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Nonlinear stability for active suspensions

Michele Coti Zelati, Helge Dietert, and David Gérard-Varet

ABSTRACT. This paper is devoted to the nonlinear analysis of a kinetic model introduced by Saintillan and Shelley to describe suspensions of active rodlike particles in viscous flows. We investigate the stability of the constant state $\Psi(t,x,p) = \frac{1}{4\pi}$ corresponding to a distribution of particles that is homogeneous in space (variable $x \in \mathbb{T}^3$) and uniform in orientation (variable $p \in \mathbb{S}^2$). We prove its nonlinear stability under the optimal condition of linearized spectral stability, without any addition of spatial diffusion. The mathematical novelty and difficulty compared to previous linear studies comes from the presence of a quasilinear term in x due to nonlinear convection. A key feature of our work, which we hope to be of independent interest, is an analysis of enhanced dissipation and mixing properties of the advection diffusion operator

$$\partial_t + (p + u(t,x)) \cdot \nabla_x - \nu \Delta_p$$

on $\mathbb{T}^3 \times \mathbb{S}^2$ for a given appropriately small vector field u.

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1. The model and main results

Microswimmers are organisms (*e.g.* bacteria) or objects operating in a fluid at the microscale. They exhibit inherent self-propulsion driven by mechanisms such as flagellar motion, cilia beating, or synthetic

fluid flow inducers. Beyond understanding the individual behaviour of microswimmers [14], special attention was paid recently to their collective motion, with impact on fluid mixing, rheological properties or self-organization of biological active matter [9, 11, 17, 18, 20].

We focus in this paper on one popular model, due to Saintillan and Shelley [19], describing dilute suspensions of self-propelled rodlike particles. This model, relevant to the dynamical behavior of bacteria, is a coupled fluid-kinetic model, detailing the interaction of the particles with the surrounding fluid medium and encapsulating the propulsion mechanisms governing their motion. It reads

$$\begin{aligned} &\partial_t \Psi + (U_0\, p + u)\cdot \nabla_x \Psi + \nabla_p \cdot \Big(\mathbb{P}_{p^\perp} \left[(\gamma E(u) + W(u))p \right] \Psi \Big) = \nu \Delta_p \Psi, \\ &-\Delta_x u + \nabla_x q = \iota \nabla_x \cdot \int_{\mathbb{S}^2} \Psi(t,x,p)\, p \otimes p \,\mathrm{d}p, \\ &\nabla_x \cdot u = 0, \end{aligned} \tag{1.1}$$

where

$$E(u) = \frac{1}{2}\left[\nabla_x u + (\nabla_x u)^T\right] \qquad \text{and} \qquad W(u) = \frac{1}{2}\left[\nabla_x u - (\nabla_x u)^T\right] \tag{1.2}$$

are the symmetric and skew-symmetric parts of $\nabla_x u$, respectively. The unknowns of the model are $\Psi = \Psi(t,x,p)$, the distribution of rodlike particles in space and orientation, and $u = u(t,x)$, $q = q(t,x)$ the fluid velocity and pressure. The space variable x is taken in a periodic box $\mathbb{T}_L := (\mathbb{R}/L\mathbb{Z})^3$ of size $L > 0$, while the orientation of the particles is given by $p \in \mathbb{S}^2$. The first equation in (1.1) describes the evolution of Ψ under three effects:

- transport by the fluid velocity $u(t,x)$ and by the self-propulsion $U_0\, p$ with velocity $U_0 > 0$.
- rotation by angular velocity $\mathbb{P}_{p^\perp}\left[(\gamma E(u) + W(u))p\right]$, where $\mathbb{P}_{p^\perp}$ denotes projection tangentially to the sphere. This expression for the angular velocity is due to Jeffery [13] and corresponds to the angular velocity of a slender particle in a Stokes flow: interactions are neglected as the suspension is assumed to be dilute. The constant $\gamma \in [-1,1]$ is related to the geometric properties of the particle.
- rotational diffusion, with small coefficient $\nu > 0$.

Finally, the last two equations in (1.1) are Stokes equations for the fluid flow. They incorporate an additional stress Σ, where

$$\Sigma = \iota \nabla_x \cdot \int_{\mathbb{S}^2} \Psi(t,x,p)\, p \otimes p \,\mathrm{d}p$$

reflects the constraint exerted by the particles on the flow. It is obtained by a continuous approximation of the sum of all single particle contributions, modeled as dipoles of opposite forces along p. Parameter $\iota \neq 0$ distinguishes between two types of swimmers: pullers (resp. pushers) correspond to $\iota > 0$ (resp. $\iota < 0$). We refer to [19] or to the introduction in [8] for more on the derivation of the model.

Of special interest is the stability of the constant state equilibrium $\Psi^{iso} = 1/4\pi$, which corresponds to a distribution of particles homogeneous in space and uniform in orientation. In particular, physicists are interested in loss of stability, with possible emergence of collective patterns and rheological changes. It is therefore natural to work with

$$\psi = \Psi - \Psi_{iso} = \Psi - \frac{1}{4\pi}.$$

Moreover, one can put the system in dimensionless form, introducing

$$t := \frac{U_0 t}{L}, \quad x := \frac{x}{L}, \quad u := \frac{u}{U_0}, \quad q := \frac{qL}{U_0}, \quad \nu := \frac{\nu L}{U_0}, \quad \iota := \frac{\iota L}{U_0}.$$

Then, for $x \in \mathbb{T}$, $p \in \mathbb{S}^2$,

$$\partial_t \psi + (p+u)\cdot\nabla_x\psi - \frac{3\gamma}{4\pi}(p\otimes p) : E(u) + \nabla_p \cdot \left(\mathbb{P}_{p^\perp}\left[(\gamma E(u) + W(u))p\right]\psi\right) = \nu\Delta_p\psi, \tag{1.3a}$$

$$-\Delta_x u + \nabla_x q = \iota \nabla_x \cdot \int_{\mathbb{S}^2} \psi(t,x,p)\, p\otimes p \,\mathrm{d}p, \tag{1.3b}$$

$$\nabla_x \cdot u = 0. \tag{1.3c}$$

Since the seminal paper [19], system (1.3) has been the matter of several numerical and theoretical studies. In particular, simulations align well with experimental observations of suspensions of bacteria.

