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# Interval KKL Observer for Uncertain Discrete Time-Varying Nonlinear Systems <sup>★</sup>

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## Abstract

Kazantzis-Kravaris/Luenberger (KKL) or Nonlinear Luenberger observers are powerful theoretical tools used to estimate the state of a very large class of nonlinear systems. Such observers rely on the existence of an injective transformation towards new coordinates in which the original system dynamics becomes an exponentially stable linear filter of the output, and thus easy to estimate. The injectivity of the transformation enables to use its left inverse in order to recover the original system state estimate in the original coordinates. The existence of such an injective transformation may be guaranteed under a backward-distinguishability assumption of the original system.

The present paper proposes an interval observer, i.e. a lower and an upper bound of the actual state, based on the KKL principle, for a discrete-time time-varying nonlinear system subject to additive bounded disturbances and measurement noise. The design of this observer mainly relies on the existence of uniform Lipschitz backward distinguishability assumption enabling to bound the state regarding the uncertainties using an interval observer for the linear filter in the new coordinates. In the original coordinates, the inclusion of the system state into the estimated interval and the boundedness of the errors are guaranteed after a finite time. Some practical considerations regarding implementation of a such observer are addressed and a numerical example illustrates the resulting performances.

*Key words:* State estimation, uncertain nonlinear system, nonlinear observer, interval observer, set-membership.

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## 1 Introduction

State estimation is a crucial step in the design and implementation of model-based control laws and diagnostic processes. Usually, this step is achieved using observers that compute some reliable estimations on the basis of known inputs and measurements. Many methods have been developed to solve this problem, with for instance the well known Kalman filters [11] and Luenberger observers [16] initially developed for linear systems in the 60s. Since then, numerous works have been carried out

to handle more and more complex systems that take into account nonlinearities and uncertainties. Among the well established techniques that can be found in the literature, the most famous ones are the  $H_\infty$  filtering, the extended and unscented Kalman filters [26], sliding mode observers, high-gain observers [14]. Another important framework that emerged in the late 60s is the set-membership framework which assumes bounded errors and aims to compute an estimated set that is guaranteed to contain the unknown system state [25]. Several types of set can be considered but the most common in the literature are based on ellipsoids [7], zonotopes [5] or intervals. The latter will be of interest for this work. One of the pioneer article regarding interval observers has been proposed by Gouz e [10] in the early 2000s. It consists in computing a minimal and a maximal bounds that enclose the state of the system using Luenberger like observers. Such observers are designed under the assumption of unknown but bounded uncertainties. Their

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<sup>★</sup> This paper was not presented at any IFAC meeting. Corresponding author L. Meyer

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design relies on the monotone systems theory [27] which allows to obtain the cooperativity (or nonnegativity) of the interval estimation errors dynamics by a proper design [8] [15]. In practice, it is difficult to find suitable observer gains that ensure at the same time the stability of the interval observer and the cooperativity of the estimation errors. However this issue has been overcome for Linear Time Invariant (LTI), Linear Time Varying (LTV), Linear Parameter Varying (LPV) systems, as well as some particular classes of nonlinear systems using a change of coordinates that can be either time-varying or not [23] [6]. Indeed, such transformation allows to change a stable dynamics into a stable and cooperative one. Even though the problem of interval observers design for nonlinear systems has been studied over the last years [19][21][23][13][20], it still remains a challenge for several reasons but especially because of the lack of a generic method. However a trend emerges from the mentioned articles which is to find bounding functions for the nonlinearity assuming its Lipschitzness. For instance in [21] and [13], a mixed-monotone decomposition of the nonlinearity, which requires its Lipschitzness, is used and an example of bounding functions is given in [21] based on the Lipschitz constant. In [19], the trajectory of the system is assumed bounded such that the Müller's Theorem proves the existence of bounding functions and a rule based on the monotonicity is given to find them. In [23], bounding functions for the nonlinearity are directly assumed. However they used the interesting idea to transform the original nonlinear system, assumed locally observable, into a system linearized by an output injection term. This technique reminds of the approach introduced in the late 90s by Kazantzis and Kravaris [12] for state estimation of nonlinear systems. The so called Kazantzis-Kravaris/Luenberger (KKL) observers, or nonlinear observers, rely on a powerful theory based on a transformation that changes the nonlinear dynamics (of dimension  $n_x$ ) into an exponentially stable linear filter of the output in some new coordinates (of dimension  $n_z \geq n_x$ ). In this way, a simple Luenberger observer is able to estimate the state in the new coordinates and an inverse transformation could allow to recover the estimation in the original base. However, in order to be able to inverse the transformation, it needs to be injective. The existence of such an injective transformation is shown in [2] under a backward distinguishability assumption, i.e. given two different state values at a certain time, there exists a time in the past where their respective outputs were different. In this case, the filter must be complex-valued of dimension  $n_x + 1$  and almost any choice of distinct complex eigenvalues are suitable. A closed form of the transformation can be computed at each time based on the nonlinear functions that describe the system and the observer matrices. KKL observers position themselves as an alternative to the extensions of Kalman filters to nonlinear systems, with the advantage to guarantee (semi-)global asymptotic stability contrary to the local convergence provided by the extended/unscented Kalman filter. More-

over, some stronger injectivity results have then been obtained [1] [3] and one in particular is about the existence of a uniform Lipschitz injective transformation which is obtained under uniform Lipschitz backward distinguishability [28], i.e. the backward distinguishability maps are uniformly Lipschitz injective. Such a design offers two main features which are an arbitrarily fast convergence and an input-to-state (ISS) stability of the estimation error. Thus, it can be seen as a discrete-time counterpart of high-gain observers. The robustness regarding the additive disturbances of the arbitrarily fast KKL observer, or in other words the boundedness of the estimation error, gives an opportunity for some adaptations to the set-membership framework.

We introduce here an Interval KKL (IKKL) observer design which is a new type of interval observer approach for discrete uncertain nonlinear systems. A specific transformation is used to obtain a linear filter in some new coordinates. An interval observer is then designed in the new coordinates to enclose the linear filter dynamics while taking into account the uncertainties bounds. Then, in order to recover the estimated interval in the original coordinates, some bounding functions of the inverse transformation are given based on its Lipschitzness. The proposed IKKL brings several innovations compared to the nonlinear interval observers of the literature. First, the design is relatively generic and may be applied to a large class of discrete uncertain nonlinear systems that respect the uniform Lipschitz backward distinguishability assumption. This assumption could seem restrictive but is actually satisfied by many real systems with appropriate measurements. Second, instead of directly looking for bounding functions of the nonlinearities, our approach uses the Lipschitzness of the direct and inverse transformations to enclose the system state. Third, it will be shown that a knowledge about the initial conditions may be dispensed. The paper is structured as follows. Section 2 presents the preliminaries that introduce some material required to design IKKL Observer. Then, the problem statement is established in Section 3. The new IKKL Observer is presented in Section 4. In Section 5, some practical considerations are detailed to ease the implementation of an IKKL Observer. Finally, Section 6 illustrates the theoretical design through simulation examples on LTV and nonlinear systems.

### Notations

The set of real and natural numbers are respectively denoted  $\mathbb{R}$  and  $\mathbb{N}$ . Consider also  $\mathbb{R}_{\geq 0} = [0, +\infty)$ ,  $\mathbb{R}_{> 0} = (0, +\infty)$  and  $\mathbb{N}_{> 0} = \mathbb{N} \setminus \{0\}$ . The set of real matrices with  $n \times p$  elements is denoted by  $\mathcal{M}_{n,p}(\mathbb{R})$ .  $I_n \in \mathcal{M}_{n,n}(\mathbb{R})$  ( $\mathbf{I}$  when there is no ambiguity) denotes the identity matrix.  $\mathbf{1}$  is a vector of ones with appropriate dimensions.  $0_{n,p} \in \mathcal{M}_{n,p}(\mathbb{R})$  ( $\mathbf{0}$  when there is no ambiguity) denotes the null matrix with  $n$  rows and  $p$  columns. For any matrix  $A \in \mathcal{M}_{n,m}(\mathbb{R})$ , let set  $A^+ = \max\{\mathbf{0}, A\}$  and  $A^- = A^+ - A$  (such that  $A = A^+ - A^-$ , with

$A^+ \geq 0$  and  $A^- \geq 0$ ). The operators  $\leq, \geq, <$  and  $>$  are understood component wise for vectors and matrices. For any finite set of matrices  $(A_i)_{i=1\dots m} \subset \mathcal{M}_{n,p}(\mathbb{R})$  (resp any finite set of vectors  $(v_i)_{i=1\dots m} \subset \mathbb{R}^p$ ), the *min* and *max* operators,  $\min_{i=1\dots m}\{A_i\}$  and  $\max_{i=1\dots m}\{A_i\}$  (resp.  $\min_{i=1\dots m}\{v_i\}$  and  $\max_{i=1\dots m}\{v_i\}$ ), are understood component-wise. The operator *diag* applied to a vector returns a diagonal matrix where the components of the vector are on the main diagonal. For matrices  $(A_i)_{i=1\dots m}$ ,  $\text{diag}(A_1, \dots, A_m)$  returns a block diagonal matrix whose diagonal blocks are the matrices  $A_i$ . For a vector norm  $|\cdot|$ , the induced matrix norm is denoted  $\|\cdot\|$ .  $\ell_\infty$  denotes the set of bounded inputs  $x$  such that  $|x| < \infty$ . The composition of two functions  $f$  and  $g$  is  $f \circ g$  provided that it is well defined, i.e for all  $x$  in the domain of  $g$ ,  $g(x)$  is in the domain of  $f$ .

