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# Limit theorems for multivariate self-similar symmetric stable moving average processes: a study with $p$ -variations

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## Abstract

We consider the class of moving average symmetric  $\alpha$ -stable processes, for  $1 < \alpha < 2$ . These processes are  $H$ -self-similar ( $0 < H < 1$ ) with stationary increments, indexed by  $\mathbb{R}^d$ , and driven by a symmetric  $\alpha$ -stable random measure  $M_\alpha$ . Our aim is to identify them by estimating the Hurst parameter  $H$ , using estimators derived from  $p$ -variations and a wavelet decomposition.

**Keywords:** Stable distributions, H-Self-similar with stationary increments (H-sssi) processes, Wavelet basis, Sums of independent random variables, Random measures  
Primary 60G18, 60G10, 65T60, 60F05 ; Secondary 60G50, 60G57

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## 1. Introduction

Self similar processes hold a crucial role in modeling complex data; Particularly noteworthy is the Fractional Brownian Motion (F.B.M.), that has been extensively studied since Mandelbrot's pioneering work [20]. Such a process is recognized for its self-similarity, characterized by the index  $0 < H < 1$ , and satisfies to the property

$$(X(\lambda x), x \in \mathbb{R}) \stackrel{(d)}{=} (\lambda^H X(x), x \in \mathbb{R}), \forall \lambda > 0 \quad (1.1)$$

where  $\stackrel{(d)}{=}$  stands for equality in distribution. This process is known for its two integral representations: the moving average and the harmonizable representations. Despite their different formulations, these representations are considered equivalent due to their distribution equality up to a constant factor for the Gaussian framework. Various methods evaluating the effectiveness of the estimations are discussed in [16]. Still in the context of Gaussian models, generalizations have been proposed by Benassi et al. [8, 7], and Lévy-Véhel et al. [23], [2], introducing processes that are self-similar

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with stationary increments ( $H$ -sssi). These processes can be described and identified by either a moving average or a real harmonizable representation, underscoring the sufficiency of estimating  $H$  for one of the two representation due to their equivalence. Expanding beyond Gaussian models, the  $\alpha$ -stable  $H$ -sssi non-Gaussian processes, as detailed in [26] or more recently in [24] can take two different forms, moving average and harmonizable processes respectively denoted by  $X$  and  $Z$ . The estimation of the Hurst parameter  $H$  for stable processes is a significant issue as published in recent works as [17]. When  $0 < \alpha < 2$  these representations do not share proportional laws, as highlighted in [26, p.358] and [9]. This requires separate analyses for each model.

This work proposes an estimation strategy for  $H$  for the moving average process. The challenge of estimating  $H$  in stable processes underscores the complexities arising from their infinite variance nature. Despite these challenges, the study of stable distributions is motivated by their practical relevance. Many models, especially in imaging or financial contexts, are based on Gaussian distributions. Recently, Bianchi et al. [13, 14] published several very interesting works on multi-fractional Brownian motion and time varying Hurst exponent applied in finance. In [11, 12], the study also delves into the time-varying Hurst exponent, highlighting its applications within the field of behavioral finance and market efficiency. Similarly, self-similar stable processes find application in physics for modeling long-range dependencies and high variability, as demonstrated by Taquu et al.'s work on network traffic [27, 29], further validating the significance of these models. In finance, Lévy-Véhel et al. suggested modeling asset returns using  $\alpha$ -stable distributions with  $1 < \alpha < 2$  in [6], challenging the Gaussian assumption prevalent in the Capital Asset Pricing Model. Other applications in wireless communications modeling can be found in [3, 4]. Self-similar stable processes are also used in physics as a model for self-similar processes with long-range dependence (the so-called Joseph effect) and infinite variance or high variability (Noah effect); recently, estimation results have been obtained for processes with long range dependence in [5]. A pioneering work of Taquu et al. provide a limit Lévy stable motion when modeling the network traffic (see [27, 29]). Other development can be also found in [22] and references within. Physicists have also proposed in [1] a wavelet based estimator for the Hurst parameter of the moving average stable process in dimension 1. They used averages of the logarithm of the absolute value of coefficients of the discrete wavelet transform in order to come back to the  $L^2$ -case. Several other works have also been developed for this topic more details can be found for instance in [28, 15, 22].

Before diving into the main topic of our work, let's briefly revisit the Gaussian framework. In [10], Benassi, Jaffard, and Roux introduce the concept of elliptic self-similar processes. They show that every centered elliptic self-similar Gaussian process can be represented through a moving average model. A prominent method for estimating the Hurst parameter  $H$  involves the use of quadratic variations. Importantly, for such Gaussian processes, it is adequate to perform the estimation along a single direction on the unit sphere  $\mathbb{S}_{d-1}$  in  $\mathbb{R}^d$ . Let  $f$  be a function  $\mathbb{R} \rightarrow \mathbb{R}$ , take  $N > 0$ ,  $0 \leq l \leq N - 1$  and consider the quadratic variations defined from

$$V_{N,2}(f) = \frac{1}{N} \sum_{l=0}^{N-1} \left| f\left(\frac{l+1}{N}\right) - f\left(\frac{l}{N}\right) \right|^2. \quad (1.2)$$

Then, taking  $f = Y_u = (Y_u(t), t \in \mathbb{R})$  the  $H$ -sssi Gaussian process defined by  $Y_u(t) = Y(tu)$ ,  $t \in \mathbb{R}$ , with  $u \in \mathbb{S}_{d-1}$  (see [8]). Hence define  $\widehat{H}_N^u$ ,  $N \geq 1$  as follows

$$\widehat{H}_N^u = \frac{-1}{2 \log N} \log V_{N,2}(Y_u), \quad (1.3)$$

$\widehat{H}_N^u$  almost surely converges to  $H$  as  $n$  tends to infinity, independently of  $u \in \mathbb{S}_{d-1}$ .

This paper's objective is to express an estimator for the Hurst index  $H$  with  $p$ -variation for the Stable non-gaussian moving average process  $X$ , supplemented by an unbiased estimator and its rate of convergence. The structure of the document is outlined as follows: Section 2 presents the main findings, including a Law of Large Numbers and outcomes articulated in three distinct manners, depending on the value of exponent  $p$ . Indeed, this method stems from the fact that the convergence rate and the applicable domain—either Gaussian or Stable non-Gaussian—are influenced by how  $p$  relates to  $\frac{\alpha}{2}$ . Section 3 details the approach for determining the domain of attraction and convergence rates through the development of triangular arrays. Then Section 4 describes the methodology for creating triangular arrays via a wavelet decomposition of the moving average process  $X$ . Subsequently, Section 5 synthesizes all preceding steps with the proofs of the main results to formulate the proposed estimator for the  $H$ -index, alongside providing an unbiased estimator. Finally, Section 6 offers concluding remarks and envisages the extension of this work to real harmonizable processes with a nod to potential applications in economics and finance. The reader can find in [Appendix A](#), [Appendix B](#) and [Appendix C](#) all the intermediary proofs and the theoretical developments on Triangular arrays, wavelet decomposition and auxiliary lemmas.

## 2. Main results - Law of Large Numbers and Estimators

### 2.1. Toward the $\alpha$ -stable framework

Let  $M_\alpha$  be an independently scattered symmetric  $\alpha$ -stable (S $\alpha$ S) stochastic measure, with the terminology of Samorodnitsky and Taqqu [26, p.281]. We focus on the family of the moving average  $\alpha$ -stable processes defined by a stochastic integral as follows:

$$X(x) = \int_{\mathbb{R}^d} \left( |x-y|^{H-d/\alpha} Q\left(\frac{x-y}{|x-y|}\right) - |y|^{H-d/\alpha} Q\left(\frac{y}{|y|}\right) \right) M_\alpha(dy) \quad (2.1)$$

where  $Q$  is an even function defined on  $\mathbb{S}_{d-1}$  the unit sphere of  $\mathbb{R}^d$  verifying  $c \leq Q \leq \frac{1}{c}$  for some  $0 < c < 1$ . It has been demonstrated in [26] that the process  $X = (X(x), x \in \mathbb{R}^d)$  exhibits self-similarity and possesses stationary increments, characterized by the Hurst parameter  $H$  with  $0 < H < 1$ . These attributes of  $\alpha$ -stability,  $H$ -self-similarity, and stationary increments of  $X$  stem from the corresponding properties of the random measure  $M_\alpha$ . It have been further explored in [9] that for  $1 < \alpha < 2$ , thanks to a Plancherel like formula, the process  $X$  admits the following harmonizable like representation

$$X(x) = \int_{\mathbb{R}^d} f_{\alpha^*}(x, \xi) \widehat{M}_\alpha(d\xi) \quad , \quad x \in \mathbb{R}^d \quad (2.2)$$

where  $\widehat{M}_\alpha$  is the Fourier transform of  $M_\alpha$ , the term  $\alpha^* = \frac{\alpha}{\alpha-1}$  is the  $\alpha$ -conjugate, and the integrated function is given by:

$$f_{\alpha^*}(x, \xi) = \frac{\exp(ix \cdot \xi) - 1}{|\xi|^{H+d/\alpha^*} Q^*\left(\frac{\xi}{|\xi|}\right)}$$

where  $Q^*$  is also an even function defined on  $\mathbb{S}_{d-1}$  and verifying  $c \leq Q^* \leq \frac{1}{c}$ .

