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A class of open surfaces with algorithmically solvable homeomorphism problem

Sylvain Maillot

April 2, 2024

Abstract

We introduce a new class of possibly noncompact n -dimensional manifolds without boundary associated to finite data which we call topological automata. This class is large enough to contain many interesting examples of open 2-dimensional and 3-dimensional manifolds of interest to low-dimensional topologists. Our main result is that the homeomorphism problem in this class is decidable for $n = 2$.¹

1 Introduction

A fundamental question in manifold topology is the *homeomorphism problem*, which asks, in a given dimension n , whether there exists an algorithm to decide if two compact triangulated n -dimensional manifolds are homeomorphic. It is easy to show that the answer is yes for $n = 1$ and $n = 2$, using the classification of these manifolds. For $n = 3$, the answer, although apparently not known in full generality (see the comments in [3]), is yes for orientable, irreducible 3-manifolds. This relies on Perelman's geometrization theorem ([22, 23, 24], see also [13, 21, 10, 6].) When $n \geq 4$, by a theorem of A. A. Markov [18] the answer is negative (see also [7].)

When the manifolds under consideration are no longer assumed to be compact, a difficulty arises: in each dimension $n \geq 2$ there are uncountably many open n -manifolds up to homeomorphism (see Appendix A.)

In this article, we introduce for each n a new class of possibly noncompact n -manifolds, called *automata n -manifolds*, for which the homeomorphism problem makes sense. Our main result, Theorem 1.5 below, states that this problem is algorithmically solvable for $n = 2$.

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We first introduce the notion of *topological n -automaton*, which is a finite data set from which an automaton n -manifold will be constructed. The definition is as follows.

Definition 1.1. Let $n \geq 1$ be an integer. A *topological n -automaton* is a triple $\mathcal{X} = ((X_0, \dots, X_p), (C_1, \dots, C_p), (f_1, \dots, f_q))$ where

- p, q are nonnegative integers;
- for each k , X_k is a compact connected triangulated n -dimensional manifold-with-boundary, called a *building block*, or a *state*;
- for each $k \geq 1$, C_k is a connected component of ∂X_k , hereafter called the *incoming boundary component* of X_k . The other boundary components of the building blocks, including all boundary components of X_0 , if any, are called the *outcoming boundary components*;
- for each i , there exist k, l with $k \leq l$ such that f_i is a simplicial homeomorphism from some outcoming boundary component of X_k onto C_l ; f_i is called an *arrow*;

subject to the condition that every outcoming boundary component is the domain of exactly one arrow.

An example of topological 2-automaton is given in Figure 1.

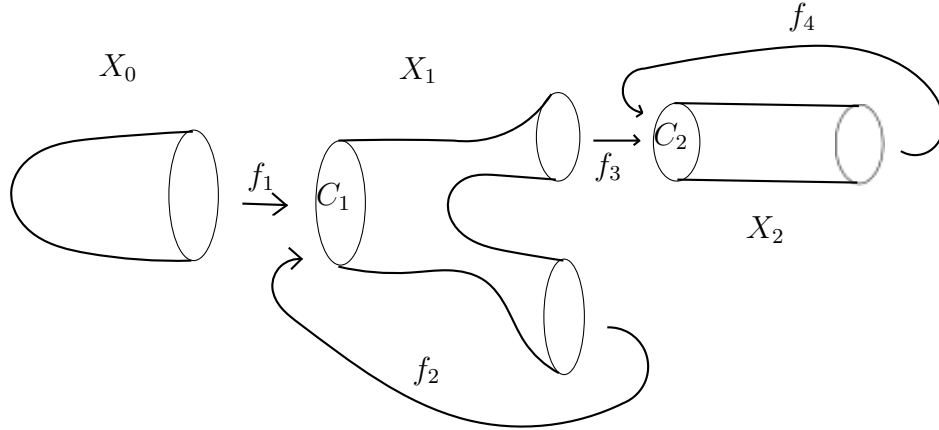


Figure 1: A topological 2-automaton \mathcal{X}

Remark 1.2. In order for decision problems about topological automata to be well-defined, one needs to be a bit more precise: letting \mathcal{X} be a topological

n -automaton, fix a triangulation of each building block X_k , i.e. a homeomorphism between X_k and the geometric realization of a simplicial complex A_k , in such a way that each f_i is conjugate to a simplicial map between some subcomplexes of the A_k 's. Then \mathcal{X} can be encoded by finite data that can be manipulated using one's favorite mathematical software.

To any topological n -automaton \mathcal{X} we associate an n -dimensional manifold without boundary $M(\mathcal{X})$ by starting with X_0 and 'following the arrows' to attach copies of the various building blocks. The idea is simple, but the formal definition is rather awkward, since we must specify an order in which the building blocks are attached, and keep track of the copies of the building blocks that have already been used. To this effect, note that by the last condition, the ordering of the f_i 's induces a linear ordering L on the set of outgoing boundary components.

Let \mathcal{X} be a topological n -automaton. We first define inductively a sequence of triples $\{(M_s, \phi_s, L_s)\}_{s \in \mathbb{N}}$ where for each s , M_s is a compact n -manifold-with-boundary, ϕ_s is a collection of homeomorphisms from the components of ∂M_s to outgoing boundary components of \mathcal{X} , and L_s is a linear ordering of the components of ∂M_s . For each $1 \leq k \leq p$ we generate a sequence of copies of X_k by setting $X_k^m := X_k \times \{m\}$ for each nonnegative integer m . By abuse of notation, we still denote by f_i the homeomorphisms between the various components of the ∂X_k^m 's induced by f_i . We define M_0 to be X_0 , ϕ_0 to be the set of identity maps of the components of ∂X_0 , and L_0 is induced by L .

Assume that the triple (M_s, ϕ_s, L_s) has been defined. Take the first component C of ∂M_s , as given by L_s . Using ϕ_s , we can identify C to some outgoing boundary component of some X_k . Then by hypothesis there is a unique f_i with domain C . Let $C_{k'}$ be the range of f_i . Take the first copy of $X_{k'}$ which has not already been used in the construction, and attach it to M_s along f_i . Then repeat the operation for all components of ∂M_s , the order being determined by L_s . The resulting manifold is M_{s+1} . The set of homeomorphisms ϕ_{s+1} is determined by the identification of each added manifold with some building block. The ordering L_{s+1} is deduced from L_s and L in a lexicographical manner.

By construction, there is a natural inclusion map from M_s to M_{s+1} . Thus the M_s 's form a direct system. We define $M(\mathcal{X})$ as the direct limit of this system. See Figure 2 for an example.

We say that an n -manifold M is an *automaton n -manifold* if it is homeomorphic to $M(\mathcal{X})$ for some topological n -automaton \mathcal{X} .

Remark 1.3. The requirement that the building blocks be connected implies that $M(\mathcal{X})$ is connected, even if the graph underlying the automaton is

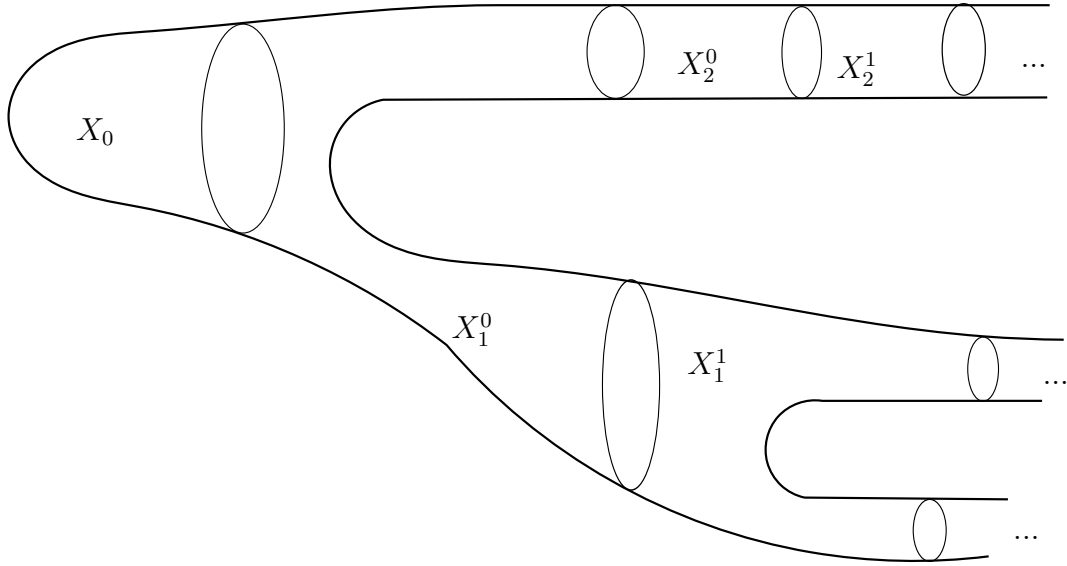


Figure 2: The surface associated to \mathcal{X}

disconnected. In fact, any vertex which is not accessible from X_0 can be discarded without changing the outcome of the construction. For example, consider a topological 2-automaton \mathcal{X} with three states X_0, X_1, X_2 , such that X_0 is a disk, X_1, X_2 are annuli, and there are three arrows: one from the only boundary component of X_0 to the incoming boundary component of X_1 , one from the outgoing boundary component of X_1 to the incoming boundary component of X_1 , one from the outgoing boundary component of X_2 to the incoming boundary component of X_2 . Then $M(\mathcal{X})$ is a plane obtained from stacking a copy of X_0 and a sequence of copies of X_1 ; the state X_2 plays no role at all.