## 2 Preliminaries

This section presents some material required to design interval KKL observers. It is divided in two parts as follows. Subsection 2.1 introduces Lipschitz functions and the concept of mixed monotone decomposition for such functions. Then, Subsection 2.2 recalls the design methodology of interval observers for discrete linear systems.

### 2.1 Lipschitzness and mixed monotone decomposition

The Lipschitz continuity is a type of uniform continuity for functions. Intuitively, a Lipschitz function is bounded in its variation. A globally Lipschitz function is defined as follows.

**Definition 1** (Globally Lipschitz function). *A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^q$  is globally Lipschitz with respect to  $x$  if*

$$\forall (x_a, x_b) \in \mathbb{R}^n \times \mathbb{R}^n, |f(x_a) - f(x_b)| \leq L|x_a - x_b|,$$

where  $L \in \mathbb{R}_{>0}$  is called the Lipschitz constant of  $f$ .

Those functions can be decomposed in a sum of two monotone Lipschitz functions using the following mixed-monotone decomposition property which is recall from [21, Properties 3 and 4].

**Property 1** (Mixed monotone decomposition [21]). *Consider a differentiable global Lipschitz function  $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}^q$  with Lipschitz constant  $c$ . Then, the following properties hold :*

- (1)  *$f$  can be written as the difference of two differentiable global Lipschitz functions  $\alpha$  and  $\beta$  which are increasing functions of  $x$ .*

$$f(x) = \alpha(x) - \beta(x)$$

*The newly introduced functions can be given by  $\alpha(x) = c \sum_{i=1}^n x_i \mathbf{1}$  and  $\beta(x) = \alpha(x) - f(x)$ .*

- (2) *There exists a differentiable global Lipschitz function  $\tilde{f}(x_a, x_b) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^q$  such that*

- $\tilde{f}(x, x) = f(x)$
- $[\frac{\partial \tilde{f}}{\partial x_a}] \geq 0$  and  $[\frac{\partial \tilde{f}}{\partial x_b}] \leq 0$

*Such a mapping  $\tilde{f}$  can be written as:*

$$\tilde{f}(x_a, x_b) = \alpha(x_a) - \beta(x_b).$$

- (3) *Using the notations introduced in items 1 and 2, then for all  $(\underline{x}, \bar{x}) \in \mathbb{R}^n \times \mathbb{R}^n$  such that  $\underline{x} \leq x \leq \bar{x}$ ,*

$$\tilde{f}(\underline{x}, \bar{x}) \leq f(x) \leq \tilde{f}(\bar{x}, \underline{x}).$$

This property will be used in Theorem 3 proof to recover the inclusion property of the proposed observer in the original coordinates from the linear filter coordinates.

### 2.2 Interval observer for discrete-time linear systems

Interval observers provide an upper and a lower bound for the state of an uncertain systems. Their design relies on the monotone system theory and on the concept of cooperativity (or nonnegativity) of the estimation errors dynamics. This guarantees the inclusion of the system state in the estimated bounds [8]. The main ideas to design an interval observer for discrete time linear systems are presented thereafter.

First, the following Lemma gives a fundamental interval order relation, obtained from interval arithmetic, that allows to compute the bounds of the product between a matrix and an interval vector.

**Lemma 1** (Order Relation [9]). *Let  $x \in \mathbb{R}^n$  be a vector such that  $x \in [\underline{x}, \bar{x}]$  with  $\underline{x}, \bar{x} \in \mathbb{R}^n$  and  $A$  be a matrix with appropriate dimensions. Then,*

$$A^+ \underline{x} - A^- \bar{x} \leq Ax \leq A^+ \bar{x} - A^- \underline{x}.$$

Consider a discrete-time uncertain linear system such as

$$x_{k+1} = Ax_k + v_k, \quad y_k = Cx_k + w_k, \quad (1)$$

where  $x_k \in \mathbb{R}^{n_x}$  and  $y_k \in \mathbb{R}^{n_y}$  are the state and output vectors,  $v_k \in \mathbb{R}^{n_x}$  and  $w_k \in \mathbb{R}^{n_y}$  are respectively the additive disturbances and measurement noise. The pair  $(A, C) \in \mathcal{M}_{n_x, n_x}(\mathbb{R}) \times \mathcal{M}_{n_y, n_x}(\mathbb{R})$  is observable.

A standard assumption in the set-membership framework is that the uncertainties are assumed unknown but bounded by known values. Thus, consider  $\underline{x}_0, \bar{x}_0 \in \mathbb{R}^{n_x}$  and consider also, for all  $k \in \mathbb{N}$ ,  $\underline{v}_k, \bar{v}_k \in \ell_\infty^{n_x}$  and  $\underline{w}_k, \bar{w}_k \in \ell_\infty^{n_y}$  such that

$$\underline{x}_0 \leq x_0 \leq \bar{x}_0, \quad \underline{v}_k \leq v_k \leq \bar{v}_k, \quad \underline{w}_k \leq w_k \leq \bar{w}_k \quad (2)$$

Then the following theorem based on the bounded uncertainties assumption recalls the design of an interval observer for discrete-time linear systems.

**Theorem 1** (Discrete-Time Interval Observer [23]). *Consider system (1) with  $x \in \ell_\infty^{n_x}$  and let (2) holds. Assuming that there exists a matrix  $L$  such that  $A - LC$  is Schur and nonnegative, the system*

$$\begin{cases} \bar{x}_{k+1} = (A - LC)\bar{x}_k + Ly_k - (L^+ \underline{w}_k - L^- \bar{w}_k) + \bar{v}_k \\ \underline{x}_{k+1} = (A - LC)\underline{x}_k + Ly_k - (L^+ \bar{w}_k - L^- \underline{w}_k) + \underline{v}_k \end{cases} \quad (3)$$

is an interval observer for (1) and satisfies the inclusion property

$$\forall k \in \mathbb{N}, \underline{x}_k \leq x_k \leq \bar{x}_k. \quad (4)$$

Moreover,  $\underline{x}, \bar{x} \in \ell_\infty^{n_x}$  provided that  $A - LC$  is Schur stable.

As a reminder, in discrete-time, a matrix is called Schur stable if all its eigenvalues have absolute value less than one, and is called cooperative (or nonnegative) if all its elements are nonnegative.

**Remark 1.** *The proof of Theorem 1 consists in showing the stability and the cooperativity of the error dynamics, which hold provided that  $A - LC$  is Schur and nonnegative. It is usually difficult to find a gain  $L$  such that  $(A - LC)$  satisfies this both properties. A solution to overcome this issue is to find a gain  $L$  that guarantees the stability and to use a linear change of coordinates that will ensure the cooperativity in the new base [23][5].*

### 3 Problem Statement

Consider the discrete-time time-varying uncertain nonlinear system

$$\begin{cases} x_{k+1} = f_k(x_k) + v_k \\ y_k = h_k(x_k) + w_k \end{cases} \quad (5)$$

where  $f_k : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x}$  and  $h_k : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_y}$  are the dynamics and output maps,  $x_k \in \mathbb{R}^{n_x}$  and  $y_k \in \mathbb{R}^{n_y}$  are the state and output vectors at discrete time  $k \in \mathbb{N}$ ,  $v_k \in \mathbb{R}^{n_x}$  and  $w_k \in \mathbb{R}^{n_y}$  are respectively the additive disturbances and measurement noise.

**Remark 2.** *The following results also hold for non-autonomous systems of the form*

$$\begin{cases} x_{k+1} = f_k(x_k, u_k) + v_k \\ y_k = h_k(x_k, u_k) + w_k \end{cases} \quad (6)$$

where  $(u_k)_{k \in \mathbb{N}}$  is a known sequence of inputs and the following assumptions hold uniformly with respect to the inputs.

A point-wise estimation of the state of system (5) can be computed using an arbitrarily fast KKL observer as presented in [28]. It relies on the uniform Lipschitz backward distinguishability of the system (5) defined as follows.