Our aim is to extend the estimator (1.3) to the case of  $S\alpha S$ -stable  $H$ -sssi processes, with  $1 < \alpha < 2$  and by replacing the power 2 in the second hand side of the quadratic variation (1.2) by the power of  $p$  for any  $0 < p < \alpha$ . The technical idea consist in using the wavelet decomposition of random measure  $M_\alpha$ . Using the stochastic integral representation (2.2), it comes that the processes  $X$  can be written in the following wavelet decomposition:

$$X(x) = \sum_{\lambda \in \Lambda_d} 2^{-jH} \Phi_\alpha^v(2^j x - k) \eta_\lambda^\alpha, \quad (2.3)$$

where  $(\eta_\lambda^\alpha, \lambda \in \Lambda_d)$  is a sequence of  $S\alpha S$  random variables and  $\Phi_\alpha^v$  are the adapted wavelets calculated from the wavelets decomposition of (2.2). They are defined by the two following integral representations:

$$\eta_\lambda^\alpha = \int_{\mathbb{R}^d} \Psi_{\lambda, \alpha} M_\alpha(dx) \quad \text{and} \quad \Phi_\alpha^v(x) = \int_{\mathbb{R}^d} f_{\alpha^*}(x, \xi) \widehat{\Psi}^v(\xi) d\xi \quad (2.4)$$

where  $\widehat{\Psi}^v(\xi)$  is the Fourier transform of the mother wavelet and

$$\Psi_{\lambda, \alpha}(x) = 2^{j \frac{d}{\alpha}} \Psi^v(2^j x - k) \quad (2.5)$$

In [Appendix A](#), we recall all the ingredients and the theoretical developments on the wavelets that leads to these equations.

## 2.2. Main result: Law of Large Numbers - Almost sure consistency

We aim at providing an estimator of the Hurst index  $H \in ]0, 1[$  for  $S\alpha S$  moving average process  $X$  (2.1). We define the following  $p$ -variations,  $0 < p < \alpha$  with  $N = 2^n, n \in \mathbb{N}^*$

$$V_{n,p}^u(X) = \frac{1}{2^n} \sum_{l=0}^{2^n-1} \left| X\left(\frac{l+1}{2^n}u\right) - X\left(\frac{l}{2^n}u\right) \right|^p \quad (2.6)$$

for  $u$  belonging to  $\mathbb{S}_{d-1}$ ,  $0 \leq l \leq 2^n - 1$ ,  $1 < \alpha < 2$ . For the sake of simplicity, we will next denote the normalized increments of the process  $X$  by:

$$\xi_l^n(X) = 2^{nH} \left( X\left(\frac{l+1}{2^n}u\right) - X\left(\frac{l}{2^n}u\right) \right) \quad (2.7)$$

so that

$$2^{nHp} V_{n,p}^u(X) = \frac{l}{2^n} \sum_{l=0}^{2^n-1} |\xi_l^n|^p. \quad (2.8)$$

**Theorem 1.** *Estimator of  $H$*

Let  $M_\alpha$  be a stochastic symmetric  $\alpha$ -stable measure, let  $X$  be the moving average stochastic processes associated to  $M_\alpha$  by (2.1) and  $V_{n,p}^u(X)$  the associated  $p$ -variations defined by (2.6) with  $0 < p < \alpha$ . We take

$$\widehat{H}_{n,p}^u = \frac{-1}{np} \log_2 V_{n,p}^u(X), \quad (2.9)$$

then

$$\lim_{n \rightarrow \infty} \widehat{H}_{n,p}^u = H \quad \mathbb{P}a.s. \quad \forall u \in \mathbb{S}_{d-1}. \quad (2.10)$$

This estimator (2.9) has a bias. In order to get an unbiased estimator for  $H$ , we form hereafter quotient of  $p$ -variations (2.12) as stated in Corollary 1 (developed and proved in subsection 5.3).

**Corollary 1.** *Unbiased estimator of  $H$*

Under the same assumptions as in Theorem 1, we form the following  $p$ -variations  $V_{n,p}^{u,r}$ , for any  $n \in \mathbb{N}^*$ ,  $r \in \mathbb{N}^*$

$$V_{n,p}^{u,r}(X) = \frac{1}{2^n - 2^r + 1} \sum_{l=0}^{2^n-2^r} \left| X\left(\frac{l+2^r}{2^n}u\right) - X\left(\frac{l}{2^n}u\right) \right|^p, \quad (2.11)$$

where  $V_{n,p}^{u,0}(X) = V_{n,p}^u(X)$  (2.6). Now define the estimator

$$\tilde{H}_{n,p}^{u,r} = \frac{1}{rp} \log_2 \frac{V_{n,p}^{u,r}(X)}{V_{n,p}^{u,0}(X)}, \quad (2.12)$$

then  $\lim_{n \rightarrow \infty} \tilde{H}_{n,p}^{u,r} = H \quad \mathbb{P}a.s.$ ,  $\forall u \in \mathbb{S}_{d-1}$  and  $E \left| \tilde{H}_{n,p}^{u,r} \right| = H$ .

2.3. *Secondary results: Asymptotic behavior of the estimator*

The approach used to get estimator (2.9) and consequently unbiased estimator (2.12) is based on establishing a Law of Large Numbers and getting the convergence rate. Such results articulate into three distinct parts, since the convergence rate and the domain of attraction can be either Gaussian or Stable non-Gaussian, depending on the position of value of exponent  $p$  related to  $\frac{\alpha}{2}$ . We distinguish three cases with different behaviors: (i)  $0 < p < \frac{\alpha}{2}$ , a critical case (ii)  $p = \frac{\alpha}{2}$ , and (iii)  $\frac{\alpha}{2} < p < \alpha$ , as we describe in the following statements.

**Theorem 2.** *Domain of attraction*

Let  $0 < p < \alpha$ , consider variations  $V_{n,p}^{u,r}$  (2.11). Then, given  $a_n = \frac{2^n}{M_n}$  with  $M_n = 2K_n 2^{L_n}$ , where  $K_n \sim \log_2(n)$ ,  $L_n \sim \log_2(n)$ , with  $n \in \mathbb{N}^*$ , we get:

- (i) if  $0 < p < \frac{\alpha}{2}$ , then  $(a_n)^{\frac{1}{2}} \frac{V_{n,p}^{u,1} - 2^{Hp} V_{n,p}^{-u,0}}{V_{n,p}^{-u,0}}$  belongs to the domain of attraction of a normal law,
- (ii) if  $p = \frac{\alpha}{2}$ , then  $(a_n \log_2(n))^{\frac{1}{2}} \frac{V_{n,p}^{u,1} - 2^{Hp} V_{n,p}^{-u,0}}{V_{n,p}^{-u,0}}$  belongs to the domain of attraction of a normal law,
- (iii) if  $\frac{\alpha}{2} < p < \alpha$ , then  $(a_n)^{\frac{1}{\tilde{\alpha}}} \frac{V_{n,p}^{u,1} - 2^{Hp} V_{n,p}^{-u,0}}{V_{n,p}^{-u,0}}$  belongs to the domain of attraction of a symmetric stable law with stability index  $\tilde{\alpha} = \frac{\alpha}{p}$  ( $1 < \tilde{\alpha} < 2$ ).