- For $\iota > 0$ (pullers), no coherent behaviour is observed, which can be interpreted as stability of Ψ^{iso}.
- For $\iota < 0$ (pushers), one observes formation of patterns when hydrodynamic interactions are increased.

Such observations are valid for small rotational diffusion. They were confirmed analytically in [19], *in the special case* $\nu = 0$, for the linearization of (1.3) around $\psi = 0$ (that is the linearization of (1.1) around $\Psi = \Psi^{iso}$). Spectral stability was studied, through a mode by mode Fourier analysis in variable x and refined in subsequent works [12, 16]. Specifically, all perturbations located at mode $k \in 2\pi\mathbb{Z}^3_*$ decay if and only if

$$\frac{\gamma|\iota|}{|k|} < \Gamma_c$$

where the threshold Γ_c is given by:

- if $\iota > 0$, $\Gamma_c = +\infty$ (unconditional stability)
- if $\iota < 0$, $\Gamma_c = \frac{4}{3\pi b_c^2(1-b_c^2)}$ with $b_c \approx 0.623$ the unique positive root of the function

$$s(b) = 2b^3 - \frac{4}{3}b + (b^4 - b^2)\ln\frac{1-b}{1+b}.$$

In particular, all modes decay under the condition

$$\frac{\gamma|\iota|}{2\pi} < \Gamma_c. \tag{1.4}$$

In the recent work [1], such linear stability for $\Gamma < \Gamma_c$ in the non-diffusive case $\nu = 0$ was revisited and linked to a mixing phenomenon, that is a transfer from low to high frequencies in p. This phenomenon is related to the free-transport operator $\partial_t + p\cdot\nabla_x$, itself related to the self-propulsion of the particles. It leads to the decay of integral quantities in p, notably to the decay of the right-hand side of the Stokes equation in (1.1) and from there to the decay of the velocity field u. As $\nu = 0$, explicit computations are possible, and the stability analysis comes down to the decay of Fourier transforms *on the sphere* (as the orientation $p \in \mathbb{S}^2$ substitutes to the usual velocity variable $v \in \mathbb{R}^d$ in other kinetic models). One key feature is that the decay of such Fourier transforms is limited, leading to a weaker mixing than in classical kinetic models, where the orientation variable $p \in \mathbb{S}^2$ is replaced by a usual velocity variable $v \in \mathbb{R}^d$.

In our recent paper [8], we carried a sharp stability study of the linearized version of (1.3) for both $\nu = 0$ and $0 < \nu \ll 1$. We provided slightly more accurate stability results in the case $\nu = 0$, but more importantly, were able to show that the linear stability criterion (1.4) is still the right one for small $\nu > 0$. The introduction of rotational diffusion makes the analysis much harder, as all explicit expressions are lost. We will recall elements of this analysis below.

The purpose of this paper is to extend the linear stability result of [8] to the full nonlinear model (1.3). We present in the next paragraph our main result, explain the main difficulties, notably in comparison to previous linear results. We also provide the general strategy of proof.

1.1. Statement and sketch of proof. Our main result can be stated as follows.

THEOREM 1. *Let $s > \frac{7}{2}$. Assume (1.4). There exists $C_0, \nu_0, \delta_0 > 0$ depending on γ and ι such that for all $\nu \leq \nu_0$, for all initial data ψ^{in} satisfying*

$$\|\psi^{in}\|_{H^s_x L^2_p} \leq \delta_0 \nu^{\frac{3}{2}} \tag{1.5}$$

system (1.3) has a unique global in time solution ψ that satisfies

$$\sup_{t\geq 0} \|\psi(t)\|^2_{H^s_x L^2_p} + \nu \int_0^\infty \|\nabla_p \psi(t)\|^2_{H^s_x L^2_p} \,\mathrm{d}t \leq C_0\, \nu^{-1} \|\psi^{in}\|^2_{H^s_x L^2_p}. \tag{1.6}$$

This theorem shows nonlinear stability of the solution Ψ^{iso} of (1.1) (or equivalently of the solution $\psi = 0$ of (1.3)), under a smallness condition on the initial datum that is explicit in terms of ν. In other words, the nonlinear stability threshold (1.5) provides an estimate of the basin of attraction of the uniform equilibrium Ψ^{iso}, depending on the strength of rotational diffusion. To determine the optimal threshold is an interesting open problem, that was resolved recently in other contexts, such as the stability of Couette flow in the Navier-Stokes equations (see [**3**, **15**, **23**] and references therein) or the stability of global Maxwellians in the Vlasov-Poisson-Landau system [**4**]. See also [**22**] for an optimal stability criterion for nematic suspensions through an entropy method.