**Definition 2** (Uniform Lipschitz Backward Distinguishability [28]). *The system (5) is uniformly Lipschitz backward distinguishable on a set  $\mathcal{X}$  if for each output  $y_i$ ,  $i \in \{1, 2, \dots, n_y\}$ , there exists  $m_i \in \mathbb{N}_{>0}$  such that for all  $k \geq \bar{m} := \max_i m_i$ , the sequence of backward distinguishability maps  $(\mathcal{O}_k^{bw})_{k \in \mathbb{N}}$  defined as*

$$\mathcal{O}_k^{bw}(x) = (\mathcal{O}_{1,k}^{bw}(x), \mathcal{O}_{2,k}^{bw}(x), \dots, \mathcal{O}_{n_y,k}^{bw}(x))$$

where  $\mathcal{O}_{i,k}^{bw}(x) \in \mathbb{R}^{m_i}$  is given by

$$\mathcal{O}_{i,k}^{bw}(x) = \begin{bmatrix} (h_{i,k-1} \circ f_{k-1}^{-1})(x) \\ (h_{i,k-2} \circ f_{k-2}^{-1} \circ f_{k-1}^{-1})(x) \\ \vdots \\ (h_{i,k-m_i} \circ f_{k-m_i}^{-1} \circ \dots \circ f_{k-1}^{-1})(x) \end{bmatrix}$$

is uniformly Lipschitz injective on  $\mathcal{X}$ , i.e., there exist  $c_0 \in \mathbb{R}_{>0}$  such that for all  $k \geq \bar{m}$  and for all  $(x_a, x_b) \in \mathcal{X} \times \mathcal{X}$ ,

$$|\mathcal{O}_k^{bw}(x_a) - \mathcal{O}_k^{bw}(x_b)| \geq c_0 |x_a - x_b|.$$

The observer design requires the following assumption.

#### Assumption 1.

- (1) *The solutions of interest of (5), initialized in a set  $\mathcal{X}_0$ , remain in a compact set  $\mathcal{X} \supseteq \mathcal{X}_0$  in positive time.*
- (2) *For all  $k \in \mathbb{N}$ ,  $f_k$  is invertible and both the sequences  $(f_k^{-1})_{k \in \mathbb{N}}$  and  $(h_k)_{k \in \mathbb{N}}$  are uniformly Lipschitz, i.e. there exist  $c_f, c_h \in \mathbb{R}_+$  such that for all  $k \in \mathbb{N}$  and for all  $(x_a, x_b) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$ ,*

$$\begin{aligned} |f_k^{-1}(x_a) - f_k^{-1}(x_b)| &\leq c_f |x_a - x_b|, \\ |h_k(x_a) - h_k(x_b)| &\leq c_h |x_a - x_b|. \end{aligned}$$

- (3) *The system (5) is uniformly Lipschitz backward distinguishable on  $\mathcal{X}$  for some  $m_i \in \mathbb{N}_{>0}, i \in \{1, 2, \dots, n_y\}$ .*

Lemma 2 and Theorem 2 recall the results obtained in [28, Theorems 1-4] for the design of an arbitrarily fast KKL observer.

**Lemma 2** ([28]). *Let be  $A \in \mathcal{M}_{n_z, n_z}(\mathbb{R})$  and  $B \in \mathcal{M}_{n_z, n_y}(\mathbb{R})$ . Under Assumption 1, given any  $T_0 : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_z}$ , the sequence  $(T_k)_{k \in \mathbb{N}}$  such that each  $T_k : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_z}$*

is given by

$$T_k(x) = A^k(T_0 \circ f_0^{-1} \circ f_1^{-1} \circ \dots \circ f_{k-1}^{-1})(x) + \sum_{j=0}^{k-1} A^{k-j-1} B(h_j \circ f_j^{-1} \circ f_{j+1}^{-1} \circ \dots \circ f_{k-1}^{-1})(x), \quad (7)$$

verifies the following equation for all  $k \in \mathbb{N}$  and  $x \in \mathcal{X}$ :

$$T_{k+1}(f_k(x)) = AT_k(x) + Bh_k(x). \quad (8)$$

**Theorem 2** (Arbitrarily fast KKL Observer [28]). *Consider system (5). Under Assumption 1, define  $n_z = \sum_{i=1}^{n_y} m_i$ . Then, there exists a  $\gamma^* \in \mathbb{R}_{>0}$  such that for any  $0 < \gamma < \gamma^*$ , there exists  $k^* \in \mathbb{N}$  such that the sequence  $(T_k)_{k \in \mathbb{N}}$  defined by (7), initialized with a globally Lipschitz map  $T_0$  and with*

$$A = \gamma \tilde{A} = \gamma \text{diag}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_{n_y}) \in \mathcal{M}_{n_z, n_z}(\mathbb{R}), \quad (9a)$$

$$B = \text{diag}(\tilde{B}_1, \tilde{B}_2, \dots, \tilde{B}_{n_y}) \in \mathcal{M}_{n_z, n_y}(\mathbb{R}), \quad (9b)$$

where for each  $i \in \{1, 2, \dots, n_y\}$ ,  $(\tilde{A}_i, \tilde{B}_i) \in \mathcal{M}_{m_i, m_i}(\mathbb{R}) \times \mathbb{R}^{m_i}$  is a controllable pair with  $\tilde{A}_i$  Schur stable, is uniformly Lipschitz injective on  $\mathcal{X}$  for all  $k \geq k^*$ . Moreover, there exists a sequence of left inverse  $(T_k^*)_{k \geq k^*}$  with  $T_k^* : \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_x}$  such that the observer

$$\hat{z}_{k+1} = A\hat{z}_k + By_k, \quad \hat{x}_k = T_k^*(\hat{z}_k), \quad (10)$$

initialized at  $\hat{z}_0 \in T_0(\mathcal{X})$ , is an arbitrarily fast observer of (5) that provides an exponentially stable estimation error. Thus, there exist positive scalars  $\bar{c}$ ,  $\bar{c}_v$  and  $\bar{c}_w$  such that for all  $k \geq k^*$ ,

$$|x_k - \hat{x}_k| \leq \frac{\bar{c}(\gamma \|\tilde{A}\|)^k}{\gamma^{\bar{m}-1}} |x_0 - \hat{x}_0| + \frac{1}{\gamma^{\bar{m}-1}} \sum_{j=0}^{k-1} (\gamma^* \|\tilde{A}\|)^{k-j-1} (\bar{c}_v v_j + \bar{c}_w w_j). \quad (11)$$

The upper bound of the high-gain parameter is given by

$$\gamma^* = \min \left\{ \frac{1}{\max_i \max |\text{eig}(\tilde{A}_i)|}, \frac{1}{\max_i \|\tilde{A}_i\| c_f}, \frac{c_c c_o}{\max_i \|\tilde{A}_i\| c_f c_c c_o + \max_i \|\tilde{B}_i\| c_h c_f \max_i (\|\tilde{A}_i\| c_f)^{m_i}} \right\}. \quad (12)$$

The number of steps required to obtain the uniform in-

jectivity of the sequence  $(T_k)_{k \in \mathbb{N}}$  is given by

$$k^* = \begin{cases} \bar{m}, & \text{if } c_T = 0, \\ \max \left( \bar{m}, \left\lfloor \frac{(\bar{m}-1) \ln \gamma + \ln \tilde{c} - \ln c_T}{\ln \gamma + \ln(\max_i \|\tilde{A}_i\| c_f)} + 1 \right\rfloor \right), & \text{otherwise.} \end{cases} \quad (13)$$

where  $\tilde{c} = c_c c_o - \max_i \|\tilde{B}_i\| c_h c_f \frac{\gamma^{\max_i (\|\tilde{A}_i\| c_f)^{m_i}}}{1 - \gamma^{\max_i \|\tilde{A}_i\| c_f}}$ ,  $c_c \in \mathbb{R}_{>0}$  is such that  $\|\mathcal{C}_i\| \geq c_c > 0$  for all  $i \in \{1, \dots, n_y\}$  with  $\mathcal{C}_i$  is the controllability matrix of the pair  $(A_i, B_i)$ .

**Remark 3.** *This result is closely related to a high-gain result in discrete time where  $\gamma$  plays the role of the high-gain parameter [14]. Thus,  $\gamma$  should be taken sufficiently small to guarantee uniform Lipschitz injectivity of  $(T_k)_{k \in \mathbb{N}}$  and increasing the convergence speed. However, it will worsen the sensitivity to the disturbances and measurement noise and the peaking phenomenon will appear.*

Within this framework, the aim is to design an interval observer for the system (5) assuming that the uncertainties are bounded as follows.

**Assumption 2.** *There exist known bounds  $\underline{v}, \bar{v} \in \ell_\infty^{n_x}$ ,  $\underline{w}, \bar{w} \in \ell_\infty^{n_y}$ , and for all  $k \in \mathbb{N}$ ,  $\underline{v}_k, \bar{v}_k \in \ell_\infty^{n_x}$  and  $\underline{w}_k, \bar{w}_k \in \ell_\infty^{n_y}$  such that*

$$\underline{v} \leq \underline{v}_k \leq v_k \leq \bar{v}_k \leq \bar{v}, \quad \underline{w} \leq \underline{w}_k \leq w_k \leq \bar{w}_k \leq \bar{w}. \quad (14)$$

However, instead of designing an interval observer directly in the original  $x$ -coordinates, an enclosure of the state of the system is first designed in the filter  $z$ -coordinates where the dynamics is linear (and thus, similarly to Theorem 1). The obtained enclosure is then recovered in the  $x$ -coordinates.

