A Faber-Krahn inequality for the Laplacian with drift under Robin boundary condition

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Abstract

We prove a Faber-Krahn inequality for the Laplacian with drift under Robin boundary condition, provided that the β parameter in the Robin condition is large enough. The proof relies on a compactness argument, on the convergence of Robin eigenvalues to Dirichlet eigenvalues when β goes to infinity, and on a strict Faber-Krahn inequality under Dirichlet boundary condition. We also show the existence and uniqueness of drifts v satisfying some L^∞ constraints and minimizing or maximizing the principal eigenvalue of $-\Delta + v \cdot \nabla$ in a fixed domain and with a fixed parameter $\beta > 0$ in the Robin condition.

Contents

1 Introduction

Throughout this paper, $d \geq 1$ is an integer. For all $x \in \mathbb{R}^d$, denote by $|x|$ the Euclidean norm of x and define

$$e_r(x) := \frac{x}{|x|} \text{ for all } x \in \mathbb{R}^d \setminus \{0\}.$$

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Let $\Omega \subset \mathbb{R}^d$ be a bounded domain (connected open set) of class C^2, with outward unit normal ν on $\partial\Omega$. If $v \in L^\infty(\Omega, \mathbb{R}^d)$ is a bounded measurable vector field, set

$$\|v\|_\infty := \||v|\|_\infty .$$

We are interested in the principal eigenvalue of the operator $-\Delta + v \cdot \nabla$ in Ω under Robin boundary condition on $\partial\Omega$. More precisely, let $\beta > 0$. Then, by [15, Theorem A.4] and Krein-Rutman theory [2], let $\lambda_1^\beta(\Omega, v)$ denote the principal eigenvalue of the problem

$$\begin{cases} -\Delta\varphi^\beta_{\Omega,v} + v \cdot \nabla\varphi^\beta_{\Omega,v} = \lambda_1^\beta(\Omega, v)\varphi^\beta_{\Omega,v} & \text{in } \Omega, \\ \dfrac{\partial\varphi^\beta_{\Omega,v}}{\partial\nu} + \beta\varphi^\beta_{\Omega,v} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

This principal eigenvalue is simple, the corresponding eigenfunction $\varphi^\beta_{\Omega,v}$ is positive in Ω and none of the other eigenvalues corresponds to a positive eigenfunction (see the discussion after [15, Theorem 1.3]). By $W^{2,p}$ elliptic regularity ([15, Theorem A.29]), the function $\varphi^\beta_{\Omega,v}$ belongs to $W^{2,p}(\Omega)$ for all $p \in [1, \infty)$ and then to $C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in (0, 1)$. The first line in (1.1) is therefore understood almost everywhere in Ω. The Hopf lemma shows that $\varphi^\beta_{\Omega,v} > 0$ on $\partial\Omega$, which, in turn, implies that

$$\min_{\overline{\Omega}} \varphi^\beta_{\Omega,v} > 0.$$

We usually normalize $\varphi^\beta_{\Omega,v}$ by

$$\max_{\overline{\Omega}} \varphi^\beta_{\Omega,v} = 1. \tag{1.2}$$

Moreover, there holds

$$\lambda_1^\beta(\Omega, v) > 0.$$

Fix $\tau \geq 0$ and $m > 0$. We are interested in the infimum of $\lambda_1^\beta(\Omega, v)$ when Ω and v vary under the constraints $|\Omega| = m$ (throughout the paper, $|A|$ denotes the n-dimensional Lebesgue measure of A for all measurable sets $A \subset \mathbb{R}^d$) and

$$\|v\|_\infty \leq \tau. \tag{1.3}$$

In the sequel, Ω^* stands for the Euclidean ball centered at 0 such that $|\Omega^*| = |\Omega|$.

Our main result states that, when Ω is not a ball, $\lambda_1^\beta(\Omega, v)$ is (strictly) greater than the corresponding quantity in Ω^* when $v = \tau\, e_r$, provided that β is large enough:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded C^2 domain and $\tau \geq 0$. Assume that Ω is not a ball. Then there exist $\beta_0 > 0$ and $\varepsilon > 0$ with the following property:*

$$\forall\, \beta \geq \beta_0,\ \forall\, v \in L^\infty(\Omega, \mathbb{R}^d) \text{ such that } \|v\|_\infty \leq \tau,\ \lambda_1^\beta(\Omega, v) \geq \lambda_1^\beta(\Omega^*, \tau e_r) + \varepsilon. \tag{1.4}$$

When $\tau = 0$, *i.e.* when $-\Delta + v \cdot \nabla = -\Delta$ is merely (minus) the Laplacian, it was proved in [9, 10, 13, 14] that, for all $\beta > 0$,

$$\lambda_1^\beta(\Omega, 0) \geq \lambda_1^\beta(\Omega^*, 0) \tag{1.5}$$

and equality holds if and only if $\Omega = \Omega^*$ up to translation. When $\tau \neq 0$, Theorem 1.1 provides on the one hand a quantified strict inequality if Ω is not a ball, but the conclusion is only established above some threshold for β, contrary to [10, Theorem 1.1], and it actually can not hold for all $\beta > 0$, since

$$\lim_{\beta \to 0} \lambda_1^\beta(\Omega, v) = \lim_{\beta \to 0} \lambda_1^\beta(\Omega^*, \tau e_r) = 0$$

for each $v \in L^\infty(\Omega, \mathbb{R}^d)$ (as follows from Lemma 2.2 below). On the other hand, when $\Omega = \Omega^*$, the uniqueness part in Theorem 1.2 below ensures that, for all $v \in L^\infty(\Omega^*, \mathbb{R}^d)$ with $\|v\| \leq \tau$, if $v \neq \tau e_r$, then $\lambda_1^\beta(\Omega^*, v) > \lambda_1^\beta(\Omega^*, \tau e_r)$ for all $\beta > 0$.

The following question nevertheless remains open:

Open problem 1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded C^2 domain, $\tau \geq 0$ and $v \in L^\infty(\Omega, \mathbb{R}^d)$ with $\|v\|_\infty \leq \tau$. Does the inequality*

$$\lambda_1^\beta(\Omega, v) \geq \lambda_1^\beta(\Omega^*, \tau e_r)$$

hold for all $\beta > 0$?