Example 1.4. Let M be an n -manifold that admits a manifold compactification, i.e. there exists a compact manifold \bar{M} such that M is homeomorphic to $\bar{M} \setminus \partial\bar{M}$. Let N_1, \dots, N_p be the boundary components of \bar{M} . Then M can be presented by a topological n -automaton with $p + 1$ states as follows: $X_0 = \bar{M}$, and for every $k \geq 1$, the state X_k is $N_k \times [0, 1]$ and the arrows are the obvious ones. In particular,

- a) Every closed n -manifold is an automaton manifold;
- b) Every 1-manifold is an automaton manifold;
- c) Every finite-type surface is an automaton 2-manifold.

In this paper we are concerned with $n = 2$, i.e. surfaces. For us, the interesting case is that of surfaces of infinite type. These have been studied at least since the 1970s since they appear as leaves of foliations on compact 3-manifolds (see e.g. [25, 9, 11].) They are also studied from the point of view of Teichmüller theory (see e.g. [14, 19] and the references therein) and translation surfaces [26].

Recently they have been the subject of much attention through the study of their mapping class groups; we refer to the survey by Aramayona and Vlamis [2].

Our main theorem is the following:

Theorem 1.5. *There is an algorithm which takes as input two topological 2-automata $\mathcal{X}_1, \mathcal{X}_2$ and decides whether $M(\mathcal{X}_1)$ and $M(\mathcal{X}_2)$ are homeomorphic.*

The proof of Theorem 1.5 relies essentially on the classification theorem for possibly noncompact surfaces, due to Kerékjártó and Richards [28]. We now recall the statement of this theorem.

Let M be a connected, possibly noncompact surface without boundary. The classical invariants which are used in the statement of the classification theorem are the genus $g(M) \in \mathbf{N} \cup \{\infty\}$, the orientability class $o(M)$, and the triple $(E(M), E'(M), E''(M))$ where $E(M)$ is the space of ends of M , $E'(M)$ is the closed subspace of nonplanar ends, and $E''(M)$ is the closed subspace of nonorientable ends. Two such triples (E, E', E'') and (F, F', F'') are called *equivalent* if there is a homeomorphism $f : E \rightarrow F$ such that $f(E') = F'$ and $f(E'') = F''$. Here is the statement of the classification theorem:

Theorem 1.6 (Classification of surfaces, Kerékjártó, Richards [28]). *Let M_1 and M_2 be two connected surfaces without boundary. Then M_1 and M_2 are homeomorphic if and only if they have same genus and orientability class, and $(E(M_1), E'(M_1), E''(M_1))$ is equivalent to $(E(M_2), E'(M_2), E''(M_2))$.*

Recall that a surface F is *planar* if every embedded circle in F separates. A planar surface F has genus 0, is orientable, and $E'(F) = E''(F) = \emptyset$. Thus planar surfaces are classified by their space of ends.

A topological 2-automaton is called *planar* if all of its building blocks are planar. This is equivalent to requiring that the associated surface should be planar.

We now give some examples and non-examples of automata surfaces. All of them are orientable, so $E'' = \emptyset$. According to [2], five infinite-type surfaces are sufficiently important to have received names in the literature. They are all automata surfaces:

- Example 1.7.** a) The *Loch Ness monster surface* [25] has exactly one end, which is nonplanar. It is presented by an automaton with two states: X_0 is a disk; X_1 is an annulus with one handle.
- b) *Jacob's ladder surface*² has exactly two ends, both nonplanar. It is presented by an automaton with three states: X_0 is an annulus; X_1 and X_2 are annuli with one handle.
- c) The *Cantor tree surface* ('arbre de Cantor' [11]) is the complement of a Cantor set in S^2 . It is presented by an automaton with two states: X_0 is a disk; X_1 is a pair of pants.
- d) The *blooming Cantor tree surface* ('arbre de Cantor fleuri' [11]) has a Cantor set's worth of ends, none of which is planar. It is presented by an automaton with two states: X_0 is a disk; X_1 is a pair of pants with one handle.
- e) The *flute surface* [4]: it is the planar surface with space of ends the ordinal $\omega + 1$. This is the example depicted in Figures 1 and 2.

In addition, the complement of a Cantor set in \mathbf{R}^2 is presented by an automaton with three states: X_0 is an annulus; X_1 is an annulus; X_2 is a pair of pants. This is the example mentioned in a blog post by D. Calegari and studied in J. Bavard's PhD thesis [5] that sparked interest in 'big mapping class groups'. Likewise, the three surfaces depicted on Figure 1 of the paper [17] by Mann and Rafi are automata surfaces; informally, it is the very idea of a 'surface that can be represented by a simple picture' which the notion of automaton surface endeavors to capture.

The simplest example of a surface which is *not* an automaton surface is the planar surface with space of ends $\omega^\omega + 1$ (see Example 3.9 below.) More generally, if M is a planar surface with countably many ends, then M is an automaton surface if and only if each point of $E(M)$ has finite Cantor-Bendixson rank. For similar reasons, Example 2.3 in [16] is not an automaton surface, and neither are most of the examples given in our Appendix A.

The paper is structured as follows: in Sections 2 and 3, we deal with the planar case, i.e. we give an algorithm to decide whether the surfaces associated to two planar topological 2-automata are homeomorphic. In Section 2, we show how to associate to a planar topological 2-automaton \mathcal{X} a combinatorial object $T(\mathcal{X})$, called an *admissible tree*, such that the structure of the

²We follow the terminology of Aramayona and Vlamis [2], itself borrowed from Ghys [11]; Philips and Sullivan [25] used that name for a Riemannian surface that is homeomorphic to the Loch Ness monster surface, but to quasiisometric to it.

space of ends of $M(\mathcal{X})$ can be read off $T(\mathcal{X})$. In Section 3, we introduce the notion of a *reduced tree*, which is a kind of normal form for $T(\mathcal{X})$, and prove Theorem 1.5 in the planar case. In Section 4, we explain how to generalize this construction in order to prove Theorem 1.5 in full generality. Finally, in Appendix A, we indicate a construction of uncountably many open surfaces which are pairwise nonhomeomorphic.

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2 From automata to admissible trees

In this section, we simply call *2-automaton* a planar topological 2-automaton. To any such 2-automaton \mathcal{X} , we shall associate a decorated graph $G(\mathcal{X})$ which contains enough information to recover the space of ends of the surface associated to \mathcal{X} . Then we show how to associate to a decorated graph G a decorated tree of a special form, called an *admissible tree*, which also contains this information. This will allow us to reduce the topological classification of planar 2-automata to the classification of admissible trees up to some equivalence relation of topological nature. This latter classification problem will be solved in Section 3.

The structure of this section is as follows: in Subsection 2.1, we introduce decorated graphs, admissible trees and their associated topological spaces. The reader interested in the motivation can jump directly to Subsection 2.2 and refer back as needed. In Subsection 2.2, we show how to pass from a 2-automaton to a decorated graph. Finally in Subsection 2.3 we show how to turn a decorated graph into an admissible tree.

2.1 Decorated graphs and admissible trees

We introduce three formal symbols $*$, O , Θ . We call *decorated graph* a pair (G, f) , where G is a finite oriented graph endowed with a base vertex, called the *root*, and f is a map from the set of all vertices of G to the set $\{*, O, \Theta\}$, in such a way that the root has image $*$. We also assume that there is no circuit in G as an oriented graph.

For simplicity, we will sometimes simply denote this graph by G , the decoration f being understood. We say that a vertex is *of type* $*$, O , or Θ

according to its image by f . A vertex different from the root is called an *ordinary vertex*.

We associate to any decorated graph G a topological space $L(G)$ as follows: let G be a decorated graph. Set $r := n_{\text{O}} + 2n_{\text{\Theta}}$, where n_{O} (resp. $n_{\text{\Theta}}$) is the number of vertices of G of type O (resp. of type Θ). Consider the alphabet $B = \{b_1, \dots, b_r\}$ and assign to each vertex v of G either one letter (if v is of type O), or an unordered pair of letters (if v is of type Θ), or nothing. Let $L(G)$ be the set of infinite words w on B that can be obtained according to the following recipe: let $c = v_1 \cdots v_l$ be a finite injective path of length at least 2 in G starting from the root and respecting the orientations of the edges. For each $k \in \{2, \dots, l-1\}$, choose a finite (possibly empty) word in the letter(s) assigned to v_k ; then choose an infinite word in the letter(s) assigned to v_l ; then w is obtained by concatenating all those words. Finally, $L(G)$ is topologised as a subset of $B^{\mathbb{N}}$ endowed with the product topology.

Of course, in this generality it is possible that some of the letters are never used (if some vertex is not accessible from the root.) It is even possible that $L(G)$ is empty. However, the space $B^{\mathbb{N}}$ is either empty, a single point, or a Cantor set. Moreover, it is readily checked from the definition that $L(G)$ is a closed subset of $B^{\mathbb{N}}$. Thus $L(G)$ is compact, metrizable and totally disconnected. We shall see later that when G is obtained from a 2-automaton \mathcal{X} by a certain construction, then $L(G)$ is homeomorphic to the space of ends of $M(\mathcal{X})$.

Next we define a special class of decorated graphs which will play an important role.

Definition 2.1. An *admissible tree* is a decorated graph which has the following properties:

- i. It is a tree.
- ii. The only vertex of type $*$ is the root.
- iii. All edges point away from the root.

It follows that the orientations of edges is uniquely determined by the underlying structure, so in the pictures, we will not represent these orientations. The convention is that the edges are oriented from left to right.

We will use some standard terminology concerning rooted trees. The *leaves* are the degree 1 vertices which are ordinary (i.e. different from the root.) We use the words ‘ancestor’, ‘descendant’, ‘son’, ‘father’, and ‘sibling’ as in genealogy.

We now give some information on the space $L(T)$ associated to an admissible tree T . First, the notion of convergence associated to our topology is

that of compact convergence; more precisely, if w is a point of $L(T)$ and (w_n) a sequence of points of $L(T)$, then (w_n) converges to w iff for each $m \in \mathbf{N}$, we have that the prefix of length m of w_n is eventually equal to the prefix of length m of w .

