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A CANONICAL TREE DECOMPOSITION FOR ORDER TYPES, AND SOME APPLICATIONS

MATHILDE BOUVEL, VALENTIN FÉRAY, XAVIER GOAOC, AND FLORENT KOECHLIN

ABSTRACT. We introduce and study a notion of decomposition of planar point sets (or rather of their chirotopes) as trees decorated by smaller chirotopes. This decomposition is based on the concept of mutually avoiding sets (which we rephrase as *modules*), and adapts in some sense the modular decomposition of graphs in the world of chirotopes. The associated tree always exists and is unique up to some appropriate constraints. We also show how to compute the number of triangulations of a chirotope efficiently, starting from its tree and the (weighted) numbers of triangulations of its parts.

Keywords: Order type, chirotopes, modular decomposition, counting triangulations, mutually avoiding point sets, generating functions, rewriting systems.

1. INTRODUCTION

Any finite sequence $P = (p_1, p_2, \dots, p_n)$ of points in \mathbb{R}^2 gives rise to a number of combinatorial data such as the subset of extremal points, or the set of crossing-free matchings of P . Often, such structures depend on the coordinates of the points of P only through a limited number of predicates, so one can replace the (geometric) point sequence by a purely combinatorial object. For instance, the convex hulls and crossing-free matchings are determined by the map sending every triple $1 \leq i, j, k \leq n$ of indices to the orientation (positive, negative or flat) of the triangle $p_i p_j p_k$. For every n , there are finitely many such maps (called *chirotope* or *order types*, see below) and they have been extensively investigated, see *e.g.* the book of Björner et al. [BLVS⁺99].

In this paper we examine chirotopes from the point of view of the *modular decomposition method*, which is an important tool in algorithmic graph theory (see *e.g.* [HP10]). To this end, we define a notion of module in chirotopes, as a partition of the chirotope into two sets, such that no line going through two points of the same set separates points of the other set; see Figure 1. This is reminiscent of the concept of *mutually avoiding sets*, previously considered in the literature, though we use here a different perspective on such sets (see discussion and references at the end of Section 1.1). Using this notion of modules, we encode a chirotope by a tree whose nodes are themselves decorated by chirotopes, each point appearing in a single node, so that subtrees joined by an edge encode modules of the original point set.

The main contributions presented here are the following.

- We introduce a notion of *canonical chirotope tree* such that each chirotope is represented by a unique canonical chirotope tree.
- We show that the proportion of realizable chirotopes that are indecomposable tends to 1.
- We show how the number of triangulations of a chirotope given by a chirotope tree can be computed from the (weighted) numbers of triangulations of its nodes.

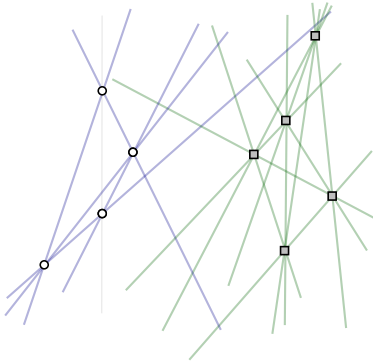


FIGURE 1. A set of 9 points which can be decomposed into two modules of size 4 and 5 points.

The first and third results also hold in the more general framework of abstract chirotopes (function mapping triples of elements either to *clockwise* or *counterclockwise*, satisfying some axioms, but not necessarily corresponding to a set of points in \mathbb{R}^2 ; see details below). We offer in Section 1.4 several new open questions raised by this concept of chirotope tree decompositions.

Throughout the paper, all point sets are finite, planar and in general position, implying in particular that no three points are ever aligned. Adapting our construction to point sets in higher dimension would need considerable work, as our definitions really use the fact that the chirotope is a map of three arguments (in general, the chirotope of a point set in \mathbb{R}^d takes $d + 1$ arguments). On the other hand, we believe that most results could be adapted to point sets that are not necessarily in general position, but we decided not to discuss such cases for simplicity.

Given a finite set X , we write $(X)_3$ for the set of triples (x, y, z) of distinct elements in X . We write $[n]$ for the set $\{1, 2, \dots, n\}$.

1.1. Context and motivation. Let us first provide some context on the objects, methods and questions that we consider.

Realizable chirotopes. The *chirotope* of a set $\mathcal{P} = \{\mathcal{P}_\ell\}_{\ell \in X}$ of points in general position labeled by X is the function

$$\chi_{\mathcal{P}} : \begin{cases} (X)_3 & \rightarrow \{-1, +1\} \\ (x, y, z) & \mapsto \begin{cases} +1 & \text{if } \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_z \text{ are in counterclockwise order,} \\ -1 & \text{if } \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_z \text{ are in clockwise order.} \end{cases} \end{cases}$$

This function is sometimes called the *labeled order type* of the point set [GP83]. We say that chirotopes of point sets are *realizable* to distinguish them from their combinatorial (abstract) generalization.

Representing realizable chirotopes. Deciding if a given function $(X)_3 \rightarrow \{-1, 1\}$ is a realizable chirotope turns out to be a challenging problem: it is equivalent to the existential theory of the reals [BLVS⁺99, Theorem 8.7.2] and NP-hard [Sho91]. Whether this problem is in NP is open, and, interestingly, a positive answer is equivalent to the existence of a (pseudorandom) generator that produces a realizable chirotope of size n in time polynomial in n while

ensuring that every element has nonzero probability¹. The space of realizable chirotopes is therefore difficult to explore, which makes it hard to test efficiently some conjectures in discrete geometry or to check experimentally a geometric algorithm in the exact geometric computing paradigm [SY17]. The present work grew out of an attempt to devise new ways of constructing and representing (large) realizable chirotopes.

Abstract chirotopes. As mentioned above, most of our results hold in a more general setting. Let us define a *chirotope* on a finite set X as a function $\chi : (X)_3 \rightarrow \{-1, 1\}$ that satisfies the following properties:

(symmetry): for any distinct $x, y, z \in X$,

$$(1) \quad \chi(x, y, z) = \chi(y, z, x) = \chi(z, x, y) = -\chi(z, y, x) = -\chi(y, x, z) = -\chi(x, z, y);$$

(interiority): for any distinct $t, x, y, z \in X$,

$$(2) \quad \chi(t, y, z) = \chi(x, t, z) = \chi(x, y, t) = 1 \quad \Rightarrow \quad \chi(x, y, z) = 1;$$

(transitivity): for any distinct $s, t, x, y, z \in X$,

$$(3) \quad \chi(t, s, x) = \chi(t, s, y) = \chi(t, s, z) = \chi(x, y, t) = \chi(y, z, t) = 1 \quad \Rightarrow \quad \chi(x, z, t) = 1.$$

It can be proven that any realizable chirotope is a chirotope. We may use the term *abstract chirotope* to mean a chirotope that is not necessarily realizable. Lastly, for readers acquainted with matroid theory, let us note that abstract chirotopes are in correspondence with the relabeling classes of acyclic uniform oriented matroids of rank 3 [BLVS⁺99, Knu92].

Notions and properties for point sets that can be expressed only through orientations of triples of points generalize to abstract chirotopes. For example, an element $x \in X$ is *extreme* in a chirotope χ on X if there exists $y \in X \setminus \{x\}$ such that $\chi(x, y, z)$ is the same for all $z \in X \setminus \{x, y\}$. With this definition, extreme elements of realizable chirotopes correspond precisely to the vertices of their convex hulls.

Triangulations. Counting the triangulations supported by a given set of n points is a classical problem in computational geometry (see the discussion in [MM16]). The fastest known algorithm is due to Marx and Miltzow [MM16] and has complexity $\mathcal{O}(n^{(11+o(1))\sqrt{n}})$. It is quite involved, and a simpler solution, due to Alvarez and Seidel [AS13], runs in time $\mathcal{O}(n^2 2^n)$.

Given a point set $P = \{\mathcal{P}_\ell\}_{\ell \in X}$, two segments $\mathcal{P}_x \mathcal{P}_y$ and $\mathcal{P}_z \mathcal{P}_t$ with distinct endpoints in P cross if and only if $\chi_P(x, y, z) = -\chi_P(x, y, t)$ and $\chi_P(z, t, x) = -\chi_P(z, t, y)$. We can therefore define the crossing of segments for abstract chirotopes. A *segment* in a chirotope χ on X is a pair of elements of X , and the segments xy and zt *cross in* χ if they satisfy the above condition. A *triangulation* of χ is an inclusion-maximal family of segments such that no two cross in χ . The algorithm of Alvarez and Seidel easily generalizes to abstract chirotopes.

¹A random generator can serve as a verifier for the problem, with the random bitstring as the certificate. Conversely, we can run a random function and a random certificate through the verifier; if accepted, we return that function, otherwise we return a fixed chirotope, e.g. n points in convex position.

Modular decomposition. The gist of modular decomposition is to break down discrete structures to allow efficient recursion. These decompositions usually start by partitioning the elements of the structure so that any two elements in a part (also called a *module*) are indistinguishable “from the outside”. Choosing this partition as coarse as possible while being nontrivial, and iterating the decomposition within each part yields a decomposition tree. These ideas originated in graph theory [Gal67], where modules gather vertices with the same neighbors outside of the module, and several hereditary classes of graphs have well-behaved modular decompositions, *e.g.* comparability graphs, permutation graphs and cographs [HP10]. We also note that a variant of modular decomposition, called split decomposition, has been thoroughly studied in graph theory; see, *e.g.*, [Rao08]. Interestingly, our chirotope tree decomposition share some features with both the modular and split decompositions: our notion of modules is closer to that of modules in graph than to that of split, but our decomposition tree are unrooted and unordered, as split decomposition trees.

Other examples of structures for which modular decompositions were developed include boolean functions, set systems and permutations [MR84, AA05]. Each structure requires an ad hoc analysis that the proposed notion of modules leads to a well-defined decomposition, where each object has a unique decomposition tree. The proportion of objects with nontrivial decomposition is often vanishingly small. Nevertheless, such decompositions proved useful *e.g.* in devising fixed-parameter polynomial algorithms for hard algorithmic problems [HP10, §7] or in studying subclasses of objects [BBR11, CFL17].

Mutual avoiding point sets. Let us discuss the link with the notion of mutually avoiding subsets, as considered in the litterature. Two planar point sets are *mutually avoiding* if no line through two points from one set separates two points from the other set. Any set of n points in general position contains two mutually avoiding subsets of size $\Omega(\sqrt{n})$ each [AEG⁺94], a bound that is asymptotically best possible [Val97], and smaller mutually avoiding subsets are actually abundant [SZ23, Theorem 1.3]. Mutually avoiding subsets have been applied to the study of crossing families [AEG⁺94] and empty k -gons [Val97].

The *modules* we consider here are mutually avoiding subsets that cover the whole chirotope. Our work goes in a different direction than the above cited works in that they are looking for mutually avoiding subsets *contained* in a chirotope, whereas we look for mutually avoiding subsets that *decompose* a chirotope. Our analysis thus applies to a more restricted class of chirotopes, the (highly) decomposable ones, but gives more information on their structure.

1.2. Our results. We now define our decomposition and state our main results.

Bowtie decomposition. The fact that a point set can be split into two modules P and Q has two interesting consequences at the level of chirotopes. First, the chirotope on point set $P \cup \{p^*\}$, which we denote $\chi_{P \cup \{p^*\}}$, is, up to relabeling, independent of the choice of the point $p^* \in Q$. Second, the chirotope $\chi_{P \cup Q}$ on point set $P \cup Q$ is completely determined by the chirotopes $\chi_{P \cup \{p^*\}}$ and $\chi_{Q \cup \{q^*\}}$ for any choices of $p^* \in Q$ and $q^* \in P$. We can use this to decompose $\chi_{P \cup Q}$ in terms of $\chi_{P \cup \{p^*\}}$ and $\chi_{Q \cup \{q^*\}}$.

Let us express this decomposition. Throughout the paper, we call *sign function* on a set X , a function $(X)_3 \rightarrow \{-1, +1\}$ that is only required to satisfy the symmetry condition (1). Let X and Y be disjoint sets, and let $x^* \notin X$ and $y^* \notin Y$. Given two sign functions χ on $X \cup \{x^*\}$ and ξ on $Y \cup \{y^*\}$, we define the *bowtie* $\kappa \stackrel{\text{def}}{=} \chi_{x^*} \bowtie_{y^*} \xi$ as the sign function on $X \cup Y$

satisfying:

$$(4) \quad \begin{cases} \kappa(x_1, x_2, x_3) = \chi(x_1, x_2, x_3) & \text{if } x_1, x_2, x_3 \text{ are all in } X; \\ \kappa(x_1, x_2, y) = \chi(x_1, x_2, x^*) & \text{if } x_1, x_2 \text{ are in } X \text{ and } y \text{ is in } Y; \\ \kappa(x, y_2, y_3) = \xi(y^*, y_2, y_3) & \text{if } x \text{ is in } X \text{ and } y_2, y_3 \text{ are in } Y; \\ \kappa(y_1, y_2, y_3) = \xi(y_1, y_2, y_3) & \text{if } y_1, y_2, y_3 \text{ are all in } Y. \end{cases}$$

With the symmetry condition (1), this completely defines κ on $(X \cup Y)_3$. Informally, this amounts to computing the sign function as in χ (resp. ξ) if the majority of the elements come from X (resp. Y), possibly replacing the only element outside X (resp. Y) by x^* (resp. y^*).

Note that for disjoint modules P and Q of $P \cup Q$, we have $\chi_{P \cup Q} = \chi_{P \cup \{p^*\}} \bowtie_{q^*} \chi_{Q \cup \{q^*\}}$ for any choice of $p^* \in Q$ and $q^* \in P$.

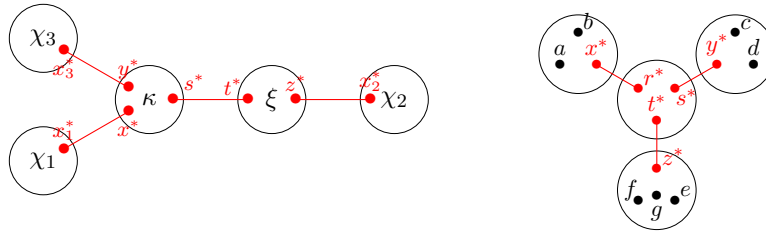
Bowtie products. While defined on sign functions, the bowtie operator also allows to combine smaller chirotopes into larger ones. This works under the following conditions.

Proposition 1.1. *Let X and Y be disjoint sets, with $|X|, |Y| \geq 2$, and let $x^* \notin X$ and $y^* \notin Y$. Let χ and ξ be chirotopes on $X \cup \{x^*\}$ and $Y \cup \{y^*\}$.*

- (i) $\chi_{x^* \bowtie_{y^*}} \xi$ is a chirotope if and only if x^* and y^* are extreme in χ and ξ .
- (ii) $\chi_{x^* \bowtie_{y^*}} \xi$ is a realizable chirotope if and only if χ and ξ are realizable and x^* and y^* are extreme in χ and ξ .
- (iii) If $\chi_{x^* \bowtie_{y^*}} \xi$ is a chirotope, then the extreme elements of $\chi_{x^* \bowtie_{y^*}} \xi$ are the elements of $X \cup Y$ extreme in χ or in ξ .

The above proposition is proved in Section 3.1. We also show (Proposition 3.11) that \bowtie has associativity and commutativity properties. We say that a chirotope κ is *decomposable* if there exist chirotopes χ and ξ , each on a strictly smaller set, such that $\kappa = \chi_{x^* \bowtie_{y^*}} \xi$. If no such χ and ξ exist we say that κ is *indecomposable*.

Chirotope trees. Formally, a *chirotope tree* is a tree whose nodes are decorated with chirotopes on disjoint ground sets of size at least 3, and whose edges select an element in the ground set of each of their extremities so that (i) no element is selected more than once, and (ii) each selected element is extreme in its chirotope. Any element selected by an edge of the chirotope tree is called a *proxy*; for illustration purposes, we write the decorating chirotope as the label of the node (if it is realizable, we may draw a realization) and we draw edges of the tree and proxy elements in red, with the edge connecting the proxy elements. Here are two examples.



Chirotope of a chirotope tree. To any chirotope tree $T = (V_T, E_T)$ we associate a sign function χ_T as follows. For a node v of T , let χ_v denote the chirotope decorating it, and let X_v denote the set of nonproxy elements of the ground set of χ_v . For $x \in \cup_{w \in V_T} X_w$, the *representative* $R_T(x, v)$ of x in the node v is x if $x \in X_v$, and otherwise it is the label of the proxy selected in χ_v by the first edge on the path in T from v to the node containing x . For distinct

$x, y, z \in \cup_{w \in V_T} X_w$, we let $v(x, y, z)$ be the intersection node of the paths in T from x to y ,² from y to z and from x to z (if two or three elements among x, y, z are in the same X_w , then we set $v(x, y, z) = w$). We define a sign function χ_T on $\cup_{w \in V_T} X_w$ by³

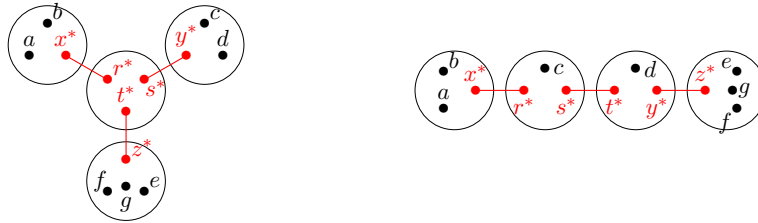
$$(5) \quad \chi_T(x, y, z) \stackrel{\text{def}}{=} \chi_v(R(x, v), R(y, v), R(z, v)) \quad \text{for } v = v(x, y, z).$$

For example, if T is the chirotope tree on the right in the previous picture we have $\chi_T(a, c, f) = \kappa(r^*, s^*, t^*) = -1$, where κ is the chirotope decorating the central node. The definition of χ_T ensures the following property: if removing an edge in a tree T produces two subtrees T_1 and T_2 , then χ_T is the bowtie product of χ_{T_1} and χ_{T_2} (see Lemma 4.1). Furthermore, this notion behaves well with those of chirotopes and of realizability.

Proposition 1.2. *For any chirotope tree T , χ_T is a chirotope. Moreover, if all chirotopes decorating a node in T are realizable, then χ_T is realizable.*

By Proposition 1.1(iii), the extreme elements of χ_T are the nonproxy extreme elements of its decorating chirotopes. Hence, if T has k nodes, then χ_T has at least $k + 2$ extreme elements. Indeed, as each node has at least 3 extreme elements and T has exactly $k - 1$ edges, therefore $2(k - 1)$ proxies, there are at least $3k - 2(k - 1) = k + 2$ extreme elements in χ_T .

Canonical chirotope trees. Let us consider a chirotope tree T and a node v of T . If $\chi_v = \kappa_{s^*} \bowtie_{t^*} \xi$ is decomposable, then v can be replaced by two nodes decorated with κ and ξ , connected by an edge, so that the resulting chirotope tree T' satisfies $\chi_{T'} = \chi_T$ (see Section 4.2 for details). Hence, a chirotope χ corresponds to many chirotope trees, at least one of which is decorated only by indecomposable chirotopes. However, even requesting the chirotopes decorating the nodes to be indecomposable does not ensure that a unique chirotope tree represents a given chirotope. Here are two trees with only indecomposable decorations and the same associated chirotope on $\{a, b, c, d, e, f, g\}$.



It turns out that the *convex* chirotopes (a chirotope is convex if all its elements are extreme) are the only source of redundancy; in the above example, the chirotope is not convex, but its restriction to $\{a, b, c, d, e\}$ is. This leads us to define a chirotope tree as *canonical* if every node is decorated by a convex or indecomposable chirotope, and if no edge connects two nodes decorated by convex chirotopes.

Theorem 1.3. *For any chirotope χ , there is a unique canonical chirotope tree T with $\chi_T = \chi$.*

We prove this theorem in Section 4.3.

²Rather: from the node containing x to the node containing y , and similarly for (y, z) and (x, z) .

³ $R(x, v)$ replaces, *i.e.* serves as a proxy for, x for computing the sign $\chi_T(x, y, z)$ in the node v .

Counting triangulations. Let \mathcal{T}_κ denote the set of triangulations of a chirotope κ on X . Given $x^* \in X$, we let $P_{\kappa, x^*}(s) = \sum_{T \in \mathcal{T}_\kappa} s^{\deg_T(x^*)}$ denote the generating polynomial of the triangulations of κ , marking the degree of x^* . We prove (see Proposition 8.1) that if $\kappa = \chi_{x^*} \bowtie_{y^*} \xi$, then

$$(6) \quad |\mathcal{T}_\kappa| = \sum_{\substack{T' \in \mathcal{T}_\chi \\ T'' \in \mathcal{T}_\xi}} \binom{\deg_{T'}(x^*) + \deg_{T''}(y^*) - 2}{\deg_{T'}(x^*) - 1} = \sum_{a, b \geq 2} \binom{a+b-2}{a-1} [s^a] P_{\chi, x^*}(s) [t^b] P_{\xi, y^*}(t).$$

More generally, given a chirotope tree T , we can compute the number of triangulations of χ_T given, for each node v of T , a collection of generating polynomials Q_v^U for the triangulations of χ_v . These polynomials are indexed by the possible restriction U of the triangulation to the set Y_v of proxy elements of χ_v , and Q_v^U enumerate the triangulations with restriction U , keeping track of the degrees of all proxy elements (it is a multivariate polynomial with one variable for each proxy element). This general principle can be used on the one hand to find explicit formulas for the number of triangulations of some families of point sets, and on the other hand to design an effective algorithm to compute the number of triangulations of a chirotope, *given its decomposition tree*.

In Section 8.3, we consider a family \mathcal{F} of chirotope trees, whose underlying trees are n -paths, with all nodes decorated by one of two chirotopes on four elements (see Section 8.3). For the chirotopes of such chirotope paths, we obtain a recurrence formula for the generating polynomials of triangulations. These recurrences can be solved via the *kernel method* from analytic combinatorics, leading to one and the same closed formula for the number of triangulations of any chirotope represented by a path in \mathcal{F} (Proposition 8.3).

For the algorithmic aspect, we recall that, if, for a node v , χ_v has n_v elements, k of which are proxies, we have to compute a family Q_v^U of k -variate polynomials. We show that the number of polynomials in the family is bounded by $\mathcal{O}(A^k)$, with $A = 6 + 4\sqrt{2} \approx 11.66$; see Lemma 9.1. In addition, these polynomials can be computed in time $\mathcal{O}(A^k 2^{n_v} n_v^{k+2})$ by a simple modification of the Alvarez-Seidel algorithm [AS13]. The following result gives a precise bound on the complexity of counting triangulations, given the polynomials Q_v^U , for all nodes v of T and all noncrossing configurations U of edges on the proxy elements of v .

Proposition 1.4. *Let T be a chirotope tree with m edges, in which each node has degree at most k , and such that χ_T has size n . The number of triangulations of χ_T can be computed from the collections of polynomials $\{Q_v^U\}$ in time $\mathcal{O}(A^k n^{2k} m)$ in the Real-RAM model.⁴*

We implemented our method for a family of trees of degree at most 3 (see Section 9.3 for the details and access to the code). This implementation is merely a proof of concept, and it is beyond the scope of this paper to optimize it or benchmark it, but we can mention that on a laptop, it took a few seconds to count the triangulations of the example of Figure 13 page 51, that assembles a set of 254 points from decorating chirotopes of size 9, and that it handles random ternary chirotope trees with 146 nodes (1024 elements) in less than 5 minutes.

Proportion of decomposable chirotopes among realizable ones. Finally, let t_n denote the number of *realizable* chirotopes of size n and d_n denote the number of those that are decomposable. Our last result gives a precise estimates of d_n , in terms of t_n and t_{n-1} . In particular,

⁴In particular, this model allows arithmetic computation on arbitrary reals in time $\mathcal{O}(1)$. We refer to Erickson et al. [EVDHM22] for a formal definition of the Real-RAM model of computation.

as announced earlier, almost all realizable chirotopes are indecomposable (as for graphs with respect to the split or modular decompositions).

Theorem 1.5. *For large n , we have $3(n - \mathcal{O}(n^{-2}))t_{n-1} \leq d_n \leq \mathcal{O}(n^{-3})t_n$.*

Since t_n is of order n^{4n} , we expect t_{n-1}/t_n to behave as $\Theta(n^{-4})$ (but only an upper bound of this order is known). If this is indeed the case, the lower bound in our theorem behaves as $\Theta(n^{-3})t_n$, i.e. our bounds are tight up to multiplicative constants. Note that we do not have an analogue result for abstract chirotopes.

1.3. Discussion and related work.

Originality with respect to other modular decomposition theories. The theory of modular decomposition [MR84] does not offer any unifying model that specializes to various structures, but rather proposes general guidelines for developing such a theory. The implementation of these guidelines usually requires analyses that are specific to the objects at hand, and the case of chirotopes is no exception: almost every step requires some specific geometric idea(s).

Among realizable chirotopes, the proportion of indecomposable ones tends to 1. This is analogue to the well-known fact that most graphs are *prime graphs* for the modular decomposition. The proof is, however, much more difficult and interesting than in the graph case because (i) the number of realizable chirotopes grows slower than the number of graphs on n vertices and (ii) this number is only known up to exponential corrections. In particular, our proof uses some geometric constructions and is specific to the realizable setting.

Let us also mention that even if most graphs are prime graphs for the modular decomposition, this decomposition has turned out useful in many ways: design of efficient algorithms [HP10], enumeration and study of specific classes [BBF⁺22], ... Similarly, for chirotopes, our tree decomposition allows one for example to construct families of large chirotopes for which enumerating triangulations is algorithmically easy (see Section 7).