Recall that, under Dirichlet boundary condition, it was proved in [17, Theorem 1.1] and [18, Remark 6.9] that, whenever (1.3) holds,

$$\lambda_1^D(\Omega, v) \geq \lambda_1^D(\Omega^*, \tau e_r), \tag{1.6}$$

where $\lambda_1^D(\Omega, v)$ stands for the principal eigenvalue of $-\Delta + v \cdot \nabla$ under Dirichlet boundary condition. Moreover, equality holds in (1.6) if and only if, up to translation, $\Omega = \Omega^*$ and $v = \tau e_r$. The inequalities (1.4)-(1.6) are called Faber-Krahn type inequalities. This terminology originates from the results of Faber [16] and Krahn [21, 22], who proved that

$$\lambda_1^D(\Omega, 0) \geq \lambda_1^D(\Omega^*, 0),$$

with equality if and only if, up to translation, $\Omega = \Omega^*$. The latter inequality means that a radially symmetric membrane which is fixed at its boundary has the lowest fundamental tone among all equimeasurable membranes, answering a conjecture of Rayleigh [27] set in dimension $d = 2$. Since these pioneering papers, much work has been done on various related optimization eigenvalue problems for elliptic operators, for instance on higher eigenvalues or functions of the eigenvalues of $-\Delta$ under Dirichlet boundary condition [3, 4, 6, 11, 12, 24, 25, 26, 30], under Neumann boundary condition [26, 28, 29], or for the first eigenvalue of Δ^2 under boundary conditions $\varphi = \frac{\partial \varphi}{\partial \nu} = 0$ on $\partial\Omega$ [5, 23]. We refer to the surveys [7, 20] for many more references on these topics.

The second main result deals with an optimization problem when the domain Ω is fixed and v varies under the constraint (1.3). Define, for all $\beta > 0$ and $\tau \geq 0$ given:

$$\underline{\lambda}^\beta(\Omega, \tau) := \inf \left\{ \lambda_1^\beta(\Omega, v) : \|v\|_\infty \leq \tau \right\} \tag{1.7}$$

and

$$\overline{\lambda}^\beta(\Omega, \tau) := \sup \left\{ \lambda_1^\beta(\Omega, v) : \|v\|_\infty \leq \tau \right\}. \tag{1.8}$$

We claim that these lower and upper bounds are positive real numbers, are uniquely reached and provide an identity relating the optimizing vector fields and the corresponding eigenfunctions:

Theorem 1.2. *[Optimization in fixed domains] Let $\Omega \subset \mathbb{R}^d$ be a bounded C^2 domain, $\tau \geq 0$ and $\beta > 0$.*

1. *There exists a unique $\underline{v} \in L^\infty(\Omega, \mathbb{R}^d)$ meeting $\|\underline{v}\|_\infty \leq \tau$ such that $\underline{\lambda}^\beta(\Omega, \tau) = \lambda_1^\beta(\Omega, \underline{v})$. One has $|\underline{v}(x)| = \tau$ for almost every $x \in \Omega$. Moreover, if $\underline{\varphi} := \varphi^\beta_{\Omega,\underline{v}}$ is the corresponding eigenfunction, then*

$$\underline{v}(x) \cdot \nabla \underline{\varphi}(x) = -\tau |\nabla \underline{\varphi}(x)| \text{ for almost every } x \in \Omega. \tag{1.9}$$

Lastly, if $\lambda \in \mathbb{R}$ and $\phi \in \bigcap_{1 \leq p < \infty} W^{2,p}(\Omega)$ satisfy

$$\begin{cases} -\Delta \phi - \tau \, |\nabla \phi| = \lambda \phi \quad \text{and} \quad \phi \geq 0 & \text{in } \Omega, \\ \dfrac{\partial \phi}{\partial \nu} + \beta \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

and $\max_{\overline{\Omega}} \phi = 1$, then $\lambda = \underline{\lambda}^\beta(\Omega, \tau)$ and $\phi = \underline{\varphi}$ in $\overline{\Omega}$. Notice that, from elliptic regularity theory applied to (1.10), since ϕ and $|\nabla \phi|$ belong to $C^{0,\alpha}(\overline{\Omega})$ for all $\alpha \in (0,1)$, the function ϕ belongs to $C^{2,\alpha}_{loc}(\Omega)$ for all $\alpha \in (0,1)$, and the first line of (1.10) holds in the classical sense in Ω.

2. *Similarly, there exists a unique $\overline{v} \in L^\infty(\Omega, \mathbb{R}^d)$ meeting $\|\overline{v}\|_\infty \leq \tau$ such that $\overline{\lambda}^\beta(\Omega, \tau) = \lambda_1^\beta(\Omega, \overline{v})$. One has $|\overline{v}(x)| = \tau$ for almost every $x \in \Omega$. Moreover, if $\overline{\varphi} := \varphi^\beta_{\Omega,\overline{v}}$ is the corresponding eigenfunction, then*

$$\overline{v}(x) \cdot \nabla \overline{\varphi}(x) = \tau \, |\nabla \overline{\varphi}(x)| \text{ for almost every } x \in \Omega. \tag{1.11}$$

Lastly, if $\lambda \in \mathbb{R}$ and $\phi \in \bigcap_{1 \leq p < \infty} W^{2,p}(\Omega)$ satisfy

$$\begin{cases} -\Delta \phi + \tau \, |\nabla \phi| = \lambda \phi \quad \text{and} \quad \phi \geq 0 & \text{in } \Omega, \\ \dfrac{\partial \phi}{\partial \nu} + \beta \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.12}$$

and $\max_{\overline{\Omega}} \phi = 1$, then $\lambda = \overline{\lambda}^\beta(\Omega, \tau)$ and $\phi = \overline{\varphi}$ in $\overline{\Omega}$. As for (1.10), the function ϕ then belongs to $C^{2,\alpha}_{loc}(\Omega)$ for all $\alpha \in (0,1)$, and the first line of (1.12) holds in the classical sense in Ω.

3. *If $\Omega = \Omega^*$, then $\underline{v} = \tau \, e_r$, $\overline{v} = -\tau \, e_r$ in Ω^* and the functions $\underline{\varphi}$ and $\overline{\varphi}$ are radially decreasing in Ω^*.*

We point out that similar properties had been derived in [17, 18] for the extremal quantities $\underline{\lambda}(\Omega, \tau)$ and $\overline{\lambda}(\Omega, \tau)$ defined like $\underline{\lambda}^\beta(\Omega, \tau)$ and $\overline{\lambda}^\beta(\Omega, \tau)$ in (1.7)-(1.8) with the Dirichlet eigenvalues $\lambda_1(\Omega, v)$ instead of the Robin ones $\lambda_1^\beta(\Omega, v)$. The asymptotic behavior as $\tau \to +\infty$ of the eigenfunctions associated with $\underline{\lambda}(\Omega, \tau)$ was analyzed in [19].