The proof of this Theorem, as well as the one of the following Corollary on the convergence rate of the estimator deduced from (2.12), is based on Proposition 1 and Proposition 2, with, in addition, an application of the delta-Method.

**Corollary 2.** *convergence rate of the estimator*

Consider  $\tilde{H}_{n,p} = \tilde{H}_{n,p}^{u,1}$  defined from (2.12), we have:

- (i) if  $0 < p < \frac{\alpha}{2}$ , then  $(a_n)^{\frac{1}{2}} (\tilde{H}_{n,p} - H)$  converges in law toward a normal law,
- (ii) if  $p = \frac{\alpha}{2}$ , then  $(a_n \log_2(n))^{\frac{1}{2}} (\tilde{H}_{n,p} - H)$  converges in law toward a normal law,
- (iii) if  $\frac{\alpha}{2} < p < \alpha$ , then  $(a_n)^{\frac{1}{\tilde{\alpha}}} (\tilde{H}_{n,p} - H)$  converges in law toward a  $\tilde{\alpha}$ -stable non-Gaussian law, where  $\tilde{\alpha} = \frac{\alpha}{p}$ .

From now, in the remainder parts of this article, we give the necessary steps to prove these main and secondary results. The idea is to write the estimator as normalised sum. By the wavelet decomposition (2.3) we get that the increments (2.7) of the process can be written

$$\xi_l^n = 2^{nH} \sum_{j=-\infty}^{\infty} 2^{-jH} \sum_{k,v} \left( \Phi_{\alpha}^v \left( 2^j \frac{(l+1)u}{2^n} - k \right) - \Phi_{\alpha}^v \left( 2^j \frac{lu}{2^n} - k \right) \right) \eta_{j,k,v}^{\alpha}. \quad (2.13)$$

If those increments were independent, we would prove that normalized  $p$ -variations  $2^{npH} V_{n,p}^u(X)$  (2.6) almost surely converges to a constant. But  $p$ -variations (2.6) are expressed as the sum of non independent terms. The idea is hence to use results from triangular arrays limit theorems - the results on convergent triangular arrays are given in Appendix B. In order to comply with the usual notations, let  $\mathcal{T} = ((T_{n,l} = |\xi_l^n|^p, 0 \leq l \leq 2^n - 1), n \in \mathbb{N}^*)$ . We study in section 3 its domain of attraction in each case (i), (ii) and (iii) (proposition 1) and derive from it a Law of Large Numbers in proposition 2. For this purpose we have to show that the set  $\mathcal{T}$  have the triangular array's structure with the suitable normalising constants For this purpose, we need to rearrange the double sum in (2.13) so that its components have the structure of dependence of triangular arrays. This is done into three steps:

- First, we use a truncated wavelet decomposition, detailed in Section 4

- so that we adapt the supports of the corresponding wavelets to be in disjoint sets and then use the independent increments of the  $\alpha$ -stable random measure  $M_\alpha$ ,
- then apply the appropriated normalizing constant to obtain the L.L.N. of proposition 2

Then in section 5 those intermediary results and steps are grouped together and we prove the convergence of estimators (2.9) and (2.12).

### 3. Law of Large Numbers for Triangular Arrays

Given  $(i_n, n \in \mathbb{N}^*)$  a sequence of positive numbers, we define a triangular array by setting  $\mathcal{T} = (\{T_{n,l}, 0 \leq l \leq i_n - 1\}, n \in \mathbb{N}^*)$  where random variables  $T_{n,l}$  are independent identically distributed (i.i.d.) column by column, along each row  $n$ .

**Definition 1.**  $\mathcal{T}$  is a convergent array if it satisfies the following equality in law

$$\lim_{n \rightarrow \infty} \frac{1}{B_n} \sum_{l=0}^{i_n-1} T_{n,l} \stackrel{(d)}{=} V$$

for a given positive sequence  $(B_n, n \in \mathbb{N}^*)$  and for a given random variable  $V$ .

**Definition 2.**  $\mathcal{T}$  admits a Law of Large Numbers (L.L.N.) if

$$\exists c \in \mathbb{R} \text{ such that } \lim_{n \rightarrow \infty} \frac{1}{i_n} \sum_{l=0}^{i_n-1} T_{n,l} = c \quad \mathbb{P}a.s.$$

Set  $(\xi_l^n, l \in \mathbb{N})$  a sequence of i.i.d. random variables with common S $\alpha$ S law for a given  $n$ . Let

$$\mathcal{T}^{(p)} = \left( \left\{ \tau_{n,l}^{(p)}, 0 \leq l \leq 2^n - 1 \right\}, n \in \mathbb{N}^* \right)$$

be a triangular array with variables  $\tau_{n,l}^{(p)}$  defined along a given line  $n$  by

$$\tau_{n,0}^{(p)} \stackrel{(d)}{=} |\xi_0^n|^p - E |\xi_0^n|^p$$

for  $0 < p < \alpha$ . Let  $F$  (respectively  $F_\tau$ ) denote the common distribution function of variables  $\xi_l^n$  (respectively  $\tau_{n,l}^{(p)}$ ).

**Proposition 1.**  $\mathcal{T}^{(p)}$  is a convergent array with normalization  $(B_n^{(p)}, n \in \mathbb{N}^*)$

(i) if  $0 < p < \frac{\alpha}{2}$  with  $B_n^{(p)} = 2^{\frac{n}{2}}$ , then  $F_\tau$  belongs to the domain of attraction of a normal law,

(ii) if  $p = \frac{\alpha}{2}$  with  $B_n^{(p)} = (2^n \log(2^{\frac{n}{2}}))^{\frac{1}{2}}$ , then  $F_\tau$  belongs to the domain of attraction of a normal law,

(iii) if  $\frac{\alpha}{2} < p < \alpha$  with  $B_n^{(p)} = 2^{\frac{n}{\alpha}}$ , then  $F_\tau$  belongs to the domain of attraction of a symmetric stable law with stability index  $\tilde{\alpha} = \frac{\alpha}{p}$  ( $1 < \tilde{\alpha} < 2$ ).



**Proposition 2.** *In these three cases,  $\mathcal{T}^{(p)}$  admits a Law of Large Numbers*

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} \sum_{l=0}^{2^n-1} \tau_{n,l}^{(p)} = 0 \quad \mathbb{P}a.s. \quad (3.1)$$

**Remark 1.** *In Gaussian (i) and critical cases (ii), we can work with any  $N > 0$  but stable case (iii) require us to work with an exponential sub-sequence  $(2^n, n \in \mathbb{N}^*)$  as it will be explained in commentary after lemma 5.*

In what follows, when there exists no possible confusion, we just note  $\tau_{n,l}$  and  $B_n$  to lighten. The proofs of propositions 1 and 2 are in [Appendix B](#).

#### 4. Truncated Adapted Wavelets decomposition

Let  $X$  be the moving average  $\alpha$ -stable process defined by (2.1). Set  $u \in \mathbb{S}_{d-1}$ , consider  $p$ -variations (2.6). Take  $l \in \{0, 2^n - 1\}$  and let  $(\xi_l^n(X), n \in \mathbb{N}^*)$  be the sequence of stationary random variables defined by (2.7) and normalized  $p$ -variations (2.8). Let  $\mathcal{T}$  be the triangular array  $\mathcal{T} = ((T_{n,l}^u, 0 \leq l \leq 2^n - 1), n \in \mathbb{N}^*)$  where  $T_{n,l}^u = |\xi_l^n|^p, 0 < p < \alpha$ . When there is no possible confusion, we denote  $\xi_l^n(X)$  and  $\xi_l^n(Z)$  by  $\xi_l^n$  for short. Variables we now consider are correlated. Our aim is to come down to results for independent variables. For this, we use the wavelet decomposition (2.3) of the moving average process  $X$ . By (2.4), normalized increments around a given point  $l$  can be written as done in (2.13). If those normalized increments were independent along each line  $n$ , then we could apply proposition 2. They aren't thought. Our aim is thus to amount to the first case of arrays of independent random variables. The principle of reduction to the independent case is the following : consider wavelets with pairwise disjoint supports, as measure  $M_\alpha$  is independently scattered, the corresponding random variables  $\eta_\lambda^\alpha$  defined by (2.4) are independent for a fixed  $j$ . For this, we will use truncated variables.