Recursive constructions of point sets. Recursive constructions of point sets abound in discrete geometry, and with some care several of them directly translate into recursively defined canonical chirotope trees (e.g. Horton sets). The closest predecessor to (and inspiration for) this work is the recursive decomposition of *chains* used by Rutschmann and Wettstein [RW23, Theorem 15] to produce a new lower bound on the maximal number of triangulations of a set of n points in the plane. Their representation applies to a smaller subset of order types (the *chains*, of which there are only exponentially many of size n), and decomposes every chain into parts of constant size. It allows to count the triangulations of a chain of size n in $\mathcal{O}(n^2)$ time based on ideas similar to the proof of Eq. (6), but tailored to the setting of chains. Another notion of *decomposability* of point sets was proposed by Balko et al. [BKLP17] to study Ramsey-type questions; that notion combines conditions on orientations and comparisons between x -coordinates, which more restrictive than our setting, and these authors focus on point sets that are fully decomposable, in the sense that every basic building block consists of a single point.

Counting triangulations. It is interesting to compare our method for counting triangulations to the experimental results for general point sets of Alvarez and Seidel [AS13], and for chains by Rutschmann and Wettstein [RW23]. An extensive comparison is beyond the scope of this paper, all the more that the methods apply to different classes of point sets and operate on different types of input. We nevertheless note the following points.

- The general method of Alvarez and Seidel takes as input an arbitrary point set. They tested three methods on high-memory hardware for various types of point sets, and none could handle any example of size 35 or more in less than 10 minutes.
- Our method takes as input a (chirotope presented by a) chirotope tree. We tested it on randomly-generated trees with decorating chirotopes of size 9 summing up to sets of ~ 1000 points; it took less than 5 minutes on a laptop to count the triangulations.
- The method of Rutschmann and Wettstein is based on formulas specific to the class of chains and could handle examples of size 2^{21} .

To us, this is an evidence that our approach is relevant for point sets that are highly decomposable. This is similar to some of the usual benefits of modular decomposition: some hard problems enjoy simple and effective solutions for instances that decompose well.

1.4. Perspectives. Let us conclude this introduction by sampling some directions of enquiry that this work opens.

- (1) Here we used canonical chirotope trees as a tool to put together large chirotopes out of smaller ones, and we therefore considered the chirotope tree as given. One could, however, start with some point set and set out to compute the decomposition tree. This raises the following computational questions. How efficiently can one find a partition of a given (realizable) chirotope into two mutually avoiding parts or decide that none exists? Or, going even further, how efficiently can one compute the canonical chirotope tree of a given (realizable) chirotope?
- (2) Can crossing-free structures besides triangulations be counted efficiently from the chirotope tree? What about other statistics of a chirotope such as the number of k -sets or the number of crossing pairs?
- (3) What is the proportion of indecomposable (abstract) chirotopes? (Our proof of Theorem 1.5 uses that $t_n/t_{n-1} \geq \Omega(n^4)$ and we are not aware of an analogue of this in the abstract setting.)

1.5. Outline of the paper. The paper is organized as follows.

- Sections 2 and 3 introduce the notions of modules and bowties, and in particular proves Proposition 1.1.
- Section 4 introduces chirotope trees and canonical chirotope trees, and in particular proves Proposition 1.2 and Theorem 1.3.
- Sections 5 and 6 estimate the number of decomposable chirotopes, and in particular prove Theorem 1.5.
- Sections 7, 8 and 9 show how to count the triangulations of a chirotope presented by a chirotope tree, and in particular prove Proposition 1.4.

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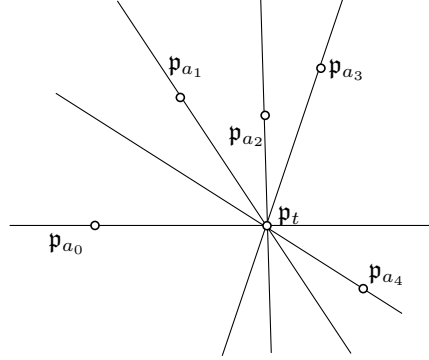


FIGURE 2. Illustration of the proof of Lemma 2.1 for a realizable chirotope. The points \mathbf{p}_{a_i} corresponding to the elements a_i are seen from \mathbf{p}_t in clockwise order (since $\chi(t, a_i, a_{i+1}) = -1$ for all i by construction). Given \mathbf{p}_{a_i} , either the line $\mathbf{p}_{a_i}\mathbf{p}_t$ splits the point set $\{\mathbf{p}_{a_j}, j < i\}$ in two non trivial parts (which happens here for $i = 4$), and we can find a triangle containing \mathbf{p}_t (namely $\mathbf{p}_{a_0}\mathbf{p}_{a_3}\mathbf{p}_{a_4}$ in the picture), or all points $\{\mathbf{p}_{a_j}, j < i\}$ are on the same side of $\mathbf{p}_{a_i}\mathbf{p}_t$ (which happens for $i = 2$ and $i = 3$ here), and we let $\mathbf{p}_{a_{i+1}}$ be any point on the other side (there is necessarily one, since \mathbf{p}_t is not extreme). Since the set is finite, we cannot be always in the latter situation and we eventually find a triangle containing \mathbf{p}_t .

2. PRELIMINARY: SOME PROPERTIES OF EXTREME POINTS

Recall that an element $x \in X$ is *extreme* in a chirotope χ on X if there exists $y \in X \setminus \{x\}$ such that $\chi(x, y, z)$ is the same for all $z \in X \setminus \{x, y\}$. We start with a standard extension to abstract chirotopes of Carathéodory's theorem: any point not extreme in a set X is contained in a triangle with vertices in X (see [BLVS⁺99, Theorem 9.2.1(1)]).

Lemma 2.1. *Let χ be a chirotope on X . An element $t \in X$ is not extreme in χ if and only if there exist $a, b, c \in X \setminus \{t\}$ such that $\chi(t, a, b) = \chi(t, b, c) = \chi(t, c, a) = 1$.*

Proof. Assume that there exist $a, b, c \in X \setminus \{t\}$ such that $\chi(t, a, b) = \chi(t, b, c) = \chi(t, c, a) = 1$. For the sake of contradiction, assume in addition that t is extreme in χ . By definition, this means that there exists $s \in X$ such that for all $x \in X \setminus \{t, s\}$, $\chi(t, s, x) = 1$. We cannot have $s = a$ as $\chi(t, a, c) = -1$. Similarly, $s \notin \{b, c\}$. Then, the tuple $\{s, t, a, b, c\}$ contradicts the transitivity axiom.

Conversely, assume that t is not extreme in χ . Let us choose $a_0, a_1 \in X$ such that $\chi(t, a_1, a_0) = 1$. Since t is not extreme, there exists $a_2 \in X \setminus \{t, a_0, a_1\}$ such that $\chi(t, a_1, a_2) = -1$. If $\chi(t, a_2, a_0) = -1$ then $a = a_0$, $b = a_2$, $c = a_1$ are as desired. So assume otherwise. Since t is not extreme, there exists $a_3 \in X \setminus \{t, a_2\}$ such that $\chi(t, a_2, a_3) = -1$; note that $a_3 \notin \{t, a_0, a_1, a_2\}$. If one of $\chi(t, a_3, a_0)$ or $\chi(t, a_3, a_1)$ equals -1 then we have our triple ($a \in \{a_0, a_1\}$, $b = a_3$ and $c = a_2$). So assume otherwise and using that t is not extreme, pick some $a_4 \in X \setminus \{t, a_3\}$ such that $\chi(t, a_3, a_4) = -1$. Due to the previous consideration, a_4 must be different from a_0, a_1 and a_2 . And so on and so forth: since X is finite we eventually find a triple as in the lemma. See Figure 2 for an illustration. \square

We now observe, for later use, that an extreme element in a chirotope defines a total order on the other elements.

Lemma 2.2. *Let χ be a chirotope on X , let $\text{ext}(\chi)$ be the set of extreme elements of χ , and take $a \in \text{ext}(\chi)$. The relation $p <_a q \Leftrightarrow \chi(a, p, q) = 1$ defines a strict total order on $X \setminus \{a\}$. Moreover, letting e_1, \dots, e_r denotes the elements of $\text{ext}(\chi) \setminus \{a\}$ as ordered by $<_a$, the cyclic order (a, e_1, \dots, e_r, a) does not depend on the choice of $a \in \text{ext}(\chi)$.*

Proof. To prove the first assertion, we only need to check the transitivity of the relation $<_a$. Let us consider three elements $x, y, z \in X \setminus \{a\}$ such that $x <_a y$ and $y <_a z$, and suppose by contradiction that $z <_a x$. Then we have $\chi(a, x, y) = \chi(a, y, z) = \chi(a, z, x) = 1$ contradicting that a is extreme, by Lemma 2.1.

Let C_a denote the cyclic order (a, e_1, \dots, e_r, a) . To prove the second assertion, it suffices to prove that $C_{e_1} = C_a$. For every $i \in \{2, \dots, r\}$, $e_1 <_a e_i$. Therefore $\chi(e_1, e_i, a) = \chi(a, e_1, e_i) = 1$. Hence a is the maximal element for $<_{e_1}$. Furthermore, let $2 \leq i < j \leq r$. As $e_1 <_a e_i <_a e_j$, we have $\chi(e_i, e_j, a) = \chi(e_i, a, e_1) = 1$. Then as e_i is extreme, $\chi(e_i, e_1, e_j) = -1$ by Lemma 2.1. Hence $e_i <_{e_1} e_j$ and $C_{e_1} = (e_1, \dots, e_r, a, e_1) = C_a$. \square

Lemma 2.2 defines a unique cyclic order on the extreme elements of a chirotope. Geometrically, this corresponds to the anticlockwise order on the convex hull. For any extreme element a , it is hence possible to define its previous and next extreme element for this cyclic order, denoted respectively a_- and a_+ . Note that a_+ (resp. a_-) is the minimum (resp. maximum) element of the total order $<_a$.

Observation 2.3. Recall that, by definition, an element a is extreme in a chirotope χ on X if and only if there exists y such that $\chi(a, y, z) = 1$ for all z in $X \setminus \{a, y\}$. With the notation introduced above, this element y is a_+ : in particular it is unique. We also have the following: a is extreme in χ if and only if there exists y' such that $\chi(a, y', z) = -1$ for all z in $X \setminus \{a, y'\}$. When it exists, this element is also unique and corresponds to a_- with the notation introduced above.

The last lemma of this section is of a more geometric nature, and only applies to realizable chirotopes. Given a set \mathcal{L} of lines in \mathbb{R}^2 , a *cell of the arrangement of \mathcal{L}* is a connected component of $\mathbb{R}^2 \setminus \bigcup_{\ell \in \mathcal{L}} \ell$.

Lemma 2.4. *Let $\chi : (X)_3 \rightarrow \{\pm 1\}$ be a realizable chirotope and let $x \in X$ be extreme in χ . Then there exists a realization $\mathcal{P} = \{\mathbf{p}_x, x \in X\}$ of χ such that \mathbf{p}_x is in an unbounded cell of the arrangement of the set of lines $\{(\mathbf{p}_y \mathbf{p}_z) : y, z \in X \setminus \{x\}, y \neq z\}$.*

Note that this is not true for *any* realization of χ but only for *some well-chosen* realization of χ : see Figure 3 for an example of two realizations of the same chirotope, one being as in the lemma, but not the other.

Proof. This proof relies on the notion of projective transform, which we now recall; see *e.g.* the book of Samuel [Sam88] or Richter-Gebert [RG11] for a primer on projective geometry. For a point $\mathbf{p} \in \mathbb{R}^2$ let $\hat{\mathbf{p}}$ be the column vector $\hat{\mathbf{p}} \stackrel{\text{def}}{=} (p_x, p_y, 1)^t$, and conversely, given $u \in \mathbb{R}^3$ with $u_z \neq 0$ let $\bar{u} \stackrel{\text{def}}{=} (u_x/u_z, u_y/u_z)$. A projective transform is a map $f : \mathbf{p} \mapsto \overline{A\hat{\mathbf{p}}}$ with $A \in \mathbb{R}^{3 \times 3}$ a nonsingular matrix. Such a transform is undefined on a line of \mathbb{R}^2 (those points \mathbf{p} such that $(A\hat{\mathbf{p}})_z = 0$); this line is said to be “sent to infinity”. Other lines of \mathbb{R}^2 are sent to lines in \mathbb{R}^2 . Indeed, a line is lifted to a 2-dimensional vector space by the application $\mathbf{p} \mapsto \hat{\mathbf{p}}$

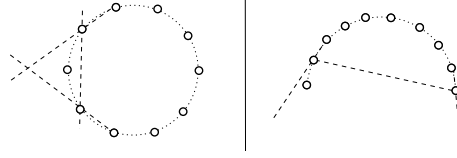


FIGURE 3. On the left, 10 points uniformly distributed on a circle; on the right, the same number of points uniformly distributed on a semicircle. Both configurations have the same chirotope. In the circle case, each point is in a bounded cell of the line arrangement defined by the other points (as illustrated for the left-most point). On the other hand in the semi-circle case, the left-most point is in an unbounded region of the line arrangement defined by the other points.

(taking the cone generated by the image), a 2-dimensional vector space is sent to another 2-dimensional vector space by A (since A is nonsingular), and finally the projection $u \mapsto \bar{u}$ maps back 2-dimensional vector spaces to lines (or to infinity in the case of the plane $u_z = 0$).

For any $\mathbf{p}, \mathbf{q}, \mathbf{r}$ in \mathbb{R}^2 we have $\chi(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \text{sign det}(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{r}})$. Indeed, $\text{sign det}(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{r}}) = +1$ if and only if the basis $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{r}})$ of \mathbb{R}^3 is positively oriented, which is equivalent to the points $(\mathbf{p}, \mathbf{q}, \mathbf{r})$ in \mathbb{R}^2 being in counterclockwise order.

Now, let f be a projective transform with matrix $A \in \mathbb{R}^{3 \times 3}$. We have

$$\begin{aligned} \chi(f(\mathbf{p}), f(\mathbf{q}), f(\mathbf{r})) &= \text{sgn det}(\widehat{f(\mathbf{p})}, \widehat{f(\mathbf{q})}, \widehat{f(\mathbf{r})}) \\ &= \text{sgn}((A\hat{\mathbf{p}})_z (A\hat{\mathbf{q}})_z (A\hat{\mathbf{r}})_z) \text{sgn det}(A\hat{\mathbf{p}}, A\hat{\mathbf{q}}, A\hat{\mathbf{r}}) \\ &= \text{sgn}((A\hat{\mathbf{p}})_z (A\hat{\mathbf{q}})_z (A\hat{\mathbf{r}})_z) \text{sgn}(\det A) \chi(p, q, r) \end{aligned}$$

The sign of the affine function $\mathbf{p} \mapsto (A\hat{\mathbf{p}})_z$ is determined by the position of \mathbf{p} lies with respect to the line $(A\hat{\mathbf{p}})_z = 0$, which is the line sent to infinity by f . Consequently, if \mathcal{P} is a point set, ℓ a line that does not split \mathcal{P} , and f a projective transform sending ℓ to infinity, then the map f either preserves all orientations of three points in \mathcal{P} or reverses all of them.

Let us now consider a realizable chirotope χ on X and let x be extreme in χ , as in the lemma. We choose an arbitrary realization \mathcal{Q} of χ and call $\mathbf{q} \stackrel{\text{def}}{=} \mathbf{q}_x$ the point labeled by x . Let us choose a line ℓ such that

- ℓ does not contain any point of \mathcal{Q} and all points of \mathcal{Q} are on the same side of ℓ ,
- there exists a point \mathbf{r} in ℓ (in particular $\mathbf{r} \notin \mathcal{Q}$) such that the segment $[\mathbf{q}\mathbf{r}]$ does not intersect any line spanned by two points of $\mathcal{Q} \setminus \{\mathbf{q}\}$.

(Such a choice is always possible since \mathbf{q} is extreme.) Let f be a projective transform sending ℓ to infinity. Since the map f sends lines to lines and since $f(\mathbf{r})$ is at infinity, f sends the half-open segment $[\mathbf{q}\mathbf{r})$ to a ray $[f(\mathbf{q})f(\mathbf{r}))$ going to infinity. From the second condition above, this ray does not meet any line spanned by two points of $f(\mathcal{P} \setminus \{\mathbf{p}\})$. Hence $f(\mathbf{p})$ is in an unbounded cell of the arrangement defined by $f(\mathcal{P} \setminus \{\mathbf{p}\})$. In particular, if f preserves the orientation of all triples in \mathcal{Q} , then $\mathcal{P} = f(\mathcal{Q})$ is a realization of χ with the desired property. On the other hand, if f reverses all orientations of triples in \mathcal{Q} , then the mirror image of $f(\mathcal{Q})$ is a realization of χ with that property. \square

3. BOWTIES AND MODULES

In this section, we study the bowtie operation \bowtie defined in the introduction, introduce formally a notion of modules, and analyze the connection between these two notions.

3.1. The bowtie operation. We recall that the bowtie operation \bowtie is defined in Eq. (4). It takes as input two sign functions χ and ξ with marked elements x^* and y^* and outputs a sign function κ , denoted $\chi_{x^*} \bowtie_{y^*} \xi$. Our first goal is to prove Proposition 1.1, which we split into several statements for convenience.

We first note that χ and ξ are essentially isomorphic to restrictions of κ (e.g., to obtain χ , we restrict κ to $(X \cup \{y\})_3$ where y is an arbitrary element of Y , and substitute y^* for y). Therefore, if κ is a chirotope, then χ and ξ are chirotopes, and if furthermore κ is realizable, then χ and ξ are also realizable. Here are the converse statements.

Proposition 3.1. *Let χ and ξ be chirotopes on $X \cup \{x^*\}$ and $Y \cup \{y^*\}$, with $|X|, |Y| \geq 2$. The sign function $\chi_{x^*} \bowtie_{y^*} \xi$ is a chirotope if and only if x^* and y^* are extreme in χ and ξ .*

Proof. Let $\kappa \stackrel{\text{def}}{=} \chi_{x^*} \bowtie_{y^*} \xi$. Assume that x^* is not extreme in χ . From Lemma 2.1, there exist $a, b, c \in X$ such that

$$(7) \quad \chi(x^*, a, b) = \chi(x^*, b, c) = \chi(x^*, c, a) = +1.$$

Let now y and y' be two distinct elements of Y . By definition of κ , we have

$$(8) \quad \kappa(y, y', a) = \kappa(y, y', b) = \kappa(y, y', c) = \xi(y, y', y^*),$$

which we can assume to be $+1$, up to switching y and y' . Furthermore, Eq. (7) implies

$$(9) \quad \kappa(y, a, b) = \kappa(y, b, c) = \kappa(y, c, a) = +1.$$

Eqs. (8) and (9) together violates the transitivity axiom of chirotopes, hence, κ is not a chirotope. Similarly if y^* is not extreme in ξ , then κ is not a chirotope either.

Conversely, let us assume that x^* and y^* are extreme in χ and ξ , and check that κ is a chirotope. The function κ is clearly a sign function, and we need to check the interiority and transitivity axioms. Let us start with the interiority axiom. Assume for the sake of contradiction that there exist distinct $a, b, c, t \in X \cup Y$ such that

$$(10) \quad \kappa(t, a, b) = \kappa(t, b, c) = \kappa(t, c, a) = -\kappa(a, b, c).$$

We cannot have all of a, b, c, t in X as χ satisfies the interiority axiom. Similarly, we cannot have a single element in Y , as replacing it by x^* would yield a quadruple violating the interiority axiom for χ . Symmetric arguments imply that we must have two elements in X and two in Y ; by symmetry we can assume that $a, b \in X$ and $c, t \in Y$. But then $\kappa(a, b, c) = \chi(a, b, x^*) = \kappa(a, b, t)$, contradicting Eq. (10). Hence κ satisfies the interiority axiom.

Now to the transitivity axiom. Take any distinct $a, b, c, s, t \in X \cup Y$ and assume for the sake of contradiction that

$$(11) \quad \kappa(t, s, a) = \kappa(t, s, b) = \kappa(t, s, c) = \kappa(a, b, t) = \kappa(b, c, t) = \kappa(c, a, t) = 1.$$

As above, we can assume that X and Y contain respectively two and three elements of $\{a, b, c, s, t\}$.

- We cannot have $\{s, t\} \subseteq X$ and $\{a, b, c\} \subseteq Y$ as this would imply $\xi(a, b, y^*) = \xi(b, c, y^*) = \xi(c, a, y^*) = 1$, and thus y^* would not be extreme in ξ by Lemma 2.1.

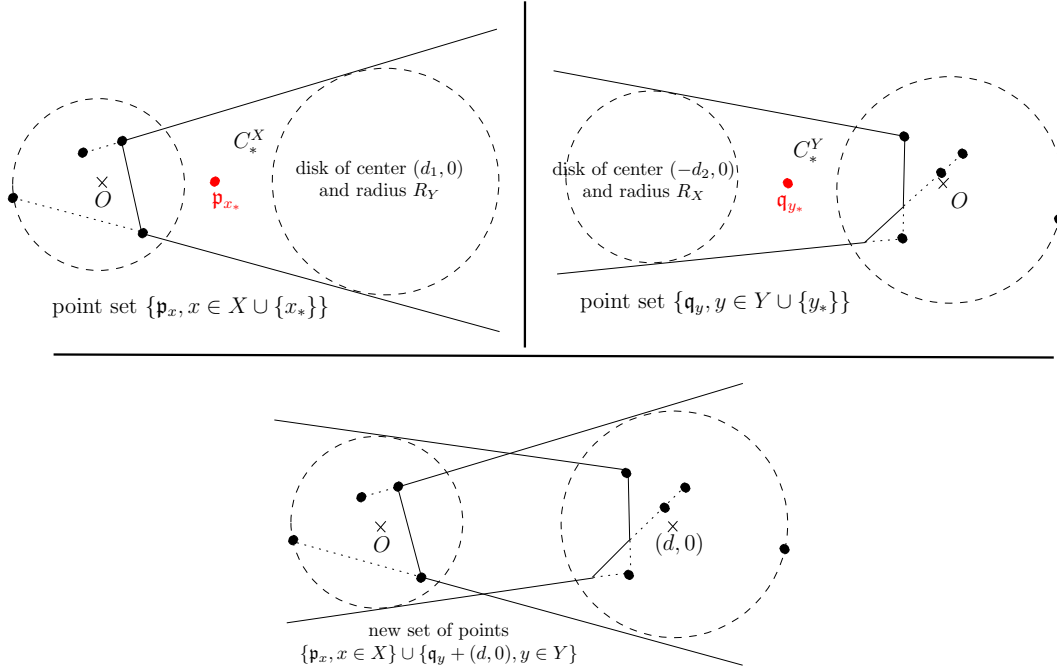


FIGURE 4. Illustration of the proof of Proposition 3.2. Marked points are drawn in red.

- We cannot have $\{s, a\} \subseteq X$ and $\{b, c, t\} \subseteq Y$ as this would imply $\kappa(c, t, a) = \xi(c, t, y^*) = \kappa(c, t, s)$, contradicting Eq. (11).
- We cannot have $\{t, a\} \subseteq X$ and $\{b, c, s\} \subseteq Y$ as this would imply $\kappa(a, t, b) = \chi(a, t, x^*) = \kappa(a, t, c)$, contradicting Eq. (11).
- We cannot have $\{b, c\} \subseteq X$ and $\{a, s, t\} \subseteq Y$ as this would imply $\kappa(a, t, b) = \xi(a, t, y^*) = \kappa(a, t, c)$, contradicting Eq. (11).

The other partitions are ruled out using the cyclic symmetry of (a, b, c) . \square

Proposition 3.2. *Let χ and ξ be realizable chirotopes on $X \cup \{x^*\}$ and $Y \cup \{y^*\}$, with $|X|, |Y| \geq 2$. The sign function $\chi_{x^*} \bowtie_{y^*} \xi$ is a realizable chirotope if and only if x^* and y^* are extreme in, respectively, χ and ξ .*

Proof. Let $\kappa \stackrel{\text{def}}{=} \chi_{x^*} \bowtie_{y^*} \xi$. One direction follows from Proposition 3.1, as if x^* or y^* is not extreme, then κ is not even a chirotope. So let us assume that χ and ξ are realizable and that x^* and y^* are extreme in χ and ξ , and build a point set realizing κ . The reader may want to look at Figure 4 while reading the proof.

Since x^* is extreme in χ , by Lemma 2.4 there exists a point set \mathcal{P} labeled by $X \cup \{x^*\}$ with chirotope χ such that \mathbf{p}_{x^*} is in an unbounded cell C_*^X of the arrangement of the set of lines $\{(\mathbf{p}_x \mathbf{p}_{x'}), x, x' \in X\}$. Up to rotating and translating this point set, we can assume that \mathbf{p}_{x^*} is on the positive real axis and that the half-line $[\mathbf{p}_{x^*}, +\infty)$ does not cross any line $(\mathbf{p}_x \mathbf{p}_{x'})$ with $x, x' \in X$. We can further assume that no line $(\mathbf{p}_x \mathbf{p}_{x'})$ with $x, x' \in X$ is horizontal, since the points are in generic position. We set $R_X = \max_{x \in X} \|\mathbf{p}_x\|$. Similarly, there exists a point

set \mathcal{Q} labeled by $Y \cup \{y^*\}$ with chirotope ξ such that \mathbf{q}_{y^*} is on the negative real axis and the half-line $(-\infty, \mathbf{q}_{x^*}]$ does not cross any line $(\mathbf{q}_y, \mathbf{q}_{y'})$ with $y, y' \in Y$. We set $R_Y = \max_{y \in Y} \|\mathbf{q}_y\|$.

Now, let \mathbf{p} be the point with coordinate $(d, 0)$ for some $d > \|\mathbf{p}_{x^*}\|$. The distance between \mathbf{p} and the union of the lines $(\mathbf{p}_x \mathbf{p}_{x'})$ with $x, x' \in X$ goes to infinity as d goes to $+\infty$: indeed it is positive, piecewise affine and could only remain constant if the cell C_*^X were bounded by horizontal lines (which we ruled out). Thus, for d large enough, say $d > d_1$, the disk of radius R_Y and center $(d, 0)$ entirely lies in C_*^X . Similarly for d large enough, say $d > d_2$, the disk of radius R_X and center $(-d, 0)$ entirely lies in C_*^Y .