Let T be an admissible tree, and v be an ordinary vertex of T . We denote by L_v the subset of $L(T)$ consisting of the words that have infinitely many letters associated to v . By definition, $L(T)$ is the disjoint union of the L_v 's. If v' is a descendant of v , then any $w \in L_v$ is a limit of a sequence (w_n) of points of $L_{v'}$, obtained by taking the n first letters of w and completing by an infinite word in $L_{v'}$.

We denote by $T(v)$ the subtree consisting of v and its descendants. We call this tree the *subtree generated by v* . Note that $T(v)$ is not an admissible tree (or even a decorated graph) since its “root” is v , which does not have type $*$. However, such subtrees will play an important role in inductive arguments. It follows from the definition of the topology that the union U of the $L_{v'}$'s for $v' \in T(v)$ is an open subset of $L(T)$. In particular, it is a neighborhood of any point of L_v . This gives a partial converse to the above observation: if $w \in L_v$ and (w_n) is a sequence converging to w , then w_n eventually belongs to U .

2.2 From planar 2-automata to decorated graphs

Let \mathcal{X} be a planar 2-automaton. The goal of this subsection is to construct a decorated graph $G(\mathcal{X})$ such that $L(G(\mathcal{X}))$ is homeomorphic to $E(M(\mathcal{X}))$.

We start by forming the directed graph G' whose vertices are the building blocks and whose edges are the arrows. An arrow is called a *loop* if its domain and range lie in the same building block.

In order to describe the space of ends of the surface $M(\mathcal{X})$, take a finite alphabet A with as many letters as arrows in \mathcal{X} , and assign a letter to each arrow. We give the set $A^{\mathbf{N}}$ of infinite words in A the product topology. The space of ends of the surface $M(\mathcal{X})$ is homeomorphic to the subset $E \subset A^{\mathbf{N}}$ consisting of words which can be read by starting from X_0 and following the arrows.

To each prefix $w = a_1 \cdots a_n$ of each word in E we can associate the number k such that w leads to X_k , and the subset E_w of E of words whose prefix is w and whose other letters correspond to loops around X_k . There are three cases: if there is no such loop, then E_w is empty; if there is one, then E_w is a singleton; otherwise, E_w is a Cantor set, no matter how many loops around X_k there are. This follows from the well-known fact that the Cantor set is the only totally disconnected compactum without isolated points. Hence we see that the number of loops around some building block is irrelevant as soon

as it is greater than or equal to 2.

We are now ready to define the decorating graph $G(\mathcal{X})$: there is a vertex of $G(\mathcal{X})$ for each building block X_k , the root being X_0 . Edges of $G(\mathcal{X})$ correspond to arrows f_i which are not loops. The image of a vertex by f is $*$ (resp. O , resp. Θ) if there is no loop (resp. one loop, resp. two or more loops) around the corresponding building block. The condition $k \leq l$ in the fourth point of the definition of a topological automaton ensures that the oriented graph $G(\mathcal{X})$ does not contain any circuit. (There may, however, exist circuits in $G(\mathcal{X})$ as an unoriented graph.)

It follows from the above description of $E(M(\mathcal{X}))$ that this space is homeomorphic to the space $L(G(\mathcal{X}))$ defined in Subsection 2.1.

2.3 From decorated graphs to admissible trees

Let G be a decorated graph. We describe a procedure which yields an admissible tree T such that $L(G)$ is homeomorphic to $L(T)$. We do not need to know that G comes from an automaton, although this is the case we are interested in.

The construction of T is done in the following three steps. At each step, one checks that the homeomorphism type of $L(T)$ does not change.

Step 1 If some vertices of G are not accessible from the root, then we remove them.

Step 2 If G is not a tree, then it contains two distinct injective paths connecting the root to some vertex y . We choose a ‘minimal’ such y , i.e. farthest from the root as possible, so that the subgraph Y consisting of y and the vertices accessible from it is a tree.

Let e_1, \dots, e_p be the edges going ‘into’ y . For each i , denote by x_i the other end of e_i . We construct a new decorated graph by introducing for each $2 \leq i \leq p$ a copy (Y_i, y_i) of (Y, y) , deleting e_i and replacing it by an edge e'_i connecting x_i to y_i . An example of this operation is represented in Figure 3.

By repeating this operation finitely many times, we obtain a decorated graph T' which is a tree.

Step 3 If T' is not admissible, pick a vertex x of type $*$ which is not the root. Let e be the edge leading to x (by the previous construction, there must be exactly one). Let x' be the initial vertex of e , and let e_1, \dots, e_p be the edges with initial vertex x (if any.) Modify T' by deleting the vertex x and the edge e , and replacing each e_i by an edge e'_i with initial vertex x' and

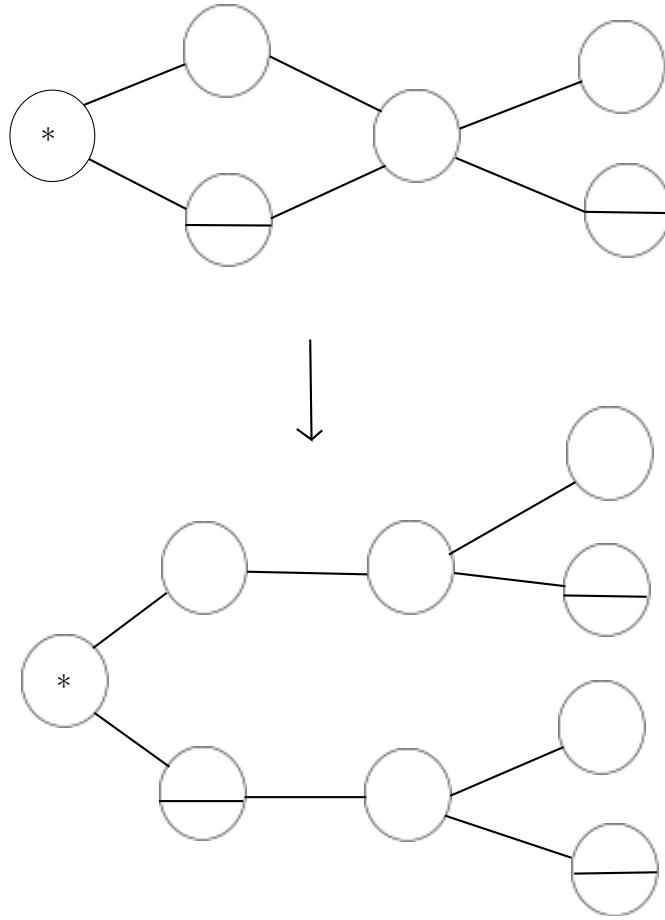


Figure 3: Making the graph G into a tree

same terminal vertex as e_i . See Figure 4 for an example of this operation. (Informally, it corresponds to ‘collapsing’ the edge e .) Repeating this finitely many times, we obtain the admissible tree T .

To sum up, starting with a 2-automaton \mathcal{X} , we can produce a decorated graph $G = G(\mathcal{X})$ such that $L(G)$ is homeomorphic to $E(M(\mathcal{X}))$, and then an admissible tree T such that $L(T)$ is homeomorphic to $L(G)$, hence also to $E(M(\mathcal{X}))$.

Given two planar automata $\mathcal{X}_1, \mathcal{X}_2$, we can compute the associated admissible trees T_1, T_2 , and our problem is reduced to checking whether or not $L(T_1)$ and $L(T_2)$ are homeomorphic. A solution of the latter problem is given in the next section. For the general case, we need more structure, see Section 4.

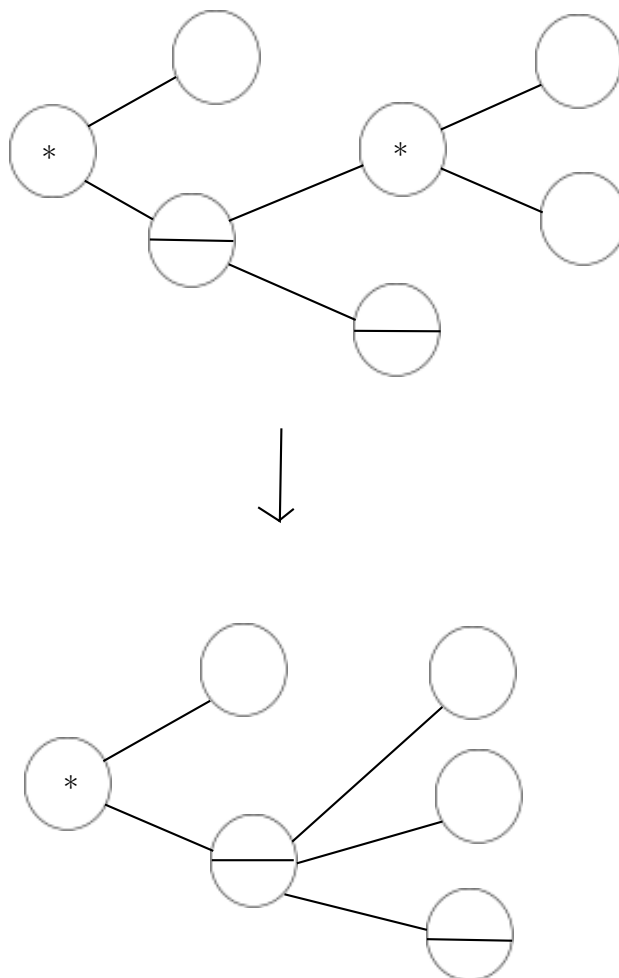


Figure 4: Making the tree admissible

3 Classification of admissible trees

3.1 Reduced admissible trees

Let T_1, T_2 be two admissible trees. We say that T_1 and T_2 are *topologically equivalent*, or just *equivalent*, if the associated spaces $L(T_1)$ and $L(T_2)$ are homeomorphic. We say that they are *isomorphic* if there is a type-preserving bijection between the set of vertices of T_1 and that of T_2 which respects the graph structure.