##### 4.1. Definition of truncated process

Decomposition ([Appendix A.4](#)) is made of two essential components : functions  $\Phi_\alpha^v$  ([A.7](#)) and random variables  $\eta_\lambda^\alpha$  ([A.1](#)) for  $X$ . In order to work with compact support adapted wavelets  $\Phi_\alpha^v$ , we cut them with the help of  $\chi_K$ , the  $\mathcal{C}^\infty$  function equals to 1 between  $-K$  and  $+K$  (with radius  $K > 0$ ), decreasing on  $(-K - \varepsilon, -K)$  and  $(K, K + \varepsilon)$ , and equals to 0 elsewhere, that is

$$\Phi_\alpha^{v,K} = \Phi_\alpha^v \times \chi_K \text{ so that } \lim_{K \rightarrow \infty} \Phi_\alpha^{v,K} = \Phi_\alpha^v. \quad (4.1)$$

By a misuse of notation, we still denote by  $\chi_K$  function  $\xi \mapsto \chi_K(|\xi|)$ . Next, we will consider  $K_n$  which slowly tends to infinity with  $n$ . Using such functions  $\Phi_\alpha^{v,K}$ , we hence work with a finite number of wavelets. We denote by  $\Phi_{\lambda,\alpha}^K$  the functions constructed from the  $\Phi_\alpha^{v,K}$ 's by dilatation and translation as in ([A.6](#)). They are with compact support. In the same way, we construct

$$\Psi_\alpha^{v,K} = \Psi_\alpha^v \times \chi_K \text{ so that } \lim_{K \rightarrow \infty} \Psi_\alpha^{v,K} = \Psi_\alpha^v. \quad (4.2)$$

We denote by  $\Psi_{\alpha,\lambda}^K$  the corresponding wavelets defined as in (Appendix A.1) and by  $\eta_\lambda^{\alpha,K}$  the corresponding random variables formed as in (A.1). We still call  $X$  the resulting process.

#### 4.1.1. Cut off on adapted wavelets

Given  $\varepsilon(\Phi_\alpha^v)$  the error made when using  $\Phi_\alpha^{v,K}$  instead of initial function  $\Phi_\alpha^v$  in the expression of  $X$ . By construction (4.1), we have

$$\Phi_\alpha^{v,K}(x) = \Phi_\alpha^v(x) \quad \text{if } |x| \leq K, \quad 0 \quad \text{else.}$$

Besides, as  $\Phi_\alpha^v$  is with rapid decrease, it comes

$$\varepsilon(\Phi_\alpha^v) = \max_{x \in \mathbb{R}^d} |(\Phi_\alpha^v - \Phi_\alpha^{v,K})(x)| = \max_{|x| > K} \frac{C_{\beta_1}}{1 + |x|^{\beta_1}}$$

We deduce from this relation that the error which has repercussions on  $|\xi_t^n|^p$  is

$$\varepsilon_p(\Phi_\alpha^v) = \mathcal{O}(K^{-\beta_1 p}). \quad (4.3)$$

#### 4.1.2. Cut off on wavelets

Let now  $\varepsilon(\eta_\lambda^\alpha)$  denote the error made when using  $\Psi_{\lambda,\alpha}^K$  instead of initial function  $\Psi_{\lambda,\alpha}$  in the construction of  $\eta_\lambda^\alpha$  and  $\varepsilon_p(\eta_\lambda^\alpha)$  the error which affects  $|\xi_t^n|^p$ . Set  $R_{K,\lambda}$  the cut part of  $\eta_\lambda^\alpha$ ,

$$\varepsilon_p(\eta_\lambda^\alpha) = \|R_{K,\lambda}^K\|_{L^p(\mathbb{P})}.$$

As  $\eta_\lambda^\alpha$  is constructed from the random measure  $M_\alpha$  by (A.1), we write

$$\varepsilon_p(\eta_\lambda^\alpha) = \left( E |M_\alpha(\Psi_{\lambda,\alpha} - \Psi_{\lambda,\alpha}^K)|^p \right)^{\frac{1}{p}}$$

with

$$(\Psi_\alpha^v - \Psi_\alpha^{v,K})(x) = 0 \quad \text{if } |x| \leq K, \quad \Psi_\alpha^v(x) \quad \text{else.}$$

Thus (A.3) and (Appendix A.1) yield that

$$\begin{aligned} (\varepsilon_p(\eta_\lambda^\alpha))^p &= (c_{p,\alpha})^p \left( \int_{|2^j x - k| > K} 2^{jd} |\Psi_\alpha^v(2^j x - k)|^\alpha dx \right)^{\frac{p}{\alpha}} \\ &\leq (c_{p,\alpha})^p (C_{\beta_2})^p \left( \int_{|y| > K} \frac{dy}{1 + |y|^{\alpha\beta_2}} \right)^{\frac{p}{\alpha}} \end{aligned}$$

for all  $\beta_2 > 0$  by property of rapid decrease of functions  $\Psi^v$ , with  $C_{\beta_2} > 0$ . Then by the change of variable  $\rho = |y|$ , it comes

$$(\varepsilon_p(\eta_\lambda^\alpha))^p \leq c(p, \alpha, \beta_2) \left( \int_K^\infty \frac{1}{1 + \rho^{\alpha\beta_2}} \rho^{d-1} d\rho \right)^{\frac{p}{\alpha}}.$$

We therefore conclude that for  $\beta_2 > d/\alpha$ ,

$$\varepsilon_p(\eta_\lambda^\alpha) = \mathcal{O}\left(K^{(d-\beta_2\alpha)\frac{1}{\alpha}}\right). \quad (4.4)$$

Last, we rapidly check that those two terms of error introduced by the use of both truncated wavelets and truncated adapted wavelets have a negligible contribution in  $|\xi_l^n|^p$ . From (4.3), (4.4) and Borel-Cantelli's lemma under conditions  $\beta_1 > \frac{1}{p}(m+1)$ ,  $\beta_2 > \frac{1}{p}(m+1) + \frac{d}{\alpha}$ ,  $\forall m > 0$ , we get from some random rank

$$\varepsilon_p(G_{\lambda,\alpha}) + \varepsilon_p(\eta_\lambda^\alpha) \leq K^{-m}.$$

Hence we can work with truncated normalized increments still written  $\xi_l^n$  with an abuse of notation, without loss of information since  $K$  large enough.

#### 4.2. Decomposition of the increments

For a fixed  $n$ , we decompose each term  $\xi_l^n$ ,  $0 \leq l \leq 2^n - 1$  in three components; respectively low, middle and high frequency:

$$\xi_l^n = \xi_{l,-}^n + \xi_{l,0}^n + \xi_{l,+}^n. \quad (4.5)$$

Define

$$I_n^\delta = \begin{cases} ]-\infty; n - L_n[ & \text{if } \delta = -1 \\ [n - L; n + L_n] & \text{if } \delta = 0 \\ ]n + L_n; \infty[ & \text{if } \delta = 1. \end{cases}$$

First we show that the middle-component  $\xi_{l,0}^n$  is the only significant one. For this, we must prove that the low and high frequency components are negligible. We first prove that those increments can be considered as being independent.

##### 4.2.1. Low and High Frequency components

Set  $L > 0$ . Let  $\xi_{l,-}^n$  be the low frequency (L.F.) component of  $\xi_l^n$  and  $\xi_{l,+}^n$  be the high frequency (H.F.) component,  $n \in \mathbb{N}^*$ ,  $0 \leq l \leq 2^n - 1$ , defined by

$$\xi_{l,\delta}^n = 2^{nH} \sum_{\substack{\lambda \in \Lambda_d \\ j \in I_n^\delta}} \left( \Phi_{\lambda,\alpha}^K \left( \frac{l+1}{2^n} u \right) - \Phi_{\lambda,\alpha}^K \left( \frac{l}{2^n} u \right) \right) \eta_\lambda^{\alpha,K} \quad (4.6)$$

**Lemma 1.** (A)  $\sup_{0 \leq l \leq 2^n - 1} \left( E \left| \xi_{l,-}^n(X) \right|^p \right)^{\frac{1}{p}} \leq c 2^{-L(1-H)},$

(B)  $\sup_{0 \leq l \leq 2^n - 1} \left( E \left| \xi_{l,+}^n(X) \right|^p \right)^{\frac{1}{p}} \leq c' 2^{-LH},$  as  $n \rightarrow \infty$ , where  $c, c' > 0$ .