We take $d > \max(d_1, d_2)$ and consider the point set

$$\{\mathbf{p}_x, x \in X\} \cup \{\mathbf{q}_y + (d, 0), y \in Y\},$$

i.e. we take all points \mathbf{p}_x (except \mathbf{p}_{x^*}) as they are, and all points \mathbf{q}_y (except \mathbf{q}_{y^*}) translated by the vector $(d, 0)$. Let $\kappa : (X \cup Y)_3 \rightarrow \{\pm 1\}$ be the chirotope of this point set. Clearly, κ is a realizable chirotope, and our goal is now to prove that $\kappa = \chi_{x^* \bowtie_{y^*}} \xi$. Obviously,

$$\begin{cases} \kappa(x_1, x_2, x_3) = \chi(x_1, x_2, x_3) & \text{if } x_1, x_2, x_3 \text{ are all in } X; \\ \kappa(y_1, y_2, y_3) = \xi(y_1, y_2, y_3) & \text{if } y_1, y_2, y_3 \text{ are all in } Y. \end{cases}$$

Take x_1, x_2 in X and y in Y . The point $\mathbf{q}_y + (d, 0)$ is in the circle of radius R_Y and center $(d, 0)$, and is therefore in C_*^X ; in other words, $\mathbf{q}_y + (d, 0)$ is in the same cell of the arrangement of the lines $\{(\mathbf{p}_x \mathbf{p}_{x'}), x, x' \in X\}$ as \mathbf{p}_{x^*} , which implies that $\kappa(x_1, x_2, y) = \chi(x_1, x_2, x^*)$. The case where y_1, y_2 are in Y and x in X is handled similarly, after translating all the points by $(-d, 0)$ (which does not change the chirotope). In this case, $\kappa(x, y_1, y_2) = \xi(y^*, y_1, y_2)$. Altogether, this shows that $\kappa = \chi_{x^* \bowtie_{y^*}} \xi$, proving that $\chi_{x^* \bowtie_{y^*}} \xi$ is realizable. \square

Propositions 3.1 and 3.2 emphasize the special role of extreme points for the bowtie operation. The following proposition will therefore be useful to iterate bowtie operations.

Proposition 3.3. *Let χ and ξ be chirotopes on disjoint sets $X \cup \{x^*\}$ and $Y \cup \{y^*\}$, with $|X|, |Y| \geq 2$, and assume that x^* and y^* are extreme in χ and in ξ . The extreme elements of $\chi_{x^* \bowtie_{y^*}} \xi$ are the extreme elements of χ other than x^* and those of ξ other than y^* .*

Proof. Let $\kappa \stackrel{\text{def}}{=} \chi_{x^* \bowtie_{y^*}} \xi$. Let x_e be extreme in χ , different from x^* . There exists $x'_e \in X$ such that

$$\forall x \in X \cup \{x^*\} \setminus \{x_e, x'_e\}, \quad \kappa(x_e, x'_e, x) = \chi(x_e, x'_e, x) = 1.$$

If $x'_e \neq x^*$, then for all $y \in Y$ we have $\kappa(x_e, x'_e, y) = \chi(x_e, x'_e, x^*) = 1$ and x_e is extreme in κ . If $x'_e = x^*$, then we use the equivalent definition (see Observation 2.3) that x_e is extreme in χ if and only if there exists $x''_e \in X$ such that

$$\forall x \in X \cup \{x^*\} \setminus \{x_e, x''_e\}, \quad \chi(x_e, x''_e, x) = -1,$$

and now that $x''_e \neq x^*$ we are back to the previous case, *mutatis mutandis*. Altogether, the extreme elements of χ different from x^* are extreme for κ . Similarly, the extreme elements of ξ different from y^* are extreme for κ .

Conversely, let z_e be extreme in κ , and assume w.l.o.g. that $z_e \in X$. By definition, there exists $z'_e \in X \cup Y$ such that for all $z \in X \cup Y \setminus \{z_e, z'_e\}$, we have $\kappa(z_e, z'_e, z) = 1$. On the one hand, if $z'_e \in X$, then $\chi(z_e, z'_e, x) = \kappa(z_e, z'_e, x) = 1$ for all $x \in X \setminus \{z_e, z'_e\}$. In addition, choosing any element $y \in Y$, we have $\chi(z_e, z'_e, x^*) = \kappa(z_e, z'_e, y) = 1$, and we recognize that

z_e is extreme in χ . On the other hand, if $z'_e \in Y$, then for all $x \in X \setminus \{z_e\}$, we have $\chi(z_e, x^*, x) = \kappa(z_e, z'_e, x) = 1$, and z_e is again extreme in χ . \square

Propositions 3.1 to 3.3 correspond to Proposition 1.1 in the introduction. We conclude this section with two additional results, used respectively in Sections 4 and 7.

For the first result, we recall that a chirotope is called convex if all its elements are extreme. The following statement is an immediate corollary of Proposition 3.3.

Corollary 3.4. *Let κ be a chirotope with a nontrivial decomposition $\kappa = \chi_{x^*} \bowtie_{y^*} \xi$. The chirotope κ is convex if and only if both χ and ξ are convex.*

3.2. Modules. Let us now consider inverting the bowtie operation: given a chirotope κ on ground set Z , we want a bipartition $X \uplus Y$ of Z , and chirotopes χ and ξ on ground sets $X \cup \{x^*\}$ and $Y \cup \{y^*\}$ (for new elements x^* and y^*) such that $\kappa = \chi_{x^*} \bowtie_{y^*} \xi$. We call such a decomposition $\kappa = \chi_{x^*} \bowtie_{y^*} \xi$ a *bowtie decomposition* of κ .

These considerations lead naturally to the following notion.

Definition 3.5. *A subset $X \subseteq Z$ is a module of a chirotope κ on Z if for every $x, x' \in X$ and $y, y' \in Z \setminus X$, one has $\kappa(x, x', y) = \kappa(x, x', y')$ and $\kappa(x, y, y') = \kappa(x', y, y')$. The bipartition $\{X, Z \setminus X\}$ of Z is called a modular bipartition of κ .*

Note that X is a module of κ if and only if $Z \setminus X$ is a module of κ . The notion of module is chosen so that the following observation holds.

Observation 3.6. Let κ be a chirotope on ground set Z and $\{X, Y\}$ a bipartition of Z . There exist chirotopes χ and ξ on disjoint ground sets $X \cup \{x^*\}$ and $Y \cup \{y^*\}$ such that $\kappa = \chi_{x^*} \bowtie_{y^*} \xi$ if and only if X is a module of κ . Moreover, when the chirotopes χ and ξ exist, they are unique up to the choice of the labels for x^* and y^* .

Remark 3.7. If one of X or Y has size at most 1, then $\{X, Y\}$ is always a modular bipartition of κ , but the size of χ or ξ is at least that of κ . We refer to such modular bipartitions, modules or bowtie decomposition as *trivial*. Note that in a nontrivial bipartition $\{X, Y\}$ of κ , both X and Y have size at least 2, which implies in particular that both χ and ξ in the decomposition $\kappa = \chi_{x^*} \bowtie_{y^*} \xi$ have ground sets of size at least 3.

Remark 3.8. When κ is realizable, its nontrivial modular bipartitions have a simple geometric interpretation: X is a nontrivial module if there exists a realization of κ such that no line through two points with labels in X separates two points with labels outside X , and conversely no line through two points with labels outside X separates two points with labels in X . This explains the connection with the notion of *mutually avoiding sets* given in the introduction: a modular bipartition is a pair of mutually avoiding sets covering the whole chirotope.

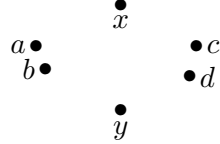
The next lemma studies the stability of modules under intersection, and will be useful in the next section.

Lemma 3.9. *Let X_1, X_2 be two modules of a chirotope κ with ground set Z . If $X_1 \cup X_2 \neq Z$, then $X_1 \cap X_2$ is a module of κ .*

Proof. Let $x, x' \in X_1 \cap X_2$ and $y, y' \in Z \setminus (X_1 \cap X_2)$. Let $p \in Z \setminus (X_1 \cup X_2)$. Since y is outside $X_1 \cap X_2$, the set $\{y, p\}$ is disjoint from X_1 or from X_2 . In either case, we have $\kappa(x, x', y) = \kappa(x, x', p)$. A symmetric argument yields $\kappa(x, x', p) = \kappa(x, x', y')$, so we have $\kappa(x, x', y) = \kappa(x, x', y')$.

It remains to argue that $\kappa(x, y, y') = \kappa(x', y, y')$. This is clear if both y and y' are outside of X_1 , as X_1 is a module, so assume otherwise. Suppose, without loss of generality, that $y \in X_1 \setminus X_2$, and $y' \in X_2 \setminus X_1$. As $x, x', y \in X_1$ and $p, y' \notin X_1$ we have $\kappa(x, y, y') = \kappa(x, y, p)$ and $\kappa(x', y, y') = \kappa(x', y, p)$. As $p, y \notin X_2$ and $x, x' \in X_2$, $\kappa(x, y, p) = \kappa(x', y, p)$. Altogether, we get $\kappa(x, y, y') = \kappa(x', y, y')$ as desired. \square

Remark 3.10. The condition $X_1 \cup X_2 \neq Z$ in Lemma 3.9 is indeed necessary. To see this, consider the chirotope induced by the following point set, where $X_1 = \{a, b, x, y\}$ and $X_2 = \{c, d, x, y\}$ are nontrivial modules but $X_1 \cap X_2 = \{x, y\}$ is not a module:



3.3. Commutativity and associativity.

Proposition 3.11. *Let χ, ξ, κ be chirotopes on disjoint ground sets $X \cup \{x^*\}, Y \cup \{y_1^*, y_2^*\}$ and $Z \cup \{z^*\}$, with $|X|, |Z| \geq 2, |Y| \geq 1$. If the starred elements are extreme in their respective chirotopes, then $\chi_{x^*} \bowtie_{y_1^*} \xi = \xi_{y_1^*} \bowtie_{x^*} \chi$ and $(\chi_{x^*} \bowtie_{y_1^*} \xi)_{y_2^*} \bowtie_{z^*} \kappa = \chi_{x^*} \bowtie_{y_1^*} (\xi_{y_2^*} \bowtie_{z^*} \kappa)$.*

Proof. The commutativity comes from the symmetry of the definition of the bowtie operation. Let us focus on the associativity condition. Let $\lambda_1 = (\chi_{x^*} \bowtie_{y_1^*} \xi)_{y_2^*} \bowtie_{z^*} \kappa$, and $\lambda_2 = \chi_{x^*} \bowtie_{y_1^*} (\xi_{y_2^*} \bowtie_{z^*} \kappa)$, which are both sign functions on $X \cup Y \cup Z$. Let us take three elements t, u, v in $X \cup Y \cup Z$. Suppose first that t, u, v all belong to the set X . Then by definition of the bowtie $\lambda_1(t, u, v) = [(\chi_{x^*} \bowtie_{y_1^*} \xi)_{y_2^*} \bowtie_{z^*} \kappa](t, u, v) = (\chi_{x^*} \bowtie_{y_1^*} \xi)(t, u, v) = \chi(t, u, v)$, and similarly $\lambda_2(t, u, v) = \chi(t, u, v)$. The cases where t, u, v are all in Y , or all in Z , are similar. If t, u, v all belong to different sets among X, Y and Z , for example if $t \in X, u \in Y, v \in Z$, then $\lambda_1(t, u, v) = (\chi_{x^*} \bowtie_{y_1^*} \xi)(t, u, y_2^*) = \xi(y_1^*, u, y_2^*)$, and $\lambda_2(t, u, v) = (\xi_{y_2^*} \bowtie_{z^*} \kappa)(y_1^*, u, v) = \xi(y_1^*, u, y_2^*)$. The final cases where one element is isolated in one set are very similar. \square

Remark 3.12. When using the associativity of bowties, one should pay particular attention to the ground sets of the chirotopes at play. For instance, with the notation of Proposition 3.11, writing $(\xi_{y_1^*} \bowtie_{x^*} \chi)_{y_2^*} \bowtie_{z^*} \kappa = \xi_{y_1^*} \bowtie_{x^*} (\chi_{y_2^*} \bowtie_{z^*} \kappa)$ does not make sense, as in the right-hand term y_2^* does not belong to the ground set of χ (even though the left-hand term is well-defined).

4. EXISTENCE AND UNIQUENESS OF THE CANONICAL CHIROTOPE TREE

We now examine how iterated bowtie products can be described and manipulated via trees.

4.1. Chirotope trees and their chirotopes. We first recall some definitions from the introduction. A *chirotope tree* is a tree whose nodes are decorated with chirotopes on disjoint ground sets, each of size at least 3, and whose edges $\{v_1, v_2\}$ select an element in the ground set of each of the chirotopes decorating v_1 and v_2 , in a way that (i) no element is selected more than once, and (ii) each selected element is extreme in its chirotope. To be complete, let us stress out that all trees we consider are graph-theoretical trees $T = (V_T, E_T)$, i.e. simple connected graphs without cycles. In particular, trees are always unrooted and unordered (neighbors of a given node are not ordered).

Given a chirotope tree T , one associates to it a sign function χ_T on $\bigcup_{w \in V_T} X_w$ defined by

$$(12) \quad \chi_T(x, y, z) \stackrel{\text{def}}{=} \chi_v(R(x, v), R(y, v), R(z, v)),$$

where v is the branching point of the paths from x to y , y to z and x to z , and $R(x, v)$, $R(y, v)$ and $R(z, v)$ the representatives of x , y and z in v (i.e. the proxy points of χ_v to which the path going from v to the nodes containing x , y and z are attached). If two or more elements among x , y and z belong to the same node, then v is taken to be this node, and we use the convention that $R(s, v) = s$ for s in X_v .

It is easy to see that this defines a sign function on $\bigcup_{w \in V_T} X_w$; we now prove Proposition 1.2, which states that it is always a chirotope, and furthermore realizable if all chirotopes decorating the nodes of T are realizable. Let us start with a lemma.

Lemma 4.1. *Let T be a chirotope tree and $e = \{v_1, v_2\}$ be an edge of T . Let s_1 and s_2 be the elements selected by e in v_1 and v_2 respectively. If T_1 and T_2 denote the two chirotope trees obtained by removing e from T , with $v_i \in T_i$, then $\chi_T = \chi_{T_1} \bowtie_{s_1} \chi_{T_2}$.*

Proof. First, we note that after the removal of e , s_1 and s_2 are no longer proxies in T_1 and T_2 , and the bowtie $\kappa = \chi_{T_1} \bowtie_{s_1} \chi_{T_2}$ is thus well-defined.

It remains to check that $\kappa = \chi_T$. Let X denote the ground set of κ . This set decomposes into $X = X_1 \uplus X_2$, where $X_i \uplus \{s_i\}$ is the ground set of T_i , for $i = 1, 2$. It is easy to check that given $x \in X$ and a node $v \in T$, we have

$$R_T(x, v) = \begin{cases} R_{T_1}(x, v) & \text{if } v \in T_1 \text{ and } x \in X_1, \\ R_{T_1}(s_1, v) & \text{if } v \in T_1 \text{ and } x \in X_2, \\ R_{T_2}(s_2, v) & \text{if } v \in T_2 \text{ and } x \in X_1, \\ R_{T_2}(x, v) & \text{if } v \in T_2 \text{ and } x \in X_2. \end{cases}$$

From this, it is easy to check that $\kappa(x, y, z) = \chi_T(x, y, z)$ for any x, y, z in X . For instance, if $x, y \in X_1$ and $z \in X_2$, then the node $v \stackrel{\text{def}}{=} v(x, y, z)$ lies in T_1 and

$$R_T(x, v) = R_{T_1}(x, v), \quad R_T(y, v) = R_{T_1}(y, v), \quad R_T(z, v) = R_{T_1}(s_1, v).$$

We therefore get

$$\chi_T(x, y, z) = \chi_v(R_{T_1}(x, v), R_{T_1}(y, v), R_{T_1}(s_1, v)) = \chi_{T_1}(x, y, s_1) = \kappa(x, y, z),$$

where the middle equality uses Eq. (12) for T_1 , and the last equality follows from the definition of the bowtie operation (Eq. (4)). Other cases (two elements in X_2 and one in X_1 , or the three elements in the same set) are similar or easier. This proves $\kappa = \chi_T$ as wanted. \square

The proof of Proposition 1.2 is now a straightforward induction using Lemma 4.1 and the fact that the bowtie of two (realizable) chirotopes whose starred elements are extreme is a (realizable) chirotope (see Propositions 3.1 to 3.3). Iterating this lemma, we can actually compute χ_T starting from the sign functions χ_v and iterating bowtie operations. For example, if T is the chirotope tree on Figure 5, we have

$$(13) \quad \chi_T = (\chi_1 \bowtie_{x_1^*} \chi_2 \bowtie_{x_2^*} (\chi_3 \bowtie_{x_3^*} \chi_4)) \bowtie_{x_4^*} \chi_4.$$

We note that, when iterating Lemma 4.1, we can choose at each step which edge plays the role of e in the lemma. This leads to different expressions of χ_T as a sequence of bowtie operations. Considering again the chirotope tree T on Figure 5, we could have written

$$(14) \quad \chi_T = \chi_1 \bowtie_{x_1^*} \chi_2 \bowtie_{x_2^*} ((\chi_3 \bowtie_{x_3^*} \chi_4) \bowtie_{x_4^*} \chi_4) \bowtie_{x_3^*} \chi_3$$

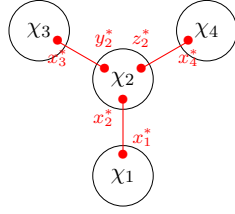


FIGURE 5. An example of chirotope tree.

The equivalence of Eqs. (13) and (14) follows from the commutativity and associativity properties of the bowtie operation (Proposition 3.11).

Remark 4.2. From Eq. (12), it follows immediately that the sign function χ_T associated to a chirotope tree T does not depend on the labels of its proxies. Accordingly, in the following, we consider chirotope trees up to relabeling of their proxies.

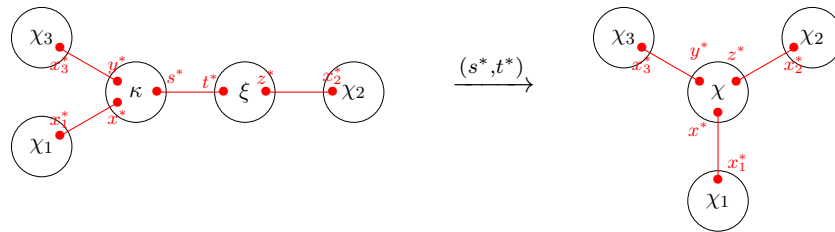
4.2. Two Operations on chirotope trees. We now adapt two standard operations on trees (edge contraction and vertex split) for chirotope trees in a way that leaves the associated chirotopes invariant.

4.2.1. *Contraction.*

Definition 4.3. Let T be a chirotope tree and $e = \{v, v'\}$ be an edge in T that selects s^* and t^* in χ_v and $\chi_{v'}$, respectively. The contraction of e in T is the tree T' obtained from T by merging the nodes v and v' into a new node v_{new} , decorated with the chirotope $\chi_{new} \stackrel{def}{=} \chi_v \bowtie_{s^* t^*} \chi_{v'}$. We denote this transformation by $T \xrightarrow{c} T'$.

Note that, by Propositions 3.1 and 3.2, χ_{new} is a chirotope, and is realizable if both χ_v and $\chi_{v'}$ are realizable. Furthermore, by Proposition 3.3, the extreme elements of χ_v and $\chi_{v'}$ different from s^* and t^* are extreme elements of χ_{new} so that the proxy elements corresponding to other edges of T are still extreme in their respective chirotopes in T' ; it follows that T' is indeed a chirotope tree.

Example 4.4. Here is an example of a contraction of an edge in a chirotope tree (with $\chi = \kappa \bowtie_{s^* t^*} \xi$).

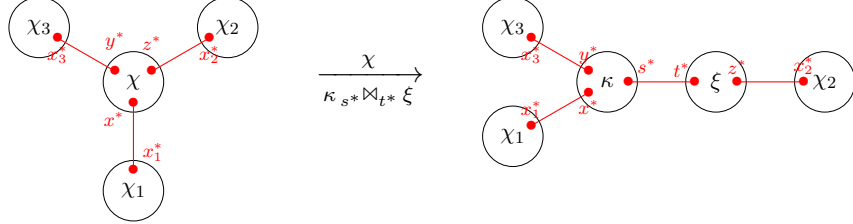


4.2.2. *Split.*

Definition 4.5. Let T be a chirotope tree with a node v_0 whose decoration $\chi_{v_0} = \chi$ has a nontrivial bowtie decomposition $\chi = \kappa \bowtie_{s^* t^*} \xi$. The split of T according to $\chi = \kappa \bowtie_{s^* t^*} \xi$ is the tree T' obtained from T by replacing v_0 by two nodes v_1 and v_2 that are decorated by κ and ξ , respectively, and form an edge selecting s^* in v_1 and t^* in v_2 . Denote this transformation by $T \xrightarrow[\kappa \bowtie_{s^* t^*} \xi]{\chi} T'$.

Note that for any edge $e = \{s_{v_0}^e, s_w^e\}$ connecting v_0 to another vertex w in T , the element $s_{v_0}^e$ belongs to the ground set of either κ or ξ . Therefore, e is naturally seen as an edge of T' , connecting either v_1 or v_2 to w , depending on the set in which $s_{v_0}^e$ lies.

Example 4.6. Here is an example of a split of a tree according to the bowtie decomposition $\chi = \kappa_{s^*} \bowtie_{t^*} \xi$.



4.2.3. *Properties.* The contraction and split decomposition operations are inverse of one another in the sense that, with the notation of Definitions 4.3 and 4.5,

$$(15) \quad T \xrightarrow[\kappa_{s^*} \bowtie_{t^*} \xi]{\chi} T' \quad \Rightarrow \quad T' \xrightarrow{(s^*, t^*)} T, \quad \text{and} \quad T \xrightarrow{(s^*, t^*)} T' \quad \Rightarrow \quad T' \xrightarrow[\chi_{v_{s^*} \bowtie_{t^*} \chi_{v'}}]{\chi_{new}} T.$$

We next show that performing an edge contraction or a vertex split on a chirotope tree does not change its associated chirotope.

Proposition 4.7. *If $T \xrightarrow{e} T'$ or $T \xrightarrow[\kappa_{s^*} \bowtie_{t^*} \xi]{\chi} T'$, then $\chi_T = \chi_{T'}$.*

Proof. We first prove the statement under the hypothesis that $T \xrightarrow{e} T'$. We proceed by induction on the number of nodes in T . If T has two nodes, then the result follows from Lemma 4.1. Assume the result holds for all chirotope trees with $n \geq 2$ nodes, and let T be a chirotope trees with $n + 1$ nodes, let e be an edge of T , and let T' be such that $T \xrightarrow{e} T'$. Since T has at least three nodes, there is an edge $f \neq e$ in T , and that edge also exist in T' . Removing f (from T) splits T into two parts T_1 and T_2 , and removing f (from T') splits T' into two parts T'_1 and T'_2 ; without loss of generality, we assume that e belongs to T_1 and T'_1 , so that $T'_2 = T_2$ and $T_1 \xrightarrow{e} T'_1$. By the induction hypothesis, $\chi_{T_1} = \chi_{T'_1}$. Also applying Lemma 4.1 twice yields

$$\chi_T = \chi_{T_1} \bowtie_{s_1} \chi_{T_2}, \quad \chi_{T'} = \chi_{T'_1} \bowtie_{s_1} \chi_{T'_2},$$

where $f = \{s_1, s_2\}$. Altogether this implies $\chi_T = \chi_{T'}$, and the lemma is proved under the hypothesis that $T \xrightarrow{e} T'$.

If $T \xrightarrow[\kappa_{s^*} \bowtie_{t^*} \xi]{\chi} T'$, then Eq. (15) and the first statement proved in this proposition also ensures that $\chi_T = \chi_{T'}$, concluding the proof. \square

4.3. **Canonical chirotope trees.** We recall from the introduction that a chirotope tree is called *canonical* if every node is decorated by a convex or indecomposable chirotope, and if no edge connects two nodes decorated by convex chirotopes. The goal of the section is to prove Theorem 1.3, which we copy here for convenience.

Theorem 1.3. For every chirotope χ , there exists a unique canonical chirotope tree T such that $\chi_T = \chi$.

Suppose that we start with the trivial chirotope tree with a single node decorated with χ , T_χ^{tr} , and, for as long as we can and in any order, we split the nodes decorated by decomposable

nonconvex chirotopes, and contract the edges joining nodes decorated by convex chirotopes. The gist of the proof of Theorem 1.3 is to prove that (i) any such sequence of operations terminates, (ii) any canonical tree T with $\chi_T = \chi$ can be obtained by such a sequence of operations, and (iii) all sequences of operations produce the same canonical tree. A convenient setting to carry out this analysis is the theory of rewriting systems.

4.3.1. *The rewriting system.* Using the contraction and split operations, we define a rewriting system on chirotope trees.

Definition 4.8. *We consider the set of chirotope trees equipped with the following rewriting rules:*

- $T \xrightarrow{\diamond} T'$ if T' is obtained from T by contracting an edge of T between two convex chirotopes,
- $T \xrightarrow{\boxtimes} T'$ if T' is obtained from T by splitting a nonconvex node of T ,

We write $T \Rightarrow T'$ if $T \xrightarrow{\diamond} T'$ or $T \xrightarrow{\boxtimes} T'$.