Determining whether two given admissible trees are (topologically) equivalent is the problem we want to solve. The problem of determining whether two given admissible trees are isomorphic is combinatorial in nature and

easy to solve algorithmically simply by enumerating all maps from the set of vertices of T_1 to that of T_2 until a suitable bijection is found (or not.)

We will show how to modify an admissible tree without changing its topological equivalence class, until it belongs to a special class of admissible trees, called *reduced*, for which topological equivalence will turn out to be equivalent to isomorphism. In order to motivate the construction, we first give some simple examples of pairs of admissible trees which are equivalent but fail to be isomorphic.

Let T_1 be a tree with two vertices: $*$ and its son v of type Θ . The space $L(T_1)$ is the space of all infinite words on a two-letter alphabet, which is a Cantor set. Let T_2 be a tree with three vertices: $*$, v_1 and v_2 , where both v_1 and v_2 are of type Θ and are sons of $*$. Then $L(T_2)$ is a disjoint sum of two Cantor sets, which is homeomorphic to a Cantor set. Thus T_1, T_2 are equivalent but nonisomorphic.

Let T_3 be a tree with three vertices: $*$, its only son v , and the only son w of v , with both v, w of type O . The space $L(T_3)$ can be described as follows: the alphabet is $\{a, b\}$ with the letter a corresponding to v and b to w . Then the elements of $L(T_3)$ are the (infinite) word $aaa \cdots$ and all words of the type $a^kbbb \cdots$ for $k \in \mathbf{N}$. Topologically, this is a space with a single accumulation point and a sequence of isolated points converging to it.

Let T_4 be a tree with vertices $*, v, w_1, w_2$, with v the only son of $*$, w_1, w_2 two sons of v , and every ordinary vertex has type O . The space $L(T_4)$ consists of an accumulation point and two sequences of isolated points converging to it. This space is homeomorphic to $L(T_3)$ although T_3 is not isomorphic to T_4 .

Finally if T_5 is a tree with vertices $*$, its only son v , and the only son w of v , with both v, w of type Θ , then $L(T_5)$ has no isolated point. Hence it is a Cantor set. More generally, any tree all of whose leaves are of type Θ have the same property, which produces many nonisomorphic equivalent trees.

We now introduce three moves that can be used to simplify an admissible tree without changing its equivalence class. Let T be an admissible tree. Recall that when v is a vertex of T , we denote by $T(v)$ the subtree consisting of v and its descendants.

Move 1 Let v be a vertex of T of type Θ . Let v' be the father of v . Assume that v' is not the root, and that v is the only son of v' . Remove the edge between v and v' and replace v' by v .

Move 2 Let v_1 be a vertex of T . Let v_2, v_3 be two descendants of v_1 such that v_2 is a son of v_1 , and v_3 is not. Assume that the subtrees $T(v_2)$ and

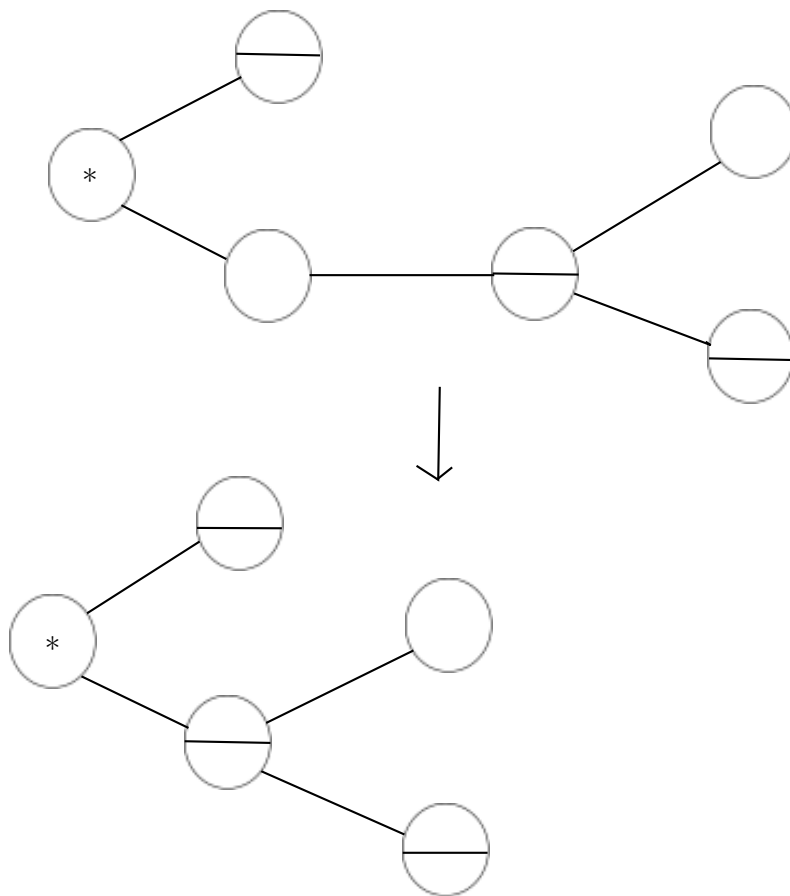


Figure 5: Move 1

$T(v_3)$ are isomorphic. Remove $T(v_2)$.

Move 3 Let v_1 be a vertex of T . Let v_2, v_3 be two distinct sons of v_1 . Assume that v_1 is not the root, or that both v_2 and v_3 are of type Θ . Further assume that the subtrees $T(v_2)$ and $T(v_3)$ are isomorphic. Remove $T(v_3)$.

Simple examples of the three moves are represented in Figures 5, 6, and 7 respectively.

Definition 3.1. Let T be an admissible tree. For $i \in \{1, 2, 3\}$, we say that T has Property (R_i) if Move i cannot be performed. We say that T is *reduced* if it has property (R_i) for all i .

We leave to the reader the tedious but elementary task of checking that Moves 1–3 do not change the homeomorphism type of the associated space $L(T)$. We then have the following proposition:

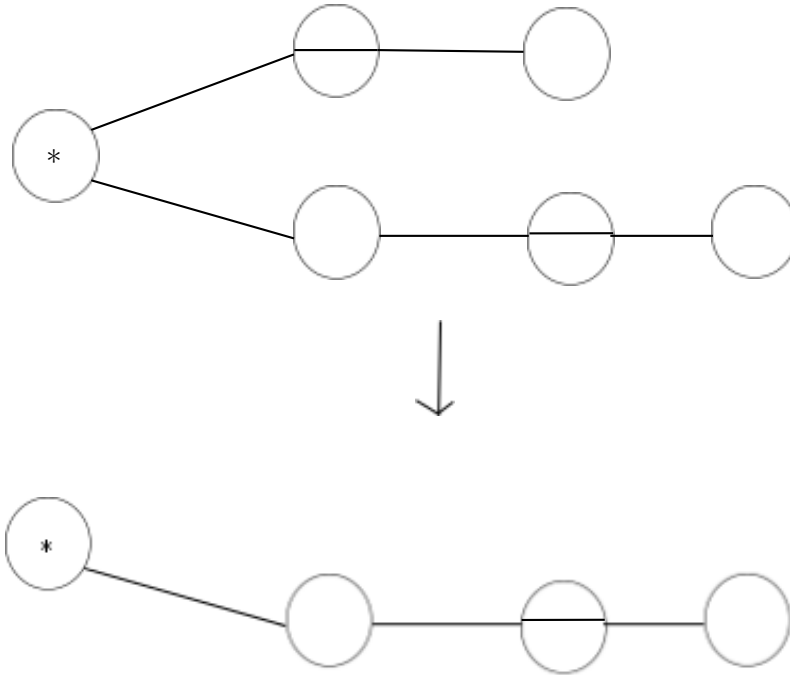


Figure 6: Move 2

Proposition 3.2. *There is an algorithm which to any admissible tree T associates a reduced admissible tree equivalent to T .*

Proof. Starting with T , apply Move 1 as much as possible, then Move 2 as much as possible, then Move 3 as much as possible, then Move 1 again etc. Each time a move is applied, the number of vertices of the tree goes down, hence the process eventually stops. \square

The main result of this section is

Theorem 3.3. *Two reduced admissible trees T_1, T_2 are equivalent if and only if they are isomorphic.*

The ‘if’ part is clear, so we only have to prove the ‘only if’ part. To this end, we develop some theory in Subsection 3.2, to be applied in Subsection 3.3.