## 4 Main Results

The following theorem states the design of an Interval KKL observer for the uncertain nonlinear system (5).

**Theorem 3** (Interval KKL Observer). *Let Assumptions 1 and 2 hold. Define  $n_z = \sum_{i=1}^{n_y} m_i$ . Consider  $n_y$  controllable pairs  $(\tilde{A}_i, \tilde{B}_i) \in \mathcal{M}_{m_i, m_i} \times \mathbb{R}^{m_i}$  with  $\tilde{A}_i$  Schur stable and nonnegative and define the pair  $(A, B)$  as in (9). Consider  $\gamma^*$  given in (12), and for any  $0 < \gamma < \gamma^*$ , consider  $k^*$  given in (13). Then, the sequence  $(T_k)_{k \in \mathbb{N}}$  defined in (7) and initialized with a globally Lipschitz  $T_0$  is uniformly Lipschitz injective on  $\mathcal{X}$  for all  $k \geq k^*$ , and admits a uniformly Lipschitz sequence of left inverse that can be extended in a uniformly Lipschitz sequence  $(T_k^*)_{k \geq k^*}$ , with  $T_k^* : \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_x}$ . Moreover, the system*

defined by

$$\begin{cases} \bar{z}_{k+1} = A\bar{z}_k + B y_k - (B^+ \underline{w}_k - B^- \bar{w}_k) + c_L V_k \mathbf{1} \\ \underline{z}_{k+1} = A\underline{z}_k + B y_k - (B^+ \bar{w}_k - B^- \underline{w}_k) - c_L V_k \mathbf{1}, \end{cases} \quad (15)$$

and initialized by any  $\underline{z}_0$  and  $\bar{z}_0$  satisfying  $T_0(\mathcal{X}_0) \subseteq [\underline{z}_0, \bar{z}_0]$ , with  $c_L \in \mathbb{R}_{>0}$  the uniform Lipschitz constant of  $(T_k)_{k \in \mathbb{N}}$  and  $V_k = \max\{|\underline{v}_k|, |\bar{v}_k|\} \in \mathbb{R}_{>0}$ , satisfies:

$$\forall k \in \mathbb{N}, \quad \underline{z}_k \leq z_k \leq \bar{z}_k, \quad (16)$$

where  $z_k = T_k(x_k)$ . Besides, the system defined by

$$\begin{cases} \bar{x}_k = \tilde{T}_k^*(\bar{z}_k, \underline{z}_k) \\ \underline{x}_k = \tilde{T}_k^*(\underline{z}_k, \bar{z}_k), \end{cases} \quad (17)$$

with for all  $k \geq k^*$ ,  $\tilde{T}_k^* : \mathbb{R}^{n_z} \times \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_x}$  is defined by

$$\tilde{T}_k^*(z_a, z_b) = c_{L^*} \sum_{i=1}^{n_z} (z_{a,i} - z_{b,i}) \mathbf{1} + T_k^*(z_b), \quad (18)$$

where  $c_{L^*} \in \mathbb{R}_{>0}$  is the uniform Lipschitz constant of  $(T_k^*)_{k \geq k^*}$ , is an interval observer for system (5) for all  $k \geq k^*$ , i.e. the following inclusion properties hold

$$\forall k \geq k^*, \quad \underline{x}_k \leq x_k \leq \bar{x}_k, \quad (19)$$

Finally, the total interval estimation error  $e_k = \bar{e}_k - \underline{e}_k$  in the  $x$ -coordinates is bounded.

Few remarks are drawn here :

#### Remark 4.

- (1) The major distinctions of Theorem 3 compared to Theorem 2 is that instead of a point-wise estimation, an enclosure for the system state is obtained for some known bounds on the uncertainties. Moreover, the matrices  $\tilde{A}_i$  must be taken nonnegative to ensure this enclosure, but it's not a restriction as they can be freely chosen.
- (2) The sequence  $(T_k^*)_{k \geq k^*}$  is an extension on  $\mathbb{R}^{n_z}$  of the sequence of left inverse  $(T_k^{-1})_{k \geq k^*}$  defined on  $T_k(\mathcal{X})$ . The existence of such an extension with the same Lipschitz constant is ensured according to [18] (see also Theorem 3 proof).
- (3) By taking  $T_0$  identically null, and  $\bar{z}_0 = \underline{z}_0 = 0$ , the assumption on the bounded initial conditions  $T_0(\mathcal{X}_0) \subseteq [\underline{z}_0, \bar{z}_0]$  holds whatever  $\mathcal{X}_0$ . Thus, with such a choice, the knowledge of initial conditions can actually be dispensed, which is very beneficial, as this knowledge (which is required for nearly every cooperative interval observers) is not always easy to obtain in practical situations.
- (4) According with Theorem statement the state estimation is guaranteed only after  $k^*$  steps which is

a limitation. However, as it can be seen in the expression of  $k^*$  in (13), if  $T_0$  is taken identically null (as advised in the previous item),  $k^* = \bar{m}$ , which is usually quite small (as it will be illustrated on an example in Section 5).

The following Lemmas, used in the proof of Theorem 2, deals with the Lipschitzness of the sequences  $(T_k)_{k \in \mathbb{N}}$  and  $(T_k^*)_{k \geq k^*}$ , and gives the expressions of the associated Lipschitz constants ( $c_L$  and  $c_{L^*}$ ) whose knowledge is necessary for a practical use of the proposed interval observer (the proofs are given in Appendix).

**Lemma 3.** Let Assumption 1 holds, and let consider the sequence  $(T_k)_{k \in \mathbb{N}}$  defined in Theorem 3. Consider  $(c_{f,k})_{k \in \mathbb{N}}$ ,  $(c_{h,k})_{k \in \mathbb{N}}$  the sequences of Lipschitz constants of respectively  $f_k^{-1}$  and  $h_k$  (in particular, for all  $k \in \mathbb{N}$ ,  $c_{f,k} \leq c_f$  and  $c_{h,k} \leq c_h$ ). Then, for any  $k \in \mathbb{N}$ ,  $T_k$  is a globally Lipschitz function with a Lipschitz constant

$$\begin{aligned} c_{L,k} &= c_T (\gamma \max_i \|\tilde{A}_i\| c_{f,k})^k \\ &+ \max_i \|\tilde{B}_i\| c_{h,k} c_{f,k} \frac{1 - (\gamma \max_i \|\tilde{A}_i\| c_{f,k})^k}{1 - \gamma \max_i \|\tilde{A}_i\| c_{f,k}}, \end{aligned} \quad (20)$$

where  $c_T$  is the Lipschitz constant of  $T_0$ . Moreover, the sequence  $(T_k)_{k \in \mathbb{N}}$  is uniformly Lipschitz with a uniform Lipschitz constant

$$c_L = c_T + \frac{\max_i \|\tilde{B}_i\| c_h c_f}{1 - \gamma \max_i \|\tilde{A}_i\| c_f}, \quad (21)$$

and in particular, for all  $k \in \mathbb{N}$ ,  $c_{L,k} \leq c_L$ .

**Lemma 4.** Let Assumption 1 holds, and let consider the sequence  $(T_k)_{k \in \mathbb{N}}$  and the integer  $k^*$  as defined in Theorem 3. Let consider the same notations as in Lemma 3 and let consider  $(c_{o,k})_{k \in \mathbb{N}}$  the sequence of injectivity coefficients of  $\mathcal{O}_k^{bw}$  (that follows from Definition 2) (in particular for all  $k \in \mathbb{N}$ ,  $c_{o,k} \geq c_o$ ). Then, the sequence  $(T_k)_{k \in \mathbb{N}}$  is uniformly Lipschitz injective on  $\mathcal{X}$  for all  $k \geq k^*$ , and admits a uniformly Lipschitz sequence of left inverse that can be extended on  $\mathbb{R}^{n_z}$  in a uniformly Lipschitz sequence  $(T_k^*)_{k \geq k^*}$ . More precisely, for any  $k \geq k^*$ ,  $T_k^*$  is a globally Lipschitz function with a Lipschitz-constant

$$c_{L,k}^* = \frac{1}{c_k \gamma^{\bar{m}-1}} \quad (22)$$

where  $c_k = c_N (c_c c_{o,k} - c_{\mathcal{R},k} - c_{\mathcal{I},k})$ , with  $c_{\mathcal{R},k} = \max_i \|\tilde{B}_i\| c_{h,k} c_{f,k} \frac{\gamma \max_i (\|\tilde{A}_i\| c_{f,k})^{m_i}}{1 - \gamma \max_i \|\tilde{A}_i\| c_{f,k}} (1 - (\min_i (\gamma \|\tilde{A}_i\| c_{f,k})^{k-m_i}))$ ,  $c_{\mathcal{I},k} = c_T \gamma^{k-\bar{m}-1} (\max_i \|\tilde{A}_i\| c_{f,k})^k$ , and  $c_N \in \mathbb{R}_{>0}$  a

constant depending on the chosen norm (more on this in Subsection 5.3). Moreover, the uniform Lipschitz constant of the sequence  $(T_k^*)_{k \geq k^*}$  is given by

$$c_{L^*} = \frac{1}{c\gamma^{m-1}}, \quad (23)$$

where  $c = c_N \left( c_c c_o - \max_i \|\tilde{B}_i\| c_h c_f \frac{\gamma^{\max_i (\|\tilde{A}_i\| c_f)^{m_i}}}{1 - \gamma^{\max_i \|\tilde{A}_i\| c_f}} \right)$ . In particular, for all  $k \in \mathbb{N}$ ,  $c_{L,k} \leq c_L$ .