As usual, for any rewriting rule \rightarrow , we denote by \rightarrow^* the rewriting rule which consists in any number (possibly zero) of successive applications of \rightarrow . By definition, a chirotope tree is canonical if the rewriting rule \Rightarrow does not apply to it.

4.3.2. *Termination.* The existence of a canonical chirotope tree representing any given chirotope χ follows from the termination property of \Rightarrow . The proof of that property uses the notion of multiset ordering. Given two finite multisets M and N of positive integers, we write $M >_{mul} N$ if there exist multisets $X \subseteq M$ and Y such that $N = (M \setminus X) \uplus Y$ and $\max(X) > \max(Y)$. It is standard that $>_{mul}$ defines a well-founded order (i.e. without infinite decreasing subsequences) on multisets of \mathbb{N} .

Lemma 4.9. *Every rewriting sequence $T_0 \Rightarrow T_1 \Rightarrow \dots \Rightarrow T_n \Rightarrow \dots$ is finite.*

Proof. Let us associate with a chirotope tree T a pair $(N(T), C(T))$, where $N(T)$ is the multiset of sizes of nonconvex chirotopes in T , and $C(T)$ is the number of convex chirotopes in T . Suppose that $T \Rightarrow T'$. If $T \xrightarrow{\diamond} T'$, then $C(T') = C(T) - 1$ and $N(T') = N(T)$; indeed, by Corollary 3.4, we replace two nodes decorated by convex chirotopes, by a single one, also decorated by a convex chirotope. If $T \xrightarrow{\boxtimes} T'$, the split operation replaces one node with a nonconvex decoration by two new nodes with decorations of smaller sizes. Thus we have $N(T) >_{mul} N(T')$ (in this case, $C(T)$ can change as one of the new node can have a convex decoration). In both cases, we have $(N(T'), C(T')) <_{lex} (N(T), C(T))$, where $<_{lex}$ denotes the lexicographic order corresponding to the pair of orders $(<_{mul}, <_{\mathbb{N}})$. This lexicographic order is well-founded (since both original orders are well-founded), implying that applying the rewriting rule \Rightarrow always terminates. \square

4.3.3. *Accessibility.* The next step is to show that the rewriting rule can turn T_{χ}^{tr} into any canonical chirotope tree whose associated chirotope is χ .

Lemma 4.10. *Let T be a canonical chirotope tree with $\chi_T = \chi$. Then $T_{\chi}^{tr} \Rightarrow^* T$.*

Proof. We start from T and contract edges one by one, until we have a tree with a single node. Since contractions do not modify the associated chirotope, we end up with T_{χ}^{tr} . Corollary 3.4 asserts that the node obtained by an edge contraction is decorated by a convex chirotope (if and) only if the two merged nodes are decorated by convex chirotopes. Since the tree we

start with is canonical, an immediate induction yields that none of the trees appearing in this sequence of contractions has an edge joining two nodes decorated by convex chirotopes. Each contraction we performed is therefore the inverse of a split operation allowed in \Rightarrow , proving the claim. \square

4.3.4. *Confluence.* It remains to show that, starting from T_χ^{tr} , independently of the choice of rewriting operations, iterating \Rightarrow as much as possible always yields the same output. This property is known as *confluence* in the theory of rewriting rules. It is well-known (Newman's lemma) that, for a terminating rewriting rule (like \Rightarrow , see Lemma 4.9), confluence is implied by the following *local confluence* property.

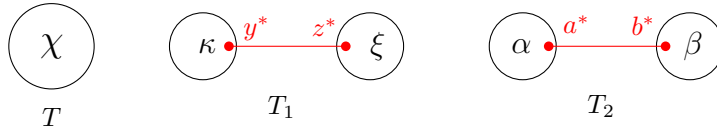
Proposition 4.11. *If T, T_1, T_2 are three chirotope trees such that $T \Rightarrow T_1$ and $T \Rightarrow T_2$, then there exists a chirotope tree T' such that $T_1 \Rightarrow^* T'$ and $T_2 \Rightarrow^* T'$.*

Proof. The statement is immediate if the transformations $T \Rightarrow T_1$ and $T \Rightarrow T_2$ operate on disjoint sets of nodes: indeed, each operation can be performed after the other one, and the resulting chirotope tree does not depend on the order of operations. If the transformations operate on nondisjoint sets of nodes then they must be of the same type as $\overset{\diamond}{\rightarrow}$ operates on two nodes decorated by convex chirotopes whereas any of the nodes on which $\overset{\bowtie}{\rightarrow}$ operates is decorated by a nonconvex chirotope. If $T \overset{\diamond}{\rightarrow} T_1$ and $T \overset{\diamond}{\rightarrow} T_2$, the local confluence follows from the associativity of \bowtie . So let us consider we are in the last case, with

$$T \xrightarrow[\kappa_{y^* \bowtie_{z^*} \xi}]{\chi} T_1 \quad \text{and} \quad T \xrightarrow[\alpha_{a^* \bowtie_{b^*} \beta}]{\chi} T_2,$$

and χ a nonconvex chirotope decorating a node v of T . The analysis turns out to be considerably more intricate.

Let $G \stackrel{\text{def}}{=} X \uplus S$ denote the ground set of χ , with X denoting the nonproxy elements of the ground set, and S its proxy elements. Throughout the analysis, we only make modifications in the tree that are local to the node labeled by χ . Accordingly we will only represent in pictures the node decorated with χ and the nodes replacing it after various operations. For example T, T_1 and T_2 are visualized as follows:

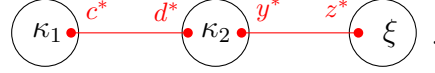


Let $\{Y, Z\}$ be the modular bipartition of $X \cup S$ associated to the bowtie decomposition $\chi = \kappa_{y^* \bowtie_{z^*} \xi}$, and $\{A, B\}$ be the modular bipartition of $X \cup S$ associated with $\chi = \alpha_{a^* \bowtie_{b^*} \beta}$. We assume that $\{A, B\} \neq \{Y, Z\}$ as otherwise $T_1 = T_2$ and the statement trivially follows. Since χ is nonconvex, Corollary 3.4 ensures that α or β is nonconvex, and the same goes for κ or ξ .

Let us first assume that $A \subsetneq Y$, and therefore $Z \subsetneq B$. If κ is convex, then α is convex too, hence β is not convex. Therefore, up to exchanging the role of (A, B) and (Y, Z) , we can assume that $A \subsetneq Y$ and that κ is not convex.

From $A \subsetneq Y$, it comes that the pair $\{A, \{y^*\} \cup Y \setminus A\}$ is then a non trivial modular bipartition of κ , as χ and κ coincide on Y , and y^* is a proxy representing the elements of Z .

Let T_3 be the split of T_1 according to the corresponding bowtie decomposition $\kappa = \kappa_1 c^* \bowtie_{d^*} \kappa_2$. Since κ is not convex, we have $T_1 \Rightarrow T_3$. The tree T_3 is visualized as follows:



By associativity (Proposition 3.11), we have $\chi = (\kappa_1 c^* \bowtie_{d^*} \kappa_2) y^* \bowtie_{z^*} \xi = \kappa_1 c^* \bowtie_{d^*} (\kappa_2 y^* \bowtie_{z^*} \xi)$. This means that κ_1 (resp. $\kappa_2 y^* \bowtie_{z^*} \xi$) is the chirotope associated to the module A (resp. B) in the corresponding bowtie decomposition. Hence, we have $\kappa_1 = \alpha$ and $(\kappa_2 y^* \bowtie_{z^*} \xi) = \beta$ (up to renaming the proxy elements c^* in a^* and d^* in b^*). This implies

$$T_2 \xrightarrow[\kappa_2 y^* \bowtie_{z^*} \xi]{\beta} T_3.$$

If β is not convex, then $T_2 \Rightarrow T_3$ and the statement follows with $T' = T_3$. If β is convex, then both κ_2 and ξ are, and $T_3 \Rightarrow T_2$, proving the statement with $T' = T_2$.

A schematic representation of the argument is presented on Figure 6.

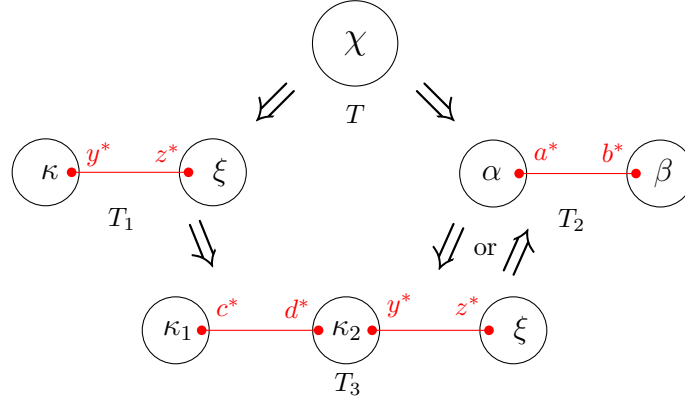


FIGURE 6. Diamond diagram proving local confluence in the case $A \subsetneq Y$.

The cases $A \subsetneq Z$, $B \subsetneq Y$, $B \subsetneq Z$ are similar to the case $A \subsetneq Y$, so from now on we can assume that $A \cap Y$, $A \cap Z$, $B \cap Y$, $B \cap Z$ are all nonempty. Lemma 3.9 then implies that each of $A \cap Y$, $A \cap Z$, $B \cap Y$ and $B \cap Z$ is a nontrivial module of χ . We use this to further decompose T_1 and T_2 in ways that eventually reduce to the same term. The specific decomposition depends on which of $A \cap Y$, $A \cap Z$, $B \cap Y$, $B \cap Z$ contain nonextremal elements. We spell out two representative cases and leave the remaining, routine, check to the reader.

Case 1: each of $A \cap Y$, $A \cap Z$, $B \cap Y$, $B \cap Z$ contains at least one nonextremal element. Many different chirotopes now enter the stage, so we record their ground sets and proxies in tables to help the reader keep track of them.

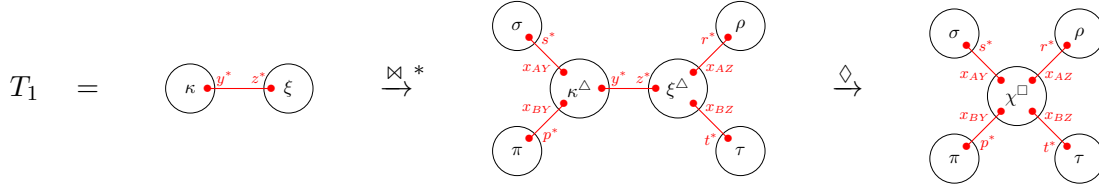
The ground set of κ is $Y \cup \{y^*\} = (A \cap Y) \cup (B \cap Y) \cup \{y^*\}$ and χ and κ coincide on Y . It follows that $A \cap Y$ and $B \cap Y$ are disjoint nontrivial modules of κ , and κ therefore decomposes into $\kappa = \sigma s^* \bowtie_{x_{AY}} \kappa^\Delta \bowtie_{x_{BY}} p^* \pi$.

Chirotope	Ground set	Proxies
κ	$Y \cup \{y^*\}$	y^*
σ	$(A \cap Y) \cup \{s^*\}$	s^*
π	$(B \cap Y) \cup \{p^*\}$	p^*
κ^Δ	$\{x_{AY}, x_{BY}, y^*\}$	x_{AY}, x_{BY}, y^*

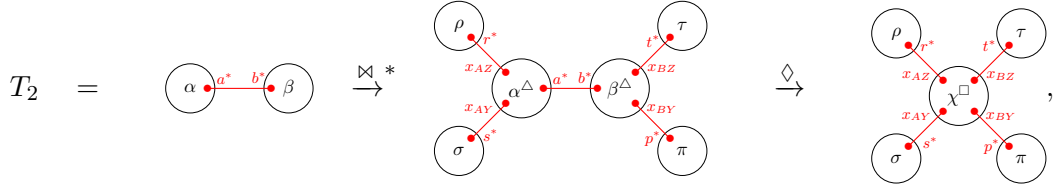
Chirotope	Ground set	Proxies
ξ	$Z \cup \{z^*\}$	z^*
τ	$(B \cap Z) \cup \{t^*\}$	t^*
ρ	$(A \cap Z) \cup \{r^*\}$	r^*
ξ^Δ	$\{x_{AZ}, x_{BZ}, z^*\}$	x_{AZ}, x_{BZ}, z^*

Similarly, the ground set of ξ is $Z \cup \{z^*\} = (A \cap Z) \cup (B \cap Z) \cup \{z^*\}$ and χ and ξ coincide on Z . It follows that $A \cap Z$ and $B \cap Z$ are disjoint nontrivial modules of ξ , and ξ therefore decomposes into $\xi = \tau \bowtie_{t^*} \bowtie_{x_{BZ}} \xi^\Delta \bowtie_{x_{AZ}} \bowtie_{r^*} \rho$.

Altogether, we can transform T_1 into



where the last transformation is a contraction of the central edge between the two convex chirotopes κ^Δ and ξ^Δ ; the resulting node is decorated with a convex chirotope on four elements, all proxies; we denote this chirotope by χ^\square , and its proxy elements are $x_{AY}, x_{AZ}, x_{BZ}, x_{BY}$. Similarly, we have



with the same chirotope χ^\square as before, and all other appearing chirotopes recalled or defined in the tables below.

Chirotope	Ground set	Proxies
α	$A \cup \{a^*\}$	a^*
σ	$(A \cap Y) \cup \{s^*\}$	s^*
ρ	$(A \cap Z) \cup \{r^*\}$	r^*
α^Δ	$\{x_{AY}, x_{AZ}, a^*\}$	x_{AY}, x_{AZ}, a^*

Chirotope	Ground set	Proxies
β	$B \cup \{b^*\}$	b^*
π	$(B \cap Y) \cup \{p^*\}$	p^*
τ	$(B \cap Z) \cup \{t^*\}$	t^*
β^Δ	$\{x_{BY}, x_{BZ}, b^*\}$	x_{BY}, x_{BZ}, b^*

We stress that the chirotopes σ, ρ, τ, π and χ^\square are the same in both decompositions, as they are restriction of our initial chirotope χ to the same sets (replacing proxy elements by arbitrary elements in the sets that they represent). The two trees obtained at the far right of both derivations are identical. Since each of $A \cap Y$, $A \cap Z$, $B \cap Y$ and $B \cap Z$ contains a nonextremal point, the chirotopes σ, ρ, τ and π are non convex, and all transformations in the decompositions of T_1 and T_2 given above are valid rules for \Rightarrow , and the statement follows.

Case 2: each of $A \cap Z$, $B \cap Y$, $B \cap Z$ contains a nonextremal element, but $A \cap Y$ does not. We now define $\kappa^\diamond \stackrel{\text{def}}{=} \sigma_{s^*} \bowtie_{x_{AY}} \kappa^\Delta$ and $\alpha^\diamond \stackrel{\text{def}}{=} \sigma_{s^*} \bowtie_{x_{AY}} \alpha^\Delta$. We note that $\kappa^\diamond \bowtie_{y^*} \bowtie_{z^*} \xi^\Delta = \alpha^\diamond \bowtie_{a^*} \bowtie_{b^*} \beta^\Delta$, since both are restrictions of χ on $(A \cap Y) \cup \{x_{AZ}, x_{BZ}, x_{BY}\}$ (replacing proxy elements by arbitrary elements in the sets that they represent). We denote this chirotope by

directly a uniform random unlabeled realizable chirotope on n elements lead to two different probability distributions. However, symmetry can be controlled, and Theorem 1.5 implies the following.

Corollary 5.1. *Let $\bar{\chi}_n$ be a uniform random unlabeled realizable chirotope on n elements. Then, $\bar{\chi}_n$ is indecomposable with probability $1 - \mathcal{O}(n^{-2})$.*

Proof. Let t_n^u denote the number of unlabeled realizable chirotopes on n elements and d_n^u the number of decomposable ones. Since a chirotope on n elements has at most n symmetries [GW23, Theorem 1.5], we have

$$\frac{t_n}{n!} \leq t_n^u \leq \frac{t_n}{(n-1)!} \quad \text{and} \quad \frac{d_n}{n!} \leq d_n^u \leq \frac{d_n}{(n-1)!}.$$

Theorem 1.5 asserts that $d_n/t_n = \mathcal{O}(n^{-3})$, from which we get

$$\frac{d_n^u}{t_n^u} \leq n \frac{d_n}{t_n} = \mathcal{O}(n^{-2}). \quad \square$$

5.2. Some classical estimates on the number of realizable chirotopes.

We start by recalling essentially known results about the number t_n of realizable chirotopes on $\{1, \dots, n\}$. A general reference on this topic is [GP93, Section 6]. The first statement that we need is the following nonasymptotic version of the case $d = 2$ in [GP93, Theorem 6.6].

Proposition 5.2. *For all $n \geq 2$, we have $t_n \leq e^{2n}(n-1)^{4n}$.*

Proof. We reproduce here the argument essentially given in [GP93]. Let $\mathbf{p}_i = (p_{i,x}, p_{i,y})$, for $1 \leq i \leq n$, represent an unknown set of n points in \mathbb{R}^2 , indexed by $\{1, \dots, n\}$. As already noticed in the proof of Lemma 2.4, for $i < j < k$ in $\{1, \dots, n\}$, we have

$$\chi(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k) = \text{sgn} \left[\det \begin{pmatrix} p_{i,x} & p_{j,x} & p_{k,x} \\ p_{i,y} & p_{j,y} & p_{k,y} \\ 1 & 1 & 1 \end{pmatrix} \right].$$

The determinant in the right-hand side is a polynomial of degree 2 in the variables $\{p_{i,x}, p_{i,y}, 1 \leq i \leq n\}$, which we denote $D_{i,j,k}$. A realizable chirotope on $\{1, \dots, n\}$ corresponds to a *sign pattern* of the collection $(D_{i,j,k})_{i < j < k}$ of polynomials, i.e. a choice of a sign for each $D_{i,j,k}$ which can be realized by some real specialization of the variables $\{p_{i,x}, p_{i,y}, 1 \leq i \leq n\}$. From a result of Warren [War68], the number of such sign patterns for general collection of polynomials is bounded by $(\frac{4edm}{N})^N$, where d is the degree of the polynomials, m their number and N the number of variables. In our case $d = 2$, $m = \binom{n}{3}$ and $N = 2n$, and we get

$$t_n \leq \left(4e \frac{2\binom{n}{3}}{2n} \right)^{2n}$$

The lemma follows using the simple bound $\binom{n}{3} \leq n(n-1)^2/6$. □

The following lemma is implicit in the discussion preceding Theorem 6.1 in [GP93].

Lemma 5.3. *There exists a universal constant K in $(0, 1/8]$ such that, for all $n \geq 3$, we have $t_{n+1} \geq K n^4 t_n$.*

Sketch of proof. The number t_{n+1} rewrites as the sum, over all realizable chirotopes χ on $[n]$, of the number of extensions of χ , meaning the number of realizable chirotopes on $[n+1]$ whose restriction to (triples of) $[n]$ is χ . Fix a geometric realization P of χ ; the number of extensions of χ is bounded from below by the number of chirotopes realized by 1-point extensions of P . This is exactly the number of cells in the arrangement of the $\binom{n}{2}$ lines through pairs of points of P , which is [Zas97, p. 65]

$$\binom{\binom{n}{2}}{2} + \binom{n}{2} + 1 - n \binom{n-2}{2} = \frac{n^4}{8} - \frac{r(n)}{8},$$

with $r(n) = 6n^3 - 23n^2 + 26n - 8 \leq \frac{n^4}{2}$ for $n \geq 3$, by a simple function analysis. The statement follows from this bound by taking $K = \frac{1}{16}$. \square

5.3. Some basic relations between numbers of chirotopes. Let t_n^* be the number of realizable chirotopes on $\{1, \dots, n\}$, in which n is an extreme element. We now relate d_n , t_n and t_n^* , starting with the last two.

Lemma 5.4. *For each $n \geq 3$, we have $\frac{3}{n}t_n \leq t_n^* \leq \frac{4}{n}t_n$.*

Proof. For a chirotope χ , we denote by $\text{ext}(\chi)$ its number of extreme elements. By symmetry, we have:

$$\sum_{\chi \in \mathcal{X}_n} \text{ext}(\chi) = |\{(\chi, i) \in \mathcal{X}_n \times [n] : i \text{ is extreme in } \chi\}| = nt_n^*.$$

Hence,

$$\frac{t_n^*}{t_n} = \frac{1}{nt_n} \sum_{\chi \in \mathcal{X}_n} \text{ext}(\chi) = \frac{1}{n} \mathbb{E}[\text{ext}(\chi)],$$

where χ is chosen uniformly at random in \mathcal{X}_n . By [GW23, Theorem 1.2],

$$\mathbb{E}[\text{ext}(\chi)] = 4 - \frac{8}{n^2 - n + 2},$$

which is between 3 and 4. \square

Now, let us turn our attention to d_n .

Lemma 5.5. *For each $n \geq 5$, we have*

$$(16) \quad 3n(t_{n-1} - d_{n-1}) \leq d_n \leq \sum_{\substack{n_1+n_2=n \\ n_1, n_2 \geq 2}} \binom{n}{n_1} t_{n_1+1}^* t_{n_2+1}^*.$$

Proof. Let \mathcal{X}_n^* denote the set of realizable chirotopes on $[n] \stackrel{\text{def}}{=} \{1, 2, \dots, n\}$ such that n is extreme, and \mathcal{D}_n the set of decomposable realizable chirotopes of size n . We define an application ϕ that associates to a pair (χ, M) – where $\chi \in \mathcal{D}_n$ and $M \subseteq [n]$ is a nontrivial module of χ – a tuple $\phi(\chi, M) = (\chi_1, \chi_2, M)$ constructed as follows. Letting $n_1 \stackrel{\text{def}}{=} |M|$ and $n_2 \stackrel{\text{def}}{=} n - n_1$,

- χ_1 is the chirotope obtained by replacing the elements of $[n] \setminus M$ by a single element labeled $n_1 + 1$, and relabeling M from 1 to n_1 consistently with the order of M ;
- χ_2 is the chirotope obtained, similarly, by replacing the elements of M by a single element labeled $n_2 + 1$, and relabeling $[n] \setminus M$ from 1 to n_2 consistently.

From Proposition 3.2 and Observation 3.6, it holds that n_1+1 and n_2+1 are extreme elements in χ_1 and χ_2 respectively, i.e. $\chi_1 \in \mathcal{X}_{n_1+1}^*$ and $\chi_2 \in \mathcal{X}_{n_2+1}^*$. We claim that ϕ is bijective. Indeed, take a triple (χ_1, χ_2, M) , with $\chi_1 \in \mathcal{X}_{n_1+1}^*$, $\chi_2 \in \mathcal{X}_{n_2+1}^*$ and $M \subseteq [n]$. Let us define χ_1^M as the chirotope over $M \cup \{x^*\}$ obtained by renaming, in χ_1 , the (extreme) element n_1+1 as x^* and $[n_1]$ as M , increasingly. The chirotope $\chi_2^{[n] \setminus M}$ is defined similarly with label set $([n] \setminus M) \cup \{y^*\}$. Finally, we set $\chi = \chi_1^M \bowtie_{x^*y^*} \chi_2^{[n] \setminus M}$ and define $\Psi(\chi_1, \chi_2, M) = (\chi, M)$. One readily checks that ϕ and Ψ are inverse of each other, showing that ϕ is bijective.

As a consequence, we have

$$d_n \leq |\{(\chi, M) : \chi \in \mathcal{D}_n, M \text{ nontrivial module of } \chi\}| = \sum_{\substack{n_1+n_2=n \\ n_1, n_2 \geq 2}} \binom{n}{n_1} t_{n_1+1}^* t_{n_2+1}^*,$$

proving the upper bound in Eq. (16).

On the other hand, the chirotopes $\chi_1 \bowtie_{x^*y^*} \chi_2$ are different for any $\chi_1 \in \mathcal{X}_3$ and $\chi_2 \in \mathcal{X}_{n-1} \setminus \mathcal{D}_{n-1}$ (this can easily be seen *e.g.* via Theorem 1.3). Since there are 2 chirotopes on $\{1, 2, 3\}$, it follows that $d_n \geq 2 \binom{n}{2} i_{n-1}^*$ where i_{n-1}^* is the number of indecomposable realizable chirotopes on $[n-1]$ in which $n-1$ is an extreme point. Using the same argument as in Lemma 5.4, we have

$$i_{n-1}^* \geq \frac{3}{n-1} (t_{n-1} - d_{n-1}),$$

concluding the proof of the lower bound in Eq. (16). \square

Combining Lemmas 5.4 and 5.5, we get

$$(17) \quad d_n \leq 16 \sum_{\substack{n_1+n_2=n \\ n_1, n_2 \geq 2}} \binom{n}{n_1} \frac{t_{n_1+1} t_{n_2+1}}{(n_1+1)(n_2+1)}.$$

5.4. Bounding the sum. The goal is to prove that the upper bound in Eq. (17) behaves as $\mathcal{O}(n^{-3}t_n)$. We fix some ε in $(0, 1/2)$ independent of n and split the sum into two parts.

$$(18) \quad d_n^{\text{small}} = \sum_{\substack{n_1+n_2=n \\ 2 \leq \min(n_1, n_2) < \varepsilon n}} \binom{n}{n_1} \frac{t_{n_1+1} t_{n_2+1}}{(n_1+1)(n_2+1)}, \text{ and}$$

$$(19) \quad d_n^{\text{large}} = \sum_{\substack{n_1+n_2=n \\ \varepsilon n \leq n_1, n_2 \leq (1-\varepsilon)n}} \binom{n}{n_1} \frac{t_{n_1+1} t_{n_2+1}}{(n_1+1)(n_2+1)}.$$

The part d_n^{small} can be bounded using the classical estimates on the number of realizable chirotopes given in Section 5.2.