**Remark 2.** As functions  $G_{\lambda,\alpha}^K$  are with compact support, we sum for each fixed  $l$  on a finite number of  $k$ . Take  $y \in \mathbb{R}^d$ , set  $N_k$  the number of translations  $k \in \mathbb{Z}^d$  such that  $y - k \in \mathcal{B}(0, K)$ , then  $N_k \leq (2K+1)^d$ .

Set  $\Phi_{u,\lambda}(t) = \Phi_{\lambda,\alpha}^K(tu)$ ,  $t \in \mathbb{R}$  and  $\Phi_{u,k}^v(t) = \Phi_\alpha^{v,K}(tu - k)$  by a misuse of notation. In what follows, we often denote the difference  $\Phi_{\lambda,\alpha}^K\left(\frac{l+1}{2^n}u\right) - \Phi_{\lambda,\alpha}^K\left(\frac{l}{2^n}u\right)$  by  $\nabla\Phi_{u,\lambda}(l/2^n)$  and we use the notation  $\xi_l^{n,\delta}$  where  $\delta \in \{-1; +1\}$  for short. The proof of lemma 1 is given in [Appendix C](#).

#### 4.2.2. The middle frequency component ( $n - L \leq j \leq n + L$ )

To study the behavior of the initial process (before truncation), we have to make  $K$  tends to infinity. We hence consider a slowly varying cut  $K_n$  which tends to infinity with  $n$ . As said before, we also make the frequency cut  $L$  tend to infinity. Take for instance  $K_n = \beta \log_2(n)$  and  $L_n = \gamma \log_2(n)$ , with  $\beta, \gamma > 0$ . Note  $H_\delta = H$  or  $1 - H$  respectively when  $\delta = 1$  (which corresponds to H.F.) or  $\delta = -1$  (for L.F.). Notice that the use of  $K_n$  and  $L_n$  leads no modification in lemma 1 to the extent that  $2^{-L_n H_\delta}$  is the dominant term when compared to  $(2K_n + 1)^d$  as  $n$  tends to infinity. Then the normalized increments (2.7) considered in the middle-frequency band are written as follows

$$\xi_{l,\delta}^{n,L_n,K_n}(X) = 2^{nH} \sum_{\substack{j \in I_n^\delta \\ k,v}} 2^{-jH} \left( \Phi_{j,k,\alpha}^{v,K_n}(2^{-n}(l+1)u) - \Phi_{j,k,\alpha}^{v,K_n}(2^{-n}lu) \right) \eta_{j,k,v}^{\alpha,K_n}. \quad (4.7)$$

Taking such a varying cut  $L_n$ , other components are asymptotically negligible as given by the following lemma

**Lemma 2.** *If  $K_n, L_n$  are chosen under conditions  $\beta > 0, \gamma > (1 + \nu)/H_\delta p, \nu > 0$  then  $\lim_{n \rightarrow \infty} \frac{1}{2^n} \sum_{l=0}^{2^n-1} \left| \xi_{l,\delta}^{n,L_n,K_n} \right|^p = 0$  P.a.s. for all  $0 < p < \alpha$  with  $\delta \in \{-1; +1\}$ .*

The proof of lemma 2 is given in Appendix C. We deduce from lemma 2 that high and low frequency components are negligible in expression (4.5). Note  $\varepsilon_p^\delta$  the error made in the expression of  $\left| \xi_{l,\delta}^{n,L_n,K_n} \right|^p$  when neglecting HF ( $\delta = 1$ ) and LF ( $\delta = -1$ ) components. We have  $\varepsilon_p^\delta = \mathcal{O}(2^{-L_n H_\delta p})$  which is negligible in the decomposition when taking a frequency cut  $L_n \xrightarrow{n \rightarrow \infty} \infty$ . To amount to a study of independent random variables, we now base our study on this middle frequency range as it is the only significant one and we use form  $\xi_{l,0}^{n,L_n,K_n}$  which is very close to  $\xi_l^n$  from a rank  $n_\varepsilon$ . When there exists no possible confusion, we note  $\xi_{l,0}^{n,L_n,K_n} =: \xi_l^{n,L_n,K_n}$ .

### 5. Estimators of $H$ for $X$

Once again, we consider the triangular array

$$\mathcal{T} = \left( (T_{n,l}^u, 0 \leq l \leq 2^n - 1), n \in \mathbb{N}^* \right)$$

where  $T_{n,l}^u = |\xi_l^n|^p$  satisfying  $T_{n,l}^u \simeq \left| \xi_l^{n,L_n,K_n} \right|^p$  with  $L_n$  and  $K_n$  two slowly growing functions as previously. Consider variables (4.7) in the middle frequency band centered around frequency  $n$  of length  $2L_n$ . The lower frequency we meet is  $j = n - L_n$  where there are  $2^{n-L_n}$  truncated wavelets with their supports of length  $2K_n 2^{-(n-L_n)}$  (so that we have, along one direction  $u_i \in \mathbb{S}_{d-1}$ ,  $-K_n \leq 2^j x_i - k \leq K_n$  where  $x = \Sigma x_i u_i$ ). From now, set  $M_n := 2K_n 2^{L_n}$ .

### 5.1. Sub sampling

Take  $\gamma \in (0; M_n)$  and let us set  $\theta_{\gamma,l}^n = \xi_{m_{l,\gamma}(n)}^{n,L_n,K_n}$  with  $m_{l,\gamma}(n) = \gamma + lM_n$ ,  $l \in (0; \frac{2^n}{M_n} - 1)$ . We thus form packets  $W_\gamma = \{\theta_{\gamma,l}^n, 0 \leq l \leq \frac{2^n}{M_n} - 1\}$  of  $\frac{2^n}{M_n}$  increments made from disjoint compact support wavelets. Furthermore as random measure  $M_\alpha$  is independently scattered, those random variables  $\theta_{\gamma,l}^n$  are independent within each packet  $W_\gamma$ . This enables us to amount to the case of arrays of independent random variables studied in section 3. Finally, take  $s \in (0; 1)$  and define  $\gamma_n(s)$  its relative position in  $(0; M_n)$  by setting  $s = \frac{\gamma_n(s)}{M_n} + \varepsilon_n$  with  $|\varepsilon_n| < \frac{1}{M_n}$ . Then we can apply proposition 2 in each packet which yields the following convergence result toward

$$\sigma_X(u) = (E|X(u)|^p)^{\frac{1}{p}}, \quad u \in \mathbb{S}_{d-1}, \quad 0 < p < \alpha \quad (5.1)$$

**Lemma 3.** For each  $s \in (0; 1)$  and  $L_n, K_n$  defined as in lemma 2, we consider the sub-variation:

$$2^{nHp} V_{n,p}^{u,\gamma_n(s)} = \frac{M_n}{2^n} \sum_{l=0}^{2^n/M_n-1} \left| \theta_{\gamma_n(s),l}^n \right|^p.$$

for  $u \in \mathbb{S}_{d-1}$ . Then

$$\lim_{n \rightarrow \infty} 2^{nHp} V_{n,p}^{u,\gamma_n(s)} = \sigma_X^p(u) \quad \mathbb{P} a.s. \quad (5.2)$$

independently of  $s$ , where  $\sigma_X(u)$  is the constant defined by (5.1), independent of  $H$  and  $s$ .

The proof of lemma 3 is given in [Appendix C](#).

### 5.2. Convergence of $p$ -variations

We now have to control family  $\{\theta_{\gamma_n(s),l}^n, 0 \leq l \leq \frac{2^n}{M_n} - 1\}$  when  $n$  tends to infinity. Recall that when we make  $n$  tends to infinity, then  $L_n$  and  $K_n$  slowly tend to infinity so that we work with initial increments  $\xi_l^n$ . Consider  $p$ -variations defined by the following relation

$$V_{n,p}^u = \frac{1}{M_n} \sum_{\gamma_n(s)=0}^{M_n-1} V_{n,p}^{u,\gamma_n(s)}. \quad (5.3)$$

Then, parameter  $H$  being known, we obtain an estimation of (5.1) by this last lemma:

**Lemma 4.** Under conditions of lemma 3, we have

$$\lim_{n \rightarrow \infty} 2^{nHp} V_{n,p}^u = \sigma_X^p(u) \quad \mathbb{P} a.s. \quad (5.4)$$

where  $\sigma_X$  is the constant (5.1).