3.2 Decomposing a totally disconnected space

To provide motivation for the notions introduced in this subsection, we first define inductively the *combinatorial depth* of an ordinary vertex in an admissible tree. Let T be an admissible tree. Recall that the *leaves* of T are the

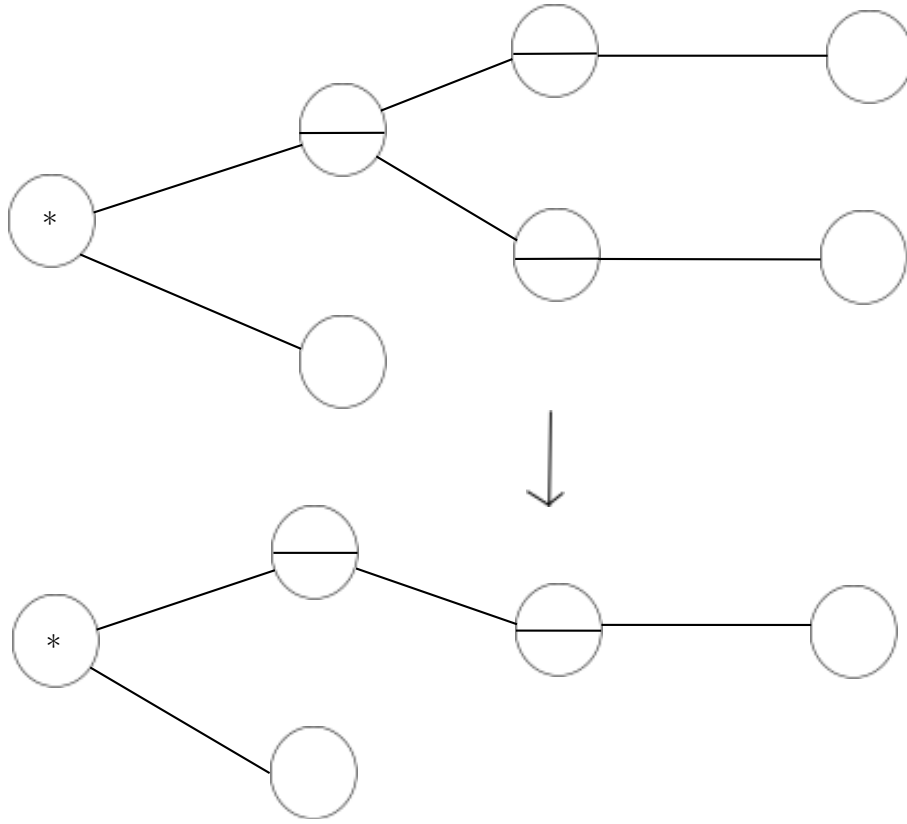


Figure 7: Move 3

vertices of T which are ordinary and have degree 1 (i.e., have no descendant.) We say that an ordinary vertex of T has *combinatorial depth* 0 if it is a leaf of T . Let T' be the admissible tree obtained from T by removing its leaves. Then the leaves of T' are declared to have combinatorial depth 1, and so on. Since T is finite, every ordinary vertex of T has combinatorial depth p for some $p \in \mathbf{N}$.

We now define the (topological) *depth* of a point in a compact, metrizable, totally disconnected space X . When $X = L(T)$ for a reduced admissible tree T , this will be *a posteriori* closely related to combinatorial depth in T , although we will not prove that fact, since we will not need it. The important point is that the definition is topological rather than combinatorial.

First we need some preliminary definitions. For brevity, we simply say that X is a *space* if it is a compact, metrizable, totally disconnected topological space. A *pointed space* is a pair (X, x) where X is a space and x a point of X .

Definition 3.4. Two pointed spaces (X, x) and (X', x') are *topologically equivalent* (or just *equivalent*) if there exists a triple (U, U', f) , where U is a neighborhood of x , U' is a neighborhood of x' , and f is a homeomorphism from U to U' such that $f(x) = x'$.

We sometimes say that x is *equivalent* to x' if there is no ambiguity concerning the ambient spaces.

We define inductively on a nonnegative integer p what it means for a point in a space to have *depth* p :

Definition 3.5. Let (X, x) be a pointed space. We say that x has *depth* 0 if it is isolated, or has a neighborhood homeomorphic to the Cantor set.

Assume that the phrase “to have depth k ” has been defined for all k less than some number p . Let Y be the set of points of X that do not have depth $< p$. Then a point $x \in X$ is said to have *depth* p if it belongs to Y , and either it is isolated in Y , or it has a neighborhood K in Y which is a Cantor set, and such that for every $y \in K$, the pairs (X, x) and (X, y) are equivalent.

We say that x has *finite depth* if it has depth p for some $p \in \mathbf{N}$. Otherwise it has *infinite depth*.

Remark 3.6. a) Let T be an admissible tree. We shall see shortly that every point of $L(T)$ has finite depth, which can be bounded from above by the combinatorics of T . This is a rather straightforward consequence of the finiteness of T . Obtaining lower bounds for the depth is more difficult and this is what much of the proof of Theorem 3.3 is about.

b) Let X be a space and p be a nonnegative integer. Then it is easy to see by induction on p that the set of points of X of depth at most p (resp. at least p) is open (resp. closed). We shall denote this subset by $X^{\leq p}$ (resp. $X^{\geq p}$.)

c) The notion of depth is similar to the Cantor-Bendixson rank; in fact, for countable spaces it is equal to the Cantor-Bendixson rank whenever it is finite (see below.)

We have the following useful lemma, which tells us that depth depends only on the local topology:

Lemma 3.7. *For every $p \in \mathbf{N}$ the following holds: let (X, x) and (X', x') be pointed spaces which are equivalent to each other and such that x has depth p ; then x' also has depth p .*

Proof. The proof is by induction on p . For $p = 0$ the statement is immediate from the definitions.

Suppose that the statement is true for all $p' < p$. Let (X, x) and (X', x') be equivalent pointed spaces such that x has depth p . Let (U, U', f) be a triple as in the definition of topological equivalence. Since $X^{\leq p}$ is open, we may, up to replacing U and U' by smaller neighborhoods, assume that U contains only points of depth $\leq p$.

If x' had depth $< p$, then applying the induction hypothesis and reversing the roles of X, X' , we could deduce that x has depth $< p$. Hence x' has depth at least p (possibly infinite.) Denote by V (resp. V') the subset of U (resp. U') consisting of points of depth $< p$. From the induction hypothesis, we know that f induces a bijection from V to V' .

By hypothesis, the point x belongs to $X^{\geq p}$, the point x' belongs to $X'^{\geq p}$, and the pairs $(X^{\geq p}, x)$ and $(X'^{\geq p}, x')$ are equivalent. If x is isolated in $X^{\geq p}$, then x' is isolated in $X'^{\geq p}$, so x' has depth p . Otherwise, there is a Cantor set $K \subset X^{\geq p}$ which is a neighborhood of x , and all of whose points are equivalent to x in X . We may assume that $K \subset U$ and take its image K' under f . Thus we have produced a Cantor set K' containing x' , and with the property that every $y' \in K'$ is equivalent to $f^{-1}(y')$, hence to x' , hence to x . This shows that x' has depth p . \square

Remark 3.8. As a by-product of the proof we see that if (X, x) and (X', x') are equivalent pairs, then we can choose the triple (U, U', f) so that U (resp. U') contains only points of depth $\leq p$, and f is depth-preserving.

Let us illustrate the notions of depth and topological equivalence on a few examples. First, when X does not contain any Cantor set (equivalently, when X is countable), the points of depth 0 are exactly the isolated points, the points of depth 1 are the accumulation points which are isolated among accumulation points, and so on. By taking appropriate countable ordinals with the order topology, one can produce for each $p \in \mathbf{N}$ examples where there exists a point of depth p , and all points have depth $\leq p$. One can also produce examples which have points of infinite depth:

Example 3.9. The ordinal $X = \omega^\omega + 1$ has points of arbitrarily large finite depths, and exactly one point of infinite depth. By Remark 3.14 below, there is no admissible tree T such that $L(T)$ is homeomorphic to X . This in turn implies that if $f : X \rightarrow S^2$ is an embedding, then $S^2 \setminus f(X)$ is a surface which is not an automaton surface.

We now turn to an example which does contain Cantor sets.

Example 3.10. Let T_6 be an admissible tree with four vertices $*, v_1, v_2, v_3$, where v_1 is the only son of $*$ and v_2, v_3 are sons of v_1 , and such that v_1, v_2 have type O and v_3 has type Θ . This is one of the simplest reduced trees.

We introduce the alphabet $\{a, b, c, d\}$ with the letter a corresponding to v_1 , the letter b to v_2 , and the letters c, d to v_3 . Then $L(T_6)$ is partitioned into three subsets $\{a^\infty\}$, U and V , where a^∞ is the infinite word $aaa \cdots$, the set U contains all words consisting of zero-or-more a 's followed by infinitely many b 's, and V contains all words consisting of zero-or-more a 's followed by an infinite word on the letters c, d .

The points of U are exactly the isolated points, which have depth 0. Each point of V is contained in some clopen Cantor set, thus also has depth 0. The point a^∞ has none of the above properties, so it does not have depth 0. Hence $L(T_6)^{\geq 1} = \{a^\infty\}$, and a^∞ , being isolated in $L(T_6)^{\geq 1}$, has depth 1 in $L(T_6)$.

Remark 3.11. In this example, the topological equivalence classes are exactly the sets $\{a^\infty\}$, U and V . If X is a countable space and $x \in X$ a point of depth 1, then the pair $(L(T_6), a^\infty)$ is not equivalent to the pair (X, x) . Indeed, a^∞ is the limit of a sequence of nonisolated points of depth 0, whereas x is not.

More generally, if T is an admissible tree and v a leaf of T , then the words of $L(T)$ which contain infinitely many letters corresponding to v have depth 0. The converse does not hold in general. For instance the space $L(T_5)$ considered in Subsection 3.1 is a Cantor set, so every point has depth 0. We shall see, however, that this phenomenon cannot happen if T is reduced; this will be an important step in the proof of Theorem 3.3.

3.3 Proof of Theorem 3.3

Let T be an admissible tree. Let v be an ordinary vertex of T . Recall that we denote by $T(v)$ the subtree generated by v (i.e. consisting of v and its descendants,) and by L_v the set of words in $L(T)$ which have infinitely many letters corresponding to v . In the sequel, we shall abusively denote by $L(T(v))$ the language associated to the admissible tree obtained from $T(v)$ by adding a root. Let us denote by S_v the subset of $L(T(v))$ consisting of elements which have at most finitely many letters corresponding to v . The following lemma shows that it makes sense to talk about the depth of an ordinary vertex, and that topological equivalence induces an equivalence relation between ordinary vertices of admissible trees.