The paper is organized as follows. In Section 2, we provide comparisons results between Robin, Dirichlet and Neumann eigenvalues in a fixed domain and for a given drift, and prove convergence of the Robin eigenvalues when $\beta \to +\infty$ (resp. when $\beta \to 0$) to the corresponding Dirichlet (resp. Neumann) eigenvalues. Section 3 is devoted to the proof of Theorem 1.2. Finally, we establish Theorem 1.1 is Section 4.

2 Comparisons and convergence results between Robin, Dirichlet and Neumann principal eigenvalues

This section is concerned with some comparisons and convergence results for Robin and Dirichlet principal eigenvalues in a given domain Ω. The results will be used in the proofs of the main Theorems 1.1 and 1.2.

We first start with an auxiliary comparison lemma between sub- and supersolutions.

Lemma 2.1. *Let $\mu \in \mathbb{R}$, $\beta \geq 0$, and $v \in L^\infty(\Omega, \mathbb{R}^d)$. Let $\psi, \varphi \in W^{2,p}(\Omega)$ for all $1 \leq p < \infty$, such that $\psi \geq 0$ and $\varphi \geq 0$ in Ω, $\|\psi\|_\infty = \|\varphi\|_\infty = 1$, and*

$$\begin{cases} -\Delta\psi + v \cdot \nabla\psi \geq \mu\psi & \text{a.e. in } \Omega, \\ -\Delta\varphi + v \cdot \nabla\varphi \leq \mu\varphi & \text{a.e. in } \Omega. \end{cases}$$

Assume also that

$$\frac{\partial\psi}{\partial\nu} + \beta\psi \geq 0 \geq \frac{\partial\varphi}{\partial\nu} + \beta\varphi \text{ on } \partial\Omega.$$

Then $\psi = \varphi$ in $\overline{\Omega}$.

Proof. The argument is reminiscent of the proof of [17, Lemma 2.1]. Remember first that ψ and φ belong to $C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in (0,1)$. Furthermore, $\psi > 0$ in Ω from the interior strong maximum principle (otherwise, ψ would be identically 0 in Ω, contradicting $\|\psi\|_\infty = 1$). Observe now that $\psi > 0$ on $\partial\Omega$. Indeed, if there exists $x_0 \in \partial\Omega$ such that $\psi(x_0) = 0$, then the Hopf lemma shows that $\frac{\partial\psi}{\partial\nu}(x_0) < 0$, which is impossible by the boundary condition satisfied by ψ. Thus, being continuous in $\overline{\Omega}$, ψ is bounded below by a positive constant, so that there exists $\gamma > 0$ such that $\gamma\psi > \varphi$ in Ω. Define

$$\gamma^* := \inf\{\gamma > 0 : \gamma\psi > \varphi \text{ in } \Omega\}$$

and $w := \gamma^*\psi - \varphi$. Note that, since $\varphi \geq 0$ in Ω and $\|\varphi\|_\infty = 1$, $\gamma^* > 0$. The function w is nonnegative in $\overline{\Omega}$,

$$\frac{\partial w}{\partial\nu} + \beta w \geq 0 \text{ on } \partial\Omega$$

and

$$-\Delta w + v \cdot \nabla w - \mu w \geq 0 \text{ a.e. in } \Omega.$$

If $w > 0$ in Ω, then, as before, w is bounded below by a positive constant in Ω, so there exists $\delta > 0$ such that $w > \delta\varphi$ in Ω, which entails in turn

$$\frac{\gamma^*}{1+\delta}\,\psi > \varphi \text{ in } \Omega,$$

contradicting the definition of γ^*, since $\gamma^* > 0$. Therefore, there exists $x_0 \in \Omega$ such that $w(x_0) = 0$, and since $w \geq 0$ in Ω, the strong maximum principle indicates that $w(x) = 0$ everywhere in Ω and then in $\overline{\Omega}$ by continuity, meaning that $\gamma^*\psi = \varphi$ in $\overline{\Omega}$. The condition $\|\varphi\|_\infty = \|\psi\|_\infty = 1$ finally yields $\varphi = \psi$ in $\overline{\Omega}$. □

Let now $\Omega \subset \mathbb{R}^d$ be a bounded C^2 domain and $v \in L^\infty(\Omega, \mathbb{R}^d)$. Denote by $\lambda_1^D(\Omega, v)$ the principal eigenvalue of $-\Delta + v\cdot\nabla$ in Ω under Dirichlet boundary condition and by $\varphi^D_{\Omega,v} \in \bigcap_{1\leq p<\infty} W^{2,p}(\Omega)$ the corresponding principal eigenfunction (which is positive in Ω) normalized by

$$\left\|\varphi^D_{\Omega,v}\right\|_\infty = 1.$$

We will show that the map $\beta \mapsto \lambda_1^\beta(\Omega, v)$ is increasing in $(0,\infty)$ and converges to $\lambda_1^D(\Omega, v)$ at infinity, and to 0 (that is, the principal eigenvalue of $-\Delta + v\cdot\nabla$ in Ω under Neumann boundary condition) as $\beta \to 0$:

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded C^2 domain and $v \in L^\infty(\Omega, \mathbb{R}^d)$. Then the map $\beta \mapsto \lambda_1^\beta(\Omega, v)$ is increasing in $(0, +\infty)$. Furthermore,*

$$\lim_{\beta\to+\infty} \lambda_1^\beta(\Omega, v) = \lambda_1^D(\Omega, v). \tag{2.1}$$

and

$$\lim_{\beta\to 0} \lambda_1^\beta(\Omega, v) = 0.$$

Proof. Let $0 < \beta_1 < \beta_2$ and assume by way of contradiction that $\lambda_1^{\beta_2}(\Omega, v) \leq \lambda_1^{\beta_1}(\Omega, v)$. Set $\varphi_1 := \varphi^{\beta_1}_{\Omega,v}$ and $\varphi_2 := \varphi^{\beta_2}_{\Omega,v}$. Both functions φ_1 and φ_2 are positive in $\overline{\Omega}$ and they satisfy

$$\begin{cases} -\Delta\varphi_1 + v\cdot\nabla\varphi_1 = \lambda_1^{\beta_1}(\Omega, v)\varphi_1 & \text{a.e. in } \Omega, \\ -\Delta\varphi_2 + v\cdot\nabla\varphi_2 = \lambda_1^{\beta_2}(\Omega, v)\varphi_2 \leq \lambda_1^{\beta_1}(\Omega, v)\varphi_2 & \text{a.e. in } \Omega, \end{cases}$$

together with

$$\frac{\partial\varphi_1}{\partial\nu} + \beta_1\varphi_1 = 0 = \frac{\partial\varphi_2}{\partial\nu} + \beta_2\varphi_2 > \frac{\partial\varphi_2}{\partial\nu} + \beta_1\varphi_2 \text{ on } \partial\Omega. \tag{2.2}$$

Lemma 2.1 applied with $(\mu, \beta, \psi, \varphi) := (\lambda_1^{\beta_1}(\Omega, v), \beta_1, \varphi_1, \varphi_2)$ then entails $\varphi_1 = \varphi_2$ in $\overline{\Omega}$, contradicting the strict inequality in (2.2). Finally,

$$\lambda_1^{\beta_1}(\Omega, v) < \lambda_1^{\beta_2}(\Omega, v),$$

and the map $\beta \mapsto \lambda_1^\beta(\Omega, v)$ is increasing in $(0, +\infty)$.