Theorem 1 has two main features. First, it shows stability under condition (1.4), which is optimal in view of the linear analysis carried in [**1**, **8**, **19**]. Second, it does not require any spatial diffusion $\kappa\Delta_x\psi$ to be added to the evolution equation in (1.3). One could as well include such term and prove stability under the same stability threshold (1.5), *independently of* κ. From this perspective, our result is quite different from the ones appeared recently in [**1**], where such translational diffusion is added to the right-hand side of (1.3a). In the case of pullers ($\iota > 0$), the authors prove the nonlinear stability of the incoherent state Ψ^{iso}, both for $x \in \mathbb{T}^d$ and $x \in \mathbb{R}^d$, under the condition

$$\|\psi^{in}\|_{H^2_x L^2_p} \ll \min(\nu, \kappa).$$

Second, in the case of pushers ($\iota < 0$), they prove nonlinear stability of the incoherent state *under the stringent assumption* $\Gamma = o(\nu^{1/2})$ instead of $\Gamma < \Gamma_c$, and under the initial constraint

$$\|\psi^{in}\|_{H^2_x L^2_p} \ll \nu^{1/2} \min(\nu^{1/2}, \kappa^{1/2}).$$

These restrictions allow to put the terms $u \cdot \nabla_x \psi$ and $\frac{3\gamma}{4\pi}(p \otimes p) : E(u)$ at the right-hand side of (1.3a) and to treat the model as a perturbation of the advection-diffusion equation

$$\partial_t \psi + p \cdot \nabla_x \psi - \nu \Delta \psi = 0.$$

Relaxing these smallness requirements forces us to adopt another strategy, that we now explain.

The general scheme of our stability proof is based on a classical bootstrap argument. We introduce the maximal time T on which various smallness assumptions on u and ψ are satisfied (see (BA0)-(BA1)-(BA2) in Section 4.1), and we show that improved smallness conditions hold up to time T, see Section 4.3. This automatically implies that T must be infinite, and that the stability estimate holds. To show improvement of these bootstrap assumptions, the key idea is to deduce from (1.3) an infinite-dimensional Volterra equation

$$u(t) + \int_0^t K(t,\tau)\, u(\tau)\, \mathrm{d}\tau = f(t)$$

where the kernel $K(t,\tau) \in L(H^s_x, H^s_x)$ and the source term $f(t) \in H^s_x$ are defined by integral quantities (with respect to p), which involve the operator solution $S_u(t,\tau)$ associated with the operator

$$\partial_t + (p+u) \cdot \nabla_x - \nu \Delta_p$$

on $\mathbb{T}^3 \times \mathbb{S}^2$. This reformulation through a Volterra equation, and expressions for K and f, are detailed in Section 4.2. From there, one proceeds in four steps:

- The first main step, interesting in its own right and further described in the next Section 1.2, establishes enhanced dissipation and mixing decay estimates for the solution of the advection-diffusion equation
$$\partial_t g + (v+p)\cdot\nabla_x g = \nu\Delta_p g, \qquad g|_{t=0} = g^{in}$$
under appropriate smallness assumption on v (to be replaced by u afterwards). Mixing estimates are notably based on the use of the vector field method.
- Once these estimates are obtained, decay in time estimates for the kernel K and for the source term F in the Volterra equation are deduced, see for instance Proposition 4.1.
- Then, by an analysis of the Volterra equation above, still under the bootstrap assumptions on u and ψ, one can improve the smallness on u and show integrable decay for it.
- Finally, with decay and smallness of u at hand, one finally goes back to the equation satisfied by ψ to improve the smallness conditions on ψ.

We stress that this strategy based on the Volterra equation for u and a sharp analysis of the advection-diffusion operator was already implemented in our linear analysis [8]. However, in the linear case, only the case $u = 0$ is considered, which allows a Fourier in x mode-by-mode analysis. Inclusion of the extra advection term requires to couple all modes, which induces several mathematical difficulties. In particular, this change is critical to the mixing estimates by the vector field method. These estimates are based on a family of fields indexed by Fourier modes k, of the form $\chi_k(p)J_k(t,\nabla_p)$ where the localization χ is *compactly supported away from* $p = -k/|k|$. This localization through χ_k is ruined by the coupling of modes.

The estimates for the advection-diffusion are independent of the active suspension model and will be explained in more detail in the next paragraph. They are hopefully of independent interest and will be proved separately in Sections 2 and 3. The remaining steps of the proof of Theorem 1 for the active suspension model will be performed in Section 4.

1.2. Advection-diffusion equations on the sphere. An important and independent part of our analysis are the mixing and enhanced dissipation properties on the sphere. These are the properties of solutions $g = g(t,x,p)$ for $x \in \mathbb{T}^3$ and $p \in \mathbb{S}^2$ solving the advection-diffusion equation
$$\partial_t g + (v+p)\cdot\nabla_x g = \nu\Delta_p g, \qquad g|_{t=0} = g^{in}, \tag{1.7}$$
where g^{in} is the assigned initial datum, $\nu \in (0,1)$ is a diffusivity parameter, and $v = v(t,x)$ is a divergence free vector-field satisfying adequate smallness and decay assumptions. We are interested in semigroup-type estimates for (1.7) that describe enhanced dissipation, as well as mixing estimates for integrated quantities.

To state our results, we expand g and v in Fourier series in x, calling $k \in \mathbb{Z}^3$ the corresponding Fourier variable. For $k \neq 0$, (1.7) becomes
$$\partial_t g_k + \mathrm{i}p\cdot k\, g_k = \nu\Delta_p g_k + |k|\, V g_k, \qquad V g_k := -\mathrm{i}\sum_{\ell\in\mathbb{Z}^3} \hat{k}\cdot v_{k-\ell} g_\ell, \tag{1.8}$$
where $\hat{k} := k/|k|$. Without loss of generality, we will assume $g_0 = 0$ initially, as this mode simply satisfies the standard heat equation
$$\partial_t g_0 = \nu\Delta_p g_0 \tag{1.9}$$
and hence such condition is preserved by the evolution. Besides providing estimates on g, we are interested in integrals of the form
$$\mathsf{V}_k[g] := \int_{\mathbb{S}^2} g_k(p)\, Z_k(p)\, \nabla(p\cdot\hat{k})\, \mathrm{d}p, \tag{1.10}$$
for an arbitrary family $\{Z_k\}_{k\neq 0}$ of smooth, possibly vector-valued, functions. Our main result is based on the assumption that
$$\sup_{t\geq 0}\|v(t)\|_{H^s} + \left(\int_0^\infty \|v(t)\|_{H^s}^2 \mathrm{d}t\right)^{\frac{1}{2}} \leq \varepsilon\nu^{\frac{5}{4}}, \tag{H}$$

for some $s > \frac{5}{2}$ and $\varepsilon \in (0,1)$ small enough.