Lemma 5.6. *For ε small enough, we have $\frac{d_n^{\text{small}}}{t_n} = \mathcal{O}(n^{-3})$ as n tends to $+\infty$.*

Proof. Assuming $\varepsilon < \frac{1}{2}$, we write

$$n^3 \frac{d_n^{\text{small}}}{t_n} = 2n^3 \sum_{\substack{n_1+n_2=n \\ 2 \leq n_1 < \varepsilon n}} \binom{n}{n_1} \frac{t_{n_1+1}}{n_1+1} \frac{t_{n_2+1}}{(n_2+1)t_n}.$$

Iterating Lemma 5.3, we get

$$\frac{t_{n_2+1}}{t_n} \leq \frac{1}{K^{n_1-1}(n_2+1)^4 \dots (n-1)^4}.$$

On the other hand, Proposition 5.2 yields:

$$\frac{t_{n_1+1}}{n_1+1} \leq e^{2n_1+2} n_1^{4(n_1+1)-1}.$$

Bringing everything together gives us:

$$n^3 \frac{d_n^{\text{small}}}{t_n} \leq 2 \sum_{\substack{n_1+n_2=n \\ 2 \leq n_1 \leq \varepsilon n}} \frac{n^4(n-1) \dots (n_2+1)}{n_1!} \frac{e^{2n_1+2} n_1^{4(n_1+1)-1}}{K^{n_1-1}(n_2+1)^5(n_2+2)^4 \dots (n-1)^4}$$

Using that for n large enough, $n^4 \leq 2(n-2)(n-1)^3$, and $n_1! \geq \left(\frac{n_1}{e}\right)^{n_1}$, we get

$$n^3 \frac{d_n^{\text{small}}}{t_n} \leq 4Ke^2 \sum_{\substack{n_1+n_2=n \\ 2 \leq n_1 \leq \varepsilon n}} \left(\frac{e}{n_1}\right)^{n_1} \frac{e^{2n_1} n_1^{4(n_1+1)-1}}{K^{n_1}(n_2+1)^4(n_2+2)^3 \dots (n-3)^3(n-2)^2}.$$

When $n_2 = n-2$, the product in the denominator $(n_2+1)^4(n_2+2)^3 \dots (n-3)^3(n-2)^2$ should be interpreted as 1 (in this case, we use $n^4 \leq 2(n-1)^4$).

Each of the $3(n-n_2-2) = 3n_1-6$ polynomial factors in the denominator are bounded from below by

$$n_2 \geq (1-\varepsilon)n \geq \frac{1-\varepsilon}{\varepsilon} n_1.$$

Hence, we get, for n large enough,

$$\begin{aligned} n^3 \frac{d_n^{\text{small}}}{t_n} &\leq 4Ke^2 \left(\frac{1-\varepsilon}{\varepsilon}\right)^6 \sum_{2 \leq n_1 \leq \varepsilon n} \left(\frac{e^3}{K} \left(\frac{\varepsilon}{1-\varepsilon}\right)^3\right)^{n_1} \frac{n_1^{4(n_1+1)-1}}{n_1^{n_1} n_1^{3n_1-6}} \\ &\leq 4Ke^2 \left(\frac{1-\varepsilon}{\varepsilon}\right)^6 \sum_{n_1 \geq 2} \left(\frac{e^3}{K} \left(\frac{\varepsilon}{1-\varepsilon}\right)^3\right)^{n_1} n_1^9. \end{aligned}$$

For ε small enough, we have a convergent series, showing that the right-hand size is bounded by a finite constant. \square

We note that the above strategy (via Lemma 5.3 and Proposition 5.2) would even fail to show that $t_{n/2+1}^2 \leq t_n$, and can therefore not be used to control d_n^{large} . In the next section, we will prove the following lemma with a completely different method.

Lemma 5.7. *For $n \geq \frac{3}{\varepsilon}$, we have $t_n \geq \frac{3K\varepsilon^6}{4} n^3 d_n^{\text{large}}$, where K is the constant introduced in Lemma 5.3.*

Proof of Theorem 1.5. Combining Eqs. (17) to (19) and Lemmas 5.6 and 5.7, we find that $\frac{d_n}{t_n} = \mathcal{O}(n^{-3})$, showing the upper bound in Theorem 1.5. The lower bound follows from Lemma 5.5, using that $\frac{d_{n-1}}{t_{n-1}} = \mathcal{O}(n^{-3})$. \square

6. QUASI-MODULES AND THE PROOF OF LEMMA 5.7

Lemma 5.7 gives a lower bound on t_n in terms of products $t_{n_1+1}t_{n_2+1}$ where $n_1+n_2 = n$. We prove it by constructing, given two small realizable chirotopes, a larger one via a generalized substitution operation “along a segment” (defined in Section 6.2). This operation is not one-to-one, but the redundancy can be controlled using a notion of “quasi-modules” (defined in Section 6.1).

6.1. Quasi-modules. Let χ be a chirotope⁵ on a set X . Let $a, b \in X$ and let Y be a subset of X disjoint from $\{a, b\}$. The line ab splits Y into $Y = Y_{-1} \uplus Y_1$, where, for $i \in \{-1, 1\}$,

$$Y_i = \{y \in Y, \chi(a, b, y) = i\}.$$

We call the set $\{Y_{-1}, Y_1\}$ the *bipartition* of Y induced by a and b . Note that although switching a and b exchanges Y_{-1} and Y_1 , the induced bipartition is the same. A bipartition is called nontrivial if *both* Y_{-1} and Y_1 are nonempty. This gives rise to a variant of the notion of modules.

Definition 6.1. *A subset W of X is called a quasi-module of a chirotope χ on X if every pair $\{a, b\}$ of elements of W defines the same bipartition of $X \setminus W$, and if this bipartition is nontrivial (meaning that neither part is empty).*

We compare the above definition to our earlier definition of module (see Definition 3.5). If $M \subset X$ is a module of a chirotope χ , then every pair of elements of M defines the same bipartition on $X \setminus M$. However, this bipartition is trivial, so a module is not a quasi-module. Another difference is that the definition of module is symmetric in M and $X \setminus M$, while that of quasi-module is not.

An example of quasi-module is given on Figure 7. On the other hand, consider a chirotope χ on X with exactly three extreme elements $\{a, b, c\}$. Each of $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$ defines the same bipartition $\{X \setminus \{a, b, c\}, \emptyset\}$. Yet, the set $\{a, b, c\}$ is not a quasi-module as it fails the nontriviality condition.

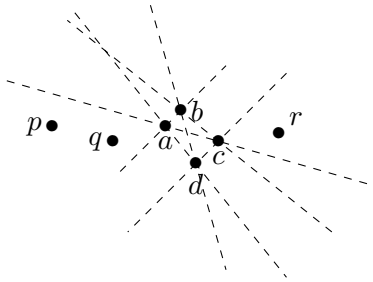


FIGURE 7. Every line of two points in $W = \{a, b, c, d\}$ defines the same bipartition of points in $\{p, q, r\}$, so W is a quasi-module. On the other hand, $\{a, b, c\}$ is not a quasi-module of that chirotope: indeed, the line ab induces the bipartition $\{\{p, q\}, \{d, r\}\}$, while the line ac induces $\{\{p, q, d\}, \{r\}\}$.

It will be useful to control the number of quasi-modules of macroscopic size of any given chirotope.

⁵Even though we are interested in realizable chirotope, the results in Section 6.1 are valid for abstract chirotopes in general, and thus stated in this setting.

Lemma 6.2. *For every $\varepsilon > 0$ and for every $n \geq 1$ such that $\varepsilon n \geq 2$, every chirotope of size n has at most n^2/ε quasi-modules of size at least εn .*

Proof. Fix a chirotope χ on a set X of size n . We will use a double-counting argument based on an upper bound on the number of pairs (W, a) , where W is a quasi-module and a a point of W .

To construct such triples, we start by choosing a point a in X . We label the elements of $X \setminus \{a\}$ as x_0, x_1, \dots, x_{n-2} as follows. We choose $x_0 \neq a$ in an arbitrary way. Then we start with the line ax_0 and rotate it in clockwise direction around a . The i -th point encountered gets the label x_i . Note that we rotate the whole line (infinite in both directions) and not only a half-line: in particular, x_1 can be on either side of the line ax_0 . An example is given on Figure 8. The resulting labelling of $X \setminus \{a\}$ depends on the chosen point x_0 (namely, changing x_0 results in a cyclic shift of the labels).

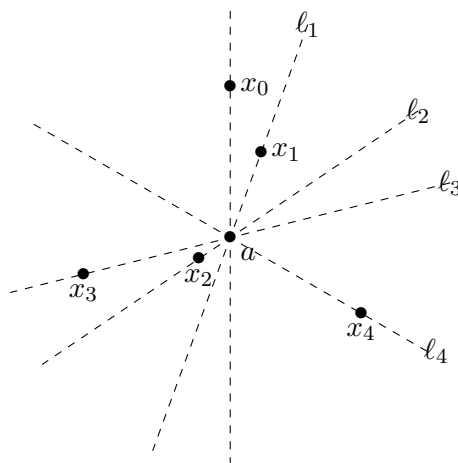


FIGURE 8. Turning the line ax_0 clockwise around a reaches ℓ_1, ℓ_2, ℓ_3 and ℓ_4 in this order, yielding the labels x_1, x_2, x_3 and x_4 .

Consider a quasi-module W containing a . Assume that W contains x_i and x_j for some $i < j$. For any h, k such that $i < h < j < k$ or $k < i < h < j$, the lines ax_i and ax_j induce different bipartitions of $\{x_h, x_k\}$, see Fig 9. This implies that either x_h or x_k is in W (possibly both). Since this holds for any h and k such that $i < h < j < k$ or $k < i < h < j$, any quasi-module W containing a is of the form

$$(20) \quad W = \{a\} \cup \{x_i, i \in I\},$$

where I is a *circular interval* of $\{0, 1, \dots, n-2\}$ (The converse is not true in general, as our construction does not control the bipartitions of $X \setminus W$ induced by lines $x_s x_t$ for x_s, x_t in W . In the example of Figure 8, $\{a, x_1, x_2, x_3\}$ is a quasi-module, but $\{a, x_3, x_4\}$ is not.) It follows that at most n^2 quasi-modules W contain a given point a . There are thus at most n^3 pairs (W, a) , where W is a quasi-module and a a point of W .

If we denote w_k the number of quasi-modules of size k of the chirotope χ , then the number of pairs (W, a) as above is $\sum_{k=0}^n k w_k$. Summing up, we have

$$\sum_{k \geq \varepsilon n} w_k \leq \frac{1}{\varepsilon n} \sum_{k=0}^n k w_k \leq \frac{n^3}{\varepsilon n} = \frac{n^2}{\varepsilon},$$

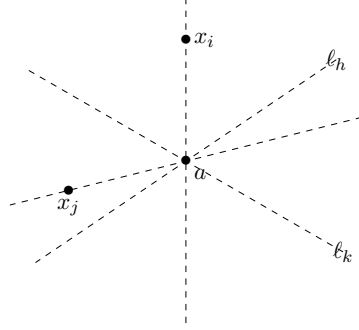


FIGURE 9. The lines ax_i and ax_j divide the plane into four quadrants. These lines induce the same bipartition of $\{x_h, x_k\}$ if and only if these two points lie in equal or opposite quadrants. The condition $i < h < j < k$ (or $k < i < h < j$) and the labelling scheme of $X \setminus \{a\}$ ensures that this is not possible if x_h and x_k lie respectively on ℓ_h and ℓ_k .

as announced. □

We conclude this section with the following lemma, which will be useful later.

Lemma 6.3. *Let χ be a chirotope on X and W be a quasi-module of χ . There exists a unique pair $\{w_1, w_2\} \subseteq W$ such that for all $a \in X \setminus W$ we have*

$$(21) \quad \forall w \in W \setminus \{w_1, w_2\}, \quad \chi(a, w_1, w) = \chi(a, w, w_2) \text{ and is independent of } w.$$

Proof. The statement is trivial if W has size two. Otherwise, fix an arbitrary point b in $X \setminus W$. We claim that b is extremal in the restriction of χ to $W \cup \{b\}$. Otherwise, by Lemma 2.1, b would be contained in a triangle $w_1w_2w_3$ of points in W i.e. one could find w_1, w_2, w_3 in W such that

$$\chi(b, w_1, w_2) = \chi(b, w_2, w_3) = \chi(b, w_3, w_1) = 1.$$

By the interiority axiom, this implies $\chi(w_1, w_2, w_3) = 1$. But from the definition of quasi-module, there must exist a point $c \in X \setminus W$ such that, for all w, w' in W , $\chi(b, w, w') = \chi(c, w', w)$. In particular,

$$\chi(c, w_2, w_1) = \chi(c, w_3, w_2) = \chi(c, w_1, w_3) = 1.$$

Using again the interiority axiom, we get $\chi(w_2, w_1, w_3) = 1$, leading to a contradiction. Hence, b is indeed extremal in the restriction of χ to $W \cup \{b\}$.

Therefore, by Lemma 2.2, the relation $w <_b w' \Leftrightarrow \chi(b, w, w') = 1$ defines a strict total order on W . Call w_1 and w_2 the minimal and maximal elements of W for this order relation. For all $w \in W \setminus \{w_1, w_2\}$, we have $w_1 <_b w <_b w_2$, i.e.

$$\chi(b, w_1, w) = \chi(b, w, w_2) = 1.$$

In particular, Eq. (21) holds for $a = b$. It also for other elements $a \in X \setminus W$, using the fact that W is a quasi-module. This proves the existence of the pair (w_1, w_2) .

The uniqueness also follows from the fact that $<_b$ is a total order. Only the pair constituted by the two extrema of this total order can satisfy Eq. (21) for $a = b$. □

We refer to the two elements $\{w_1, w_2\}$ given by Lemma 6.3 as the *antipodal elements* of the quasi-module W . For example, the antipodal elements of the weak module $\{a, b, c, d\}$ in Figure 7 are b and d .

6.2. Substituting a realizable chirotope along a segment. Let χ be a realizable chirotope on a set X and let $a \in X$ be extreme for χ . As we saw in Lemma 2.2, and immediately after, we can define a_+ (resp. a_-) as the unique element of X such that $\chi(a, a_+, x) = 1$ for every $x \in X \setminus \{a, a_+\}$ (resp. such that $\chi(a_-, a, x) = 1$ for every $x \in X \setminus \{a, a_-\}$). (Geometrically, a_-, a and a_+ are consecutive points on the convex hull of χ .) We refer to a_+ (resp. a_-) as the *successor* (resp. the *predecessor*) of a on the convex hull of χ .

Definition 6.4. Let χ be a realizable chirotope on a set X and let a be an element of X extreme for χ . Let a_+ and a_- be, respectively, the successor and predecessor of a on the convex hull of χ . A δ -squeezing of (χ, a) is a realization \mathcal{P} of χ such that:

- $\mathbf{p}_a = (0, 1)$, $\mathbf{p}_{a_+} = (-1, 0)$ and $\mathbf{p}_{a_-} = (1, 0)$,
- the y -coordinates of every other point of \mathcal{P} is in $[-\delta, \delta]$, and
- every line going through two points of $\mathcal{P} \setminus \{\mathbf{p}_a\}$ has slope in $[-\delta, \delta]$.

This definition is illustrated in Figure 10.

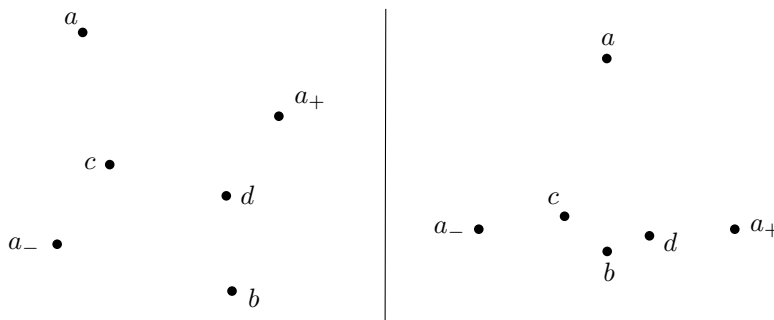


FIGURE 10. Two realization of the same chirotope χ , the second one being a δ -squeezing of (χ, a) for $\delta \approx .5$ (the slope of the line bc is close to $.5$).

Lemma 6.5. Let χ be a realizable chirotope on a set X and let a be an element of X extreme for χ . For every $\delta > 0$, the pair (χ, a) admits a δ -squeezing.

Proof. Let \mathcal{Q} be a realization of χ in which \mathbf{q}_a is in an unbounded cell Σ of the line arrangement $\{(\mathbf{q}_y, \mathbf{q}_z), z, y \in X \setminus \{a\}\}$ (such a realization exists by Lemma 2.4). Applying a translation, a rotation and a scaling, we can further assume that $\mathbf{q}_{a_+} = (-1, 0)$ and $\mathbf{q}_{a_-} = (1, 0)$ while keeping \mathbf{q}_a in an unbounded cell Σ ; note that \mathbf{q}_a (and thus Σ) lies above the x -axis. Let $\vec{u} = (u_x, u_y)$ be a direction such that $\mathbf{q}_a + \mathbb{R}_{\geq 0}\vec{u} \subset \Sigma$. Note that u_y can be chosen to be positive.

For any $\alpha \in \mathbb{R}$ and $\beta > 0$, the transform $f_{\alpha, \beta} : (x, y) \mapsto (x + \alpha y, \beta y)$ preserves orientations and leaves the x -axis fixed. Moreover, for every point (x_0, y_0) with $y_0 > 0$, there exists $\alpha \in \mathbb{R}$ such that $f_{\alpha, \frac{1}{y_0}}(x_0, y_0) = (0, 1)$ (namely, $\alpha = -\frac{x_0}{y_0}$).

Now, let $t \in \mathbb{R}_{\geq 0}$ and consider the point $\mathbf{p}(t) \stackrel{\text{def}}{=} \mathbf{q}_a + t\vec{u}$. Let α_t and β_t be such that $f_{\alpha_t, \beta_t}(\mathbf{p}(t)) = (0, 1)$ and consider the point set

$$\mathcal{P}(t) \stackrel{\text{def}}{=} \{f_{\alpha_t, \beta_t}(\mathbf{q}) : \mathbf{q} \in \{\mathbf{p}(t)\} \cup \mathcal{Q} \setminus \{\mathbf{q}_a\}\}$$

still labeled by X (with $\mathbf{p}(t)$ labeled by a and all other labels unchanged). Since $\mathbf{p}(t)$ and \mathbf{q}_a are both in Σ , $\{\mathbf{p}(t)\} \cup \mathcal{Q} \setminus \{\mathbf{q}_a\}$ has the same chirotope as \mathcal{Q} , that is χ . Since f_{α_t, β_t} preserves orientations, $\mathcal{P}(t)$ is also a geometric realization of χ .

It is easy to check that $(\alpha_t, \beta_t) \rightarrow (-u_x/u_y, 0)$ as $t \rightarrow \infty$. Consequently, the y -coordinate of every point of $\mathcal{P}(t) \setminus \{(0, 1)\}$ goes to 0 as $t \rightarrow \infty$. Also, the slope of every line going through two points of $\mathcal{P}(t) \setminus \{(0, 1)\}$ goes to 0 as $t \rightarrow \infty$. Hence, for every $\delta > 0$, for t large enough, $\mathcal{P}(t)$ is a δ -squeezing of (χ, a) . \square

We now use δ -squeezings to construct larger realizable chirotopes out of smaller ones.

- (1) We start with two realizable chirotopes χ on $X \cup \{x_1^*, x_2^*\}$ and ξ on $Y \cup \{y^*\}$, where X and Y are disjoint and $x_1^*x_2^*$ is not an extreme edge⁶ of χ , while y^* is extreme in ξ .
- (2) We pick some realization \mathcal{P} of χ with $\mathbf{p}_{x_1^*} = (-1, 0)$ and $\mathbf{p}_{x_2^*} = (1, 0)$, and some δ -squeezing \mathcal{Q} of (ξ, y^*) , for some $\delta > 0$ small enough.
- (3) We let κ denote the chirotope on $X \cup Y$ realized by the point set $(\mathcal{P} \setminus \{\mathbf{p}_{x_1^*}, \mathbf{p}_{x_2^*}\}) \cup (\mathcal{Q} \setminus \{q_{y^*}\})$.

For $\delta > 0$ small enough, Y is a quasi-module for κ and (the labels of) $(-1, 0)$ and $(1, 0)$ are its antipodal elements. We assume this is the case from now on. Let us stress that the chirotope κ depends not only on the chirotopes χ and ξ , but on the actual realizations \mathcal{P} and \mathcal{Q} (see Figure 11). We refer to the above construction as *substituting ξ into χ along the segment $x_1^*x_2^*$ squeezing ξ orthogonally to y^** .

Lemma 6.6. *For any realizable chirotope κ on Z and any quasi-module Y of κ , there exist a single chirotope χ on $(Z \setminus Y) \cup \{x_1^*, x_2^*\}$ and at most two chirotopes ξ on $Y \cup \{y^*\}$ such that κ can be obtained by the above construction (up to renaming the symbols x_1^*, x_2^* and y^*).*

Proof. Suppose κ and Y are obtained by the above construction. Let $\{x_1^*, x_2^*\}$ denote the antipodal elements of Y , defined through Lemma 6.3. Then, χ must coincide with the restriction of κ to $(Z \setminus Y) \cup \{x_1^*, x_2^*\}$, so it is unique (up to renaming the symbols x_1^* and x_2^*).

Since $x_1^*, x_2^* \in Y$ and Y is a quasi-module of κ , this pair defines a nontrivial bipartition, so there exist y_+^* and y_-^* in $(Z \setminus Y)$ such that $\chi(x_1^*, x_2^*, y_-^*) = -1$ and $\chi(x_1^*, x_2^*, y_+^*) = +1$. By construction, ξ must coincide with the restriction of κ to $Y \cup \{y_-^*\}$ or to $Y \cup \{y_+^*\}$. There is therefore at most two choices for the chirotope ξ on $Y \cup \{y^*\}$ (up to renaming the symbol y^*). \square

We now use this construction for counting purposes. Let $w_{n,k}$ denote the number of pairs (χ, W) where χ is a realizable chirotope on $[n]$ and W is a quasi-module of χ of size k . Recall that t_n^* denotes the number of realizable chirotopes on $[n]$ in which n is extreme. Let t_n^\dagger denote the number of realizable chirotopes on $[n]$ in which n and $n-1$ do not form an extreme edge.

Lemma 6.7. *For any integers $n_1 \geq 2$, $n_2 \geq 2$, we have $w_{n_1+n_2, n_2} \geq \frac{1}{2} \binom{n_1+n_2}{n_1} t_{n_1+2}^\dagger t_{n_2+1}^*$.*

Proof. Let $n = n_1 + n_2$. Start with a triple (I, χ, ξ) where $I \subset [n]$ has size n_1 , χ is a realizable chirotope on $[n_1 + 2]$ such that $n_1 + 1$ and $n_1 + 2$ do not form an extreme edge, and ξ is a realizable chirotope on $[n_2 + 1]$ in which $n_2 + 1$ is extreme. With our notations, the number of such triples is $\binom{n_1+n_2}{n_1} t_{n_1+2}^\dagger t_{n_2+1}^*$.

⁶An extreme edge is an edge between two consecutive points on the convex hull.

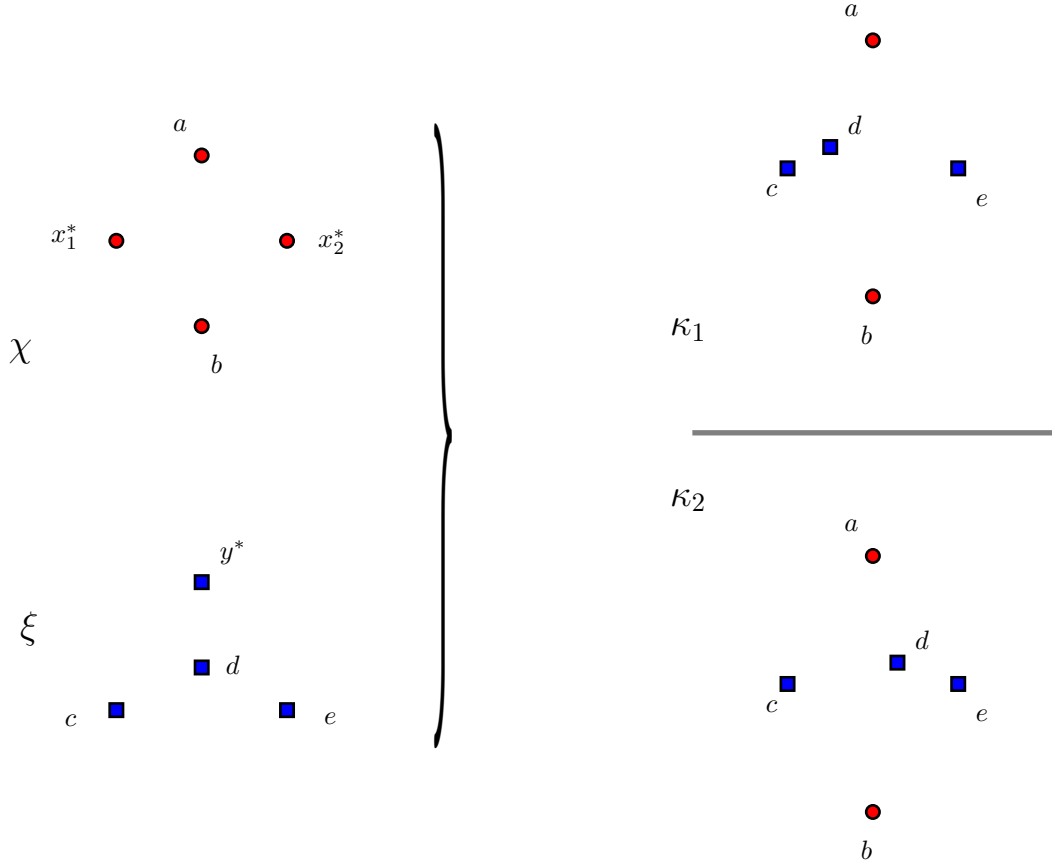


FIGURE 11. Two chirotopes κ_1 and κ_2 obtained from the same substitution (substituting ξ into χ along the segment $x_1^*x_2^*$ squeezing ξ orthogonally to y^*), but using different realizations. Observe that d is not on the same side of the line ab in κ_1 and κ_2 .