Let now $\beta > 0$ and assume by way of contradiction that $\lambda_1^\beta(\Omega, v) \geq \lambda_1^D(\Omega, v)$. Both functions $\varphi^\beta_{\Omega,v}$ and $\varphi^D_{\Omega,v}$ are positive in Ω and they satisfy

$$\begin{cases} -\Delta\varphi^\beta_{\Omega,v} + v\cdot\nabla\varphi^\beta_{\Omega,v} = \lambda_1^\beta(\Omega,v)\varphi^\beta_{\Omega,v} & \text{a.e. in } \Omega, \\ -\Delta\varphi^D_{\Omega,v} + v\cdot\nabla\varphi^D_{\Omega,v} = \lambda_1^D(\Omega,v)\varphi^D_{\Omega,v} \leq \lambda_1^\beta(\Omega,v)\varphi^D_{\Omega,v} & \text{a.e. in } \Omega. \end{cases}$$

Furthermore, the Hopf lemma implies that $\frac{\partial\varphi^D_{\Omega,v}}{\partial\nu} < 0$ on $\partial\Omega$, whence

$$\frac{\partial\varphi^\beta_{\Omega,v}}{\partial\nu} + \beta\varphi^\beta_{\Omega,v} = 0 > \frac{\partial\varphi^D_{\Omega,v}}{\partial\nu} + \beta\varphi^D_{\Omega,v} \text{ on } \partial\Omega. \tag{2.3}$$

Lemma 2.1 applied with $(\mu,\beta,\psi,\varphi) := (\lambda_1^\beta(\Omega,v),\beta,\varphi^\beta_{\Omega,v},\varphi^D_{\Omega,v})$ then entails $\varphi^\beta_{\Omega,v} = \varphi^D_{\Omega,v}$ in $\overline{\Omega}$, contradicting the strict inequality in (2.3) (or the fact that $\varphi^\beta_{\Omega,v} > 0 = \varphi^D_{\Omega,v}$ on $\partial\Omega$). Finally,

$$\lambda_1^\beta(\Omega,v) < \lambda_1^D(\Omega,v)$$

for all $\beta > 0$.

Let us now turn to the proof of (2.1). Pick up any increasing sequence $(\beta_k)_{k\in\mathbb{N}}$ of positive real numbers with $\lim_{k\to+\infty}\beta_k = +\infty$ and set $\lambda_k := \lambda_1^{\beta_k}(\Omega,v)$ for all $k\in\mathbb{N}$. The sequence $(\lambda_k)_{k\in\mathbb{N}}$ is increasing and bounded above by $\lambda_1^D(\Omega,v)$ and therefore converges to some $\mu \leq \lambda_1^D(\Omega,v)$. For all k, if φ_k is defined as $\varphi_k := \theta_k\varphi^{\beta_k}_{\Omega,v}$ with $\theta_k > 0$ such that $\|\varphi_k\|_{L^2(\Omega)} = 1$, then

$$\begin{cases} -\Delta\varphi_k + v\cdot\nabla\varphi_k = \lambda_k\varphi_k & \text{a.e. in } \Omega, \\ \dfrac{\partial\varphi_k}{\partial\nu} + \beta_k\varphi_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

We claim that the sequence $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in $H^1(\Omega)$. Indeed, for all $k\in\mathbb{N}$,

$$\begin{aligned} \lambda_k\int_\Omega\varphi_k^2 &= -\int_\Omega\varphi_k\Delta\varphi_k + \int_\Omega(v\cdot\nabla\varphi_k)\varphi_k \\ &= \int_\Omega|\nabla\varphi_k|^2 - \int_{\partial\Omega}\varphi_k\frac{\partial\varphi_k}{\partial\nu} + \int_\Omega(v\cdot\nabla\varphi_k)\varphi_k \\ &= \int_\Omega|\nabla\varphi_k|^2 + \beta_k\int_{\partial\Omega}\varphi_k^2 + \int_\Omega(v\cdot\nabla\varphi_k)\varphi_k. \end{aligned}$$

From this, we derive, for all $\varepsilon > 0$,

$$\begin{aligned} \int_\Omega|\nabla\varphi_k|^2 + \beta_k\int_{\partial\Omega}\varphi_k^2 &\leq \lambda_k\int_\Omega\varphi_k^2 + \|v\|_\infty\int_\Omega\varphi_k|\nabla\varphi_k| \\ &\leq \left(\lambda_k + \frac{1}{2\varepsilon}\|v\|_\infty\right)\int_\Omega\varphi_k^2 + \frac{\varepsilon}{2}\|v\|_\infty\int_\Omega|\nabla\varphi_k|^2. \end{aligned} \tag{2.5}$$

Provided $\varepsilon\|v\|_\infty < 2$, recalling that the sequences $(\lambda_k)_{k\in\mathbb{N}}$ and $(\|\varphi_k\|_{L^2(\Omega)})_{k\in\mathbb{N}}$ are bounded, one obtains that the sequence $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in $H^1(\Omega)$, that is, there is $M\in\mathbb{R}_+$ such that $\|\varphi_k\|_{H^1(\Omega)} = \sqrt{\|\varphi_k\|^2_{L^2(\Omega)} + \|\,|\nabla\varphi_k|\,\|^2_{L^2(\Omega)}} \leq M$ for all $k\in\mathbb{N}$. Therefore, there exists $\varphi\in H^1(\Omega)$ such that, up to a subsequence,

$$\varphi_k \rightharpoonup \varphi \text{ weakly in } H^1(\Omega),\ \ \varphi_k\to\varphi \text{ strongly in } L^2(\Omega) \ \text{ and } \ \varphi_k\to\varphi \text{ a.e. in } \Omega, \tag{2.6}$$

as $k \to +\infty$, whence

$$\|\varphi\|_{L^2(\Omega)} = 1 \quad \text{and} \quad \varphi \geq 0 \text{ a.e. in } \Omega. \tag{2.7}$$