Lemma 3.12. *Let T be an admissible tree. For every ordinary vertex v , any two elements of L_v are topologically equivalent. In particular, they have the same depth.*

Proof. Let v be an ordinary vertex of T . Let w, w' be words in L_v . By definition, w is equal to w_1w_2 where w_1 is a (possibly empty) finite word, and w_2 is an infinite word on the letter(s) associated to v . Similarly, w' has a decomposition $w'_1w'_2$.

Case 1 The vertex v has type O .

Let b_k be the letter associated to v . We have $w_2 = w'_2 = b_k^\infty$. If v is a leaf, then both w and w' are isolated, and thus equivalent. Otherwise, let U be the union of $\{w\}$ with the set of words of the form w_1w_5 with $w_5 \in S_v$. Likewise, let U' be the union of $\{w'\}$ with the set of words of the form w'_1w_5 with $w_5 \in S_v$. It follows from the definition of the topology on $L(T)$ that U (resp. U') is an open neighborhood of w (resp. w'). Moreover, the map $f : U \rightarrow U'$ defined by setting $f(w) := w'$, and for every $w_5 \in S_w$, $f(w_1w_5) := w'_1w_5$ is a homeomorphism. Hence w is equivalent to w' .

Case 2 The vertex v has type Θ .

Let b_k, b_l be the letters associated to v . Let T_{kl} be the rooted infinite dyadic tree with edges labeled by b_k, b_l . The set F of finite words on $\{b_k, b_l\}$ (i.e. the free monoid on $\{b_k, b_l\}$) can be identified with the set of finite injective paths in T_{kl} starting from the root. Likewise, the set K of infinite words on the same letters can be identified with the set of infinite injective paths in T_{kl} starting from the root. There exists an automorphism ϕ of T_{kl} (as a rooted tree) which takes every finite prefix of w_2 to the prefix of w'_2 of same length (cf. [12, Section 3].) This automorphism induces a self-homeomorphism (still denoted by ϕ) of the Cantor set K which takes w_2 to w'_2 .

We now argue as in Case 1. Let U (resp. U') be the set of words of the form w_1w_5 (resp. w'_1w_5) with $w_5 \in L(T(v))$. Then U is a neighborhood of w , U' is a neighborhood of w' , and we can define a homeomorphism $f : U \rightarrow U'$ in the following way: if $w_5 \in K$, then $f(w_1w_5) := w'_1\phi(w_5)$ (in particular, $f(w) = w'$ as required); otherwise w_5 is of the form w_3w_4 with $w_3 \in F$ and w_4 is some infinite word on the letters associated to descendants of v , and we set $f(w_1w_3w_4) := w'_1\phi(w_3)w_4$. \square

Remark 3.13. A variation on this argument proves that if T_1, T_2 are admissible trees, v_1 is an ordinary vertex of T_1 , and v_2 is an ordinary vertex of T_2 such that $T_1(v_1)$ is isomorphic to $T_2(v_2)$, then v_1 is equivalent to v_2 . The converse is not true in general; however, we shall see that it is true if both T_1 and T_2 are reduced. This will be a key step in the proof.

Remark 3.14. Let T be an admissible tree. If $x \in L_v$ for some leaf v of T , then x is isolated or belongs to some Cantor clopen subset of L . Thus v has depth 0. If v is an ordinary vertex of T which is not a leaf, and all descendants of v have finite depth, then v has finite depth, equal to M or $M + 1$, where M is the maximum of the depths of the descendants of v . This implies that every ordinary vertex has finite depth, and depth is nonincreasing along paths in the tree starting from the root and going to the leaves.

Lemma 3.15. *Let T be a reduced tree. Then the vertices of depth 0 of T are exactly its leaves.*

Proof. We have already remarked that the leaves have depth 0. Let us prove the converse by contradiction. Let T be a reduced tree, and v be a vertex of depth 0 which is not a leaf. Assume that v has minimal combinatorial depth among such vertices, i.e. is as close to the leaves as possible. All the descendants of v have depth 0, so by minimality, v has combinatorial depth 1.

If v has exactly one son v' , then v' must be of type Θ ; otherwise every point of L_v would be the limit of some sequence of isolated points, hence of nonzero depth. This contradicts Property (R₁).

Thus v has at least two sons. By Property (R₃), it has at most (hence exactly) two sons v', v'' , and these have different types. Now if, say, v' has type O, then again every point of L_v would be the limit of sequence of isolated points, giving a contradiction. \square

Proposition 3.16. *For every integer $p \geq 0$, the following assertions hold:*

- (A _{p}) *Let T be a reduced tree and v be a vertex of T of depth p . Then all descendants of v have depth $< p$.*
- (B _{p}) *Let T_1, T_2 be reduced trees. For $i = 1, 2$ let v_i be an ordinary vertex of T_i . Assume that for $i = 1, 2$, v_i has depth p , and all its descendants have depth $< p$. Further assume that v_1 is topologically equivalent to v_2 . Then $T_1(v_1)$ is isomorphic to $T_2(v_2)$.*

Proof. The proof is by double induction on p . First observe that (A₀) and (B₀) are direct consequences of Lemma 3.15. Our induction scheme is the following: prove (B₁), then (A₁), then (B₂), then (A₂) etc.

This reduces to two claims:

Claim 1 Assume $p \geq 1$. If (A_k) and (B_k) are true for all $k \leq p$, then (B_{p+1}) is true.

Let T_1, T_2, v_1, v_2 be as in the statement of (B_{p+1}) . From the induction hypothesis we know that v_1, v_2 have depth $p + 1$ and their sons have depth at most p . Furthermore, the grandsons of v_1, v_2 have depth at most $p - 1$.

Let v'_1 be a son of v_1 of depth k . For every point x of L_{v_1} , there exists a sequence of points of $L_{v'_1}$ which converges to x . Since v_1 is equivalent to v_2 , it follows that v_2 has a descendant v'_2 which is equivalent to v'_1 . We claim that v'_2 is a son of v_2 . Suppose not. Then there is a unique vertex v''_2 of T_2 which is a son of v_2 and an ancestor of v'_2 , and the depth of v'_2 is strictly between the depths of v_2 and v'_2 . Arguing as above, we see that v''_2 is equivalent to some descendant v''_1 of v_1 . By the induction hypothesis, the subtrees $T_1(v'_1)$ and $T_2(v'_2)$ are isomorphic to each other, and so are $T_1(v''_1)$ and $T_2(v''_2)$. This contradicts Property (R_2) .

To sum up, we have shown that for every son v'_1 of v_1 , there is a son v'_2 of v_2 such that v'_1 is equivalent to v'_2 . By the induction hypothesis, the subtrees $T_1(v'_1)$ and $T_2(v'_2)$ are isomorphic. By Property (R_3) , for $i = 1, 2$ we have the following property: if v', v'' are two distinct sons of v'_i , then $T_i(v')$ is not isomorphic to $T_i(v'')$. Hence we have produced a bijection between the set of sons of v_1 and the set of sons of v_2 which preserves the isomorphism type of the generated trees. This implies that $T_1(v_1)$ is isomorphic to $T_2(v_2)$. We thus have proved Claim 1.

Claim 2 Assume $p \geq 1$. If (A_k) is true for every $k < p$, and (B_k) is true for every $k \leq p$, then (A_p) is true.

Arguing by contradiction as in the proof of Lemma 3.15 and taking a counterexample (T, v) with v of minimal combinatorial depth, we may assume the following: v has depth p ; it has at least one son v_1 of depth p , possibly other sons v_2, \dots, v_r of depth p , and possibly other sons of depth $< p$. Furthermore, all descendants of the v_k 's have depth $< p$.

There exists a sequence of points of L_{v_1} converging to some point x of L_v . This shows that x is not isolated in $L(T)^{\geq p}$. Thus there exists a Cantor neighborhood $K \subset L(T)^{\geq p}$ all of whose points are equivalent in $L(T)$. This implies that v_1 is equivalent to v . By transitivity, all the v_k 's are equivalent to one another. By Property (B_p) , their associated subtrees are pairwise isomorphic. This shows that $r = 1$ (otherwise we would have a contradiction with Property (R_3) .)

We now show that v does not have a son of depth $< p$. Suppose that v' is such a son. We are assuming (A_k) for all $k < p$, so every descendant of v' has depth strictly less than the depth of v' . Since v is equivalent to v_1 ,

there must exist a son v_2 of v_1 such that v_2 is equivalent to v' . Applying (B_k) with k the depth of v' , we deduce that $T(v')$ is isomorphic to $T(v_2)$. This contradicts Property (R_2) .

Hence v_1 is the only son of v . By (R_1) , it must be of type O. Hence every point of L_{v_1} is isolated in $L(T)^{\geq p}$, but no point of L_v is. This contradicts equivalence between v and v_1 . Thus we have proved Claim 2. \square

We now turn to the proof of our main technical theorem. The proof is a variant of the argument used in the proof of Claim 1 above, with the difference that the root must be treated in a different way since Move 3 is not fully available for it.

Proof of Theorem 3.3. Let T_1, T_2 be reduced admissible trees which are topologically equivalent. Since an admissible tree is reduced to the root iff its associated space is empty, we can assume that this is not the case.

Let v_1 be a son of the root of T_1 . By topological equivalence, there exists an ordinary vertex v_2 of T_2 which is equivalent to v_1 . By Proposition 3.16, the subtrees $T_1(v_1)$ and $T_2(v_2)$ are isomorphic.

We claim that v_2 is a son of the root of T_2 . Suppose it is not. Let v'_2 be the only ancestor of v_2 which is a son of the root. By topological equivalence, there exists a vertex v'_1 of T_1 which is equivalent to v'_2 . By Proposition 3.16 again, $T_1(v'_1)$ is isomorphic to $T_2(v'_2)$. Hence $T_1(v'_1)$ has a proper subtree isomorphic to $T_1(v_1)$. This contradicts Property (R_2) .