The proof of lemma 3 is given in [Appendix C](#). Then this relation can be written under the following form

$$\lim_{n \rightarrow \infty} nHp + \log(V_{n,p}^u - 2^{-nHp} \sigma_X^p(u)) = 0 \quad \mathbb{P} a.s.$$

This enables us to get the estimator (2.9) and the main result stated in section 2.

### 5.3. unbiased estimator

From this first estimator (2.9), we now consider variations  $V_{n,p,r}^u$  (2.11) to get an unbiased estimator (2.12). With the same reasoning line, we get as in (5.4) that

$$\lim_{n \rightarrow \infty} 2^{nHp} V_{n,p,r}^u - 2^{rHp} \sigma_X^p = 0 \quad \mathbb{P}a.s. \quad (5.5)$$

Hence, by (5.4) and (5.5), it comes

$$\lim_{n \rightarrow \infty} \frac{V_{n,p,r}^u}{V_{n,p,0}^u} = 2^{rHp} \quad \mathbb{P}a.s.$$

Thus  $\tilde{H}_{n,p,r}^u$  defined by

$$\tilde{H}_{n,p,r}^u = \frac{1}{rp} \log_2 \frac{V_{n,p,r}^u}{V_{n,p,0}^u} \quad (5.6)$$

almost surely converges to  $H$  as  $n$  tends to infinity for all  $r > 0$ . We easily check that  $\tilde{H}_{n,p,r}^u$  is unbiased as we get  $E \left| \tilde{H}_{n,p,r}^u \right| = H$  by the  $H$ -sssi property of  $X$ , so Corollary 1 holds.  $\square$

## 6. Conclusion and perspectives

In this paper, we introduced an estimator along with its unbiased version for the Hurst exponent  $H$  associated with the  $\alpha$ -Stable non-Gaussian having a moving average representation (2.1), for  $1 < \alpha < 2$ . Furthermore, we showed a Law of Large Numbers for these estimations. The proposed estimator (2.9) and its unbiased version (2.12) are based on  $p$ -variations. Their rates of convergence and the conditions under which they converge to either a Gaussian or a Stable non-Gaussian distribution have been detailed. We also showed that, the relationship of the  $p$  exponent to the threshold  $\frac{\alpha}{2}$  influences the estimators' behavior and the limiting laws family.

Several perspectives may be investigated beyond this work. For instance, exploring finance applications, notably in the stable-Capital Asset Pricing Model (CAPM) that employs  $\alpha$ -Stable distributions for modeling financial asset returns where  $1 < \alpha < 2$ , would be highly valuable. Empirical investigations into self-similarity and stability have revealed the market's fractal nature, assuming symmetric  $\alpha$ -Stable asset returns. Considering the ongoing challenges in machine learning, exploring overfitting within these contexts as a continuation of the research on fractional and multifractional Brownian motion presented in [11] is pertinent.

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## Appendix A. Wavelet decomposition

### Appendix A.1. Recalls on Lemarié-Meyer wavelets

Set  $\Lambda_d = \mathbb{Z} \times \mathbb{Z}^d \times \mathbb{E}_d^*$  where  $\mathbb{E}_d^* = \{0, 1\}^d \setminus \{(0, \dots, 0)\}$ . Take  $\lambda = (j, k, v)$ ,  $\lambda \in \Lambda_d$ . Let  $(\Psi^v, v \in \mathbb{E}_d^*)$  be a generator system of an orthogonal basis of  $L^2(\mathbb{R}^d)$  made with Lemarié-Meyer wavelets and construct the so-called Littlewood-Paley basis (refer to the book of Y.Meyer [21]) of  $L^2(\mathbb{R}^d)$  made of the following wavelets

$$\Psi_\lambda(x) = 2^{j\frac{d}{2}} \Psi^v(2^j x - k).$$

Notice that  $v$  determine the choice of the mother wavelet  $\Psi^v$  (vector  $(0, \dots, 0)$  is excluded as it designs the scale function). We define an unconditional basis  $(\Psi_{\lambda, \alpha}, \lambda \in \Lambda_d)$  of  $L^\alpha(\mathbb{R}^d)$ ,  $\alpha > 0$ , constructed after  $L^\alpha$ -normalization of the usual Lemarié-Meyer basis of  $L^2(\mathbb{R}^d)$  in the following way

$$\Psi_{\lambda, \alpha}(x) = 2^{j\frac{d}{\alpha}} \Psi^v(2^j x - k)$$

as announced in (2.5) where  $2^{j\frac{d}{\alpha}}$  is the  $L^\alpha(\mathbb{R}^d)$ -normalization.

### Appendix A.2. Random measures

Using the notations of Samorodnitsky-Taquq (see [26]chapter 3), let us introduce the stochastic S $\alpha$ S measure  $M_\alpha$ , with a Lebesgue control measure  $m$ . It is an application from  $L^\alpha(\mathbb{R}^d)$  to  $L^p(\mathcal{F}, \mathbb{P})$ ,  $p < \alpha$ , such that  $M_\alpha(f) = \int_{\mathbb{R}^d} f(x) M_\alpha(dx)$  for each integrand  $f \in L^\alpha(\mathbb{R}^d)$ . Then  $(\eta_\lambda^\alpha, \lambda \in \Lambda_d)$  defined by (A.1) are given by

$$\eta_\lambda^\alpha = M_\alpha(\Psi_{\lambda, \alpha}) \tag{A.1}$$

so that it is a sequence of S $\alpha$ S random variables satisfying the following stationarity property

$$\eta_{j+r, k, v}^\alpha \stackrel{(d)}{=} \eta_{j, k, v}^\alpha \text{ and } \eta_{j, k+s, v}^\alpha \stackrel{(d)}{=} \eta_{j, k, v}^\alpha \quad \forall r \in \mathbb{Z}, s \in \mathbb{Z}^d. \tag{A.2}$$

In addition, using a known property of stable random variables ([26] p.18), we recall the following relation connecting  $\|M_\alpha(f)\|_{L^2}$  and  $\|f\|_{L^\alpha}$

$$(E|M_\alpha(f)|^p)^{\frac{1}{p}} = c_{p, \alpha} \cdot \left( \int_{\mathbb{R}^d} |f(x)|^\alpha dx \right)^{\frac{1}{\alpha}} \tag{A.3}$$

where  $c_{p, \alpha}$  is a constant.

### Appendix A.3. Adapted wavelets

Set

$$f_\beta(x, \xi) = \frac{\exp(ix \cdot \xi) - 1}{|\xi|^{H+d/\beta} S\left(\frac{\xi}{|\xi|}\right)} \tag{A.4}$$

where  $0 < H < 1$ ,  $\beta > 1$  and  $S \in \mathcal{M}_{c,1/c}$ . We check that  $f_\beta(x, \cdot) \in L^\beta(\mathbb{R}^d)$ . Then define

$$\Phi_{\lambda,\alpha}(x) = \int_{\mathbb{R}^d} f_{\alpha^*}(x, \xi) \widehat{\Psi}_{\lambda,\alpha^*}(\xi) d\xi \quad (\text{A.5})$$

where  $\alpha^*$  is the  $\alpha$ -conjugate. Such functions  $\Phi_{\lambda,\alpha}$  are called the adapted wavelets associated to  $X$ . We deduce from (Appendix A.1) that they satisfy the following scaling relation

$$\Phi_{\lambda,\alpha}(x) = 2^{-jH} \Phi_\alpha^v(2^j x - k) \quad (\text{A.6})$$

where  $\Phi_\alpha^v$  is defined from the mother wavelet  $\Psi^v$  by

$$\Phi_\alpha^v(x) = \int_{\mathbb{R}^d} f_{\alpha^*}(x, \xi) \widehat{\Psi}^v(\xi) d\xi. \quad (\text{A.7})$$

#### Appendix A.4. Wavelet decomposition of $X$

We here use the wavelet decomposition of random measure  $M_\alpha$  as done in [9]. It comes that the moving average process  $X$  admits the wavelet decomposition announced in (2.3)

$$X(x) = \sum_{\lambda \in \Lambda_d} 2^{-jH} \Phi_\alpha^v(2^j x - k) \eta_\lambda^\alpha$$

where  $(\eta_\lambda^\alpha, \lambda \in \Lambda_d)$  is a sequence of S $\alpha$ S random variables defined by (A.1) and function  $\Phi_{\lambda,\alpha}$  is given by (A.5).