Moreover, since $\lim_{k\to+\infty} \beta_k = +\infty$, (2.5) shows that

$$\lim_{k\to+\infty} \mathrm{tr}(\varphi_k) = 0 \text{ strongly in } L^2(\partial\Omega), \tag{2.8}$$

where $\mathrm{tr} : H^1(\Omega) \to L^2(\partial\Omega)$ denotes the trace operator. Since this trace operator is continuous from $H^1(\Omega)$ to $L^2(\partial\Omega)$ with the topologies induced by the norms, and since $\varphi_k \rightharpoonup \varphi$ weakly in $H^1(\Omega)$ as $k \to +\infty$, it follows that $\mathrm{tr}(\varphi_k) \rightharpoonup \mathrm{tr}(\varphi)$ weakly in $L^2(\partial\Omega)$ as $k \to +\infty$. Finally, since $\mathrm{tr}(\varphi_k) \to 0$ strongly in $L^2(\partial\Omega)$ as $k \to +\infty$ by (2.8), one gets that $\mathrm{tr}(\varphi) = 0$, meaning that $\varphi \in H^1_0(\Omega)$.

Consider now $\psi \in C^\infty_c(\Omega)$. One has

$$\begin{aligned}
\int_\Omega \nabla\varphi \cdot \nabla\psi + \int_\Omega (v \cdot \nabla\varphi)\psi &= \lim_{k\to+\infty} \int_\Omega \nabla\varphi_k \cdot \nabla\psi + \int_\Omega (v \cdot \nabla\varphi_k)\psi \\
&= \lim_{k\to+\infty} \lambda_k \int_\Omega \varphi_k\psi \\
&= \mu \int_\Omega \varphi\psi,
\end{aligned}$$

which means that φ is an $H^1_0(\Omega)$ weak solution of

$$\begin{cases} -\Delta\varphi + v \cdot \nabla\varphi = \mu\varphi & \text{in } \Omega, \\ \mathrm{tr}(\varphi) = 0 & \text{on } \partial\Omega. \end{cases}$$

Elliptic H^2 and $W^{2,p}$ estimates show that $\varphi \in W^{2,p}(\Omega)$ for all $1 \leq p < \infty$, and, since $\varphi \geq 0$ in Ω and $\|\varphi\|_{L^2(\Omega)} = 1$, the strong maximum principle entails that $\varphi > 0$ in Ω. Thus, by uniqueness of the principal eigenvalue of $-\Delta + v \cdot \nabla$ under Dirichlet boundary condition, one gets that

$$\mu = \lambda_1^D(\Omega, v),$$

which ends the proof.

Lastly, let us investigate the limit of $\lambda_1^\beta(\Omega, v)$ as $\beta \to 0$. Pick up any decreasing sequence $(\beta_k)_{k\in\mathbb{N}}$ of positive real numbers with $\lim_{k\to+\infty} \beta_k = 0$ and set $\lambda_k := \lambda_1^{\beta_k}(\Omega, v)$ for all $k \in \mathbb{N}$. The sequence $(\lambda_k)_{k\in\mathbb{N}}$ is decreasing and bounded below by 0, and therefore converges to some $\lambda \geq 0$. For all k, if φ_k is defined as $\varphi_k := \theta_k \varphi^{\beta_k}_{\Omega,v}$ with $\theta_k > 0$ such that $\|\varphi_k\|_{L^2(\Omega)} = 1$, then as above (2.4)-(2.5) still hold and there exists $\varphi \in H^1(\Omega)$ satisfying (2.6)-(2.7), up to a subsequence. Pick now any $\psi \in H^1(\Omega)$. For all $k \in \mathbb{N}$, one has

$$\lambda_k \int_\Omega \varphi_k\psi = \int_\Omega \nabla\varphi_k \cdot \nabla\psi + \beta_k \int_{\partial\Omega} \varphi_k\psi + \int_\Omega (v \cdot \nabla\varphi_k)\psi.$$

But $\beta_k \to 0$ as $k \to +\infty$ and the sequence $(\mathrm{tr}(\varphi_k))_{k\in\mathbb{N}}$ is bounded in $L^2(\partial\Omega)$ (since so is $(\varphi_k)_{k\in\mathbb{N}}$ in $H^1(\Omega)$). Hence, by (2.6), the passage to the limit as $k \to +\infty$ in the above formula leads to

$$\lambda \int_\Omega \varphi\psi = \int_\Omega \nabla\varphi \cdot \nabla\psi + \int_\Omega (v \cdot \nabla\varphi)\psi.$$

In other words, φ is an $H^1(\Omega)$ weak solution of

$$\begin{cases} -\Delta\varphi + v \cdot \nabla\varphi = \lambda\varphi & \text{in } \Omega, \\ \dfrac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Elliptic H^2 and $W^{2,p}$ estimates show that $\varphi \in W^{2,p}(\Omega)$ for all $1 \le p < \infty$, and by (2.7) the strong maximum principle and Hopf lemma entail that $\varphi > 0$ in $\overline{\Omega}$. Thus, by uniqueness of the principal eigenvalue of $-\Delta + v \cdot \nabla$ under Neumann boundary condition, one gets that $\lambda = 0$. The proof of Lemma 2.2 is thereby complete. □

3 Optimization of the principal eigenvalue in a fixed domain

This section is devoted to the proof of Theorem 1.2.

Proof of Theorem 1.2. *Part 1.* We first focus on the infimum problem and begin with the existence part. Let $(v_k)_{k\in\mathbb{N}}$ be a sequence of vector fields in $L^\infty(\Omega, \mathbb{R}^d)$ such that $\|v_k\|_\infty \le \tau$ for all k and

$$\lim_{k\to+\infty} \lambda_1^\beta(\Omega, v_k) = \underline{\lambda}^\beta(\Omega, \tau).$$