THEOREM 2. *Let $s > \frac{5}{2}$. There exists absolute constants $C_0, \varepsilon, \nu_0, \eta_1 > 0$ with the following properties. For all $\nu \leq \nu_0$, if (H) holds, then the solution to (1.7) satisfies the enhanced dissipation estimate*

$$\|g(t)\|_{H^s_x L^2_p} \leq C_0 \mathrm{e}^{-\eta_1 \nu^{\frac{1}{2}} t} \|g^{in}\|_{H^s_x L^2_p}, \tag{1.11}$$

and the mixing estimate

$$\sum_{k \neq 0} |k|^{2s+\frac{1}{2}-} |\mathsf{V}_k[g_k(t)]|^2 \leq C_0 \left(\frac{\nu^{\frac{1}{2}}}{\min\{1, \nu^{\frac{1}{2}} t\}} \right)^3 \sup_k \left(\|Z_k\|^2_{W^{1,\infty}_p} + \|Z_k\|^2_{H^2_p} \right) \|g^{in}, \nabla_p g^{in}, \nabla^2_p g^{in}\|^2_{H^s_x L^2_p}, \tag{1.12}$$

for all $t \geq 0$.

Remark 1.1. A look at the proof of the theorem shows that only the weaker condition

$$\sup_{t \geq 0} \|v(t)\|_{H^s} \leq \varepsilon \nu^{\frac{3}{4}}$$

is needed for the enhanced dissipation estimate (1.11). We remind that the expression *enhanced dissipation* refers to the fact that the typical timescale of exponential decay (in our case $\nu^{-1/2}$) is much shorter than the usual time scale ν^{-1} of the heat equation (see [5, 7]). This is related to mixing, which transfers low frequencies to high frequencies in p, on which the viscous dissipation $-\nu\Delta_p$ acts more efficiently.

The proof of Theorem 2 is quite involved and is not using special properties of the sphere $\mathbb{S}^2$. The two basic ingredients are the derivation of hypocoercive estimates, inspired by [2, 21], and the use of the vector field method, inspired by [4, 6].

Regarding hypocoercivity, the key estimate, related to the simple advection-diffusion operator

$$L_\nu = \partial_t + p \cdot \nabla_x - \nu \Delta_p$$

(that is with $u = 0$), is established in Section 2. Compared to the linear analysis in [8], we provide a more general version of this hypocoercive estimate, where

- the function g is replaced by a general tensor Y (which will later be g or Jg or $J^2 g$ for some good vector field J)
- a source term is included. This source term will later come either from the convection ($-v \cdot \nabla_x g$ or $-J(v \cdot \nabla_x g)$ or $-J^2(v \cdot \nabla_x g)$), or from commutators between J and L_ν.

This hypocoercive estimate will reveal better decay properties for the quantity $\nabla(p \cdot k)Y$, which vanishes at $p = \pm \hat{k}$.

Actually, considering $Y = g$ and the source term $-v \cdot \nabla_x g$ would be enough for the exponential decay (1.11). However, the derivation of polynomial decay estimates for integral quantities, uniformly in ν, requires appropriate vector fields.

Introduction of such vector fields and subsequent estimates are developed in Section 3. Like in [8], after Fourier transform in x, the vector field takes the form

$$J_k = \alpha_k \nabla_p + \mathrm{i} \left(\frac{|k|}{\nu} \right)^{1/2} \beta_k \nabla_p (p \cdot \hat{k})$$

where

$$(\alpha_k, \beta_k) = (\alpha_{k,\nu}, \beta_{k,\nu})(t) = (\alpha, \beta)(\nu^{1/2} |k|^{1/2} t)$$

for well-chosen functions α, β (with a slightly different and slightly better choice than the one in [8]). The evolution of the coefficients α_k, β_k is chosen to have a better commutator with L_ν. More precisely, one key feature is that the most annoying term in this commutator vanishes near $p = \hat{k}$. This helps, as the hypocoercive estimate yields better decay properties for quantities that vanish at $p = \pm \hat{k}$. Still, this implies to introduce an additional cut-off $\chi_k(p)$, where χ is localized away from $p = -\hat{k}$.

The problem is that such localization is partly destroyed by the convection term, which is non-local in Fourier. This forces to conduct two types of estimates, suboptimal non-localized ones and optimal localized ones. This process is carried in Paragraphs 3.1 and 3.2. The suboptimal nonlocal bounds are the main limitation in the stability threshold (1.5).

Once this analysis is done, the abstract source term F is replaced by the convection term $(-u\cdot\nabla_x)g$. Due to coupling of Fourier modes, only summation over modes yields closed estimates, as seen in Paragraph 3.3. Finally, mixing estimates for integral quantities are given in Paragraph 3.4.

1.3. Notation. We write $a \lesssim b$ if $a \leq Cb$ for a constant C only depending on the constants γ and ι, and write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$. Note that in Sections 2 and 3 which only deal with equation (1.7) and variations around it, notation $a \lesssim b$ will refer to $a \leq Cb$ for an absolute constant C. For any s, we also denote $H^s = H^s_x(\mathbb{T}^3)$, $\mathcal{H}^s = H^s_x(\mathbb{T}^3; L^2_p(\mathbb{S}^2))$, and finally $\mathbb{Z}^3_* = \mathbb{Z}^3 \setminus \{0\}$.

2. Advection-diffusion with forcing: hypocoercivity

Equation (1.8) can be seen as a forced advection-diffusion equation, as long as we impose mild assumptions on the right-hand side. In this section, we derive energy estimates for a slight generalization of (1.8).