Let x_1^*, x_2^* and y^* be formal symbols not in $[n]$. For each such triple (I, χ, ξ) , we do the following operation:

- We let $\tilde{\chi}$ denote the chirotope on $I \cup \{x_1^*, x_2^*\}$ obtained from χ by relabeling $[n_1]$ increasingly into I and mapping $n_1 + 1 \mapsto x_1^*$ and $n_1 + 2 \mapsto x_2^*$.
- We let $\tilde{\xi}$ denote the chirotope on $([n] \setminus I) \cup \{y^*\}$ obtained from ξ by relabeling $[n_2]$ increasingly into $[n] \setminus I$ and mapping $n_2 + 1 \mapsto y^*$.
- We substitute $\tilde{\xi}$ into $\tilde{\chi}$ along the segment $x_1^*x_2^*$ squeezing $\tilde{\xi}$ orthogonally to y^* .

The result of this operation is a pair (κ, W) where κ is a realizable chirotope on $[n]$ and W is a quasi-module. By Lemma 6.6, each pair (κ, W) is the image of at most two triples (I, χ, ξ) , which concludes the proof. \square

We can now complete the proof of Lemma 5.7, that is, that for $n \geq \frac{3}{\varepsilon}$,

$$t_n \geq \frac{3K\varepsilon^7}{4} n^3 d_n^{\text{large}} \quad \text{where} \quad d_n^{\text{large}} = \sum_{\substack{n_1+n_2=n \\ \varepsilon n \leq n_1, n_2 \leq (1-\varepsilon)n}} \binom{n}{n_1} \frac{t_{n_1+1} t_{n_2+1}}{(n_1+1)(n_2+1)}.$$

Proof of Lemma 5.7. Lemma 5.4 gives that $t_n^* \geq \frac{3}{n}t_n$. Furthermore, adapting the proof of Lemma 5.4 yields

$$\frac{t_n^\dagger}{t_n} = \frac{2}{n(n-1)} \mathbb{E}[\text{nee}(\chi)]$$

where $\text{nee}(\chi)$ denotes the number of pairs of points of χ which are not extreme edges. For $n \geq 5$, we have

$$\mathbb{E}[\text{nee}(\chi)] \geq \binom{n}{2} - n \geq \frac{1}{2} \binom{n}{2}$$

and hence $t_n^\dagger \geq \frac{1}{2}t_n$. Now, by Lemmas 6.2 and 6.7 we have

$$\begin{aligned} \frac{n^2}{\varepsilon} t_n &\geq \sum_{\substack{n_1+n_2=n \\ \varepsilon n \leq n_1, n_2 \leq (1-\varepsilon)n}} w_{n,n_2} \geq \sum_{\substack{n_1+n_2=n \\ \varepsilon n \leq n_1, n_2 \leq (1-\varepsilon)n}} \frac{1}{2} \binom{n}{n_1} t_{n_1+2}^\dagger t_{n_2+1}^* \\ &\geq \sum_{\substack{n_1+n_2=n \\ \varepsilon n \leq n_1, n_2 \leq (1-\varepsilon)n}} \frac{3}{4(n_2+1)} \binom{n}{n_1} t_{n_1+2} t_{n_2+1}. \end{aligned}$$

(The inequality $t_{n_1+2}^\dagger \geq \frac{1}{2}t_{n_1+2}$ requires $n_1+2 \geq 5$, which is ensured by the assumption $n \geq \frac{3}{\varepsilon}$.) By Lemma 5.3, there exists a universal constant $K > 0$ such that, for all $n_1 \geq 2$, we have $t_{n_1+2} \geq K(n_1+1)^4 t_{n_1+1}$. Substituting into the previous inequality and using that $n_1+1 \geq \varepsilon n$, we obtain as desired

$$\begin{aligned} t_n &\geq \frac{\varepsilon}{n^2} \frac{3K}{4} \sum_{\substack{n_1+n_2=n \\ \varepsilon n \leq n_1, n_2 \leq (1-\varepsilon)n}} \binom{n}{n_1} (n_1+1)^5 \frac{t_{n_1+1} t_{n_2+1}}{(n_1+1)(n_2+1)} \\ &\geq \varepsilon^6 n^3 \frac{3K}{4} \sum_{\substack{n_1+n_2=n \\ \varepsilon n \leq n_1, n_2 \leq (1-\varepsilon)n}} \binom{n}{n_1} \frac{t_{n_1+1} t_{n_2+1}}{(n_1+1)(n_2+1)}. \quad \square \end{aligned}$$

7. A BIJECTION TO DECOMPOSE TRIANGULATIONS THROUGH ONE BOWTIE PRODUCT

In this section, we consider a decomposable chirotope $\kappa = \chi_{x^*} \bowtie_{y^*} \xi$, and relate the triangulations of κ to those of χ and ξ . We show that each triangulation of κ projects into triangulations of χ and ξ , and that conversely every pair of triangulations of χ and ξ can be obtained as projections of (possibly many) triangulations of κ . We establish a bijection between these pre-images and maximal⁷ sets of noncrossing edges between appropriate sets.

Note: We recall from the introduction that the notion of crossing edges can be read on the chirotope, giving meaning to the notion that xy and zt cross in a (possibly abstract) chirotope χ , if x, y, z and t are elements of the ground set of χ . If there is no ambiguity on which chirotope χ we are considering, we shall simply say that xy and zt cross.

7.1. Statement of the bijection. In the whole section, we let $\kappa = \chi_{x^*} \bowtie_{y^*} \xi$ where χ and ξ are chirotopes on $X \cup \{x^*\}$ and $Y \cup \{y^*\}$, with X and Y disjoint, $x^* \notin X$ being extreme in χ and $y^* \notin Y$ being extreme in ξ . As a preamble, note that for x_1, x_2, x_3, x_4 in X , the pairs x_1x_2 and x_3x_4 cross in χ if and only if they cross in κ , so we shall simply say that x_1x_2 and x_3x_4 cross, without making the underlying chirotope explicit.

⁷In what follows, inclusion-maximality is equivalent to being of maximal cardinality.

Let $\pi_{Y \rightarrow x^*}$ denote the map that sends every element of Y to x^* . We extend this map to sets of edges by putting, for any triangulation T of κ ,

$$\pi_{Y \rightarrow x^*}(T) \stackrel{\text{def}}{=} \{\{\pi_{Y \rightarrow x^*}(a), \pi_{Y \rightarrow x^*}(b)\} : \{a, b\} \in T\} \setminus \{\{x^*, x^*\}\}.$$

We define $\pi_{X \rightarrow y^*}$ analogously. Given a set T of edges of $\chi_{x^*} \bowtie_{y^*} \xi$, not necessarily a triangulation, and two sets A and B , we let T_{AB} denote the set of edges of T with one element in A and the other in B ; we say that such an edge is *between* A and B . As illustrated in Figure 12, any triangulation T of κ decomposes into triangulations of χ and ξ and a set T_{XY} of noncrossing edges. The aim of this section is to prove that this decomposition is a bijection.

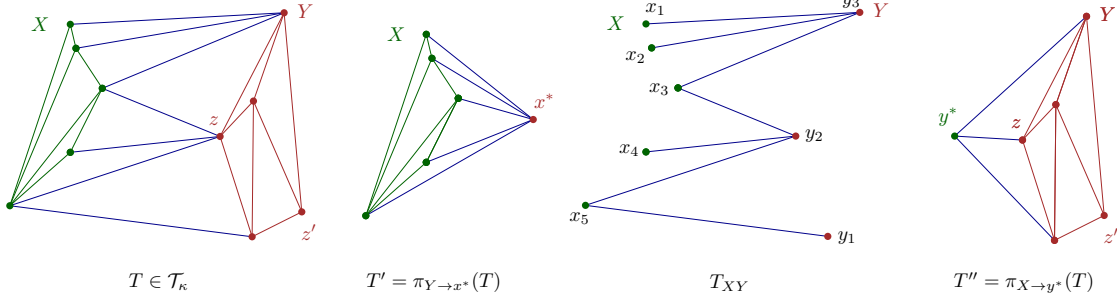


FIGURE 12. A triangulation T of $\kappa = \chi_{x^*} \bowtie_{y^*} \xi$, the projection of T on $X \cup \{x^*\}$, the set T_{XY} , and the projection of T on $Y \cup \{y^*\}$.

Recall that \mathcal{T}_α denotes the set of triangulations of a chirotope α . Given $T \in \mathcal{T}_\alpha$ and an element x of the ground set of α , we write $N_T(x)$ for the neighborhood of x in T and $\deg_T(x)$ for the degree of x in T . We extend the notation $N_T(x)$ and $\deg_T(x)$ in the obvious manner when T is a (not necessarily maximal) set of noncrossing edges.

Proposition 7.1. *For any triangulation T of κ , the sets $T' \stackrel{\text{def}}{=} \pi_{Y \rightarrow x^*}(T)$ and $T'' \stackrel{\text{def}}{=} \pi_{X \rightarrow y^*}(T)$ are triangulations of χ and ξ , respectively, and T_{XY} is a maximal set of noncrossing edges between $N_{T'}(x^*)$ and $N_{T''}(y^*)$.*

Conversely, for any triangulations T' of χ and T'' of ξ and any maximal set H of noncrossing edges between $N_{T'}(x^)$ and $N_{T''}(y^*)$, there exists a unique triangulation $T \in \mathcal{T}_\kappa$ such that $T' = \pi_{Y \rightarrow x^*}(T)$, $T'' = \pi_{X \rightarrow y^*}(T)$ and $H = T_{XY}$.*

The proof of the bijection studies in particular the geometric structure of the sets of noncrossing edges T_{XY} for any triangulation of κ ; from this structure we derive the following combinatorial lemma that will be useful for counting triangulations. (We use the convention $\binom{n}{-1} = \delta_{n,-1}$ for any integer n , where δ is the usual Kronecker symbol.)

Lemma 7.2. *Let $X' \subseteq X$, $Y' \subseteq Y$ be two nonempty subsets of elements of κ , let $0 \leq k \leq |Y'|$, and let z_1, z_2, \dots, z_k be distinct elements of Y' . Then for $i_1, \dots, i_k \geq 1$ there are exactly*

$$\binom{|X'| + |Y'| - (i_1 + \dots + i_k) - 2}{|Y'| - (k + 1)}$$

maximal sets H of noncrossing edges between X' and Y' and satisfying $\deg_H(z_j) = i_j$ for every $j \in [k]$.

In particular, for $k = 0$, we obtain that for any nonempty subsets $X' \subseteq X$ and $Y' \subseteq Y$ there are exactly $\binom{|X'|+|Y'|-2}{|Y'|-1}$ maximal sets H of noncrossing edges in κ .

The rest of this section is devoted to proving Lemma 7.2 and Proposition 7.1.

7.2. Set of noncrossing edges between X and Y . For a chirotope α on a set A , we write $\text{ext}(\alpha)$ for the set of elements of A that are extreme in α . As we saw in Lemma 2.2, the relation defined by

$$\text{for } a, b, c \in A, b <_{\alpha, a} c \Leftrightarrow \alpha(a, b, c) = 1$$

defines a strict total order on $X \setminus \{a\}$ if a is extreme for α . This order allows to characterize crossings of edges between the modules X and Y of κ .

Lemma 7.3. *For any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ with $y_1 <_{\xi, y^*} y_2$, the edges x_1y_1 and x_2y_2 cross in $\chi_{x^*} \bowtie_{y^*} \xi$ if and only if $x_1 <_{\chi, x^*} x_2$.*

Proof. By definition, we have $\chi(x^*, x_1, x_2) = \kappa(x_2, y_2, x_1) = -\kappa(x_1, y_1, x_2)$ and similarly, we have $\xi(y^*, y_1, y_2) = -\kappa(x_2, y_2, y_1) = \kappa(x_1, y_1, y_2)$. We also have $\xi(y^*, y_1, y_2) = 1$ by hypothesis. Hence if $x_1 <_{\chi, x^*} x_2$, then $\kappa(x_2, y_2, x_1) = 1 = -\kappa(x_2, y_2, y_1)$ and $\kappa(x_1, y_1, x_2) = -1 = -\kappa(x_1, y_1, y_2)$, so that x_1y_1 and x_2y_2 cross in κ . Conversely, if x_1y_1 and x_2y_2 cross, in particular $\kappa(x_1, y_1, y_2) = -\kappa(x_1, y_1, x_2)$, hence $1 = \xi(y^*, y_1, y_2) = -\chi(x_1, x^*, x_2) = \chi(x^*, x_1, x_2)$ and $x_1 <_{\chi, x^*} x_2$. This proves the lemma. \square

Lemma 7.4. *Let $X' \subseteq X$, $Y' \subseteq Y$ be two nonempty subsets of elements of κ . Let us write $a = |X'|$, $b = |Y'|$, and $y_1 <_{\xi, y^*} \dots <_{\xi, y^*} y_b$ the elements of Y' . The map sending a set H of edges to $(\deg_H(y_1), \dots, \deg_H(y_b))$ is a bijection between maximal sets of noncrossing edges between X' and Y' and sequences $(d_1, \dots, d_b) \in (\mathbb{N}_{\geq 1})^b$ satisfying $d_1 + \dots + d_b = a + b - 1$.*

For example the set of noncrossing edges of the third picture in Figure 12 is associated with the triple $(1, 3, 3)$.

Proof. Let us write $x_1 <_{\chi, x^*} \dots <_{\chi, x^*} x_a$ the elements of X' , ordered by $<_{\chi, x^*}$. Let $\Lambda_{a,b}$ denote the set of sequences $(d_1, \dots, d_b) \in (\mathbb{N}_{\geq 1})^b$ satisfying $d_1 + \dots + d_b = a + b - 1$.

Let H be a maximal set of noncrossing edges between X' and Y' . For $j \in [b]$, we let \mathcal{I}_j^H denote the set of indices i such that x_iy_j is an edge of H . By Lemma 7.3, for $j < j'$, elements of $\mathcal{I}_{j'}^H$ are smaller than or equal to elements of \mathcal{I}_j^H . But, using again that H is maximal, we know that every element of X' or Y' is the endpoint of an edge, so that each \mathcal{I}_j^H is a non-empty interval, say $[m_j, M_j]$. Moreover, consecutive \mathcal{I}_j^H must have a common endpoint (otherwise H would not be maximal). In other words, we have

$$1 = m_b \leq M_b = m_{b-1} \leq M_{b-1} = \dots = m_2 \leq M_2 = m_1 \leq M_1 = a.$$

Since $\deg_H(y_j) = M_j - m_j + 1 = m_{j-1} - m_j + 1$ (with the convention $m_0 = a$), it follows that the map sending H to $(\deg_H(y_1), \dots, \deg_H(y_b))$ takes value in $\Lambda_{a,b}$.

Conversely, it is clear that the values $\deg_H(y_j) = M_j - m_j + 1$ uniquely determine the tuple (m_1, \dots, m_b) , which in turn uniquely determines $(\mathcal{I}_1^H, \dots, \mathcal{I}_b^H)$ (recall that $M_1 = a$ and $M_j = m_{j-1}$ for $2 \leq j \leq b$), which determines H . Thus the map $H \mapsto (\deg_H(y_1), \dots, \deg_H(y_b))$ is a bijection. \square

We can now prove Lemma 7.2.

Proof of Lemma 7.2. As before, let $a = |X'|$, $b = |Y'|$, and let $y_1 <_{\xi, y^*} \dots <_{\xi, y^*} y_b$ be the elements of Y' . We write $z_j = y_{h_j}$ for some index h_j . By Lemma 7.4, we need to count the number of sequences (d_1, \dots, d_b) satisfying $d_1 + \dots + d_b = a + b - 1$ with $d_i \in \mathbb{Z}_{\geq 1}$, and $d_{h_j} = i_j$ for $j \leq k$ (we allow $k = 0$). These sequences are in bijection with the sequences (n_1, \dots, n_{b-k}) of $b - k$ positive integers summing to $a + b - 1 - (i_1 + \dots + i_k)$. They are sometimes called compositions of $a + b - 1 - (i_1 + \dots + i_k)$ into $b - k$ parts, and are known to be counted by $\binom{a+b-(i_1+\dots+i_k)-2}{b-k-1}$. (With the convention $\binom{n}{-1} = \delta_{n,-1}$, the formula is also correct when $k = b$.) \square

7.3. Proof of the bijection. The goal of this section is to prove Proposition 7.1. We use the notation in the first part of its statement, namely: T is a triangulation of $\kappa = \chi_{x^*} \bowtie_{y^*} \xi$, where χ and ξ are chirotopes on $X \cup \{x^*\}$ and $Y \cup \{y^*\}$, with X and Y disjoint, $x^* \notin X$ being extreme in χ and $y^* \notin Y$ being extreme in ξ . We also recall that $T' = \pi_{Y \rightarrow x^*}(T)$ and $T'' = \pi_{X \rightarrow y^*}(T)$. Additionally we set $X' = N_{T'}(x^*)$ and $Y' = N_{T''}(y^*)$. The set T_{XY} is a subset of T and hence noncrossing by definition. We start by showing that the same is true for $\pi_{Y \rightarrow x^*}(T)$ (and symmetrically $\pi_{X \rightarrow y^*}(T)$).

Lemma 7.5. $\pi_{Y \rightarrow x^*}(T)$ (resp. $\pi_{X \rightarrow y^*}(T)$) is a set of noncrossing edges.

Proof. Recall that whether two edges cross or not in a chirotope α depends only on the restriction of the chirotope α to the four endpoints of these two edges. As χ and κ coincide on X , no two edges of T_{XX} can cross in T' . Furthermore, by definition of $\pi_{Y \rightarrow x^*}$ and κ , the existence of two crossing edges x^*x_1 and x_2x_3 of T' would lead to the existence of an edge $y_1x_1 \in T$ crossing the edge $x_2x_3 \in T$, for some $y_1 \in Y$. This is impossible since T is a set of noncrossing edges in κ . This proves the lemma, the case of $\pi_{X \rightarrow y^*}(T)$ being symmetrical. \square

We still need to prove that T_{XY} , $\pi_{Y \rightarrow x^*}(T)$ and $\pi_{X \rightarrow y^*}(T)$ are maximal. For this, we use cardinality arguments. As a consequence of Lemma 7.4, any maximal set of noncrossing edges between X' and Y' has size $|X'| + |Y'| - 1$. In particular

$$(22) \quad |T_{XY}| \leq |X'| + |Y'| - 1.$$

Besides, a classical consequence of Euler's formula, via double counting, is that any triangulation Δ of a realizable chirotope of size n with h extreme elements has exactly

$$(23) \quad |\Delta| = 3n - h - 3$$

edges. In particular, any family of noncrossing edges of a realizable chirotope of size n with h extreme elements has at most $3n - h - 3$ edges. Eq. (23) also holds for abstract chirotopes since every abstract chirotope can be realized by a *generalized configuration*, that is a set of points in the plane together with an arrangement of pseudolines, each one containing exactly two points [FG17, Theorems 5.2.6].

Let us come back to the setting of Proposition 7.1, and let h_X (resp. h_Y) denote the number of elements of X (resp. Y) that are extreme in κ . As T is a triangulation of κ , we have

$$|T| = 3(|X| + |Y|) - (h_X + h_Y) - 3.$$

By Lemma 7.5, $T' = \pi_{Y \rightarrow x^*}(T)$ and $T'' = \pi_{X \rightarrow y^*}(T)$ are families of noncrossing edges of χ and ξ , hence they have at most as many edges as triangulations of χ and ξ , i.e., using Eq. (23), we have

$$|T'| \leq 3(|X| + 1) - (h_X + 1) - 3 = 3|X| - h_X - 1 \quad \text{and} \quad |T''| \leq 3|Y| - h_Y - 1.$$

By definition of T' , T'' , X' and Y' , we further have

$$|T'| = |T_{XX}| + \deg_{T'}(x^*) = |T_{XX}| + |X'|, \quad |T''| = |T_{YY}| + |Y'|.$$

Hence, recalling Eq. (22), we have the following inequality

$$|T| = |T_{XX}| + |T_{YY}| + |T_{XY}| \leq |T_{XX}| + |T_{YY}| + |X'| + |Y'| - 1 = |T'| + |T''| - 1.$$

so that

$$|T| \leq |T'| + |T''| - 1 \leq 3(|X| + |Y|) - (h_X + h_Y) - 3 = |T|.$$

This implies that every inequality we have proved so far is in fact an equality. In particular, T' (resp. T'') contains the maximal possible number of noncrossing edges of χ (resp. ξ), ensuring that it is a triangulation of χ (resp. ξ); and T_{XY} is a *maximal* set of noncrossing edges between X' and Y' . This proves the first part of Proposition 7.1.

It remains to prove that for any triangulations T' and T'' of, respectively, χ and ξ , and any maximal set H of noncrossing edges between X' and Y' , there exists a unique triangulation $T \in \mathcal{T}_\kappa$ such that $T' = \pi_{Y \rightarrow x^*}(T)$, $T'' = \pi_{X \rightarrow y^*}(T)$ and $H = T_{XY}$.

Let us first argue that such a triangulation T is unique. Indeed, if $\pi_{Y \rightarrow x^*}(T) = T'$ then $T_{XX} = T'_{XX}$ and the edges of T between X and X are determined. Similarly, the edges of T between Y and Y must be determined by $T_{YY} = T''_{YY}$, and the edges between X and Y must be determined by $T_{XY} = H$.

To prove the existence of T , we consider the set of edges $E \stackrel{\text{def}}{=} T'_{XX} \cup T''_{YY} \cup H$. By construction, E satisfies $\pi_{Y \rightarrow x^*}(E) = T'$, $\pi_{X \rightarrow y^*}(E) = T''$ and $E_{XY} = H$, so it suffices to prove that E is a triangulation. We first argue that the size of E satisfies the Eq. (23). Indeed, we have

$$\begin{aligned} |E| &= |T'_{XX}| + |T''_{YY}| + |H| \\ &= (|T'| - \deg_{T'}(x^*)) + (|T''| - \deg_{T''}(y^*)) + (|X'| + |Y'| - 1) \quad \text{by Eq. (22)} \\ &= (|T'| - |X'|) + (|T''| - |Y'|) + (|X'| + |Y'| - 1) \\ &= |T'| + |T''| - 1. \end{aligned}$$

Let h_X (resp. h_Y) denote the number of elements of X (resp. Y) that are extreme in κ . Since T' and T'' are triangulations of respectively χ and ξ , by Eq. (23),

$$|T'| = 3(|X| + 1) - (h_X + 1) - 3 = 3|X| - h_X - 1 \quad \text{and} \quad |T''| = 3|Y| - h_Y - 1.$$

We also have $|T'| = |T'_{XX}| + \deg_{T'}(x^*) = |T'_{XX}| + |X'|$ and $|T''| = |T''_{YY}| + \deg_{T''}(y^*) = |T''_{YY}| + |Y'|$. Therefore,

$$|E| = 3|X| - h_X - 1 + 3|Y| - h_Y - 1 - 1 = 3(|X| + |Y|) - (h_X + h_Y) - 3.$$

We recognize Eq. (23). It now remains to argue that the edges in E are pairwise noncrossing. By hypothesis, the edges in $T'_{XX} \subseteq T'$ are pairwise noncrossing, and similarly for the edges in T''_{YY} , or in H . An edge in T'_{XX} cannot cross an edge in T''_{YY} as X and Y are mutually avoiding sets in κ . Finally, an edge of H cannot cross in κ an edge of T'_{XX} (resp. T''_{YY} by symmetry): suppose by contradiction that $x_1y_1 \in H$ is crossing $x_2x_3 \in T'_{XX}$ in κ . Then it would imply that the edges x_1x^* and x_2x_3 are two crossing edges in T' , which is impossible as T' is a triangulation of χ . Altogether, E is a triangulation of κ . This ends the proof of Proposition 7.1.

8. COUNTING TRIANGULATIONS

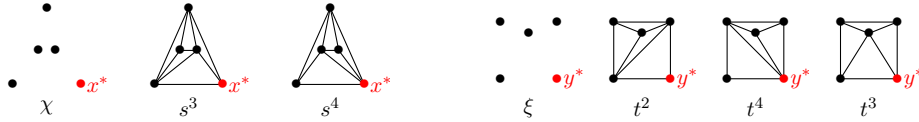
We now explain how to count the triangulations of a chirotope presented by a chirotope tree from information on the triangulations of the chirotopes decorating the nodes.

8.1. Counting the triangulations of a bowtie. Let us first consider the case of a single bowtie $\kappa = \chi_{x^*} \bowtie_{y^*} \xi$. Proposition 7.1 describes the *set* of triangulations of κ in terms of the *sets* of triangulations of χ and ξ . With Lemma 7.2, however, we can *count* the former knowing only the generating polynomials of the latters marking the degrees of the proxies x^* and y^* .

Let $P_{\chi, x^*}(s)$ and $P_{\xi, y^*}(t)$ denote the generating polynomials of the triangulations of χ and ξ recording, respectively, the degrees of the elements x^* and y^* :

$$P_{\chi, x^*}(s) = \sum_{T' \in \mathcal{T}_\chi} s^{\deg_{T'}(x^*)} \quad \text{and} \quad P_{\xi, y^*}(t) = \sum_{T'' \in \mathcal{T}_\xi} t^{\deg_{T''}(y^*)}.$$

Here are examples with $P_{\chi, x^*}(s) = s^3(s+1)$ and $P_{\xi, y^*}(t) = t^2(1+t+t^2)$.