*Proof of Theorem 3.* The proof is divided in two parts. In the first one, it will be proven that (15) is an interval observer for system (5) by proving the cooperativity and boundedness of the upper and lower estimation errors dynamics in the  $z$ -coordinates. Then, the second part will show that the same properties can be recovered in the  $x$ -coordinates.

First, using Lemma 2 and equation 8, the dynamics of (5) in the  $z$ -coordinates is written as

$$\begin{aligned} z_{k+1} &= T_{k+1}(x_{k+1}) \\ &= T_{k+1}(f_k(x_k) + v_k) \\ &= T_{k+1}(f_k(x_k)) + T_{k+1}(f_k(x_k) + v_k) \\ &\quad - T_{k+1}(f_k(x_k)) \\ &= Az_k + Bh_k(x_k) + T_{k+1}(f_k(x_k) + v_k) \\ &\quad - T_{k+1}(f_k(x_k)) \\ &= Az_k + By_k - Bw_k + T_{k+1}(f_k(x_k) + v_k) \\ &\quad - T_{k+1}(f_k(x_k)). \end{aligned}$$

Define the upper error in the  $z$ -coordinates as  $\bar{\varepsilon}_k = \bar{z}_k - z_k$ , and let prove by induction that  $\bar{\varepsilon}_k \geq 0$  for any  $k \in \mathbb{N}$ . According to Theorem's hypothesis, it comes that  $z_0 = T_0(x_0) \in T_0(\mathcal{X}_0) \subset [\underline{z}_0, \bar{z}_0]$ , and thus  $\bar{\varepsilon}_0 = \bar{z}_0 - z_0 \geq 0$ . Then, let assume that  $\bar{\varepsilon}_k \geq 0$  and let prove that  $\bar{\varepsilon}_{k+1} \geq 0$ . The following dynamics holds:

$$\begin{aligned} \bar{\varepsilon}_{k+1} &= \bar{z}_{k+1} - z_{k+1} \\ &= A\bar{z}_k + By_k - (B^+ \underline{w}_k - B^- \bar{w}_k) + c_L V_k \mathbf{1} \\ &\quad - \left( Az_k + By_k - Bw_k + T_{k+1}(f_k(x_k) + v_k) \right. \\ &\quad \left. - T_{k+1}(f_k(x_k)) \right) \\ &= A\bar{\varepsilon}_k - (B^+ \underline{w}_k - B^- \bar{w}_k) + Bw_k \\ &\quad + c_L V_k \mathbf{1} + T_{k+1}(f_k(x_k)) - T_{k+1}(f_k(x_k) + v_k). \end{aligned}$$

Note that, according to Lemma 3  $(T_k)_{k \in \mathbb{N}}$  is uniformly Lipschitz with Lipschitz constant  $c_L$  for all  $k \in \mathbb{N}$ , thus  $|T_k(x_a) - T_k(x_b)| \leq c_L |x_a - x_b|$  for all  $(x_a, x_b) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$ . Note also that under Assumption 2,  $|v_k| \leq V_k = \max\{|\underline{v}_k|, |\bar{v}_k|\}$ . Therefore, for all  $k \in \mathbb{N}$ :

$$\begin{aligned} &T_{k+1}(f_k(x_k) + v_k) - T_{k+1}(f_k(x_k)) \\ &\leq |(T_{k+1}(f_k(x_k) + v_k) - T_{k+1}(f_k(x_k)))| \mathbf{1} \\ &\leq c_L |v_k| \mathbf{1} \\ &\leq c_L V_k \mathbf{1}. \end{aligned}$$

Thus,  $c_L V_k \mathbf{1} - (T_{k+1}(f_k(x_k) + v_k) - T_{k+1}(f_k(x_k))) \geq 0$  for all  $k \in \mathbb{N}$ . Moreover, according to Assumption 2 and Lemma 1,  $-(B^+ \underline{w}_k - B^- \bar{w}_k) + Bw_k \geq 0$  for all  $k \in \mathbb{N}$ . Then,  $A$  being nonnegative, it comes that  $\bar{\varepsilon}_{k+1} \geq 0$ . Hence, it has been proven that for all  $k \in \mathbb{N}$ ,  $\bar{\varepsilon}_k \geq 0$ . The same reasoning applied to the lower error  $\underline{\varepsilon}_k = z_k - \underline{z}_k$  results in  $\underline{\varepsilon}_k \geq 0$  for all  $k \in \mathbb{N}$  and proves the inclusion property (16) in the  $z$ -coordinates.

Moreover, from the expression of  $\bar{\varepsilon}_{k+1}$  one gets that for all  $k \in \mathbb{N}_{>0}$ ,

$$\begin{aligned} \bar{\varepsilon}_k &= A^k \bar{\varepsilon}_0 + \sum_{j=0}^{k-1} A^{k-j-1} \left( -(B^+ \underline{w}_j - B^- \bar{w}_j) + Bw_j \right. \\ &\quad \left. + c_L V_j \mathbf{1} - (T_{j+1}(f_j(x_j) + v_j) - T_{j+1}(f_j(x_j))) \right). \end{aligned}$$

Therefore, for all  $k \in \mathbb{N}_{>0}$ ,

$$\begin{aligned} |\bar{\varepsilon}_k| &= \left| A^k \bar{\varepsilon}_0 + \sum_{j=0}^{k-1} A^{k-j-1} \left( -(B^+ \underline{w}_j - B^- \bar{w}_j) + Bw_j \right. \right. \\ &\quad \left. \left. + c_L V_j \mathbf{1} - (T_{j+1}(f_j(x_j) + v_j) - T_{j+1}(f_j(x_j))) \right) \right| \\ &\leq (\gamma \|\tilde{A}\|)^k |\bar{\varepsilon}_0| + \sum_{j=0}^{k-1} (\gamma \|\tilde{A}\|)^{k-j-1} \left( \|B\| |\bar{w}_j - \underline{w}_j| \right. \\ &\quad \left. + (c'_N + 1) c_L V_j \right), \end{aligned} \quad (24)$$

where  $c'_N \in \mathbb{R}_{>0}$  depends on the chosen norm (more on this in Subsection 5.3).

Thus, as  $A$  is Schur stable,  $\bar{\varepsilon}_k$  is bounded for all  $k \in \mathbb{N}$ . The same arguments allow to conclude that  $\underline{\varepsilon}_k$  is also bounded for all  $k \in \mathbb{N}$  and the bounding term is the same except that  $\bar{\varepsilon}_0$  is replaced by  $\underline{\varepsilon}_0$ .

Hence, this proves that the interval estimation errors in the  $z$ -coordinates are bounded for all  $k \in \mathbb{N}$ .

Now, let consider a real  $0 < \gamma < \gamma^*$ , and the associated value  $k^*$  given by (13). The fact that the sequence  $(T_k)_{k \geq k^*}$  is uniformly Lipschitz injective with a left inverse that and can be extended on  $\mathbb{R}^{n_z}$  in a uniformly Lipschitz sequence  $(T_k^*)_{k \geq k^*}$  directly results from Lemma 4.

Consider now the upper error in the  $x$ -coordinates defined as

$$\bar{e}_k = \bar{x}_k - x_k = \tilde{T}_k^*(\bar{z}_k, \underline{z}_k) - T_k^*(z_k),$$

with  $\tilde{T}_k^*$  defined in (18). Note that the expression of  $\tilde{T}_k^*$  results from applying the second item of Property 1 to  $T_k^*$  which is globally Lipschitz for all  $k \geq k^*$ . Therefore, from the third item of Property 1, it immediately comes that  $\bar{e}_k \geq 0$ . In the same way, it is proved that  $\underline{e}_k \geq 0$  for all  $k \geq k^*$ . which proves the inclusion property (19) in the  $x$ -coordinates.