We have shown that the isomorphism types of the trees generated by the sons of the roots of T_1, T_2 are the same. By rule (R_3) , each type of subtree with root of type Θ can occur at most once. This need not be true for types of subtrees with ancestor of type O. However, each son of the root of type O contributes a single point, so the number of such points is invariant under homeomorphism. This concludes the proof of Theorem 3.3. \square

4 The general case

In this section, we indicate how to adapt the proof from the planar case to the general case.

We begin with some simple observations. Let \mathcal{X} be a topological 2-automaton. Then every handle (resp. cross-cap) in $M(\mathcal{X})$ must come from a handle (resp. cross-cap) in some building block. Conversely, any handle (resp. cross-cap) in some building block X_k will give either infinitely many handles (resp. cross-caps) in $M(\mathcal{X})$ or just one, according to whether there is a loop on X_k or higher in the hierarchy, or there is no such loop.

It follows that we can determine the genus (finite or infinite) and orientability class of $M(\mathcal{X})$ directly on the automaton. In the sequel, we focus on the triple (E, E', E'') , using the machinery of decorated graphs. For brevity, we say that (E, E', E'') is a *triple of spaces* if E is a compact, metrizable, totally disconnected space, E' is a closed subset of E , and E'' is a closed subset of E' . Two such objects (E, E', E'') and (F, F', F'') are *equivalent* if there is a homeomorphism $f : E \rightarrow F$ such that $f(E') = F'$ and $f(E'') = F''$.

4.1 Decorated graphs and admissible trees

In order to suitably generalize our notion of decorated graph, we work with the extended set of symbols $S := \{*, *^h, *^c, O, O^h, O^c, \Theta, \Theta^h, \Theta^c\}$. The ‘meaning’ of these symbols will be explained next. For the moment, let us just say that the superscript h stands for ‘handle’, and ‘c’ is for cross-cap.

A *decorated graph* is a pair (G, f) , where G is a finite oriented graph without oriented cycles, endowed with a base vertex, called the *root*, and f is a map from the set of all vertices of G to the set S , in such a way that the root has image $*$, $*^h$, or $*^c$.

We would like to associate to a decorated graph G a triple of spaces $(L(G), L'(G), L''(G))$. However, there is a difficulty, which we now explain by means of an example.

Let G be a decorated tree with three vertices v_1, v_2, v_3 with v_1 the root, v_2 a son of v_1 , and v_3 a son of v_2 , and such that v_1 (resp. v_2 , resp. v_3) has type $*$ (resp. O , resp. O^h .)

In order to define $L(G)$, we ignore the superscripts, so in this case we assign the letter a to v_2 and the letter b to v_3 , and get a space with a single accumulation point a^∞ and a sequence of isolated points of the form $a^k b^\infty$. Since only v_2 has a superscript h , it would be tempting to say that $L'(G)$ consists of all points of $L(G)$ except a^∞ . However, this does not work, since $L'(G)$ must be closed. The reader is invited to find a corresponding automaton (the connection between automata and decorated graphs will be explained in the next subsection), draw a picture of its associated surface, and check that every end of this surface is indeed nonplanar.

In order to overcome this issue, we make the following definition:

Definition 4.1. A decorated graph G is *coherent* if it satisfies the following properties:

- i. No vertex of type O or Θ has a descendant of type $*^h$, $*^c$, O^h , O^c , Θ^h , or Θ^c .
- ii. No vertex of type O^h or Θ^h has a descendant of type $*^c$, O^c , or Θ^c .

Let G be a coherent decorated graph. We define a triple of spaces $(L(G), L'(G), L''(G))$ as follows: set $r := n_O + 2n_\Theta$, where n_O (resp. n_Θ) is the number of vertices of G of type belonging to the set $\{O, O^h, O^c\}$ (resp. of type belonging to $\{\Theta, \Theta^h, \Theta^c\}$). Consider the alphabet $B = \{b_1, \dots, b_r\}$ and assign to each vertex v of G zero, one or two letters according to its type. Then $L(G)$ is defined as a subspace of $B^{\mathbf{N}}$ in the same way as in the planar case, ignoring the superscripts. The subset $L'(G)$ is defined as the union of the L_v 's over all vertices v of type O^h , O^c , Θ^h or Θ^c , and $L''(G)$ is the union of the L_v 's over all vertices of type O^c or Θ^c .

Definition 4.2. An *admissible tree* is a decorated graph which has the following properties:

- i. It is a tree.
- ii. It is coherent.
- iii. All edges point away from the root.
- iv. The root has type $*$, and is the only vertex with this property.
- v. There are no vertices of type $*^h$ or $*^c$.

4.2 From automata to admissible trees

Let \mathcal{X} be a topological 2-automaton. First we form the decorated graph (G, f) from the directed graph underlying the automaton by assigning to each building block X_k a symbol in S , in the following way: if there is no loop around X_k , then we put $*$ if X_k is planar, $*^h$ if X_k is orientable, but not planar, and $*^c$ if X_k is nonorientable. If there is one loop around X_k , then we put O if X_k is planar, O^h if it is orientable, but not planar, and O^c if it is nonorientable. If there are two or more loops, we put Θ if X_k is planar, Θ^h if it is orientable but not planar, and Θ^c if it is nonorientable.

Then we modify G in order to make it coherent. For brevity we say that a vertex *has type* T^h if its type belongs to the set $\{*^h, O^h, \Theta^h\}$. Likewise, we say that a vertex has type T^c if its type belongs to the set $\{*^c, O^c, \Theta^c\}$.

The procedure for making G coherent has two steps.

Step 1 For every vertex v of type O (resp. Θ) that has at least one descendant of type T^h , but no descendant of type T^c , we change the type of v to O^c (resp. Θ^c).

Step 2 For every vertex v of type O (resp. Θ , resp. O^h , resp. Θ^h) that has at least one descendant of type T^c , we change the type of v to O^c (resp. Θ^c , resp. O^c , resp. Θ^c).

The result is a coherent graph G such that $(L(G), L'(G), L''(G))$ is equivalent to $(E(M(\mathcal{X})), E'(M(\mathcal{X})), E''(M(\mathcal{X})))$.

Next we modify the decorated graph in order to make it a tree, by removing the part which is not accessible from the root, and duplicating some of its parts. This is done exactly as in the planar case, and preserves coherence.

Finally, in order to make the tree admissible, change all $*^h$ and $*^c$ to $*$ if necessary, and proceed as in the planar case.

To sum up, we have obtained an admissible tree T such that the triple $(L(T), L'(T), L''(T))$ is equivalent to $(E(M(\mathcal{X})), E'(M(\mathcal{X})), E''(M(\mathcal{X})))$.

Suppose now that we have two topological 2-automata $\mathcal{X}_1, \mathcal{X}_2$ and wish to know whether their associated surfaces are homeomorphic. We first check whether or not they have same genus and orientability class, as explained above. If they do, then we still have to check the equivalence of the triples $(L(T_i), L'(T_i), L''(T_i))$.

4.3 Reduced trees and their classification

We say that two admissible trees T_1 and T_2 are *equivalent* if the triple $(L(T_1), L'(T_1), L''(T_1))$ is equivalent to $(L(T_2), L'(T_2), L''(T_2))$. We also have the obvious extension of notion of (combinatorial) isomorphism between two admissible trees. Again, isomorphism implies equivalence; the converse does not hold, so we need to define reduced trees.

This is done using three moves which extend the moves used in the planar case. Moves 2 and 3 are defined in the same way as in the planar case, the word ‘isomorphic’ being interpreted in the appropriate generalized way. Move 1 needs the following adjustment:

Move 1' Let v be a vertex of T of type Θ (resp. Θ^h , resp. Θ^c). Let v' be the father of v . Assume that v' is not the root, and that v is the only son of v' . Further assume that v' has type O or Θ (resp. O^h or Θ^h , resp. O^c or Θ^c .) Remove the edge between v and v' and replace v' by v .

Let T be an admissible tree. If none of Moves 1', 2, 3 can be performed,

we say that T is *reduced*. Then we have the following extension of Proposition 3.2, which is proved in the same way:

Proposition 4.3. *There is an algorithm which to any admissible tree T associates a reduced admissible tree equivalent to T .*

Theorem 3.3 extends to the following result:

Theorem 4.4. *Two reduced admissible trees T_1, T_2 are equivalent if and only if they are isomorphic.*

The proof of Theorem 4.4 is a straightforward extension of that of Theorem 3.3. We need to define topological equivalence and depth of points.

Let (X, X', X'') be a triple of spaces. We think of points of X as colored by one of three colors according to the partition $X = X'' \sqcup (X' \setminus X'') \sqcup (X \setminus X')$. Thus, if (X, X', X'') and (Y, Y', Y'') are triples of spaces, a homeomorphism $f : X \rightarrow Y$ induces an equivalence of triples iff it is color-preserving. If U is an open subset of X , then the restriction $f : U \rightarrow f(U)$ is still color-preserving. We say that a subset $A \subset X$ is *monochromatic* if all points of A belong to the same of these three subsets.

Definition 4.5. Let (X, X', X'') and (Y, Y', Y'') be triples of spaces. Let $x \in X$ and $y \in Y$. We say that x is *equivalent* to y if there exists a triple (U, V, f) where U is a neighborhood of x , V a neighborhood of y , and $f : U \rightarrow V$ is a color-preserving homeomorphism which takes x to y .

A point $x \in X$ has *depth* 0 if either x is isolated, or x has an open, monochromatic neighborhood homeomorphic to the Cantor set. Assuming we know what ‘to have depth k ’ means for all k less than some number p , we say that x has *depth* p if it belongs to the complement A of the set of points of depth $< p$, and either it is isolated in A , or it has an open neighborhood K in A which is a monochromatic Cantor set, and all of whose points are equivalent to x in X .

