{N}$ , write

$$\varphi_k := \varphi_{\Omega, v_k}^{\beta_k} \quad \text{and} \quad \lambda_k := \lambda_1^{\beta_k}(\Omega, v_k).$$

One has

$$\begin{cases} -\Delta \varphi_k + v_k \cdot \nabla \varphi_k = \lambda_k \varphi_k & \text{a.e. in } \Omega, \\ \frac{1}{\beta_k} \frac{\partial \varphi_k}{\partial \nu} = -\varphi_k & \text{on } \partial \Omega. \end{cases}$$

Lemma 2.2 shows that  $\lambda_k \leq \lambda_1^D(\Omega, v_k)$  for all  $k \in \mathbb{N}$ , while [8, Proposition 5.1] ensures that the sequence  $(\lambda_1^D(\Omega, v_k))_{k \in \mathbb{N}}$  is bounded (recall that  $\|v_k\|_\infty \leq \tau$  for all  $k \in \mathbb{N}$ ). Furthermore, each  $\lambda_k$  is a positive real number. Therefore, the sequence  $(\lambda_k)_{k \in \mathbb{N}}$  is bounded. Arguing as in the proof of Lemma 2.2, one concludes that the sequence  $(\varphi_k)_{k \in \mathbb{N}}$  is then bounded in  $H^1(\Omega)$ , which entails that the sequence  $(\text{tr}(\varphi_k))_{k \in \mathbb{N}}$  is bounded in  $H^{\frac{1}{2}}(\partial \Omega)$ . Therefore, together with the boundedness of the sequence  $(1/\beta_k)_{k \in \mathbb{N}}$ , [1, Theorem 15.2] implies that the sequence  $(\varphi_k)_{k \in \mathbb{N}}$  is bounded in  $W^{2,2}(\Omega)$ , and a bootstrap argument therefore shows that  $(\varphi_k)_{k \in \mathbb{N}}$  is bounded in  $W^{2,p}(\Omega)$  for all  $1 \leq p < \infty$ . Thus, there exist  $\mu \in \mathbb{R}$ ,  $\varphi \in \bigcap_{1 \leq p < \infty} W^{2,p}(\Omega)$  and  $f \in L^\infty(\Omega)$  such that, up to a subsequence,

$$\lim_{k \rightarrow +\infty} \lambda_k = \mu,$$

$$\varphi_k \rightharpoonup \varphi \text{ weakly in } W^{2,p}(\Omega) \text{ and } \varphi_k \xrightarrow[k \rightarrow +\infty]{} \varphi \text{ strongly in } C^{1,\alpha}(\overline{\Omega})$$

for all  $1 \leq p < \infty$  and all  $\alpha \in (0, 1)$ , and

$$v_k \cdot \nabla \varphi_k \rightharpoonup f \text{ weakly-* in } L^\infty(\Omega).$$

Furthermore, as in the proof of Lemma 2.2, there holds  $\text{tr}(\varphi_k) \rightarrow 0$  as  $k \rightarrow +\infty$  strongly in  $L^2(\partial\Omega)$ . One therefore has

$$\begin{cases} -\Delta\varphi + f = \mu\varphi & \text{a.e. in } \Omega, \\ \varphi \geq 0 & \text{in } \bar{\Omega}, \\ \text{tr}(\varphi) = 0 & \text{on } \partial\Omega, \\ \max_{\bar{\Omega}} \varphi = 1 \end{cases}$$

and  $f \geq -\tau |\nabla\varphi|$  a.e. in  $\Omega$ , so that

$$-\Delta\varphi - \tau |\nabla\varphi| \leq \mu\varphi \text{ a.e. in } \Omega.$$

Define

$$v(x) := \begin{cases} -\tau \frac{\nabla\varphi(x)}{|\nabla\varphi(x)|} & \text{if } \nabla\varphi(x) \neq 0, \\ 0 & \text{if } \nabla\varphi(x) = 0, \end{cases}$$

so that  $\|v\|_\infty \leq \tau$  and

$$-\Delta\varphi + v \cdot \nabla\varphi \leq \mu\varphi \text{ a.e. in } \Omega.$$

Let now  $\psi := \varphi_{\Omega, v}^D$ , so that  $\psi > 0$  in  $\Omega$  and

$$-\Delta\psi + v \cdot \nabla\psi = \lambda_1^D(\Omega, v)\psi \text{ a.e. in } \Omega.$$

If  $\mu < \lambda_1^D(\Omega, v)$ , then

$$-\Delta\varphi + v \cdot \nabla\varphi \leq \mu\varphi \leq \lambda_1^D(\Omega, v)\varphi \text{ a.e. in } \Omega,$$

and [17, Lemma 2.1] implies that  $\varphi = \psi$  in  $\bar{\Omega}$ , therefore  $\mu = \lambda_1^D(\Omega, v)$ , a contradiction. Finally,

$$\lambda_1^D(\Omega, v) \leq \mu.$$

But (4.1) and Lemma 2.2 imply that

$$\mu \leq \lambda_1^D(\Omega^*, \tau e_r),$$

and one therefore obtains

$$\lambda_1^D(\Omega, v) \leq \lambda_1^D(\Omega^*, \tau e_r),$$

which contradicts the ‘‘equality’’ statement in [17, Theorem 1.1] since  $\Omega$  is not a ball. This concludes the proof of Theorem 1.1.  $\square$

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