Moreover, for all  $k \geq k^*$ ,

$$\begin{aligned}
|\bar{e}_k| &= |\hat{T}_k^*(z_k, \bar{z}_k) - T_k^*(z_k)| \\
&= |c_{L^*} \sum_{i=1}^{n_z} (\bar{z}_{k,i} + z_{k,i}) \mathbf{1} + T_k^*(z_k) - T_k^*(z_k)| \\
&= |c_{L^*} \sum_{i=1}^{n_z} (\bar{e}_{k,i} + \underline{\varepsilon}_{k,i}) \mathbf{1} + T_k^*(z_k) - T_k^*(z_k)| \\
&\leq c_N'' c_{L^*} \sum_{i=1}^{n_z} (\bar{e}_{k,i} + \underline{\varepsilon}_{k,i}) + |T_k^*(z_k) - T_k^*(z_k)| \\
&\leq c_N'' c_{L^*} \sum_{i=1}^{n_z} (\bar{e}_{k,i} + \underline{\varepsilon}_{k,i}) + c_{L^*} |z_k - z_k| \\
&= c_N'' c_{L^*} \sum_{i=1}^{n_z} (\bar{e}_{k,i} + \underline{\varepsilon}_{k,i}) + c_{L^*} |\underline{\varepsilon}_k| \\
&= c_{L^*} \left( c_N'' \sum_{i=1}^{n_z} (\bar{e}_{k,i} + \underline{\varepsilon}_{k,i}) + |\underline{\varepsilon}_k| \right),
\end{aligned}$$

where  $c_N''$  is a constant that depends on the chosen norm. As shown before,  $\bar{e}_k$  and  $\underline{\varepsilon}_k$  are bounded for all  $k \in \mathbb{N}$ , thus it comes that  $\bar{e}_k$  is also bounded but for all  $k \geq k^*$ . The same reasoning applies to  $\underline{e}_k$ . Therefore, the interval estimation errors in the  $x$ -coordinates are bounded for all  $k \geq k^*$ .  $\square$

The following remarks are added :

**Remark 5.**

- (1) From (24), it can be noticed that the convergence speed of the errors in the  $z$ -coordinates can be made arbitrarily fast by picking  $\gamma$  closer to zero. However, it can be shown that this can increase the pessimism of the estimated intervals. Thus, the issue of picking an optimal  $\gamma$  that leads to minimal estimation errors in the  $x$ -coordinates is raised. This problem is addressed in Subsection 5.2.
- (2) The choice of the eigenvalues spacing for  $A$  and the coefficients of  $B$  are also important as it acts on the filter dynamics and its error. This issue is let to future researches.

## 5 Practical considerations

Even though KKL observers relies on a powerful theory, their main issue is their difficulty of implementation. Indeed, it is usually very challenging to find analytically the inverse transformations, but fortunately they can be approximated. Moreover, the tuning of such observers (particularly by the choice of an appropriate value for  $\gamma$ ) is generally difficult and suffers from the lack of generic methods. Finally, in the set-membership framework, the conservatism must be reduced in order to obtain more precise information about the state. This section addresses these issues.

### 5.1 Approximation of the inverse transformations

In practice, it is usually difficult to obtain an exact expression for the sequence of the inverse transformations  $(T_k^*)_{k \geq k^*}$ . There exist several ways to handle this issue. A simple way is to solve an optimization problem [1] such that, for  $k \in \mathbb{N}$  and given a  $z_k \in \mathbb{R}^{n_z}$  :

$$\begin{aligned}
\min_{x_k} & |z_k - T_k(x_k)|^2 \\
\text{s.t.} & x_k \in \mathcal{X}
\end{aligned} \tag{25}$$

Other numerical approximation approaches are given in [17]. Recently, new methods based on neural networks have been developed to learn a model of the transformations [24] [22] [4].

In the following, the influence of the approximation of the inverse transformations is studied. Consider  $\hat{T}_k^*$  being an approximation of  $T_k^*$ , i.e there exist  $\delta$  such that for all  $k \geq k^*$  and for all  $z \in \mathbb{R}^{n_z}$ ,  $|\hat{T}_k^*(z) - T_k^*(z)| \leq \delta$ . Then, the upper estimation error becomes

$$\begin{aligned}
|\bar{e}_k| &\leq c_N'' c_{L^*} \sum_{i=1}^{n_z} (\bar{e}_{k,i} + \underline{\varepsilon}_{k,i}) + |\hat{T}_k^*(z_k) - T_k^*(z_k)| \\
&\leq c_N'' c_{L^*} \sum_{i=1}^{n_z} (\bar{e}_{k,i} + \underline{\varepsilon}_{k,i}) + |T_k^*(z_k) - T_k^*(z_k)| + \delta \\
&\leq c_{L^*} \left( c_N'' \sum_{i=1}^{n_z} (\bar{e}_{k,i} + \underline{\varepsilon}_{k,i}) + |\underline{\varepsilon}_k| \right) + \delta.
\end{aligned}$$

Thus, it can be conclude that the estimation errors are robust with respect to the approximation error of the inverse transformations, as soon as this approximation error is uniformly bounded (here by  $\delta$ ).

### 5.2 Choice of $\gamma$

The section deals with the way to choose  $\gamma$  for a given pair  $(A, B)$ . As explained in the first of Remark 5, there exists a trade-off between the convergence speed and the steady state error of the estimated interval in the  $x$ -coordinates. This trade-off can be tuned by  $\gamma$  which is thus no longer only a high-gain parameter but also what can be called a conservatism parameter. In this section, a method to find an optimal  $\tilde{\gamma}$  that minimizes the steady state estimation error is proposed.

According to Assumption 2 and to the expression of the estimation errors in the  $z$ -coordinates given in the proof, the upper and lower steady state values  $\underline{\varepsilon}_\infty = \bar{\varepsilon}_\infty = \varepsilon_\infty^\pm$  are bounded by

$$|\varepsilon_\infty^\pm| \leq \frac{1}{1 - \gamma \|A\|} \left( \|\tilde{B}\| \|\bar{w} - \underline{w}\| + (c_N' + 1) c_L V \right) \tag{26}$$

where  $V = \max\{|V|, |\bar{V}|\}$ .

Therefore, following the expression of the estimation er-

rors in the  $x$ -coordinates, the upper and lower steady state values  $e_\infty = \bar{e}_\infty = e_\infty^\pm$  are bounded by

$$\begin{aligned} |e_\infty^\pm| &\leq c_L^*(2c_N''n_z + 1)|\varepsilon_\infty^\pm| \\ &\leq \frac{c_L^*(2c_N''n_z + 1)}{1 - \gamma\|\tilde{A}\|} \left( \|\tilde{B}\|\|\bar{w} - \underline{w}\| + (c_N' + 1)c_L V \right) \end{aligned} \quad (27)$$

Recalling that  $c_L^*$  and  $c_L$  depends also on  $\gamma$ , see Equations (21) and (23), one is able to find an optimal  $\tilde{\gamma}$  that minimizes the steady state estimation errors.

One simple way to obtain a numerical value is to consider the following optimization problem :

$$\begin{aligned} \min_{\gamma} \quad & |e_\infty^\pm(\gamma)|^2 \\ \text{s.t.} \quad & 0 < \gamma < \gamma^* \end{aligned} \quad (28)$$

**Remark 6.** *Instead of considering the steady state error, it is also possible but much more computationally expensive to consider the bounds of the estimation errors at each step  $k$ . The conservatism would be reduced but a minimization problem has to be solved at each time leading to an optimal sequence  $(\tilde{\gamma}_k)_{k \in \mathbb{N}}$ .*

### 5.3 Type of norm

The results presented in this paper hold for any norm of vector and induced matrix norm as all the norm are equivalent on a finite dimension vector space. The norms quantify the magnitude of vectors or matrices in different ways. The most commonly used are the  $\mathcal{L}_1$ ,  $\mathcal{L}_2$  and  $\mathcal{L}_\infty$ . Among the possibilities, the  $\mathcal{L}_\infty$  norm is advised because its deals with the worst case scenario. Thus, the choice of  $\gamma$  as described in Section 5.2 will lead to reduce the value of the largest error. In this case, the norm depending constants are  $c_N = c_N' = c_N'' = 1$ .

### 5.4 Non-uniform values

Another way to reduce the conservatism of the estimated interval is to consider the non-uniform constants given in Lemmas 3 and 4 when implementing the observer. Then, as the non-uniform constants are less conservative than the uniform ones, replacing  $c_L$  and  $c_L^*$  by  $c_{L,k}$  and  $c_{L,k}^*$  given respectively by (15) and (18) reduces the pessimism of the estimated intervals.

### Remark 7.

- (1) *The expressions of  $c_L^*$  in (23) or  $c_{L,k}^*$  in (22) results on valid Lipschitz constants of  $T_k^*$  only for  $k \geq k^*$ . However, they still can be used before because, in any case, the inclusion of the system state and the boundedness of the errors are not guaranteed until  $k^*$ . One could also set their value to zero before  $k^*$ .*
- (2) *Note that this approach can also be used to find the optimal sequence  $(\tilde{\gamma}_k)_{k \in \mathbb{N}}$ .*

## 6 Illustrative example

Consider the Lipschitz nonlinear system adapted from [29] and represented by (5) with

$$\begin{aligned} f_k(x_k) &= \begin{bmatrix} a_k x_{1,k} + x_{2,k} + 1 \\ -\frac{5}{4} \tanh(x_{1,k}) + b_k + 1 \end{bmatrix}, \\ h_k(x_k) &= \begin{bmatrix} -1 & 2.5 \end{bmatrix} x_k, \end{aligned} \quad (29)$$

where  $a_k = -0.42 + 0.01 \cos(\omega k T_d)$  and  $b_k = -1.6 - 0.01 \cos(\omega k T_d)$  where  $\omega = 10\pi$  rad/s and  $T_d = 0.001$  s is the sampling time. The measurement noise is generated as a uniform distribution such that  $\bar{w}_k = -\underline{w}_k = 0.1$ . Some small additive state disturbances are considered, also uniformly distributed and such that  $\bar{v}_k = -\underline{v}_k = 0.005$ . This is relevant as when the nonlinear dynamics is known, usually the modeling errors are low. The initial condition is set as  $x_0 = [0.5; -1]$ .