In order to motivate the previous definition, consider the following example:

Example 4.6. Let T_7 be an admissible tree with three vertices $*_0, v, w$ where v is the only son of $*_0$ and has type Θ^c , and w is the only son of v and has type Θ . In order to describe $L(T_7)$, we introduce two letters a, b corresponding to v and two letters c, d corresponding to w . Then $L(T_7) = U \cup V$ where U is the set of words containing only a 's and b 's, and V its complement.

Since V is a countable union of monochromatic Cantor sets, all of its points have depth 0. However, any point x of U has depth 1. Indeed,

although x certainly has open neighborhoods homeomorphic to the Cantor set (e.g. $L(T_7)$ itself), it does not have any monochromatic one, since x belongs to $L'(T_7)$, and there are sequences of points not in $L'(T_7)$ converging to x .

As in the planar case, one proves that two equivalent points have the same depth (cf. Lemma 3.7), that it makes sense to talk about the topological type of a vertex of an admissible tree (cf. Lemma 3.12), and that when the tree is reduced, the points of depth 0 are exactly the leaves (cf. Lemma 3.15.) The key proposition is the following extension of Proposition 3.16:

Proposition 4.7. *For every integer $p \geq 0$, the following assertions hold:*

- (A_p) *Let T be a reduced tree and v be a vertex of T of depth p . Then all descendants of v have depth $< p$.*
- (B_p) *Let T_1, T_2 be reduced trees. For $i = 1, 2$ let v_i be an ordinary vertex of T_i . Assume that for $i = 1, 2$, v_i has depth p , and all its descendants have depth $< p$. Further assume that v_1 is topologically equivalent to v_2 . Then $T_1(v_1)$ is isomorphic to $T_2(v_2)$.*

Proof. The only significant difference with the planar case being in the replacement of Move 1 by Move 1', we discuss the part of the proof concerned with that Move in detail, and give only a brief sketch for the rest.

As in the planar case, (A₀) and (B₀) are immediate from the characterization of points depth 0, and the induction is based on two claims:

Claim 1 Assume $p \geq 1$. If (A_k) and (B_k) are true for all $k \leq p$, then (B_{p+1}) is true.

Let T_1, T_2, v_1, v_2 be as in the statement of (B_{p+1}). From the induction hypothesis we know that v_1, v_2 have depth $p + 1$ and their sons have depth at most p . Furthermore, the grandsons of v_1, v_2 have depth at most $p - 1$.

Let v'_1 be a son of v_1 of depth k . From equivalence of v_1 and v_2 we see that v_2 has a descendant v'_2 which is equivalent to v'_1 . Using Property (R₂) and the induction hypothesis, we show that v'_2 is actually a son of v_2 . Using Property (R₃), we get a bijection from the set of sons of v_1 and the set of sons of v_2 which preserves the isomorphism type of the generated trees. This proves Claim 1.

Claim 2 Assume $p \geq 1$. If (A_k) is true for every $k < p$, and (B_k) is true for every $k \leq p$, then (A_p) is true.

Arguing by contradiction as in the proof of Lemma 3.15 and taking a counterexample (T, v) with v of minimal combinatorial depth, we may assume

the following: v has depth p ; it has at least one son v_1 of depth p , possibly other sons of depth p , and possibly other sons of depth $< p$. Furthermore, all descendants of the v_k 's have depth $< p$.

Let x be a point of L_v . Then x is not isolated in $L(T)^{\geq p}$, so there exists a monochromatic Cantor neighborhood $K \subset L(T)^{\geq p}$ all of whose points are equivalent in $L(T)$. Moreover, v_1 is equivalent to v . Using Properties (R₃) and (B_p) we see that v has only one son of depth p .

The possibility that v should have a son of depth $< p$ is ruled out as in the planar case. Hence v_1 is the only son of v . We now discuss according to the type of v_1 and show that each case leads to a contradiction.

Case 1 v_1 has type O, O^h, or O^c.

Then v has depth $p + 1$.

Case 2 v_1 has type Θ^c.

Then by coherence, v_1 has type O^c or Θ^c, so the tree is not reduced (using Move 1'.)

Case 3 v_1 has type Θ^h.

Then by coherence, v has type either in {O^h, Θ^h}, in which case the tree is not reduced, or in {O^c, Θ^c}, in which case v has depth $p + 1$.

Case 4 v_1 has type Θ.

Then v has type either in {O^h, Θ^h, O^c, Θ^c}, in which case the tree is not reduced, or in {O, Θ}, in which case v has depth $p + 1$. □

Finally, one deduces Theorem 4.4 from Proposition 4.7 in the same way that Theorem 3.3 was deduced from Proposition 3.16.

5 Concluding remarks

Remark 5.1. Our definition of a topological automaton is rather restrictive. It is possible to broaden it, for instance by removing the restriction that $k \leq l$ in the last condition. This creates technical problems, but does not seem to enlarge the class of manifolds significantly.

At one extreme, one may think of associating an n -dimensional manifold to a Turing machine. One simple-minded way to do this is to start with a n -disk; each time the Turing machine does something, add an n -annulus; if the machine stops, glue in a n -disk. Then the resulting n -manifold is compact

if and only if the Turing machine stops. Since the halting problem for Turing machines is undecidable, the homeomorphism problem for n -manifolds arising from this construction is undecidable.

When $n = 1$, given the fact that there are only two 1-manifolds up to homeomorphism, and that they are both automata 1-manifolds, it is tempting to dismiss this phenomenon as an artefact caused by the use of Turing machines. In higher dimensions, however, this should probably be taken seriously. Notions of *computable manifolds* have been proposed; we refer to work of Calvert-Miller [8] and Aguilar-Conde [1] and the references therein for further discussion.

It would be interesting to define intermediate classes between automata surfaces and computable surfaces and determine whether the homeomorphism problem—or other algorithmic problems—are decidable for them.

Remark 5.2. Defining the *complexity* of an automaton surface F as the number of ordinary vertices of the unique reduced admissible tree corresponding to F , it is possible to enumerate automata surfaces in order of increasing complexity in the spirit of S. Matveev’s enumeration of 3-manifolds [20]. We hope that this could be useful to test conjectures or search for a counterexample in a systematic way.

Unsurprisingly, the examples mentioned in the introduction have low complexity. The first infinite-type planar surface appearing in this enumeration that currently does not have a name has a Cantor set of ends, all of whose points are limits of sequences of isolated ends. Perhaps it should be called the Cantor flute surface?

Remark 5.3. For $n \geq 4$, the homeomorphism problem for automata n -manifolds is of course undecidable, since this is already the case for compact n -manifolds by Markov’s theorem. For $n = 3$ it is an open question; in fact it is already open in the case of 1-ended 3-manifolds (presented by topological 3-automata with only two states and two arrows). We refer the interested reader to the paper [15] for further discussion.

A Uncountably many surfaces

In this appendix we prove the following proposition.³

Proposition A.1. *There are uncountably many planar surfaces up to homeomorphism.*

³Since writing the first version of this article, we have found the reference [27]. We have decided to keep this appendix since it is referred to in the text.

Proof. We use a construction which was shown to us by Gilbert Levitt. Let C be a Cantor set embedded in $[0, 1] \times \{0\} \subset \mathbf{R}^2$. Fix an arbitrary injective convergent sequence $((x_n, 0))_{n \in \mathbf{N}}$ of points of C and let $(x_\infty, 0) \in C$ be its limit. For each natural number n , fix an increasing embedding $\iota_n : \omega^n + 1 \rightarrow [0, 1]$ such that 1 belongs to the image of ι_n ; define an embedding σ_n of $[0, 1]$ into \mathbf{R}^2 by setting $\sigma_n(t) = (x_n, (1 - t)/(n + 1))$; finally let X_n be the image of $\sigma_n \circ \iota_n$.

We now define a map $\phi : \{0, 1\}^{\mathbf{N}} \rightarrow \mathcal{P}(\mathbf{R}^2)$ by the formula

$$\phi((a_n)_{n \in \mathbf{N}}) := C \cup \bigcup_{n \in \mathbf{N}, a_n=1} X_n.$$

Note that for each $\mathbf{a} = (a_n)_{n \in \mathbf{N}}$, the space $\phi(\mathbf{a})$ is compact and totally disconnected. (For compactness, observe that C and every finite union of X_n are compact, while if (P_k) is a sequence of points of $\phi(\mathbf{a})$ that hits X_n for infinitely many n , then it admits a subsequence that converges to $(x_\infty, 0)$, since the diameter of X_n tends to 0 as n tends to infinity.)

Lemma A.2. *Let $\mathbf{a}, \mathbf{b} \in \{0, 1\}^{\mathbf{N}}$. If $\phi(\mathbf{a})$ and $\phi(\mathbf{b})$ are homeomorphic, then $\mathbf{a} = \mathbf{b}$.*

Proof. Let f be a homeomorphism from $\phi(\mathbf{a})$ to $\phi(\mathbf{b})$. Recall that a point in a totally disconnected space is called a *condensation point* if all of its neighborhoods are uncountable. Observe that the set of condensation points of $\phi(\mathbf{a})$ (resp. $\phi(\mathbf{b})$) is exactly C . Hence $f(C) = C$. The other points of $\phi(\mathbf{a})$ have finite Cantor-Bendixson rank, and the rank is preserved by f , i.e. f sends isolated points to isolated points, limits of sequences of isolated points to similar points etc. Thus if $a_n = 1$ for some n , then $f((x_n, 0)) = (x_n, 0)$ and $b_n = 1$. Conversely, if $b_n = 1$ then $a_n = 1$. \square

Proposition A.1 now follows since for every $\mathbf{a} \in \{0, 1\}^{\mathbf{N}}$ the surface $\mathbf{R}^2 \setminus \phi(\mathbf{a})$ has $\phi(\mathbf{a}) \sqcup \{*\}$ as space of ends. \square

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