## Appendix B. Convergent Triangular Arrays

First, quote this known property of stable variables ([26]p.16), for all  $x > 0$

$$P[|\xi_l^n| > x] \underset{x \rightarrow \infty}{\sim} \lambda^\alpha c_\alpha x^{-\alpha}$$

where  $\lambda > 0$  is the scale parameter of the S $\alpha$ S law,  $c_\alpha \in \mathbb{R}$  and  $a(x) \underset{x \rightarrow \infty}{\sim} b(x)$  means that  $\lim_{x \rightarrow \infty} \frac{a(x)}{b(x)} = 1$ . Then it comes that  $F_\tau$  behaves as follow as  $|x| \rightarrow \infty$

$$F_\tau(x) = 1 - (c'_\alpha + o(1)) x^{-\tilde{\alpha}} \text{ if } x > 0, 0 \text{ otherwise} \quad (\text{B.1})$$

where  $\tilde{\alpha} = \frac{\alpha}{p}$  ( $\tilde{\alpha} > 1$ ) and  $c'_\alpha \in \mathbb{R}$ .

#### Appendix B.1. Step 1 : Gaussian behavior (case (i))

Using a convergence result of Gnedenko-Kolmogorov for sums of independent random variables with finite variance ([theorem3 p.101][19]),  $F_\tau$  belongs to the domain of attraction of a normal law with normalization  $B_n = \sqrt{2^n}$  as it fulfills the two following conditions for all  $\varepsilon > 0$

$$2^n \int_{|x| \geq \varepsilon} x^2 dF_\tau(B_n x) \rightarrow 0 \text{ and } 2^n \int_{|x| < \varepsilon} x^2 dF_\tau(B_n x) \rightarrow \sigma$$

as  $n \rightarrow \infty$ , where the choice of  $B_n$  proceed from (B.1). So we asymptotically have

$$P \left[ 2^{-\frac{n}{2}} \sum_{l=0}^{2^n-1} \tau_{n,l} \geq x_n \right] \underset{n \rightarrow \infty}{\sim} \int_{x_n}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{t^2}{2\sigma^2} \right) dt \quad (\text{B.2})$$

with  $x_n \xrightarrow[n \rightarrow \infty]{} \infty$  and  $\sigma^2$  denotes finite variance of  $\tau_1^{(p)}$ . Taking  $x_n = (2\mu \ln(2^n))^{\frac{1}{2}}$  with  $\mu/\sigma > 1$ , it comes

$$\sum_{n \geq 1} \int_{x_n}^{\infty} \exp(-t^2/2\sigma^2) dt < \infty$$

since  $\int_{x_n}^{\infty} \exp(-t^2/2\sigma^2) dt \leq \exp(-x_n^2/2\sigma^2)$  as soon as  $x_n > \sigma^2$ .

Thus  $2^{-n} \sum_{l=0}^{2^n-1} \tau_{n,l} \leq x_n 2^{-\frac{n}{2}}$  from some random rank  $n_0$  by Borel-Cantelli, and (3.1) holds as  $\lim_{n \rightarrow \infty} x_n 2^{-\frac{n}{2}} = 0$ .

#### Appendix B.2. Step 2 : Critical case (case (ii))

For this case, variables  $\tau_{n,l}$  are not with finite second order moment so they don't fulfill conditions of theorem3 in [19], applied in step Appendix B.1. This no more possible to refer to next step Appendix B.3 either as the limit law would be with stability index  $\tilde{\alpha} = 2$  by Gnedenko's result [18] whereas Rosovskii's lemma only applies for  $1 < \tilde{\alpha} < 2$ . These are the reasons why we have to treat this case apart. By another result of Gnedenko-Kolmogorov on limit distributions of sums of i.i.d. random variables ([19]thm2 p.128), we get that  $F_\tau$  belongs to the domain of attraction of a normal law as it fulfills the two following conditions for all  $\varepsilon > 0$

$$2^n \int_{|x| \geq \varepsilon} dF_\tau(B_n x) \rightarrow 0$$

and

$$2^n \left[ \int_{|x| < \varepsilon} x^2 dF_\tau(B_n x) - \left( \int_{|x| < \varepsilon} x dF_\tau(B_n x) \right)^2 \right] \rightarrow 1$$

as  $n \rightarrow \infty$ , for a normalizing sequence with general term  $B_n = \sqrt{2^n \ln(\sqrt{2^n})}$ . Its form is obtained using expression (B.1) of  $F_\tau$ . We get the same kind of result as in step 1 but with a necessary logarithmic correcting term. Hence we obtain (B.2) apart from a constant. So, taking  $x_n$  as in the first step, we conclude that (3.1) holds thanks to Borel-Cantelli's lemma, as  $\lim_{n \rightarrow \infty} x_n (\ln(2^n) 2^{-n-1})^{1/2} = 0$ .

#### Appendix B.3. Step 3 : Stable behavior (case (iii))

As  $\{\tau_{n,l}, 0 \leq l \leq 2^n - 1\}$  is a sequence of centered i.i.d. random variables satisfying (B.1), a Gnedenko's result [18] gives us that  $F_\tau$  belongs to the domain of attraction of a stable law with index of stability  $\tilde{\alpha} = \frac{\alpha}{p}$  with  $B_n = 2^{n/\tilde{\alpha}}$ . As  $1 < \tilde{\alpha} < 2$ , we can apply the following result, due to Rosovskii (formula (1.2) p.454 in [25])

**Lemma 5.** Set  $s_{2^n} = \tau_{n,0} + \dots + \tau_{n,2^n}$ , and  $(B_n, n \in \mathbb{N}^*)$  a positive sequence. Then

$$P(s_{2^n} > x_n) \underset{n \rightarrow \infty}{\sim} 2^n (1 - F_\tau(x_n)) \quad (\text{B.3})$$

uniformly for all sequence  $(x_n)_{n \in \mathbb{N}^*}$  such that  $\frac{x_n}{B_n} \xrightarrow{n \rightarrow \infty} \infty$ .

Take  $x_n = n2^{n/\tilde{\alpha}}$ . As  $x_n \xrightarrow{n \rightarrow \infty} \infty$  and  $\frac{x_n}{B_n} \xrightarrow{n \rightarrow \infty} \infty$ , thanks to (B.1) and (B.3) we can write

$$P(\tau_{n,0} + \dots + \tau_{n,2^n} > n2^{n/\tilde{\alpha}}) \underset{n \rightarrow \infty}{\sim} c'_\alpha n^{-\tilde{\alpha}}.$$

Notice that we here have to work with an exponential sub-sequence  $(2^n, n \in \mathbb{N}^*)$  in order to get convergence of  $\sum_{n \geq 1} n^{-\tilde{\alpha}}$ . Then Borel-Cantelli's lemma yields that from some random rank  $n_1$ , we get  $2^{-n} \sum_{l=0}^{2^n-1} \tau_{n,l} \leq n2^{-n/\tilde{\alpha}^*}$  (with  $\tilde{\alpha}^*$  the  $\tilde{\alpha}$ -conjugate) which allows us to conclude that (3.1) holds.