2.1. A general hypocoercivity setup and enhanced dissipation. The basic starting block is to find the hypocoercive dissipation functional for a tensor Y solving the advection-diffusion equation (1.8). We will then first apply this estimate with $Y = g$ and then later with some vector fields $Y = Jg$, $Y = JJg$, for a suitable J. As a small generalization of (1.8), we study a family of $(0, n)$-tensors $(Y_k)_{k\in\mathbb{Z}^3_*}$ evolving as

$$(\partial_t + \mathrm{i}p\cdot k - \nu\Delta_p)Y_k = |k|F_k, \tag{2.1}$$

for some forcing term F_k. From the analysis in [8], we expect enhanced dissipation on the time-scale $O(\nu^{-\frac{1}{2}}|k|^{-\frac{1}{2}})$ so that we set $h = \nu^{\frac{1}{2}}|k|^{\frac{1}{2}}t$ and define the time-dependent weights

$$(a_k, b_k, c_k) := (a, b, c)(h), \quad (a'_k, b'_k, c'_k) := (a', b', c')(h)$$

for non-negative functions a, b, c to be specified below. Denoting by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the L^2 inner product and norm, for any smooth non-negative function $\chi = \chi(p)$, we define the sesquilinear form

$$\begin{aligned} E_{\chi,k}(Y_k, \tilde{Y}_k) := \langle Y_k\chi, \tilde{Y}_k\chi\rangle &+ \left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} a_k\langle\nabla_p Y_k\chi, \nabla_p\tilde{Y}_k\chi\rangle \\ &+ b_k\langle \mathrm{i}\nabla_p(p\cdot\hat{k})Y_k\chi, \nabla_p\tilde{Y}_k\chi\rangle + b_k\langle\nabla_p Y_k\chi, \mathrm{i}\nabla_p(p\cdot\hat{k})\tilde{Y}_k\chi\rangle \\ &+ \left(\frac{\nu}{|k|}\right)^{-\frac{1}{2}} c_k\langle\nabla_p(p\cdot\hat{k})Y_k\chi, \nabla_p(p\cdot\hat{k})\tilde{Y}_k\chi\rangle \end{aligned} \tag{2.2}$$

and the corresponding energy functional

$$\begin{aligned} E_{\chi,k}(Y_k) := E_{\chi,k}(Y_k, Y_k) = \|Y_k\chi\|^2 &+ \left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} a_k\|\nabla_p Y_k\chi\|^2 \\ &+ 2b_k\,\mathrm{Re}\langle\mathrm{i}\nabla_p(p\cdot\hat{k})Y_k\chi, \nabla_p Y_k\chi\rangle + \left(\frac{\nu}{|k|}\right)^{-\frac{1}{2}} c_k\|\nabla_p(p\cdot\hat{k})Y_k\chi\|^2. \end{aligned} \tag{2.3}$$

We also define the dissipation functional

$$\begin{aligned} D_{\chi,k}(Y_k) := \frac{\nu}{|k|}\|\nabla_p Y_k\chi\|^2 &+ \frac{\nu}{|k|}a_k\left(\frac{\nu}{|k|}\right)^{\frac{1}{2}}\|\nabla_p^2 Y_k\chi\|^2 \\ &+ b_k\|\nabla_p(p\cdot\hat{k})Y_k\chi\|^2 + \frac{\nu}{|k|}c_k\left(\frac{\nu}{|k|}\right)^{-\frac{1}{2}}\|\nabla_p\left(\nabla_p(p\cdot\hat{k})Y_k\right)\chi\|^2. \end{aligned} \tag{2.4}$$

The first lemma shows that for a good choice of a, b, c, these functionals provide a good estimate.

Lemma 2.1. *Let $\chi = \chi(p)$ a smooth cut-off and set*

$$a(h) = A\min(h,1), \quad b(h) = B\min(h^2,1), \quad c = C\min(h^3,1)$$

for positive constants A, B, C. There exist constants $\nu_0, B_0, M > 0$ such that for $\nu \le \nu_0$, $B < B_0$ and

$$A = B^{2/3}, \qquad C = \frac{100B^2}{A} \tag{2.5}$$

any $(0,n)$-tensor solution Y_k of (2.1) satisfies

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_{\chi,k}(Y_k) + \frac{3}{4}|k|\, D_{\chi,k}(Y_k) - |k|\,\mathrm{Re}\, E_{\chi,k}(Y_k, F_k) \\
&\quad \le |k| M c_k \left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} \|Y_k\chi\|^2 \\
&\qquad + |k|M\left[\frac{\nu}{|k|}\|Y_k\nabla\chi\|^2 + \left(\frac{\nu}{|k|}\right)^{\frac{3}{2}} a_k\|\nabla Y_k\nabla\chi\|^2 + \left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} c_k\|\nabla(p\cdot\hat{k})Y_k\nabla\chi\|^2\right].
\end{aligned}$$

Remark 2.2. As we shall see, a, b, c satisfy the condition that $b_k^2 < \frac{1}{2}a_k c_k$. This makes the quadratic form $E_{\chi,k}$ is coercive, in the sense that

$$\|Y_k\chi\|^2 + \left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} a_k\|\nabla_p Y_k\chi\|^2 + \left(\frac{\nu}{|k|}\right)^{-\frac{1}{2}} c_k\|\nabla_p(p\cdot\hat{k})Y_k\chi\|^2 \lesssim E_{\chi,k}(Y_k).$$

Remark 2.3. The statement of the lemma would still be true on any Riemannian manifold for which $(0,n)$-tensors Z satisfy $|[\nabla,\Delta]Z| \lesssim |\nabla Z|$. To illustrate this, we will not use the fact that on $\mathbb{S}^2$, $[\nabla_p, \Delta_p]Z = -\nabla_p Z$ has some extra coercivity property.