For $\kappa = \chi_{x^*} \bowtie_{y^*} \xi$, using Proposition 7.1 and Lemma 7.2, $|\mathcal{T}_\kappa|$ can be expressed in terms of $P_{\chi, x^*}(s)$ and $P_{\xi, y^*}(t)$, as explained in the following proposition.

Proposition 8.1. *Let $\kappa = \chi_{x^*} \bowtie_{y^*} \xi$ where χ and ξ are two chirotopes on disjoint ground sets $X \cup \{x^*\}$ and $Y \cup \{y^*\}$, respectively. The number of triangulations of κ is*

$$(6) \quad |\mathcal{T}_\kappa| = \sum_{\substack{T' \in \mathcal{T}_\chi \\ T'' \in \mathcal{T}_\xi}} \binom{\deg_{T'}(x^*) + \deg_{T''}(y^*) - 2}{\deg_{T'}(x^*) - 1} = \sum_{a, b \geq 2} \binom{a + b - 2}{a - 1} [s^a] P_{\chi, x^*}(s) [t^b] P_{\xi, y^*}(t).$$

Proof. Let \mathcal{T}_\bowtie denote the set of triples (T', T'', H) such that $T' \in \mathcal{T}_\chi$, $T'' \in \mathcal{T}_\xi$ and H is a maximal set of noncrossing edges between $X' \stackrel{\text{def}}{=} N_{T'}(x^*)$ and $Y' \stackrel{\text{def}}{=} N_{T''}(y^*)$. By Lemma 7.2 (case $k = 0$), for fixed $T' \in \mathcal{T}_\chi$ and $T'' \in \mathcal{T}_\xi$, there are $\binom{|X'| + |Y'| - 2}{|X'| - 1}$ sets H such that $(T', T'', H) \in \mathcal{T}_\bowtie$. Hence, recalling that $|X'| = \deg_{T'}(x^*)$ and $|Y'| = \deg_{T''}(y^*)$, we have

$$|\mathcal{T}_\bowtie| = \sum_{\substack{T' \in \mathcal{T}_\chi \\ T'' \in \mathcal{T}_\xi}} \binom{\deg_{T'}(x^*) + \deg_{T''}(y^*) - 2}{\deg_{T'}(x^*) - 1}.$$

But, from Proposition 7.1, we know that $|\mathcal{T}_\kappa| = |\mathcal{T}_\bowtie|$. This proves the first equality in the proposition. For the second equality, it suffices to group triangulations according to the degrees of x^* and y^* (which are at least 2 in any triangulation). \square

8.2. Generating polynomial of the triangulations of a bowtie. We just used the bijection of Proposition 7.1 to express $|\mathcal{T}_\kappa|$ for $\kappa \stackrel{\text{def}}{=} \chi_{x^*} \bowtie_{y^*} \xi$ from $P_{\chi, x^*}(s)$ and $P_{\xi, y^*}(t)$ (see Eq. (6)). To iterate this computation through several bowtie products, we need to determine not only $|\mathcal{T}_\kappa|$, but $P_{\kappa, z^*}(t)$ for any extreme element z^* of κ , to be used as a proxy element in a subsequent bowtie product.

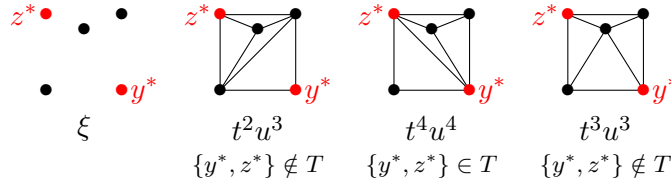
Note that at this point, χ and ξ do not play a symmetric role anymore, as z^* is an element of the ground sets of only one of them. Without loss of generality, we suppose that $z^* \in Y$, and we replace $P_{\xi, y^*}(t)$ by two bivariate polynomials:

$$Q_{\xi, y^*, z^*}^{\in} (t, u) = \sum_{\substack{T \in \mathcal{T}_{\xi} \\ \{y^*, z^*\} \in T}} t^{\deg_T(y^*)} u^{\deg_T(z^*)} \quad \text{and} \quad Q_{\xi, y^*, z^*}^{\notin} (t, u) = \sum_{\substack{T \in \mathcal{T}_{\xi} \\ \{y^*, z^*\} \notin T}} t^{\deg_T(y^*)} u^{\deg_T(z^*)}.$$

Observe that these polynomials refine the polynomials $P_{\xi, y^*}(t)$ defined earlier, in the sense that

$$(24) \quad P_{\xi, y^*}(t) = Q_{\xi, y^*, z^*}^{\in} (t, 1) + Q_{\xi, y^*, z^*}^{\notin} (t, 1).$$

Coming back to our previous example for (ξ, y^*) , it has the following triangulations



Thus $Q_{\xi, y^*, z^*}^{\in} (t, u) = t^4 u^4$ and $Q_{\xi, y^*, z^*}^{\notin} (t, u) = t^2 u^3 (1 + t)$. In the following proposition, we express $P_{\kappa, z^*}(u)$ in terms of $Q_{\xi, y^*, z^*}^{\in} (t, u)$, $Q_{\xi, y^*, z^*}^{\notin} (t, u)$ and $P_{\chi, x^*}(s)$.⁸

Proposition 8.2. *Let $\kappa = \chi_{x^*} \bowtie_{y^*} \xi$ where χ and ξ are chirotopes on $X \cup \{x^*\}$ and $Y \cup \{y^*\}$, with X and Y disjoint, $x^* \notin X$ and $y^* \notin Y$. For any $z^* \in Y$ extreme in ξ we have*

$$P_{\kappa, z^*}(u) = \sum_{a, b \geq 2} \binom{a+b-2}{a-1} [s^a] P_{\chi, x^*}(s) [t^b] Q_{\xi, y^*, z^*}^{\notin} (t, u) \\ + \sum_{a, b \geq 2} R_{a,b}(u) [s^a] P_{\chi, x^*}(s) [t^b] Q_{\xi, y^*, z^*}^{\in} (t, u), \quad \text{with } R_{a,b}(u) = \sum_{i=0}^{a-1} \binom{a+b-i-3}{b-2} u^i.$$

Proof. The proof relies again on the bijection of Proposition 7.1, and the reader is invited to refer to Figure 12. As before, for a triangulation $T \in \mathcal{T}_{\kappa}$, we let $T' = \pi_{Y \rightarrow x^*}(T)$, $T'' = \pi_{X \rightarrow y^*}(T)$ and T_{XY} be the set of edges of T between X and Y . We let $\mathcal{T}_{\kappa}^{\in}$ (resp. $\mathcal{T}_{\kappa}^{\notin}$) denote the set of triangulations of \mathcal{T}_{κ} that contain (resp. do not contain) at least one edge between z^* and an element of X . Note that T is in $\mathcal{T}_{\kappa}^{\in}$ if and only if the edge $\{y^*, z^*\}$ belongs to T'' .

For any $T \in \mathcal{T}^{\notin}$, we have $\deg_T(z^*) = \deg_{T''}(z^*)$ as $\{y^*, z^*\} \notin T''$ so that the contribution of all triangulations of \mathcal{T}^{\notin} to $P_{\kappa, z^*}(u)$ can then be computed by a straightforward adaptation of Proposition 8.1:

$$\sum_{T \in \mathcal{T}_{\kappa}^{\notin}} u^{\deg_T(z^*)} = \sum_{a, b \geq 2} \binom{a+b-2}{a-1} [s^a] P_{\chi, x^*}(s) [t^b] Q_{\xi, y^*, z^*}^{\notin} (t, u).$$

Now, assume that $T \in \mathcal{T}^{\in}$, so that $\{y^*, z^*\} \in T''$. The edges incident to z^* in T are exactly the union of those incident to z^* in T_{XY} and those incident to z^* in T'' , minus the edge

⁸Recalling also Eq. (24), the Proposition 8.2 suggests that, by iteration, knowing the polynomials Q at each node in a chirotope tree will be enough to compute the polynomial P for the chirotope described by the whole tree. This is actually true only in the special case where the tree we consider is a path (as discussed in Section 8.3). Further refinements are necessary for considering general trees, and these will be presented later, in Proposition 9.3.

$\{y^*, z^*\}$. Hence $\deg_T(z^*) = \deg_{T_{XY}}(z^*) + \deg_{T''}(z^*) - 1$. Using Lemma 7.2 (case $k = 1$) after regrouping the terms according to the value of $i = \deg_{T_{XY}}(z^*)$, we obtain:

$$\sum_{T \in \mathcal{T}_\kappa^\xi} u^{\deg_T(z^*)} = \sum_{\substack{T' \in \mathcal{T}_\chi \\ T'' \in \mathcal{T}_\xi, \{y^*, z^*\} \in T''}} u^{\deg_{T''}(z^*)} \underbrace{\sum_{i=1}^{\deg_{T'}(x^*)} \binom{\deg_{T'}(x^*) + \deg_{T''}(y^*) - i - 2}{\deg_{T''}(y^*) - 2}}_{=R_{\deg_{T'}(x^*), \deg_{T''}(y^*)}(u)} u^{i-1}.$$

Grouping the terms by the values of $a = \deg_{T'}(x^*) \geq 2$ and $b = \deg_{T''}(y^*) \geq 2$, we finally obtain

$$\sum_{T \in \mathcal{T}_\kappa^\xi} u^{\deg_T(z^*)} = \sum_{a, b \geq 2} R_{a,b}(u) [s^a] P_{\chi, x^*}(s) [t^b] Q_{\xi, y^*, z^*}^\xi(t, u),$$

concluding the proof. □

8.3. Example of application: triangulations of a family of chirotope paths. As claimed earlier, iterating Proposition 8.2, it is possible to compute the polynomial P associated to a chirotope described by a chirotope tree that is a path, knowing the polynomials Q at each node in this tree. In this section, we show an application of this fact. Specifically, we study a family of chirotope trees, where the underlying trees are simply paths (we call them *chirotope paths* for short), and where the decorations are chosen among two chirotopes, symmetric of each other. These leads, for each k , to 2^k different chirotopes (some but not all being equivalent by symmetry) with the same number of triangulations. This number of triangulations can be evaluated explicitly, combining Proposition 8.2 with the so-called *kernel method* from analytic combinatorics (see, e.g., [BFG⁺02]).

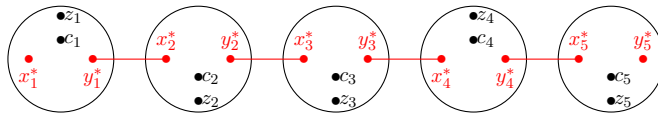
Consider the two following chirotopes on $\{x^*, y^*, z, c\}$:

$$\chi^{(0)} = \begin{array}{c} \bullet z \\ \bullet c \\ \bullet x^* \bullet y^* \end{array} \quad \text{and} \quad \chi^{(1)} = \begin{array}{c} \bullet x^* \bullet y^* \\ \bullet c \\ \bullet z \end{array}$$

Each of $\chi^{(0)}$ and $\chi^{(1)}$ has a unique triangulation, which contains the edge $\{x^*, y^*\}$ and where each vertex has degree 3. Thus,

$$\begin{aligned} P_{\chi^{(0)}, y^*}(s) &= P_{\chi^{(1)}, y^*}(s) = s^3, \\ Q_{\chi^{(0)}, x^*, y^*}^\xi(t, u) &= Q_{\chi^{(1)}, x^*, y^*}^\xi(t, u) = t^3 u^3, \\ Q_{\chi^{(0)}, x^*, y^*}^\notin(t, u) &= Q_{\chi^{(1)}, x^*, y^*}^\notin(t, u) = 0. \end{aligned}$$

For $i \in \mathbb{N}$, let $\chi_i^{(0)}$ denote a copy of $\chi^{(0)}$ relabeled by $x^* \mapsto x_i^*$, $y^* \mapsto y_i^*$, $z \mapsto z_i$, $c \mapsto c_i$. We similarly define $\chi_i^{(1)}$ to be a copy of $\chi^{(1)}$ with the same relabeling. For any word $\sigma = \sigma_1 \sigma_2 \dots \sigma_k \in \{0, 1\}^k$ we put $\chi^{(\sigma)} \stackrel{\text{def}}{=} \chi_1^{(\sigma_1)} \bowtie_{x_2^*} \chi_2^{(\sigma_2)} \bowtie_{y_2^*} \chi_3^{(\sigma_3)} \dots \bowtie_{y_{k-1}^*} \chi_k^{(\sigma_k)}$. This expression translates into a chirotope tree, actually a path, for example for $\chi^{(01101)}$:



The chirotopes $\chi^{(0)}$ and $\chi^{(1)}$ are indecomposable and nonconvex, so these trees are canonical and Theorem 1.3 ensures that $\chi^{(\sigma)} = \chi^{(\sigma')}$ if and only if $\sigma = \sigma'$. Moreover, $\chi^{(0)}$ and $\chi^{(1)}$ are realizable so $\chi^{(\sigma)}$ is realizable by Proposition 1.1 (ii) and is the chirotope of a set of $2k + 2$

elements, k of which are interior. It turns out that the number of triangulations of $\chi^{(\sigma)}$ depends only on the length of σ , and there is a closed formula for this number.

Proposition 8.3. *For every $\sigma \in \{0, 1\}^k$ we have*

$$\left| \mathcal{T}_{\chi^{(\sigma)}} \right| = \frac{3(2k+1) \binom{4k+2}{2k+1}}{(2k+2)(4k+1)} - \frac{4^k \binom{2k+2}{k+1}}{2k+1} \sim_{k \rightarrow \infty} \frac{3 - 2\sqrt{2}}{\sqrt{2\pi}} \frac{16^k}{k^{3/2}}.$$

Recall that for $\sigma \in \{0, 1\}^k$, the chirotope $\chi^{(\sigma)}$ has $2k+2$ elements. In particular, up to a multiplicative constant, $\left| \mathcal{T}_{\chi^{(\sigma)}} \right|$ has the same asymptotics as the $2k$ -th Catalan number, which is the number of triangulations of $2k+2$ points in convex position.

Proof. Let $\tau \in \{0, 1\}^{\mathbb{N}}$ be an infinite binary word, let τ_k be its k th letter, and let $\tau^{[k]}$ denote its prefix of length k . We introduce the shorthands

$$P_k(u) \stackrel{\text{def}}{=} P_{\chi^{(\tau^{[k]})}, y_k^*}(u) \quad \text{and} \quad Q(t, u) \stackrel{\text{def}}{=} Q_{\chi_k^{(\tau_k)}, x_k^*, y_k^*}^{\inset}(t, u) = t^3 u^3$$

and recall that $Q_{\chi_k^{(\tau_k)}, x_k^*, y_k^*}^{\notinset}(t, u) = 0$. Since $\chi^{(\tau^{[k+1]})} = \chi^{(\tau^{[k]})} y_k^* \bowtie_{x_{k+1}^*} \chi_{k+1}^{(\tau_{k+1})}$, we can apply Proposition 8.2 with $z^* = y_{k+1}^*$ to get

$$P_{k+1}(u) = \sum_{a, b \geq 2} R_{a,b}(u) [s^a] P_k(s) [t^b] Q(t, u) = u^3 \sum_{a=0}^{2k+1} R_{a,3}(u) [s^a] P_k(s).$$

(Note that a can only range from 2 to $2k+1$ as it is the degree of a point in a triangulation of a set of $2k+2$ points.) Using the identity $R_{a,3}(u) = \sum_{i=0}^{a-1} \binom{a-i}{1} u^i = \frac{u}{(1-u)^2} (u^a - 1) + \frac{a}{1-u}$, we get

$$P_{k+1}(u) = \frac{u^4}{(1-u)^2} \underbrace{\sum_{a=0}^{2k+1} [s^a] P_k(s) u^a}_{= P_k(u)} - \frac{u^4}{(1-u)^2} \underbrace{\sum_{a=0}^{2k+1} [s^a] P_k(s)}_{= P_k(1)} + \frac{u^3}{(1-u)} \underbrace{\sum_{a=0}^{2k+1} a [s^a] P_k(s)}_{= P'_k(1)},$$

which rewrites as

$$(25) \quad P_{k+1}(s) - \frac{s^4}{(1-s)^2} P_k(s) = -\frac{s^4}{(1-s)^2} P_k(1) + \frac{s^3}{1-s} P'_k(1).$$

Let us introduce the formal series $F(s, u) \stackrel{\text{def}}{=} \sum_{k \geq 1} P_k(s) u^k$. Note that by definition of $P_k(s)$, its specialization $F(1, u) = \sum_{k \geq 1} P_k(1) u^k$ is the generating series of the sequence recording, for each k , the number of triangulations of $\chi^{(\tau^{[k]})}$. Since $\chi^{(\tau^{[k]})}$ is realizable, we have $P_k(1) \leq 30^{2k+2}$ [Sha11]. Moreover, each P_k has degree at most $2k+1$ and nonnegative coefficients, so that, for $|s| \leq 2$, we have $|P_k(s)| \leq P_k(2) \leq P_k(1) \cdot 2^{2k+1} \leq (60)^{2k+2}$. This implies that F is analytic in the two variables s and u on the domain

$$D = \{(s, u) : |s| < 2, |u| < \frac{1}{60^2}\},$$

which contains in particular the point $(1, 0)$. Now, multiplying Eq. (25) by u^{k+1} and summing up for $k \geq 1$, we obtain the following functional equation on D :

$$(F(s, u) - us^3) - \frac{s^4}{(1-s)^2} u F(s, u) = -\frac{s^4}{(1-s)^2} u F(1, u) + \frac{s^3}{1-s} u \partial_s F(1, u),$$

which rewrites as

$$(26) \quad \left(1 - \frac{us^4}{(1-s)^2}\right) F(s, u) = us^3 \left(1 - \frac{s}{(1-s)^2} F(1, u) + \frac{1}{1-s} \partial_s F(1, u)\right),$$

where, as usual, $\partial_s F(s, u)$ denotes the derivative of F according to the first variable s .

The right-hand side of Eq. (26) is linear in $F(1, u)$ and $\partial_s F(1, u)$. Let us apply the standard *kernel method* (see, e.g., [BFG⁺02]): since $F(s, u)$ is analytic near $(1, 0)$, we can evaluate Eq. (26) for some $s(u)$ that cancels the coefficient $1 - \frac{us^4}{(1-s)^2}$, called the *kernel*, and such that $s(u)$ is close to 1 when u is close to 0. Two functions satisfy these conditions:

$$s_1(u) = \frac{-1 + \sqrt{1 + 4\sqrt{u}}}{2\sqrt{u}} \quad \text{and} \quad s_2(u) = \frac{1 - \sqrt{1 - 4\sqrt{u}}}{2\sqrt{u}}.$$

Substituting into Eq. (26) yields the following linear system:

$$(27) \quad \begin{cases} \frac{s_1(u)}{(1-s_1(u))^2} F(1, u) - \frac{1}{1-s_1(u)} \partial_s F(1, u) = 1 \\ \frac{s_2(u)}{(1-s_2(u))^2} F(1, u) - \frac{1}{1-s_2(u)} \partial_s F(1, u) = 1 \end{cases}.$$

Solving this linear system gives us the closed form

$$F(1, u) = s_1(u) + s_2(u) - s_1(u)s_2(u) - 1.$$

The rest of the proof is computational. Since $\sqrt{1-4x} = -\sum_{j \geq 0} \frac{1}{2^{j-1}} \binom{2j}{j} x^j$, we have the following expansions:

$$\begin{aligned} \sqrt{1+4x} - \sqrt{1-4x} &= \sum_{j \geq 0} \frac{(-1)^{j+1} + 1}{2j-1} \binom{2j}{j} x^j = 2x \sum_{k \geq 0} \frac{1}{4k+1} \binom{4k+2}{2k+1} x^{2k}, \\ \sqrt{1+4x} + \sqrt{1-4x} &= \sum_{j \geq 0} \frac{(-1)^{j+1} - 1}{2j-1} \binom{2j}{j} x^j = -2 \sum_{k \geq 0} \frac{1}{4k-1} \binom{4k}{2k} x^{2k}, \\ \sqrt{1-16u} &= -\sum_{k \geq 0} \frac{1}{2k-1} \binom{2k}{k} 4^k u^k. \end{aligned}$$

Hence

$$\begin{aligned} s_1(u) + s_2(u) &= \frac{\sqrt{1+4\sqrt{u}} - \sqrt{1-4\sqrt{u}}}{2\sqrt{u}} = \sum_{k \geq 0} \frac{1}{4k+1} \binom{4k+2}{2k+1} u^k, \\ 4us_1(u)s_2(u) &= -1 + \sqrt{1+4\sqrt{u}} + \sqrt{1-4\sqrt{u}} - \sqrt{1-16u}, \\ &= -1 - 2 \sum_{k \geq 0} \frac{1}{4k-1} \binom{4k}{2k} u^k + \sum_{k \geq 0} \frac{1}{2k-1} \binom{2k}{k} 4^k u^k. \end{aligned}$$

The constant terms in the last equation cancel out, so we can divide by $4u$ and we obtain

$$F(1, u) = \sum_{k \geq 0} \frac{1}{4k+1} \binom{4k+2}{2k+1} u^k + \frac{1}{2} \sum_{k \geq 1} \frac{1}{4k-1} \binom{4k}{2k} u^{k-1} - \sum_{k \geq 1} \frac{1}{2k-1} \binom{2k}{k} 4^{k-1} u^{k-1} - 1.$$

Grouping the terms with same degree yields $P_k(1) = \frac{1}{4k+1} \binom{4k+2}{2k+1} + \frac{1}{2} \frac{1}{4k+3} \binom{4k+4}{2k+2} - \binom{2k+2}{k+1} \frac{4^k}{2k+1}$ for $k \geq 1$ (recalling that $P_k(1)$ is the coefficient of u^k in $F(1, u)$, which is also the number of triangulations of $\chi^{(\tau^{[k]})}$). As $\binom{4k+4}{2k+2} / \binom{4k+2}{2k+1} = \frac{2(4k+3)}{2k+2}$, this simplifies to

$$P_k(1) = \left(\frac{1}{4k+1} + \frac{1}{2k+2} \right) \binom{4k+2}{2k+1} - \binom{2k+2}{k+1} \frac{4^k}{2k+1},$$

which gives the announced closed formula. Finally, using Catalan numbers $C_k = \frac{1}{k+1} \binom{2k}{k}$, and their asymptotics $C_k = \frac{4^k}{k^{3/2} \sqrt{\pi}} (1 + o(1))$, we can easily compute:

$$P_k(1) = \frac{3(2k+1)}{4k+1} C_{2k+1} - C_{k+1} \frac{4^k(k+2)}{2k+1} \sim_{k \rightarrow \infty} \frac{3 - 2\sqrt{2}}{\sqrt{2\pi}} \frac{16^k}{k^{3/2}}. \quad \square$$

9. GENERAL ALGORITHM FOR COUNTING TRIANGULATIONS, AND ITS COMPLEXITY

Sections 7 and 8 presented a method to count the triangulations of chirotope paths. We now generalize this method to arbitrary chirotope trees.

9.1. Chirotope with several marked elements and their triangulation polynomials.

Let ξ be a chirotope with ground set $Y \cup \{y_1^*, \dots, y_k^*\}$, where y_1^*, \dots, y_k^* are marked *extreme* elements of ξ which do not belong to Y . We have seen in Section 8.2 that, with two marked elements ($k = 2$), it is useful to introduce two bivariate polynomials Q^\inleftarrow and Q^\inrightarrow , separating the triangulations joining or not joining the marked elements. For a general k , we shall need one k -variate polynomial for each possible restriction of a triangulation to $\{y_1^*, \dots, y_k^*\}$. Such a restriction is in general a collection of noncrossing edges with extremities in $\{y_1^*, \dots, y_k^*\}$, not necessarily maximal. We will denote the set of such configurations by $\mathcal{U}_{\{y_1^*, \dots, y_k^*\}}$, or simply \mathcal{U}_k when there is no ambiguity.

Since y_1^*, \dots, y_k^* are *extreme* elements, the set $\{y_1^*, \dots, y_k^*\}$ is in convex position, and the asymptotic behaviour of $|\mathcal{U}_k|$ is known.

Lemma 9.1. *For any k and y_1^*, \dots, y_k^* in convex position, we have $|\mathcal{U}_k| = \mathcal{O}(A^k)$, where $A = 6 + 4\sqrt{2}$.*

Proof. Let U in \mathcal{U}_k be a set of noncrossing edges with extremities in $\{y_1^*, \dots, y_k^*\}$. We first note that each edge of the convex hull can be included in U or not, independently of all other edges. We therefore get that $|\mathcal{U}_k| = 2^k |\mathcal{D}_k|$, where \mathcal{D}_k is the set of collections of noncrossing edges with extremities in $\{y_1^*, \dots, y_k^*\}$, and containing all edges of the convex hull. The elements of \mathcal{D}_k are called *dissections* of the k -gon with vertices $\{y_1^*, \dots, y_k^*\}$. In particular, it is proved in [FN99, Theorem 4] that $|\mathcal{D}_k| = \mathcal{O}((3 + 2\sqrt{2})^k)$. The statement follows. \square

The collection of triangulation polynomials associated with a chirotope ξ having k marked elements $Y^* = \{y_1^*, \dots, y_k^*\}$ is defined as follows. For a set $U \in \mathcal{U}_k$ of noncrossing edges, we set

$$Q_\xi^U(y_1, \dots, y_k) = \sum_{\substack{T \in \mathcal{T}_\xi \\ T_{Y^* Y^*} = U}} \prod_{i=1}^k y_i^{\deg_T(y_i^*)},$$

where we recall from Section 8 that $T_{Y^* Y^*}$ is the subset of edges of T with both extremities in Y^* . (To simplify notation, we do not include the marked elements $\{y_1^*, \dots, y_k^*\}$, in the index

of Q .) Note that the degree of each Q_ξ^U in a variable y_i is less than the number of elements of ξ , that is $|Y| + k$. If U is not in \mathcal{U}_k , we use the convention $Q_\xi^U = 0$.