Before designing the IKKL observer, the requirements of Assumptions 1 and 2 must be checked. The first step is to compute the inverse dynamics  $f_k^{-1}$ , which is given by

$$f_k^{-1}(x_k) = \begin{bmatrix} \operatorname{arctanh}\left(-\frac{4}{5}(x_{2,k} - b_k - 1)\right) \\ x_{1,k} - a_k \operatorname{arctanh}\left(-\frac{4}{5}(x_{2,k} - b_k - 1)\right) - 1 \end{bmatrix}.$$

Then, the following bullets show that Assumption 1 holds :

- The sets  $\mathcal{X}_0 = [-0.3, 0.6] \times [-1.5, 0]$  and  $\mathcal{X} = [-0.79, 1.22] \times [-1.66, 0.22]$  verify Assumption 1.1.
- $f_k^{-1}$  and  $h_k$  are uniformly Lipschitz on  $\mathcal{X}$  with constants  $c_f = 2.8480$  and  $c_h = 2.6926$  respectively, thus Assumption 1.2 holds. Indeed, one could compute the derivative of  $f_k^{-1}$  with respect to  $x_k$  and notice that its norm is upper bounded on  $\mathcal{X}$  by 2.8480 for all  $k \in \mathbb{N}$ . Then, the output function is linear time invariant with respect to  $x_k$  thus its Lipschitz constant is equal to the norm of  $\begin{bmatrix} -1 & 2.5 \end{bmatrix}$ .
- Assumption 1.3 holds for  $\bar{m} = 2$  as the backward distinguishability map defined by

$$\mathcal{O}_k^{bw}(x) = \begin{bmatrix} (h_{k-1} \circ f_{k-1}^{-1})(x) \\ (h_{k-2} \circ f_{k-2}^{-1} \circ f_{k-1}^{-1})(x) \end{bmatrix},$$

has the norm of its derivative lower bounded by  $c_o = 2.5161$  on  $\mathcal{X}$  for all  $k \geq 2$ .

- The uncertainties are bounded and  $x_0 \in \mathcal{X}_0$  so Assumption 2 holds.

Thus, the sequence  $(T_k)_{k \in \mathbb{N}}$  can be computed according to (7) where the observer matrices are taken as  $A_1 = \operatorname{diag}(0.01, 0.1)$ , i.e. Schur stable and nonnegative, and  $B_1 = [1; 1]$  such that  $(A_1, B_1)$  is controllable leading to

a controllability constant  $c_c = 1.1$ . The initial mapping  $T_0$  is taken as 0, thus we may choose  $z_0 = 0$  and  $\bar{z}_0 = 0$ . Because of the difficulty to obtain an analytic expression of the inverse sequence  $(T_k^*)$ , a numerical approximation is performed as presented in Section 5.1 using the Matlab function *fmincon*.

The upper bound of the high-gain parameter is computed according to (12) and results in  $\gamma^* = 1.9625$ . The optimal  $\gamma$  is computed as presented in Section 5.2 leading to  $\tilde{\gamma} = 1.0536$ .

A comparison is made between the use of uniform and non-uniform Lipschitz constants of the transformations  $(T_k)$  and  $(T_k^*)$ . In the legends of the following Figures, the case where the uniform constants are used is denoted by IKKL (A) whereas IKKL (B) refers to the case where the non-uniform constants are used. The uniform constants values are  $c_L = 10.9563$  and  $c_L^* = 0.5182$ . The non-uniform constants are computed at each time step such as detailed in Section 5.4. In this particular example, the difference between both IKKL (A) and (B) will be mostly visible during the transient state because  $c_{f,k}$  and  $c_{h,k}$  are actually constant for all  $k \in \mathbb{N}$ , and  $c_{o,k}$  does not change significantly with respect to  $k$ . Moreover, note that a sequence  $\tilde{\gamma}_k$  is not considered here.

The number of steps after which  $(T_k)_{k \in \mathbb{N}}$  becomes uniformly injective and the inclusion property (19) holds is computed according to (13) and outputs  $k^* = 2$ .

The simulation duration is set to 100 ms (100 time steps).

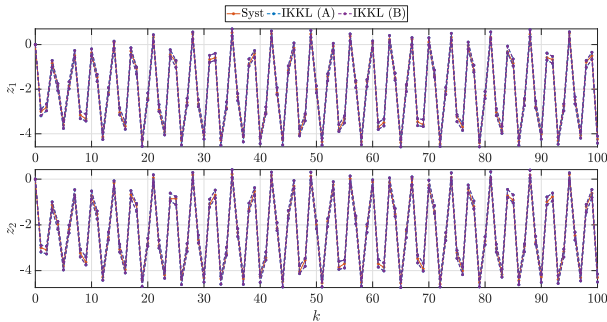


Fig. 1. The state estimations in the  $z$ -coordinates.

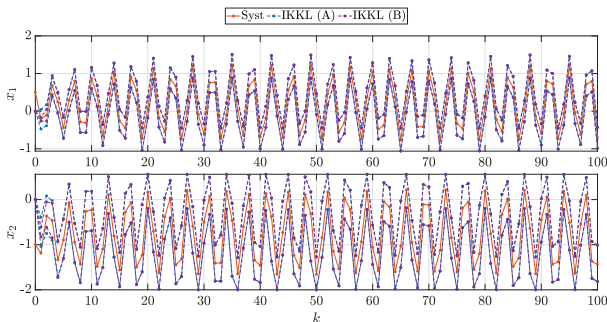


Fig. 2. The state estimations in the  $x$ -coordinates.

Figure 1 shows the state estimation in the  $z$ -coordinates by the IKKL observers. Note that in those coordinates, the inclusion of the filter state in the estimated bounds is guaranteed for all  $k \in \mathbb{N}$ . Next, Figure 2 presents

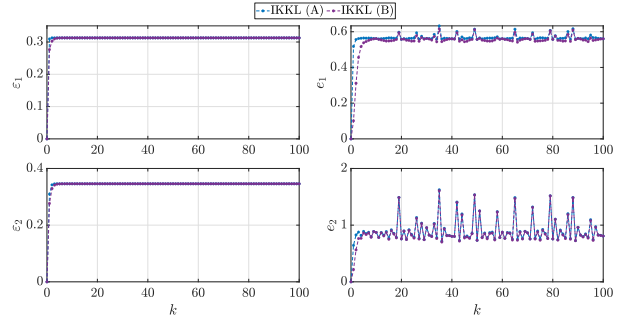


Fig. 3. The interval width in the  $z$ -coordinates (left) and the  $x$ -coordinates (right).

the state estimations for both IKKL observers in the  $x$ -coordinates where the inclusion of the system state in the estimated bounds is guaranteed for all  $k \geq k^*$ . Finally, Figure 3 illustrates the difference in the conservatism of the estimated intervals computed by both of the IKKL observers. The interval width in the  $z$ -coordinates (on the left) and the  $x$ -coordinates (on the right) are shown. As mentioned, the difference between the two approaches is more noticeable during the transient state.

## 7 Conclusions

This work presents a new interval observer for discrete-time time-varying nonlinear uncertain systems called Interval KKL observer. It is based on the KKL framework developed for such systems. The proposed design requires the uniformly Lipschitz backward distinguishability of the system and known bounds on the uncertainties. The enclosure of the system state in the estimated bounds is obtained using the Lipschitzness of both the direct and the inverse transformations and is thus guaranteed only after a certain time. Some practical considerations have been addressed for the implementation of such a design, covering the approximation methods of the inverse transformations and solutions to reduce the pessimism. The proposed Interval KKL observer has been illustrated through a numerical simulation which highlights its efficiency.

Further research may focus on fine tuning the design in order to obtain less pessimism, possibly with another way to recover the estimated intervals in the original coordinates. An adaptation to continuous systems may also be considered.

## A Proofs of Lemmas 3 and 4

This appendix gives the proofs of Lemmas 3 and 4.