PROOF OF LEMMA 2.1. Through the change of variables

$$\nu' := \frac{\nu}{|k|}, \quad t' := |k|t, \quad k' := \frac{k}{|k|}, \tag{2.6}$$

we can restrict to the case $|k| = 1$, so that $k = \hat{k}$. To lighten notations, we drop the subscript k, writing Y instead of Y_k, F instead of F_k, and so on, as well as the subscript p on the various differential operators. We find through standard estimates that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Y\chi\|^2 + \nu\|\nabla Y\chi\|^2 \le \mathrm{Re}\langle Y\chi, F\chi\rangle + 2\nu\|\nabla Y\chi\|\,\|Y\nabla\chi\|. \tag{2.7}$$

Similarly, as

$$\partial_t\nabla Y + \mathrm{i}p\cdot kY - \nu\Delta\nabla Y = \nabla F_k - i\nabla(p\cdot k)Y + \nu[\nabla,\Delta]Y,$$

we get

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla Y\chi\|^2 + \nu\|\nabla\nabla Y\chi\|^2 \le{}& \mathrm{Re}\langle\nabla Y\chi, \nabla F\chi\rangle + 2\nu\|\nabla\nabla Y\chi\|\,\|\nabla Y\nabla\chi\| \\
&+ \|\nabla Y\chi\|\,\|\nabla(p\cdot k)Y\chi\| + \nu\|[\nabla,\Delta]Y\chi\|\|\nabla Y\chi\|.
\end{aligned} \tag{2.8}$$

Also, using

$$\begin{aligned}
&\partial_t(\mathrm{i}\nabla(p\cdot k)Y) + \mathrm{i}p\cdot k(\mathrm{i}\nabla(p\cdot k)Y) - \nu\mathrm{i}\nabla(p\cdot k)\Delta Y = \mathrm{i}\nabla(p\cdot k)F, \\
&\partial_t\nabla Y + \mathrm{i}p\cdot k\nabla Y - \nu\nabla\Delta Y = \nabla F - \mathrm{i}\nabla(p\cdot k)Y,
\end{aligned}$$

we find

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re}\langle \mathrm{i}\nabla(p\cdot k)Y\chi,\nabla Y\chi\rangle+\|\nabla(p\cdot k)Y\chi\|^2\\
&\quad\le \nu\langle \mathrm{i}\nabla(p\cdot k)\Delta Y\chi,\nabla Y\chi\rangle+\nu\langle \mathrm{i}\nabla(p\cdot k)Y\chi,\nabla\Delta Y\chi\rangle\\
&\qquad+\operatorname{Re}\langle \mathrm{i}\nabla(p\cdot k)F\chi,\nabla Y\chi\rangle+\operatorname{Re}\langle \mathrm{i}\nabla(p\cdot k)Y\chi,\nabla F\chi\rangle\\
&\quad\le \nu\|\nabla\nabla Y\chi\|\left[2\|\nabla(\nabla(p\cdot k)Y)\chi\|+\|Y\chi\|+2\|\nabla(p\cdot k)Y\nabla\chi\|\right]\\
&\qquad+\operatorname{Re}\langle \mathrm{i}\nabla(p\cdot k)F\chi,\nabla Y\chi\rangle+\operatorname{Re}\langle \mathrm{i}\nabla(p\cdot k)Y\chi,\nabla F\chi\rangle.
\end{aligned}$$

Finally using

$$\partial_t(\mathrm{i}\nabla(p\cdot k)Y)+\mathrm{i}p\cdot k(\mathrm{i}\nabla(p\cdot k)Y)-\nu\Delta(\mathrm{i}\nabla(p\cdot k)Y)=\mathrm{i}\nabla(p\cdot k)F+\nu[\mathrm{i}\nabla(p\cdot k),\Delta]Y$$

together with the commutator formula (see [8])

$$\Delta(\nabla(p\cdot e)\otimes Y)=-\nabla(p\cdot e)\otimes Y-2(p\cdot e)\nabla Y+\nabla(p\cdot e)\otimes\Delta Y, \tag{2.9}$$

we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(p\cdot k)Y\chi\|^2+\nu\|\nabla(\nabla(p\cdot k)Y)\chi\|^2+\nu\|\nabla(p\cdot k)Y\chi\|^2\\
&\quad\le 2\nu\|\nabla(\nabla(p\cdot k)Y)\chi\|\,\|\nabla(p\cdot k)Y\nabla\chi\|+\operatorname{Re}\langle\nabla(p\cdot k)Y\chi,\nabla(p\cdot k)F\chi\rangle\\
&\qquad+2\nu\|\nabla(p\cdot k)Y\chi\|\,\|\nabla Y\chi\|
\end{aligned}$$

Hence, gathering these estimates leads to:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_\chi(Y)+D_\chi(Y)-\operatorname{Re}E_\chi(Y,F)\\
&\le \nu a'\|\nabla Y\chi\|^2+\nu^{\frac12}a\|\nabla Y\chi\|\,\|\nabla(p\cdot k)Y\chi\|+\nu^{\frac32}a\|[\nabla,\Delta]Y\chi\|\|\nabla Y\chi\|\\
&\quad+\nu^{\frac12}b'\operatorname{Re}\langle\mathrm{i}\nabla(p\cdot k)Y\chi,\nabla Y\chi\rangle+2\nu b\|\nabla\nabla Y\chi\|(\|\nabla(\nabla(p\cdot k)Y)\chi\|+\|Y\chi\|+\|\nabla(p\cdot k)Y\nabla\chi\|)\\
&\quad+c'\|\nabla(p\cdot k)Y\chi\|^2+2\nu^{\frac12}c\|\nabla(p\cdot k)Y\chi\|\,\|\nabla Y\chi\|\\
&\quad+2\nu\|\nabla Y\chi\|\,\|Y\nabla\chi\|+2\nu^{\frac32}a\|\nabla\nabla Y\chi\|\,\|\nabla Y\nabla\chi\|+2\nu^{\frac12}c\|\nabla(\nabla(p\cdot k)Y)\chi\|\,\|\nabla(p\cdot k)Y\nabla\chi\|.
\end{aligned}$$