Remark 9.2. We already encountered specializations of these polynomials Q_ξ^U in cases with few marked elements:

- when there is no marked element, then Q_ξ^U is just a number, and is equal to $|\mathcal{T}_\xi|$ (here, U needs to be empty);
- when there is one marked element y^* , then $Q_\xi^U(y) = P_{\xi, y^*}(y)$ (here again, U needs to be empty);
- and when there are two marked elements y_1^* and y_2^* , then $Q_\xi^U(y_1, y_2) = Q_{\xi, y_1^*, y_2^*}^\infty(y_1, y_2)$ (resp. $Q_{\xi, y_1^*, y_2^*}^\neq(y_1, y_2)$) if U is the singleton containing the edge $\{y_1^*, y_2^*\}$ (resp. if U is empty).

The next step is to explain how to compute recursively Q_ξ^U through a bowtie operation, just like Proposition 8.2 does for chirotopes with only two marked elements. To this end, for any set $\{x_1, \dots, x_r\}$ of variables and integers $a, b \geq 2$ satisfying $b \geq r$, we define

$$R_{a,b}(\{x_1, \dots, x_r\}) = \sum_{i_1, \dots, i_r \geq 1} \binom{a+b - (i_1 + \dots + i_r) - 2}{b - (r+1)} x_1^{i_1-1} \dots x_r^{i_r-1}.$$

Note that these polynomials are symmetric, justifying that we write our list of variables as a set. We recall the convention $\binom{n}{-1} = \delta_{-1}$, leading to the following simpler formula if $b = r$:

$$R_{a,r}(\{x_1, \dots, x_r\}) = \sum_{\substack{i_1, \dots, i_r \geq 1 \\ i_1 + \dots + i_r = a+r-1}} x_1^{i_1-1} \dots x_r^{i_r-1}.$$

Other particular cases are given by $R_{a,b}(\emptyset) = \binom{a+b-2}{b-1}$ and $R_{a,b}(\{u\})$ being the polynomial $R_{a,b}(u)$ from Proposition 8.2. Just like $R_{a,b}(\emptyset)$ and $R_{a,b}(\{u\})$ occurred in Proposition 8.2 to count triangulations with zero and one edge between two marked elements, respectively, the family $R_{a,b}(\cdot)$ of multivariate polynomials allows us to handle the general case with several marked elements.

Now, consider a second chirotope χ on $X \cup \{x^*\}$, assume that x^* is extreme, and build the bowtie product $\kappa = \chi_{x^*} \bowtie_{y_k^*} \xi$ on ground set $X \cup Y \cup \{y_1^*, \dots, y_{k-1}^*\}$. We want to compute the associated collection of triangulation polynomials

$$Q_\kappa^U(y_1, \dots, y_{k-1}) = \sum_{T \in \mathcal{T}_\kappa} \prod_{i=1}^{k-1} y_i^{\deg_T(y_i^*)},$$

$$T_{\{y_1^*, \dots, y_{k-1}^*\} \cup \{y_1^*, \dots, y_{k-1}^*\} = U}$$

for $U \in \mathcal{U}_{k-1}$. The bijection of Proposition 7.1 between the triangulations of κ and those of χ and ξ leads to the following formula, which generalizes Proposition 8.2.

Proposition 9.3. *Let χ , ξ and $\kappa = \chi_{x^*} \bowtie_{y_k^*} \xi$ be as above. For any U in \mathcal{U}_{k-1} , we have*

$$(28) \quad Q_\kappa^U(y_1, \dots, y_{k-1}) = \sum_{\substack{S \subseteq [k-1] \\ a, b \geq 2}} \left([s^a] P_{\chi, x^*}(s) \right) \cdot R_{a,b}(\{y_i, i \in S\}) \cdot \left([y_k^b] Q_\xi^{U \cup Y^*(S, k)}(y_1, \dots, y_k) \right),$$

where $Y^*(S, k) = \{\{y_i^*, y_k^*\}, i \in S\}$.

Proof. We write $Y_{k-1}^* = \{y_1^*, \dots, y_{k-1}^*\}$, $Y_k^* = \{y_1^*, \dots, y_k^*\}$, and $\tilde{Y} = Y \cup Y_{k-1}^*$, for short. For $T \in \mathcal{T}_\kappa$ a triangulation of κ , we let $T' = \pi_{\tilde{Y} \rightarrow x^*}(T)$ and $T'' = \pi_{X \rightarrow y_k^*}(T)$, as usual. We also recall that $T_{X\tilde{Y}}$ denotes the set of edges in T joining X to \tilde{Y} . We recall from Proposition 7.1 that the map $T \mapsto (T', T'', T_{X\tilde{Y}})$ is a bijection.

Fix U in \mathcal{U}_{k-1} . We note that T and T'' coincide on Y_{k-1}^* . Thus, the restriction of T to Y_{k-1}^* is U if and only if that of T'' is U as well. This implies that the restriction of T'' to Y_k^* is of the form $U \cup Y^*(S, k)$ for some $S \subseteq [k-1]$. To shorten notation, for a given S , we let $\mathcal{T}_\xi^{U,S}$ be the set of triangulations T'' of ξ whose restriction to Y_k^* is equal to $U \cup Y^*(S, k)$.

With the same argument as in Proposition 8.2, we see that the degree of y_i^* in T are computed as follows:

- if i is not in S , i.e. if y_i^* is not connected to y_k^* in T'' , then

$$\deg_T(y_i^*) = \deg_{T''}(y_i^*).$$

- if i is in S , i.e. if y_i^* is connected to y_k^* in T'' , then

$$\deg_T(y_i^*) = \deg_{T''}(y_i^*) + \deg_{T_{X\tilde{Y}}}(y_i^*) - 1.$$

Hence we have:

$$(29) \quad Q_\kappa^U(y_1, \dots, y_{k-1}) = \sum_{S \subseteq [k-1]} \sum_{T' \in \mathcal{T}_X} \sum_{T'' \in \mathcal{T}_\xi^{U,S}} \left(\prod_{i=1}^{k-1} y_i^{\deg_{T''}(y_i^*)} \right) \cdot \left(\sum_{T_{X\tilde{Y}}} \prod_{i \in S} y_i^{\deg_{T_{X\tilde{Y}}}(y_i^*)-1} \right).$$

In the last sum, $T_{X\tilde{Y}}$ runs over maximal sets of noncrossing edges between $N_{T'}(x^*)$ and $N_{T''}(y_k^*)$ (see the statement of Proposition 7.1). In particular, even if it is not explicit in the notation, this last sum depends on T' and T'' . Using Lemma 7.2, we see that

$$\begin{aligned} \sum_{T_{X\tilde{Y}}} \prod_{i \in S} y_i^{\deg_{T_{X\tilde{Y}}}(y_i^*)-1} &= \sum_{\substack{(d_i)_{i \in S} \\ d_i \geq 1}} \left(\prod_{i \in S} y_i^{d_i-1} \right) \left(\frac{|N_{T'}(x^*)| + |N_{T''}(y_k^*)| - (\sum_{i \in S} d_i) - 2}{|N_{T''}(y_k^*)| - (|S| + 1)} \right) \\ &= R_{|N_{T'}(x^*)|, |N_{T''}(y_k^*)|}(\{y_i, i \in S\}). \end{aligned}$$

We split the sums in Eq. (29), depending on the values of the parameters $a \stackrel{\text{def}}{=} |N_{T'}(x^*)| = \deg_{T'}(x^*)$ and $b \stackrel{\text{def}}{=} |N_{T''}(y_k^*)| = \deg_{T''}(y_k^*)$. For given $a, b \geq 2$, there are $[s^a]P_{\chi, x^*}(s)$ triangulations T' with $\deg_{T'}(x^*) = a$ and we have

$$\sum_{\substack{T'' \in \mathcal{T}_\xi^{U,S} \\ \deg_{T''}(y_k^*) = b}} \left(\prod_{i=1}^{k-1} y_i^{\deg_{T''}(y_i^*)} \right) = [y_k^b] Q_\xi^{U \cup Y^*(S, k)}(y_1, \dots, y_k).$$

This implies

$$Q_\kappa^U(y_1, \dots, y_{k-1}) = \sum_{S \subseteq [k-1]} \sum_{a, b \geq 2} ([s^a]P_{\chi, x^*}(s)) ([y_k^b] Q_\xi^{U \cup Y^*(S, k)}(y_1, \dots, y_k)) R_{a,b}(\{y_i, i \in S\}),$$

as announced. \square

9.2. A general algorithm. We now explain how to use Proposition 9.3 to count the triangulations of a chirotope presented by a chirotope tree T . We proceed as follows.

- (1) For each node v , we denote by χ_v the chirotope decorating the node v and by Y_v its set of proxy elements, and we compute the associated collection of triangulation polynomials $\{Q_{\chi_v}^U : U \in \mathcal{U}_{Y_v}\}$. These polynomials count triangulations of χ_v , keeping track of the degree of each elements of Y_v in the triangulation, and of the restriction of the triangulation to Y_v . These polynomials can be computed in parallel for all nodes v , using for each one a modified version of the Alvarez-Seidel algorithm [AS13].
- (2) As long as the chirotope tree is not reduced to a single node, we pick an arbitrary leaf and perform an edge contraction (as defined in Section 4.2) to merge it with its parent. We compute the collection of triangulation polynomials to be associated to the merged node using Proposition 9.3: we take χ to be the merged leaf, and use a single marked element x^* . In that case, as noted by Remark 9.2, $P_{\xi, x^*}(u) = Q_{\xi}^U(u)$, with U necessarily empty. For further details, see the proof of Proposition 9.4 below.

Each application of Step (2) yields a strictly smaller chirotope tree that, by Proposition 4.7, represents the same chirotope, and for which we know the collection of triangulation polynomials of every node. Hence Step (2) can indeed be repeated until the tree is reduced to a single node. When this is the case, there are no more marked elements, so the associated collection of triangulation polynomials is actually a single integer, equal to the number of triangulations of χ_T .

The complexity of this algorithm is given in the following proposition. We recall from the introduction (see also Lemma 9.1) that A stands for the constant $6 + 4\sqrt{2} \approx 11.66$.

Proposition 9.4. *Let T be a chirotope tree.*

- (i) *If a node v of T is decorated with a chirotope χ_v with n_v elements, including a set Y_v of k proxies, then $\{Q_{\chi_v}^U : U \in \mathcal{U}_{Y_v}\}$ can be computed from χ_v in time $O(A^k 2^{n_v} n_v^{k+2})$.*
- (ii) *If T has m edges, each of its node has degree at most k , and the ground set of χ_T has n elements, then the above algorithm computes the number of triangulations of χ_T from the polynomials $\{Q_{\chi_v}^U : v \text{ a node of } T, U \in \mathcal{U}_{Y_v}\}$ in time $O(A^k n^{2k} m)$.*

Proof. Statement (i) follows from a simple modification of the Alvarez-Seidel algorithm [AS13] so as to compute the degrees of marked vertices and keep track of the presence of edges between them through the aggregation.

To prove Statement (ii), let us now analyze one iteration of Step (2). Consider a leaf ℓ of the chirotope tree T , and denote by v its parent node. We let ξ denote the chirotope decorating v and denote by $Y \cup \{y_1^*, \dots, y_k^*\}$ its ground set, with $k \geq 2$ (where, as usual, the starred elements indicate the proxies). Similarly, we let χ denote the chirotope decorating the leaf ℓ and $X \cup \{x^*\}$ its ground set (a leaf has only one proxy element, denoted x^* here).

Let $\mathcal{U}_k = \mathcal{U}_{\{y_1^*, \dots, y_k^*\}}$ and $\mathcal{U}_{k-1} = \mathcal{U}_{\{y_1^*, \dots, y_{k-1}^*\}}$. For each $U \in \mathcal{U}_{k-1}$, we initialize $\tilde{Q}_{\xi}^U = 0$. Now, for every $\bar{U} \in \mathcal{U}_k$, we denote $p(\bar{U})$ the restriction of \bar{U} to $\{y_1^*, \dots, y_{k-1}^*\}$ and $S_{\bar{U}} = \{i \in [k-1], y_i^* y_k^* \in \bar{U}\}$. Then we compute

$$(30) \quad \sum_{a, b \geq 2} \left([s^a] P_{\chi, x^*}(s) \right) \cdot R_{a, b}(\{y_i, i \in S_{\bar{U}}\}) \cdot \left([y_k^b] Q_{\xi}^{\bar{U}}(y_1, \dots, y_k) \right)$$

and add it to $\tilde{Q}_\xi^{p(\bar{U})}$. By Proposition 9.3, when all $\bar{U} \in \mathcal{U}_k$ have been treated, we have $\tilde{Q}_\xi^U = Q_\xi^U$ for all $U \in \mathcal{U}_{k-1}$. This proves the correctness of our algorithm.

Let us analyze the complexity of this second step. We first consider, for given \bar{U} and a, b , the multiplication in Eq. (30). The first factor $[s^a]P_{\chi, x^*}(s)$ is a scalar, while the two other polynomials are $(k-1)$ -variate and have degree less than n in each variable (in general the degree of a polynomial Q_α^V in a variable y_i is less than the number of elements, including marked ones, in α). Therefore the last two polynomials have each at most n^{k-1} monomials, and the multiplication has complexity $\mathcal{O}(n^{2(k-1)})$ (in the Real-RAM model⁹, using a brute-force multiplication algorithm). Since we only need to consider the case $a, b \leq n$ (further terms are equal to 0), and since there are $\mathcal{O}(A^k)$ sets \bar{U} in \mathcal{U}_k (see Lemma 9.1), this step has complexity $\mathcal{O}(n^{2k} A^k)$.

Altogether, one iteration of Step (2), that is the merge of a leaf of a chirotope tree with its parent, and the update of the associated collection of polynomials, can be done in time $\mathcal{O}(n^{2k} A^k)$. Notice that the strict upper bound of n on the partial degree of these polynomials remains true, as it follows from the fact that in any chirotope tree representing a chirotope of size n , every node has at most n elements. As T has m edges, Step (2) needs to be iterated m times until the tree is reduced to a single vertex. This completes the proof of Proposition 9.4. \square

9.3. A proof-of-concept implementation. We implemented the method described in the previous section (both the modification of the Alvarez-Seidel algorithm and the recursive counting behind Proposition 1.4, restated above as Proposition 9.4) for the special case of trees T where

- every node has degree at most 3,
- every node of degree 3 is decorated by a chirotope with only 3 extreme elements, which are its proxies, and
- every node of degree 2 has its two proxies consecutive on the convex hull.

These conditions ensure that in any triangulation of a chirotope decorating a node of T , there is always an edge between any two proxies, i.e. we only have one non-zero polynomial attached to each node. We propose this as a proof of concept: we made little effort in optimizing or benchmarking it as either is beyond the scope of this paper.

9.3.1. Repository. Our implementation runs within Sagemath to take advantage of basic functions for the manipulation of multivariate polynomials, but is mostly basic python. The code in Sagemath can be found on the repository [chirotope_tree](#). The example of Figure 13 can be obtained by running the command `sage -python example.py`, which prints two numbers, first the number of triangulations of the example, and the total time (in seconds) spent on computing it (ignoring the initial time to launch sage).

⁹Recall that, in the Real-RAM model, adding or multiplying any two scalars takes constant time. For a more realistic complexity analysis, we note that the bit complexity of every number arising in the computation is $\mathcal{O}(n)$, as each such number is bounded from above by the number of triangulations, which is $\mathcal{O}(30^n)$.

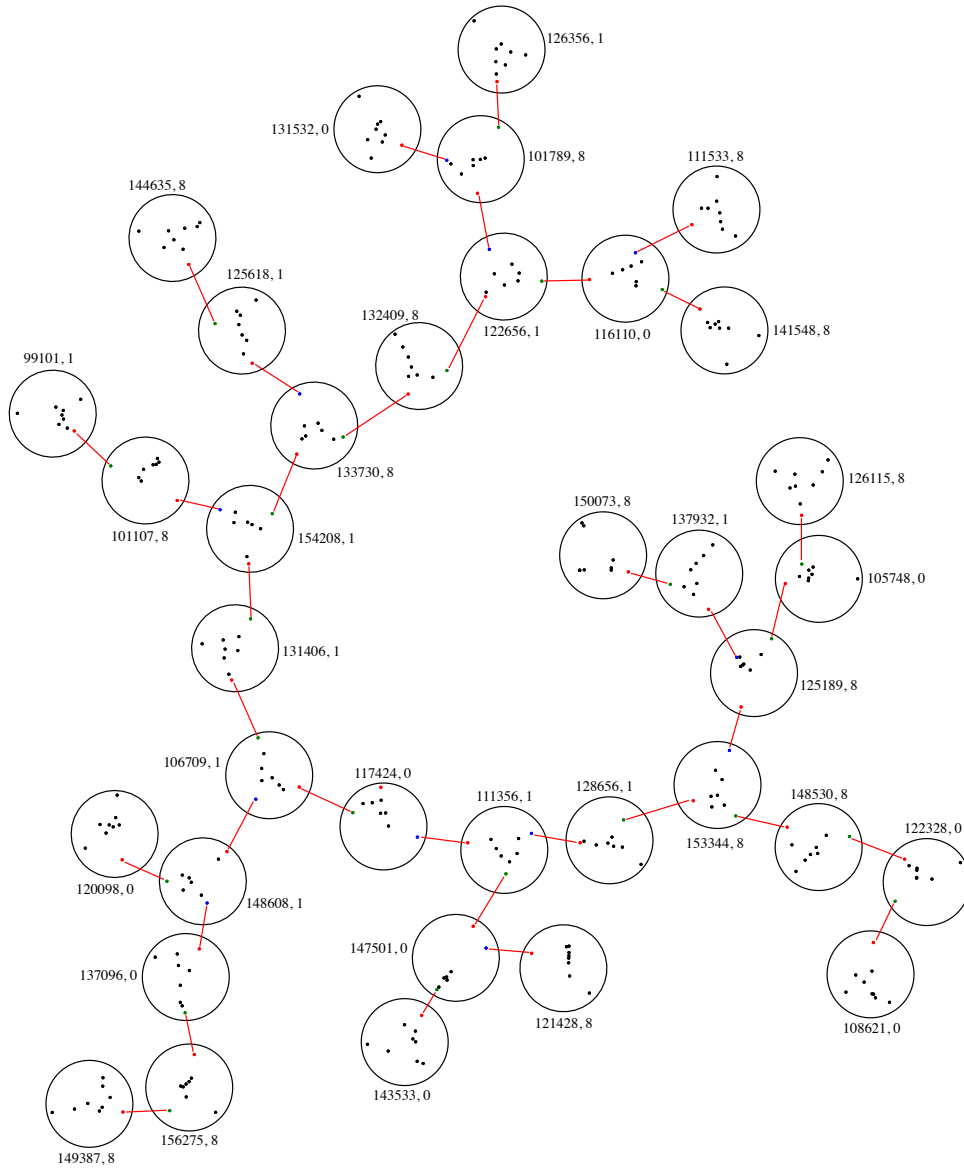


FIGURE 13. A chirotope tree with 36 nodes, each decorated with a chirotope of size 9, adding up to 254 elements. The numbers next to each node indicate the identifier of the decorating chirotope in the order type database [AAK02], as well as the label, in this database entry, of the proxy shown in red in the figure.

9.3.2. *An example.* Our code computes in a few seconds on a standard laptop that the number of triangulations of the chirotope of size 254 presented by the tree of Figure 13 is

592 966 751 293 974 711 252 579 414 478 724 131 868 483 318 559 312 640 993 804 562 350 446
 478 462 502 194 102 338 465 347 793 563 468 964 711 887 069 260 192 677 783 5079 385 675
 578 362 313 461 572 573 372 584 158 103 703 847 713 232 664 $\approx 5.92966751 \cdot 10^{180}$.

REFERENCES

- [AA05] M. H. Albert and M. D. Atkinson. Simple permutations and pattern restricted permutations. *Discrete Mathematics*, 300(1-3):1–15, 2005.
- [AAK02] O. Aichholzer, F. Aurenhammer, and H. Krasser. Enumerating order types for small point sets with applications. *Order*, 19(3):265–281, 2002.
- [ÁDKP22] P. Ágoston, G. Damásdi, B. Keszegh, and D. Pálvölgyi. Orientation of good covers. *arXiv preprint arXiv:2206.01723*, 2022.
- [AEG⁺94] B. Aronov, P. Erdős, W. Goddard, D. J. Kleitman, M. Klugerman, J. Pach, and L. J. Schulman. Crossing families. *Combinatorica*, 14(2):127–134, 1994.
- [AS13] V. Alvarez and R. Seidel. A simple aggregative algorithm for counting triangulations of planar point sets and related problems. In *Proceedings of the twenty-ninth annual symposium on Computational geometry*, pages 1–8, 2013.
- [BBF⁺22] F. Bassino, M. Bouvel, V. Féray, L. Gerin, M. Maazoun, and A. Pierrot. Random cographs: Brownian graphon limit and asymptotic degree distribution. *Random Struct. Algorithms*, 60(2):166–200, 2022.
- [BBR11] F. Bassino, M. Bouvel, and D. Rossin. Enumeration of pin-permutations. *Electronic Journal of Combinatorics*, 18, 2011.
- [BFG⁺02] C. Banderier, P. Flajolet, D. Gardy, M. Bousquet-Mélou, A. Denise, and D. Gouyou-Beauchamps. Generating functions for generating trees. *Discrete Mathematics*, 246(1–3):29–55, 2002.
- [BKLP17] M. Balko, J. Kynčl, S. Langerman, and A. Pilz. Induced Ramsey-type results and binary predicates for point sets. *Electronic Notes in Discrete Mathematics*, 61:77–83, 2017.
- [BLVS⁺99] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. M. Ziegler. *Oriented matroids*. Number 46 in Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1999.
- [CFL17] C. Chauve, E. Fusy, and J. Lumbroso. An exact enumeration of distance-hereditary graphs. In *2017 Proceedings of the Fourteenth Workshop on Analytic Algorithmics and Combinatorics (ANALCO'17)*, pages 31–45, 2017.
- [EVDHM22] J. Erickson, I. Van Der Hoog, and T. Miltzow. Smoothing the gap between NP and ER. *SIAM Journal on Computing*, 51(2):FOCS20–102–138, 2022.
- [FG17] S. Felsner and J. E. Goodman. Pseudoline arrangements. In *Handbook of Discrete and Computational Geometry*, pages 125–157. Chapman and Hall/CRC, 2017.
- [FN99] P. Flajolet and M. Noy. Analytic combinatorics of non-crossing configurations. *Discrete Math.*, 204(1-3):203–229, 1999.
- [Gal67] T. Gallai. Transitiv orientierbare Graphen. *Acta Mathematica Academiae Scientiarum Hungarica*, 18(1-2):25–66, March 1967.
- [GP83] J. E. Goodman and R. Pollack. Multidimensional sorting. *SIAM J. Comput.*, 12(3):484–507, 1983.
- [GP93] J. E. Goodman and R. Pollack. Allowable sequences and order types in discrete and computational geometry. In *New trends in discrete and computational geometry*, pages 103–134. Berlin: Springer-Verlag, 1993.
- [GW23] X. Goaoc and E. Welzl. Convex hulls of random order types. *Journal of the ACM*, 70(1):Article No. 8: 1–47, 2023.
- [HP10] M. Habib and C. Paul. A survey of the algorithmic aspects of modular decomposition. *Computer Science Review*, 4(1):42–59, February 2010.
- [Knu92] D. E. Knuth. *Axioms and hulls*, volume 606 of *Lecture Notes in Computer Science*. Springer, 1992.
- [MM16] D. Marx and T. Miltzow. Peeling and Nibbling the Cactus: Subexponential-Time Algorithms for Counting Triangulations and Related Problems. In S. Fekete and A. Lubiw, editors, *32nd International Symposium on Computational Geometry (SoCG 2016)*, volume 51 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 52:1–52:16, Dagstuhl, Germany, 2016. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.
- [MR84] R. H. Möhring and F. J. Radermacher. Substitution decomposition for discrete structures and connections with combinatorial optimization. *Annals of Discrete Math*, 19:257–356, 1984.

- [Rao08] M. Rao. Solving some NP-complete problems using split decomposition. *Discrete Appl. Math.*, 156(14):2768–2780, 2008.
- [RG11] J. Richter-Gebert. *Perspectives on projective geometry: a guided tour through real and complex geometry*. Springer, 2011.
- [RW23] D. Rutschmann and M. Wettstein. Chains, Koch chains, and point sets with many triangulations. *Journal of the ACM*, 70(3):1–26, 2023.
- [Sam88] P. Samuel. *Projective geometry*. Springer, 1988.
- [Sha11] A. Sharir, Micha and Sheffer. Counting triangulations of planar point sets. *The Electronic Journal on Combinatorics*, 18(1):1–70, 2011.
- [Sho91] P. Shor. Stretchability of pseudolines is NP-hard. *Applied Geometry and Discrete Mathematics-The Victor Klee Festschrift*, 1991.
- [SY17] V. Sharma and C. K. Yap. Robust geometric computation. In *Handbook of Discrete and Computational Geometry*, pages 1189–1223. Chapman and Hall/CRC, 2017.
- [SZ23] A. Suk and J. Zeng. A positive fraction Erdős–Szekeres theorem and its applications. *Discrete & Computational Geometry*, pages 1–18, 2023.
- [Val97] P. Valtr. On mutually avoiding sets. In *The Mathematics of Paul Erdős II*, pages 324–328. Springer, 1997.
- [War68] H. E. Warren. Lower bounds for approximation by nonlinear manifolds. *Transactions of the American Mathematical Society*, 133(1):167–178, 1968.
- [Zas97] T. Zaslavsky. Facing up to arrangements: face-count formulas for partitions of space by hyperplanes. *Memoirs of American Mathematical Society*, 154:1–95, 1997.

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