For all $k \in \mathbb{N}$, define $\lambda_k := \lambda_1^\beta(\Omega, v_k)$ and let $\varphi_k := \varphi^\beta_{\Omega,v_k}$ be the corresponding eigenfunction, normalized with $\max_{\overline{\Omega}} \varphi_k = 1$. Since $(v_k)_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega, \mathbb{R}^d)$ and the sequence $(\lambda_k)_{k\in\mathbb{N}}$ is bounded, $W^{2,p}$ elliptic estimates ([15, Theorem A.29]) show that the sequence $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in $W^{2,p}(\Omega)$ for all $1 \le p < \infty$. Up to a subsequence, there exist $\varphi \in \bigcap_{1\le p<\infty} W^{2,p}(\Omega)$ and $f \in L^\infty(\Omega)$ such that, as $k \to +\infty$,

$$\varphi_k \rightharpoonup \varphi \text{ weakly in } W^{2,p}(\Omega)$$

for all $1 \le p < \infty$,

$$\varphi_k \to \varphi \text{ strongly in } C^{1,\alpha}(\overline{\Omega})$$

for all $\alpha \in (0, 1)$, and

$$v_k \cdot \nabla\varphi_k \overset{*}{\rightharpoonup} f \text{ weak-}* \text{ in } L^\infty(\Omega).$$

As a consequence,

$$-\Delta\varphi + f = \underline{\lambda}^\beta(\Omega, \tau)\varphi \text{ a.e. in } \Omega$$

and

$$-\Delta\varphi - \tau\,|\nabla\varphi| \le \underline{\lambda}^\beta(\Omega, \tau)\varphi \text{ a.e. in } \Omega.$$

Moreover, $\varphi \geq 0$ in Ω, $\|\varphi\|_\infty = 1$ and

$$\frac{\partial \varphi}{\partial \nu} + \beta \varphi = 0 \text{ on } \partial\Omega.$$

Define now $v \in L^\infty(\Omega, \mathbb{R}^d)$ by

$$v(x) := \begin{cases} -\tau \dfrac{\nabla\varphi(x)}{|\nabla\varphi(x)|} & \text{if } \nabla\varphi(x) \neq 0, \\ 0 & \text{if } \nabla\varphi(x) = 0. \end{cases} \tag{3.1}$$

Notice that $\|v\|_\infty \leq \tau$, which entails that $\underline{\lambda}^\beta(\Omega, \tau) \leq \lambda_1^\beta(\Omega, v)$. On the one hand,

$$-\Delta\varphi + v \cdot \nabla\varphi = -\Delta\varphi - \tau\,|\nabla\varphi| \leq \underline{\lambda}^\beta(\Omega,\tau)\varphi \leq \lambda_1^\beta(\Omega, v)\varphi \ \text{ a.e. in } \Omega. \tag{3.2}$$

On the other hand, $\varphi^\beta_{\Omega,v} > 0$ in $\overline{\Omega}$,

$$\frac{\partial \varphi^\beta_{\Omega,v}}{\partial \nu} + \beta \varphi^\beta_{\Omega,v} = 0 = \frac{\partial \varphi}{\partial \nu} + \beta\varphi \ \text{ on } \partial\Omega$$

and $\|\varphi^\beta_{\Omega,v}\|_\infty = 1$. Lemma 2.1 applied with $(\mu, \beta, \psi, \varphi) := (\lambda_1^\beta(\Omega, v), \beta, \varphi^\beta_{\Omega,\tau}, \varphi)$ yields

$$\varphi^\beta_{\Omega,v} = \varphi \ \text{ in } \overline{\Omega}.$$

As a consequence, all inequalities in (3.2) are equalities and

$$\underline{\lambda}^\beta(\Omega, \tau) = \lambda_1^\beta(\Omega, v).$$

Furthermore, since $\varphi \in W^{2,p}(\Omega)$ for each $1 \leq p < \infty$, it follows that

$$|\nabla(\partial_{x_i}\varphi)| \times \mathbb{1}_{\{\partial_{x_i}\varphi = 0\}} = 0 \ \text{ a.e. in } \Omega$$

for each $1 \leq i \leq d$, where $\partial_{x_i}\varphi := \frac{\partial\varphi}{\partial x_i}$, whence

$$\Delta\varphi \times \mathbb{1}_{\{\nabla\varphi = 0\}} = 0 \ \text{ a.e. in } \Omega.$$

Since $-\Delta\varphi + v \cdot \nabla\varphi = \lambda_1^\beta(\Omega, v)\varphi > 0$ a.e. in Ω, one gets that the set $\{x \in \Omega : \nabla\varphi(x) = 0\}$ is negligible. Therefore, in addition to $v \cdot \nabla\varphi = -\tau|\nabla\varphi|$ a.e. in Ω, (3.1) also entails that $|v(x)| = \tau$ for almost every $x \in \Omega$. The vector field $\underline{v} := v$ and the function

$$\underline{\varphi} := \varphi = \varphi^\beta_{\Omega,v}$$

then fulfill the required conclusions of part 1 of Theorem 1.2.

Let us now turn to the uniqueness result in part 1 of Theorem 1.2. Assume that $w \in L^\infty(\Omega, \mathbb{R}^d)$ is such that $\|w\|_\infty \leq \tau$ and $\lambda_1^\beta(\Omega, w) = \underline{\lambda}^\beta(\Omega, \tau)$. One has

$$\begin{cases} -\Delta\varphi^\beta_{\Omega,v} + w \cdot \nabla\varphi^\beta_{\Omega,v} \geq -\Delta\varphi^\beta_{\Omega,v} - \tau|\nabla\varphi^\beta_{\Omega,v}| = \underline{\lambda}^\beta(\Omega,\tau)\varphi^\beta_{\Omega,v}, \\ -\Delta\varphi^\beta_{\Omega,w} + w \cdot \nabla\varphi^\beta_{\Omega,w} = \underline{\lambda}^\beta(\Omega,\tau)\varphi^\beta_{\Omega,w}, \end{cases} \tag{3.3}$$

a.e. in Ω, together with

$$\frac{\partial \varphi^\beta_{\Omega,v}}{\partial \nu} + \beta \varphi^\beta_{\Omega,v} = 0 = \frac{\partial \varphi^\beta_{\Omega,w}}{\partial \nu} + \beta \varphi^\beta_{\Omega,w} \ \text{ on } \partial\Omega.$$

Furthermore, both functions $\varphi^\beta_{\Omega,v}$ and $\varphi^\beta_{\Omega,w}$ are positive (in $\overline{\Omega}$), with L^∞ norms equal to 1. Lemma 2.1 applied with $(\mu, \beta, \psi, \varphi) := (\underline{\lambda}^\beta(\Omega,\tau), \beta, \varphi^\beta_{\Omega,v}, \varphi^\beta_{\Omega,w})$, and the vector field w instead of v, then entails

$$\varphi^\beta_{\Omega,v} = \varphi^\beta_{\Omega,w} \ \text{ in } \overline{\Omega}.$$

Consequently, the first line in (3.3) then yields

$$w \cdot \nabla \varphi^\beta_{\Omega,v} = -\tau |\nabla \varphi^\beta_{\Omega,v}| \ \text{ a.e. in } \Omega,$$

that is, $w \cdot \nabla \varphi = -\tau |\nabla \varphi|$ a.e. in Ω. Since $\nabla \varphi \neq 0$ a.e. in Ω and $\|w\|_\infty \leq \tau$, one concludes that

$$w = -\tau \frac{\nabla \varphi}{|\nabla \varphi|} \ \text{ a.e. in } \Omega,$$

that is, $w = v$ a.e. in Ω.