We are now ready to list the constraints on the coefficients a,b,c in order to absorb most of the terms of the right-hand side. It is suffices that

- $a'\le\frac{1}{100}$ to absorb the first term at the right-hand side by the term $\nu\|\nabla Y\chi\|^2$ in the dissipation functional $D_\chi(Y)$ at the left-hand side.
- $a^2\le\frac{b}{100}$ to absorb the second term by the term $\nu\|\nabla Y\chi\|^2+b\|\nabla(p\cdot k)Y\chi\|^2$.
- $a\le 1$ and ν small to absorb the third term by $\nu\|\nabla Y\chi\|^2$.
- $(b')^2\le\frac{b}{100}$ to absorb the fourth term by $\nu\|\nabla Y\chi\|^2+b\|\nabla(p\cdot k)Y\chi\|^2$.
- $b^2\frac{\nu^{\frac12}}{a}\le\frac{\nu^{\frac12}c}{100}$ to bound the fifth term by

  $$\frac{1}{10}\left(\nu^{\frac32}a\|\nabla^2Y\chi\|^2+\nu^{\frac12}c\|\nabla\left(\nabla(p\cdot k)Y\chi\right)\|^2+\nu^{\frac12}c\|Y\chi\|^2+\nu^{\frac12}c\|\nabla(p\cdot k)Y\nabla\chi\|^2\right)$$

  the first two terms being absorbed by $D_\chi(Y)$ at the left-hand side.
- $c'\le\frac{b}{100}$ to absorb the sixth term by $b\|\nabla(p\cdot k)Y\chi\|^2$.
- $c^2\le\frac{b}{100}$ to absorb the seven term by $\nu\|\nabla Y\chi\|^2+b\|\nabla(p\cdot k)Y\chi\|^2$.

These conditions are satisfied with our choice of a,b,c. The last three terms involving $\nabla\chi$ can be treated classically using Young's inequality, without further constraints on a,b,c. □

2.2. The energy inequality for g. As a first application of Lemma 2.1, we deduce an energy inequality for the solution g to (2.1). For the trivial cutoff $\chi \equiv 1$, we use the shorthand notation $E_k = E_{\chi,k}$ and $D_k = D_{\chi,k}$. By the previous lemma, taking $Y_k = g_k$, we get:

Lemma 2.4. *Assume that the scalar function g satisfies (2.1). Then with the same functions as in Lemma 2.1 we find (for possible smaller ν_0 and B_0) that*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_k(g_k) + |k|a_k\left(\frac{\nu}{|k|}\right)^{\frac{1}{2}}\|g_k\|^2 + \frac{5}{8}|k|\,D_k(g_k) - |k|\,\mathrm{Re}\,E_k(g_k,F_k) \le 0. \tag{2.10}$$

For the proof, we first recall the following interpolation result, see [8, Lemma 4.2].

Lemma 2.5. *For all $\sigma \in (0,1]$, all vectors $e \in \mathbb{S}^2$ and all complex-valued $g \in H^1(\mathbb{S}^2)$, the following inequality holds*

$$\sigma^{\frac{1}{2}}\|g\|^2 \le \frac{\sigma}{2}\|\nabla g\|^2 + 2\|\nabla(p\cdot e)\,g\|^2. \tag{2.11}$$

We now can perform the proof.

PROOF OF LEMMA 2.4. Applying Lemma 2.1 yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_k(g_k) + \frac{3}{4}|k|\,D_k(g_k) - |k|\,\mathrm{Re}\,E_k(g_k,F_k) \le |k|Mc_k\left(\frac{\nu}{|k|}\right)^{\frac{1}{2}}\|g_k\chi\|^2. \tag{2.12}$$

By the change of variables (2.6), we can again restrict to $|k| = 1$. We distinguish between long times and short times. First, for long times t such that $\nu \le b_k = b(\nu^{\frac{1}{2}}t)$, we apply the interpolation result with $\sigma = \frac{\nu}{b_k}$. It gives

$$D_k(g_k) \ge \nu\|\nabla g_k\|^2 + b_k\|\nabla(p\cdot k)g_k\|^2 \ge \frac{1}{2}\nu^{\frac{1}{2}}b_k^{\frac{1}{2}}\|g_k\|^2.$$

As $b_k^{\frac{1}{2}} \gg a_k \gg c_k$ this gives the claimed control and allows the absorption of the right-hand side in (2.12).

For the smaller times t such that $\nu \ge b_k$, we find by Poincaré inequality that

$$c_k\nu^{\frac{1}{2}}\|g_k\|^2 \lesssim c_k\nu^{\frac{1}{2}}\|\nabla g_k\|^2 + c_k\nu^{\frac{1}{2}}\|\nabla(p\cdot k)g_k\|^2.$$

As for such times $c_k \ll \nu^{\frac{1}{2}}$ and $c_k \ll b_k$, we can also absorb it in the dissipation and control the right-hand side of (2.12). □

3. Advection-diffusion with forcing: vector fields

To deduce mixing estimates for (1.7) that are uniform in the diffusivity parameter ν, we make use of vector fields adopting the strategy in [8]. Proving enhanced dissipation for g essentially relies on the energy inequality of Lemma 2.4 together with a suitable treatment of the forcing term given by advection (see Section 3.3), as done in Section 3.3.2. However, the proof of uniform-in-ν mixing estimates is more involved: the use of vector fields requires cut-offs to localize and improve basic L^2 estimates near the south pole $p = -\hat{k}$, see Remark 3.2 and Lemma 3.4.