## Appendix C. Proof of intermediary Lemmas

### Appendix C.1. Proof of Lemma 1

*Proof of (A).* As the  $\Phi_{\lambda,\alpha}$ 's are real valued, we apply the Taylor-Young equality:  $\nabla \Phi_{u,\lambda}(l/2^n) = 2^{-n} (\Phi_{u,\lambda})'(2^{-n}l) + \frac{1}{2} 2^{-2n} (\Phi_{u,\lambda})''(2^{-n}l + \theta)$  with  $\theta \in ]0, 2^{-n}[$  where  $(\Phi_{u,\lambda})^{(m)}$  designs the  $m$ th derivative of  $\Phi_{u,\lambda}$ . Hence (4.6) may be of the following form

$$\xi_{l,-}^n = 2^{nH} \sum_{\lambda \in \Lambda_d} [2^{-n} (\Phi_{u,\lambda})'(2^{-n}l) + 2^{-2n-1} (\Phi_{u,\lambda})''(2^{-n}l + \theta)] \eta_\lambda^{\alpha,K}.$$

By (A.6) it comes that  $(\Phi_{u,\lambda})^{(m)}(t) = 2^{-jH} 2^{mj} (D^m \Phi_k^v)(2^j t u)$  in one direction  $u$ . Set

$$\xi_{l,-}^{n,1} = 2^{nH} \sum_{j < n-L} 2^{-jH} \sum_{k \in \mathbb{Z}^d, v \in \mathbb{E}_d^*} 2^{j-n} (D \Phi_k^v) \left( 2^j \frac{l}{2^n} u \right) \eta_{j,k,v}^{\alpha,K},$$

$$R(l) = 2^{nH-1} \sum_{j < n-L} 2^{-jH} \sum_{k \in \mathbb{Z}^d, v \in \mathbb{E}_d^*} 2^{2(j-n)} (D^2 \Phi_k^v) \left( 2^j \left( \frac{l}{2^n} + \theta \right) u \right) \eta_{j,k,v}^{\alpha,K},$$

so that

$$\xi_{l,-}^n = \xi_{l,-}^{n,1} + R(l).$$

In a first time, consider the case where  $1 \leq p < \alpha$ . This enables us to apply the Minkowski inequality,

$$(E |\xi_{l,-}^n|^p)^{\frac{1}{p}} \leq (E |\xi_{l,-}^{n,1}|^p)^{\frac{1}{p}} + (E |R(l)|^p)^{\frac{1}{p}}. \quad (\text{C.1})$$

Set  $M = \sup_{y \in \mathbb{R}^d} |\Phi_\alpha^{v,K}(y)|$ ,  $M' = \sup_{y \in \mathbb{R}^d} \|D \Phi_\alpha^{v,K}(y)\|$ , and  $M'' = \sup_{y \in \mathbb{R}^d} \|D^2 \Phi_\alpha^{v,K}(y)\|$ .

We first treat the second order term  $R(l)$ . Since  $\alpha > p$ , by properties of  $\alpha$ -stable random variables and (A.1), there exists  $a \in \mathbb{R}^+$  such that  $(E |\eta_\lambda^{\alpha,K}|^p)^{\frac{1}{p}} = a$ . Thus the Minkowski inequality yields

$$\|R(l)\|_{L^p(\mathbb{P})} \leq \frac{a}{2} \sum_{j < n-L} 2^{-(j-n)H} 2^{2(j-n)} \sum_{k,v} \left\| D^2 \Phi_k^v \left( 2^j \left( \frac{l}{2^n} + \theta \right) u \right) \right\|.$$

From remark 2, it becomes

$$\sup_{0 \leq l \leq 2^{n-1}} (E |R(l)|^p)^{\frac{1}{p}} \leq \frac{a}{2} M'' (2K+1)^d \sum_{j > L} 2^{jH} 2^{-2j}.$$

Then by the change of variable  $j' = n - j$ , using the fact that  $H < 1$ , we obtain

$$\sup_{0 \leq l \leq 2^{n-1}} (E |R(l)|^p)^{\frac{1}{p}} \leq C''(p) (2K+1)^d 2^{-L(2-H)} \quad (\text{C.2})$$

where  $C''(p) > 0$ . Then study part  $\xi_{l,1}^{n,-}$ . The same arguments lead to

$$\sup_{0 \leq l \leq 2^{n-1}} \left( E \left| \xi_{l,-}^{n,1} \right|^p \right)^{\frac{1}{p}} \leq aM'(2K+1)^d \sum_{j > L} 2^{-j(1-H)}$$

so that

$$\sup_{0 \leq l \leq 2^{n-1}} \left( E \left| \xi_{l,-}^{n,1} \right|^p \right)^{\frac{1}{p}} \leq C'(p) (2K+1)^d 2^{-L(1-H)} \quad (\text{C.3})$$

with  $C'(p) > 0$ . In order to come down to the initial process, we have to make the frequency cut  $L$  tend to infinity. As  $2^{-L(2-H)} = o(2^{-L(1-H)})$  when  $L \rightarrow \infty$ , using (C.2) and (C.3), we can disregard the second order term in (C.1). It comes that  $E \left| \xi_{l,-}^n \right|^p = \mathcal{O}(2^{-L(1-H)p})$  which leads us to (i) with  $c := c_1(p, K) > 0$  independent of  $n$ .

Next, if  $p \leq 1$ , we use the Hölder inequality :  $\sup_l \left( E \left| \xi_{l,-}^n \right|^p \right)^{\frac{1}{p}} \leq \sup_l E \left| \xi_{l,-}^n \right|$  and we apply the results pointed out with a constant  $c_1(1, K)$ .  $\square$

*Proof of (B).* Using self-similarity of  $X$  and (A.2), we get

$$\xi_{l,+}^n \stackrel{(d)}{=} 2^{-LH} \sum_{j > 0} 2^{-jH} \sum_{k,v} (\Phi_\alpha^{v,K} (2^{j+L}(l+1)u - k) - \Phi_\alpha^{v,K} (2^{j+L}lu - k)) \eta_{j,k,v}^{\alpha,K}.$$

Thus  $\xi_{l,+}^n \stackrel{(d)}{=} 2^{-LH} V_{L,n}^+(X)$  where  $V_{L,n}^+(X)$  is a variation of process  $X$ . As previously for  $1 \leq p < \alpha$ , the Minkowski inequality applies. Then remark 3 yields that

$$\sup_l \left( E \left| V_{L,n}^+(X) \right|^p \right)^{\frac{1}{p}} \leq 2aM (2K+1)^d (2^H - 1)^{-1}$$

as  $H > 0$ . So it comes that (ii) holds with  $c' := c_2(p, K)$  independent of  $n$ .

If  $p \leq 1$ , we apply the Hölder inequality once again and we use the previous result with  $c_2(1, K)$ , thus lemma 1 holds.  $\square$

*Appendix C.2. Proof of Lemma2*

By the well-known Bienaymé-Tchebycheff inequality,

$$P \left( \frac{1}{2^n} \sum_{l=0}^{2^n-1} \left| \xi_{l,\delta}^{n,L_n,K_n} \right|^p > \varepsilon_n \right) \leq \frac{1}{2^n \varepsilon_n} \sum_{l=0}^{2^n-1} E \left| \xi_{l,\delta}^{n,L_n,K_n} \right|^p.$$

Take  $\varepsilon_n = n^{-\nu}$ ,  $\nu > 0$ . As  $2^{-n} \sum_{l=0}^{2^n-1} E \left| \xi_{l,\delta}^{n,L_n,K_n} \right|^p \leq c 2^{-L_n H_\delta p}$  by lemma 1, we conclude by Borel-Cantelli for  $L_n = \gamma \log_2(n)$  as soon as  $\gamma H_\delta p - \nu > 1$ .  $\square$

*Appendix C.3. Proof of Lemma3*

As random variables  $\theta_{\gamma_n(s),l}^n$  are independent, result (3.1) holds. It comes that

$$\lim_{n \rightarrow \infty} \frac{M_n}{2^n} \sum_{l=0}^{2^n/M_n-1} \left| \theta_{\gamma_n(s),l}^n \right|^p - E \left| \theta_{\gamma_n(s),l}^n \right|^p = 0 \quad \mathbb{P} a.s.$$

where the average  $\frac{M_n}{2^n} \sum_{l=0}^{2^n/M_n-1} E \left| \theta_{\gamma_n(s),l}^n \right|^p = E |X(u)|^p$  using (1). So constant  $\sigma_X^p(u)$  in (5.2) equals  $E |X(u)|^p$  which term depends only on  $\alpha$  and  $p$ .  $\square$

*Appendix C.4. Proof of Lemma4*

As lemma 3 holds uniformly in  $s$ , it comes that for all  $s$  we have

$$\lim_{n \rightarrow \infty} \frac{1}{M_n} \sum_{\gamma=0}^{M_n-1} 2^{nH_p} V_{n,p}^{u,\gamma} = \int_0^1 \sigma_X^p(u) ds \quad \mathbb{P} a.s.$$

so using (5.3), relation (5.4) holds as  $\sigma_X^p$  does not depend on  $s$ .  $\square$