Lastly, let $\lambda \in \mathbb{R}$ and $\phi \in \bigcap_{1 \leq p < \infty} W^{2,p}(\Omega)$ satisfy

$$\begin{cases} -\Delta \phi - \tau \, |\nabla \phi| = \lambda \phi \ \text{ and } \ \phi \geq 0 & \text{in } \Omega, \\ \dfrac{\partial \phi}{\partial \nu} + \beta \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

and $\max_{\overline{\Omega}} \phi = 1$. Define $q \in L^\infty(\Omega, \mathbb{R}^d)$ by

$$q(x) := \begin{cases} -\tau \dfrac{\nabla \phi(x)}{|\nabla \phi(x)|} & \text{if } \nabla \phi(x) \neq 0, \\ 0 & \text{if } \nabla \phi(x) = 0. \end{cases}$$

Notice that $\|q\|_\infty \leq \tau$. Since $-\tau |\nabla \phi| = q \cdot \nabla \phi$ a.e. in Ω, the nonnegativity of ϕ and the uniqueness of the pair of principal eigenvalue and principal normalized eigenfunction imply that

$$\lambda = \lambda^\beta_1(\Omega, q) \geq \underline{\lambda}^\beta(\Omega,\tau), \ \text{ and } \ \phi = \varphi^\beta_{\Omega,q} \text{ in } \overline{\Omega}.$$

Both functions $\phi = \varphi^\beta_{\Omega,q}$ and $\varphi = \varphi^\beta_{\Omega,v}$ are positive in $\overline{\Omega}$ with L^∞ norms equal to 1, and they satisfy

$$\begin{cases} -\Delta \varphi^\beta_{\Omega,q} + v \cdot \nabla \varphi^\beta_{\Omega,q} \geq -\Delta \varphi^\beta_{\Omega,q} - \tau |\nabla \varphi^\beta_{\Omega,q}| = \lambda^\beta_1(\Omega,q) \varphi^\beta_{\Omega,q}, \\ -\Delta \varphi^\beta_{\Omega,v} + v \cdot \nabla \varphi^\beta_{\Omega,v} = \underline{\lambda}^\beta(\Omega,\tau) \varphi^\beta_{\Omega,v} \leq \lambda^\beta_1(\Omega,q) \varphi^\beta_{\Omega,v}, \end{cases} \tag{3.4}$$

a.e. in Ω, together with

$$\frac{\partial \varphi^\beta_{\Omega,q}}{\partial \nu} + \beta \varphi^\beta_{\Omega,q} = 0 = \frac{\partial \varphi^\beta_{\Omega,v}}{\partial \nu} + \beta \varphi^\beta_{\Omega,v} \ \text{ on } \partial\Omega.$$

Lemma 2.1 applied with $(\mu,\beta,\psi,\varphi)=(\lambda_1^\beta(\Omega,q),\beta,\varphi^\beta_{\Omega,q},\varphi^\beta_{\Omega,v})$ then entails

$$\varphi^\beta_{\Omega,q}=\varphi^\beta_{\Omega,v}\ \text{ in }\overline{\Omega},$$

that is, $\phi=\varphi=\underline{\varphi}$ in $\overline{\Omega}$. Furthermore, all inequalities in (3.4) are equalities and

$$\underline{\lambda}^\beta(\Omega,\tau)=\lambda_1^\beta(\Omega,q),$$

whence $\lambda=\underline{\lambda}^\beta(\Omega,\tau)$. All properties in part 1 of Theorem 1.2 have now been proved.

Part 2. Notice that, for all $v\in L^\infty(\Omega)$, $\lambda_1^\beta(\Omega,v)\le\lambda_1^D(\Omega,v)$ by Lemma 2.2. Since

$$\sup_{v\in L^\infty(\Omega,\mathbb{R}^d),\,\|v\|_\infty\le\tau}\lambda_1^D(\Omega,v)<+\infty$$

by [8, Proposition 5.1], [17, Theorem 1.5] or [18, Theorem 6.6], if follows that the quantity $\overline{\lambda}^\beta(\Omega,\tau)$ defined in (1.8) is a real number. Then, arguments similar to those in part 1 above yield the conclusions of part 2.

Part 3. Consider now the case $\Omega=\Omega^*$ and denote

$$\phi:=\varphi^\beta_{\Omega^*,\tau e_r}.$$

This function ϕ is positive in $\overline{\Omega^*}$, it is of class $W^{2,p}(\Omega^*)$ for all $1\le p<\infty$, and $\max_{\overline{\Omega^*}}\phi=1$. For any $\mathcal{R}\in O(d)$, the function $\phi\circ\mathcal{R}$ satisfies the same equation as ϕ in Ω^* and the same boundary condition on $\partial\Omega^*$. The uniqueness of the pair of eigenvalue and principal normalized eigenfunction then entails that $\phi\circ\mathcal{R}=\phi$ in $\overline{\Omega^*}$ for any $\mathcal{R}\in O(d)$, that is, ϕ is radially symmetric in $\overline{\Omega^*}$. Let R denote the radius of Ω^*. For any $\sigma\in(0,R]$, there holds

$$-\Delta\phi+\tau e_r\cdot\nabla\phi=\lambda_1^\beta(\Omega^*,\tau e_r)\phi>0$$

almost everywhere in $\{x:|x|\le\sigma\}$ and ϕ is constant on the sphere $\{y:|y|=\sigma\}$. The weak maximum principle then implies that $\phi(x)\ge\phi(y)$ for all $|x|\le|y|=\sigma$, and the Hopf lemma even yields $e_r\cdot\nabla\phi(y)<0$ for all $|y|=\sigma$. As a conclusion, ϕ is radially decreasing and

$$\tau e_r\cdot\nabla\phi=-\tau|\nabla\phi|$$

everywhere in $\overline{\Omega^*}\setminus\{0\}$ and the function ϕ then fulfills (1.10) in Ω^* with $\lambda:=\lambda_1^\beta(\Omega^*,\tau e_r)$. It then follows from the last result of part 1 of the present theorem that

$$\underline{\lambda}^\beta(\Omega^*,\tau)=\lambda_1^\beta(\Omega^*,\tau e_r),$$

and the uniqueness of the vector field minimizing $\lambda_1^\beta(\Omega^*,v)$ implies that $\underline{v}=\tau e_r$.