*Proof of Lemma 3.* Using (7) and Assumption 1.2, it

comes for all  $k \in \mathbb{N}$  and for all  $(x_a, x_b) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$ :

$$\begin{aligned}
& |T_k(x_a) - T_k(x_b)| \\
& \leq c_T (\gamma \max_i \|\tilde{A}_i\|_{c_f, k})^k |x_a - x_b| \\
& + \sum_{j=0}^{k-1} (\gamma \max_i \|\tilde{A}_i\|_{c_f, k})^{k-j-1} \max_i \|\tilde{B}_i\|_{c_h, k} c_{f, k}^{k-j} |x_a - x_b| \\
& = \left( c_T (\gamma \max_i \|\tilde{A}_i\|_{c_f, k})^k \right. \\
& + \max_i \|\tilde{B}_i\|_{c_h, k} c_{f, k} \frac{1 - (\gamma \max_i \|\tilde{A}_i\|_{c_f, k})^k}{1 - \gamma \max_i \|\tilde{A}_i\|_{c_f, k}} \left. \right) |x_a - x_b| \\
& \leq \left( c_T + \frac{\max_i \|\tilde{B}_i\|_{c_h, k} c_{f, k}}{1 - \gamma \max_i \|\tilde{A}_i\|_{c_f, k}} \right) |x_a - x_b|.
\end{aligned}$$

One could identify the expressions of the non-uniform Lipschitz constants (20) and the uniform Lipschitz constant (21) of  $(T_k)_{k \in \mathbb{N}}$ .  $\square$

*Proof of Lemma 4.* This proof follows a reasoning close to the one proposed in [28] for the demonstration of the uniform Lipschitzness of  $(T_k)_{k \in \mathbb{N}}$  (needed in Theorem 2 for the existence of the proposed punctual observer). However, in [28] there is no need of knowing the explicit expressions of the Lipschitz constants as they are not used for the punctual observer implementation. The sequence  $(T_k)_{k \in \mathbb{N}}$  given by (7) can be decomposed as

$$T_k(x) = (T_{1,k}(x), \dots, T_{i,k}(x), \dots, T_{n_y,k}(x)),$$

where for each  $i \in \{1, \dots, n_y\}$ ,  $(T_{i,k})$  are defined with  $(\gamma \tilde{A}_i, \tilde{B}_i)$  instead of  $(A, B)$ . Then, using the closed form expression (7), one has for each  $i \in \{1, \dots, n_y\}$ , for all  $k \geq m_i$  and for all  $(x_a, x_b) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$ ,

$$\begin{aligned}
T_{i,k}(x_a) - T_{i,k}(x_b) &= (\mathcal{I}_{i,k}(x_a) - \mathcal{I}_{i,k}(x_b)) \\
&+ (\mathcal{R}_{i,k}(x_a) - \mathcal{R}_{i,k}(x_b)) + (\mathcal{T}_{i,k}(x_a) - \mathcal{T}_{i,k}(x_b)),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{I}_{i,k}(x_a) - \mathcal{I}_{i,k}(x_b) &= (\gamma \tilde{A}_i)^k \left( (T_0 \circ f_0^{-1} \circ \dots \circ f_{k-1}^{-1})(x_a) \right. \\
&\quad \left. - (T_0 \circ f_0^{-1} \circ \dots \circ f_{k-1}^{-1})(x_b) \right),
\end{aligned}$$

$$\begin{aligned}
& \mathcal{R}_{i,k}(x_a) - \mathcal{R}_{i,k}(x_b) \\
&= \sum_{j=0}^{k-m_i-1} (\gamma \tilde{A}_i)^{k-j-1} \tilde{B}_i \left( (h_{i,j} \circ f_j^{-1} \circ \dots \circ f_{k-1}^{-1})(x_a) \right. \\
&\quad \left. - (h_{i,j} \circ f_j^{-1} \circ \dots \circ f_{k-1}^{-1})(x_b) \right),
\end{aligned}$$

$$\begin{aligned}
& \mathcal{T}_{i,k}(x_a) - \mathcal{T}_{i,k}(x_b) \\
&= \sum_{j=k-m_i}^{k-1} (\gamma \tilde{A}_i)^{k-j-1} \tilde{B}_i \left( (h_{i,j} \circ f_j^{-1} \circ \dots \circ f_{k-1}^{-1})(x_a) \right. \\
&\quad \left. - (h_{i,j} \circ f_j^{-1} \circ \dots \circ f_{k-1}^{-1})(x_b) \right).
\end{aligned}$$

The norm of each of these terms is then bounded exploiting Assumption 1. Therefore, for all  $k \geq m_i$  and for all  $(x_a, x_b) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$ ,

$$|\mathcal{I}_{i,k}(x_a) - \mathcal{I}_{i,k}(x_b)| \leq c_T (\gamma \|\tilde{A}_i\|_{c_f, k})^k |x_a - x_b|,$$

where  $c_T$  is the Lipschitz constant of  $T_0$ ,

$$\begin{aligned}
& |\mathcal{R}_{i,k}(x_a) - \mathcal{R}_{i,k}(x_b)| \\
& \leq \sum_{j=0}^{k-m_i-1} (\gamma \|\tilde{A}_i\|)^{k-j-1} \|\tilde{B}_i\|_{c_h, k} c_{f, k}^{k-j} |x_a - x_b| \\
& = \|\tilde{B}_i\|_{c_h, k} c_{f, k} \frac{(\gamma \|\tilde{A}_i\|_{c_f, k})^{m_i}}{1 - \gamma \|\tilde{A}_i\|_{c_f, k}} \\
& \quad \left( 1 - (\gamma \|\tilde{A}_i\|_{c_f, k})^{k-m_i} \right) |x_a - x_b|,
\end{aligned}$$

and

$$|\mathcal{T}_{i,k}(x_a) - \mathcal{T}_{i,k}(x_b)| \geq \gamma^{m_i-1} c_c |\mathcal{O}_{i,k}^{bw}(x_a) - \mathcal{O}_{i,k}^{bw}(x_b)|,$$

where  $c_c = \min_i \|\mathcal{C}_i\| \in \mathbb{R}_{>0}$  which exists since  $(\tilde{A}_i, \tilde{B}_i)$  is controllable with a corresponding controllability matrix  $\mathcal{C}_i$ .

Concatenating the results for the  $n_y$  outputs, it comes that for all  $k \geq \bar{m}$  and for all  $(x_a, x_b) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$ ,

$$\begin{aligned}
|T_k(x_a) - T_k(x_b)| &\geq |\mathcal{T}_k(x_a) - \mathcal{T}_k(x_b)| \\
&\quad - |\mathcal{R}_k(x_a) - \mathcal{R}_k(x_b)| - |\mathcal{I}_k(x_a) - \mathcal{I}_k(x_b)| \\
&= c_k \gamma^{\bar{m}-1} |x_a - x_b|,
\end{aligned}$$

where  $c_k$  is defined as:

$$\begin{aligned}
c_k &= c_N \left( c_c c_{o,k} \right. \\
&\quad - \max_i \|\tilde{B}_i\|_{c_h, k} c_{f, k} \frac{\gamma \max_i (\|\tilde{A}_i\|_{c_f, k})^{m_i}}{1 - \gamma \max_i \|\tilde{A}_i\|_{c_f, k}} \\
&\quad \left. (1 - (\min_i (\gamma \|\tilde{A}_i\|_{c_f, k})^{k-m_i})) \right. \\
&\quad \left. - c_T \gamma^{k-\bar{m}-1} (\max_i \|\tilde{A}_i\|_{c_f, k})^k \right),
\end{aligned}$$

with  $c_N \in \mathbb{R}_{>0}$  is a constant that depends on the chosen norm.

Moreover, one could show that [28, Theorem 3], for all  $k \geq k^*$  with  $k^*$  given in (13),

$$c_k \geq c := c_N \left( c_c c_o - \max_i \|\tilde{B}_i\|_{c_h, k} c_{f, k} \frac{\gamma \max_i (\|\tilde{A}_i\|_{c_f, k})^{m_i}}{1 - \gamma \max_i \|\tilde{A}_i\|_{c_f, k}} \right).$$

As  $c > 0$  for  $\gamma < \gamma^*$ , this proves the (uniform) Lipschitz injectivity of  $(T_k)_{k \geq k^*}$ , which implies the existence and uniqueness of a left inverse sequence  $(T_k^{-1})_{k \geq k^*}$  such that for all  $k \geq k^*$ , one has for all  $x \in \mathbb{R}^{n_x}$ ,  $T_k^*(T_k(x)) = x$ , and for all  $z_a, z_b \in T_k(\mathcal{X})$

$$|T_k^*(z_a) - T_k^*(z_b)| \leq \frac{1}{c_k \gamma^{m-1}} |z_a - z_b| \leq \frac{1}{c \gamma^{m-1}} |z_a - z_b|,$$

which corresponds to the (uniform) Lipschitzness of  $(T_k^{-1})_{k \geq k^*}$ .

Finally, the extension on  $\mathbb{R}^{n_z}$  of the sequence  $(T_k^{-1})_{k \geq k^*}$  in a uniform Lipschitz sequence  $(T_k^*)_{k \geq k^*}$  with the same Lipschitz constants is guaranteed by [18, Theorem 1]. From the previous results, one could easily identify the expressions of the non-uniform Lipschitz constants (22) and the uniform Lipschitz constant (23) of  $(T_k^*)_{k \geq k^*}$ .  $\square$

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