Similarly, by denoting $\psi:=\varphi^\beta_{\Omega^*,-\tau e_r}$, one proves similarly that ψ is radially decreasing and one still has $\tau e_r\cdot\nabla\psi=-\tau|\nabla\psi|$, that is, $-\tau e_r\cdot\nabla\psi=\tau|\nabla\psi|$, everywhere in $\overline{\Omega^*}\setminus\{0\}$. Part 2 of the present theorem then implies that

$$\overline{\lambda}^\beta(\Omega^*,\tau)=\lambda_1^\beta(\Omega^*,-\tau e_r)$$

and $\overline{v}=-\tau e_r$. The proof of Theorem 1.2 is thereby complete. □

4 Proof of the minimization result

Let us now prove Theorem 1.1. Arguing by contradiction, assume that the conclusion does not hold. There exist then a sequence $(\beta_k)_{k\in\mathbb{N}}$ of positive numbers such that $\lim_{k\to+\infty}\beta_k = +\infty$ and a sequence of vector fields $(v_k)_{k\in\mathbb{N}}$ such that, for all $k\in\mathbb{N}$, $\|v_k\|_\infty \leq \tau$ and

$$\lambda_1^{\beta_k}(\Omega, v_k) < \lambda_1^{\beta_k}(\Omega^*, \tau e_r) + \frac{1}{k+1}. \tag{4.1}$$

For all $k\in\mathbb{N}$, write

$$\varphi_k := \varphi_{\Omega,v_k}^{\beta_k} \quad \text{and} \quad \lambda_k := \lambda_1^{\beta_k}(\Omega, v_k).$$

One has

$$\begin{cases} -\Delta\varphi_k + v_k\cdot\nabla\varphi_k = \lambda_k\varphi_k & \text{a.e. in } \Omega, \\ \dfrac{1}{\beta_k}\dfrac{\partial\varphi_k}{\partial\nu} = -\varphi_k & \text{on } \partial\Omega. \end{cases}$$

Lemma 2.2 shows that $\lambda_k \leq \lambda_1^D(\Omega, v_k)$ for all $k\in\mathbb{N}$, while [8, Proposition 5.1] ensures that the sequence $(\lambda_1^D(\Omega, v_k))_{k\in\mathbb{N}}$ is bounded (recall that $\|v_k\|_\infty \leq \tau$ for all $k\in\mathbb{N}$). Furthermore, each λ_k is a positive real number. Therefore, the sequence $(\lambda_k)_{k\in\mathbb{N}}$ is bounded. Arguing as in the proof of Lemma 2.2, one concludes that the sequence $(\varphi_k)_{k\in\mathbb{N}}$ is then bounded in $H^1(\Omega)$, which entails that the sequence $(\mathrm{tr}(\varphi_k))_{k\in\mathbb{N}}$ is bounded in $H^{\frac{1}{2}}(\partial\Omega)$. Therefore, together with the boundedness of the sequence $(1/\beta_k)_{k\in\mathbb{N}}$, [1, Theorem 15.2] implies that the sequence $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in $W^{2,2}(\Omega)$, and a bootstrap argument therefore shows that $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in $W^{2,p}(\Omega)$ for all $1\leq p<\infty$. Thus, there exist $\mu\in\mathbb{R}$, $\varphi\in\bigcap_{1\leq p<\infty} W^{2,p}(\Omega)$ and $f\in L^\infty(\Omega)$ such that, up to a subsequence,

$$\lim_{k\to+\infty}\lambda_k = \mu,$$

$$\varphi_k \underset{k\to+\infty}{\rightharpoonup} \varphi \text{ weakly in } W^{2,p}(\Omega) \text{ and } \varphi_k \underset{k\to+\infty}{\to} \varphi \text{ strongly in } C^{1,\alpha}(\overline{\Omega})$$

for all $1\leq p<\infty$ and all $\alpha\in(0,1)$, and

$$v_k\cdot\nabla\varphi_k \rightharpoonup f \text{ weakly-}* \text{ in } L^\infty(\Omega).$$

Furthermore, as in the proof of Lemma 2.2, there holds $\mathrm{tr}(\varphi_k)\to 0$ as $k\to+\infty$ strongly in $L^2(\partial\Omega)$. One therefore has

$$\begin{cases} -\Delta\varphi + f = \mu\varphi & \text{a.e. in } \Omega, \\ \varphi \geq 0 & \text{in } \overline{\Omega}, \\ \mathrm{tr}(\varphi) = 0 & \text{on } \partial\Omega, \\ \max\limits_{\overline{\Omega}}\varphi = 1 \end{cases}$$

and $f\geq -\tau\,|\nabla\varphi|$ a.e. in Ω, so that

$$-\Delta\varphi - \tau\,|\nabla\varphi| \leq \mu\varphi \text{ a.e. in } \Omega.$$

Define

$$v(x) := \begin{cases} -\tau\dfrac{\nabla\varphi(x)}{|\nabla\varphi(x)|} & \text{if } \nabla\varphi(x)\neq 0, \\ 0 & \text{if } \nabla\varphi(x) = 0, \end{cases}$$

so that $\|v\|_\infty \leq \tau$ and

$$-\Delta\varphi + v \cdot \nabla\varphi \leq \mu\varphi \text{ a.e. in } \Omega.$$

Let now $\psi := \varphi^D_{\Omega,v}$, so that $\psi > 0$ in Ω and

$$-\Delta\psi + v \cdot \nabla\psi = \lambda^D_1(\Omega, v)\psi \text{ a.e. in } \Omega.$$

If $\mu < \lambda^D_1(\Omega, v)$, then

$$-\Delta\varphi + v \cdot \nabla\varphi \leq \mu\varphi \leq \lambda^D_1(\Omega, v)\varphi \text{ a.e. in } \Omega,$$

and [17, Lemma 2.1] implies that $\varphi = \psi$ in $\overline{\Omega}$, therefore $\mu = \lambda^D_1(\Omega, v)$, a contradiction. Finally,

$$\lambda^D_1(\Omega, v) \leq \mu.$$

But (4.1) and Lemma 2.2 imply that

$$\mu \leq \lambda^D_1(\Omega^*, \tau e_r),$$

and one therefore obtains

$$\lambda^D_1(\Omega, v) \leq \lambda^D_1(\Omega^*, \tau e_r),$$

which contradicts the "equality" statement in [17, Theorem 1.1] since Ω is not a ball. This concludes the proof of Theorem 1.1. □

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