To treat the convection term in a sharp way, we need to introduce two hypocoercivity schemes in Section 3.3.3: a non-localized version (Lemma 3.5), without losses with respect to the convection term, and a localized one (Lemma 3.6), allowing the derivation of a sharp energy inequality. These results are then applied in Section 3.4 to obtain mixing estimates that are uniform in ν.

Our goal is to construct vector fields that have good commutation properties with the advection-diffusion equation (1.7). We look for vector fields of the form

$$J_k = \alpha_k\nabla_p + \mathrm{i}\left(\frac{|k|}{\nu}\right)^{\frac{1}{2}}\beta_k\nabla(p\cdot\hat{k}) \tag{3.1}$$

where we again introduce the good time-scale $h = \nu^{\frac{1}{2}}|k|^{\frac{1}{2}}t$

$$\alpha_k(t) = \alpha(h), \qquad \beta_k(t) = \beta(h)$$

for functions α, β to be specified. The factor in front of β_k is inspired by the fact that as $\nu \to 0$, we want to recover $\alpha_k \equiv 1$ and $\left(\frac{|k|}{\nu}\right)^{\frac{1}{2}} \beta_k \equiv |k|t$. The reason is that the vector field $\nabla_p + \mathrm{i}|k|t\nabla(p \cdot \hat{k})$ commutes to the free transport part. For the full operator in (1.7) we find the commutators

$$[\partial_t + \mathrm{i}p \cdot k - \nu\Delta_p, \alpha_k \nabla_p] = \nu^{\frac{1}{2}}|k|^{\frac{1}{2}}\alpha'\nabla_p - \mathrm{i}|k|\alpha\nabla(p \cdot \hat{k}) - \alpha\nu[\Delta_p, \nabla_p]$$

and

$$\begin{aligned}&\left[\partial_t + \mathrm{i}p \cdot k - \nu\Delta_p, \mathrm{i}\left(\frac{|k|}{\nu}\right)^{\frac{1}{2}} \beta_k \nabla(p \cdot \hat{k})\right] \\ &\quad = \mathrm{i}|k|\beta'\nabla(p \cdot \hat{k}) - \mathrm{i}\nu^{\frac{1}{2}}|k|^{\frac{1}{2}}\beta\nabla(p \cdot \hat{k}) + 2\mathrm{i}\nu^{\frac{1}{2}}|k|^{\frac{1}{2}}\beta\nabla\left[(p \cdot \hat{k})\cdot\right]\end{aligned}$$

so that

$$\begin{aligned}(\partial_t + \mathrm{i}p \cdot k - \nu\Delta_p)J_kY_k = -\mathrm{i}|k|(\alpha - \beta')\nabla(p \cdot \hat{k})Y_k + \nu^{\frac{1}{2}}|k|^{\frac{1}{2}}(\alpha' + 2\mathrm{i}\beta p \cdot \hat{k})\nabla_p Y_k \\ - \alpha\nu[\Delta_p, \nabla_p]Y_k + \mathrm{i}\nu^{\frac{1}{2}}|k|^{\frac{1}{2}}\beta\nabla(p \cdot \hat{k})Y_k.\end{aligned}$$

Clearly, for the $O(1)$ term at the right-hand side to disappear, the condition $\alpha = \beta'$ is necessary. A key idea from [8] is to complete this condition in such a way that the second term at the right-hand side vanishes at the *north pole* $p = \hat{k}$. The goal is to benefit from estimates for quantities $\nabla(p \cdot \hat{k})Y_k$ or $\nabla(\nabla(p \cdot \hat{k})Y_k)$ better than those for Y_k or ∇Y_k alone. In [8], we achieve this by imposing

$$\beta'(h) = \alpha(h), \qquad \alpha'(h) = -2\mathrm{i}\beta(h).$$

Together with the initial data $\alpha(0) = 1$, $\beta(0) = 0$, this provides

$$\alpha(h) = \cosh((1 - \mathrm{i})h), \qquad \beta(h) = \frac{1 + \mathrm{i}}{2}\sinh((1 - \mathrm{i})h).$$

This is a good choice up to the critical time $\nu^{-\frac{1}{2}}|k|^{-\frac{1}{2}}$. However, afterwards both terms behave asymptotically with the factor $\mathrm{e}^{(1-\mathrm{i})h}$, which is growing fast. Hence, we rather consider here

$$\alpha(h) = \mathrm{e}^{-(1-\mathrm{i})h}\cosh((1 - \mathrm{i})h) = \frac{1}{2}\left(1 + \mathrm{e}^{-2h}\mathrm{e}^{2\mathrm{i}h}\right) \tag{3.2}$$

and

$$\beta(h) = \mathrm{e}^{-(1-\mathrm{i})h}\frac{1 + \mathrm{i}}{2}\sinh((1 - \mathrm{i})h) = \frac{(1 + \mathrm{i})}{4}\left(1 - \mathrm{e}^{-2h}\mathrm{e}^{2\mathrm{i}h}\right). \tag{3.3}$$

Remark 3.1. With this new choice of α and β, we have the important property that

$$\alpha \sim 1, \quad \beta^2 \sim b$$

where b is the function appearing in Lemma 2.1 (and now fixed).

In light of (3.2)-(3.3), we compute the commutator between J_k and the advection-diffusion equation as

$$\begin{aligned}&[\partial_t + \mathrm{i}p \cdot k - \nu\Delta_p, J_k] + |k|(1 - \mathrm{i})\left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} \\ &\quad = |k|2\mathrm{i}\beta\left(\frac{\nu}{|k|}\right)^{\frac{1}{2}}\nabla_p\left[(p \cdot \hat{k} - 1)\cdot\right] + |k|\left[-\frac{\nu}{|k|}\alpha[\Delta_p, \nabla_p] - \mathrm{i}\left(\frac{\nu}{|k|}\right)^{\frac{1}{2}}\beta\nabla(p \cdot \hat{k})\right].\end{aligned}$$

Hence we find for a tensor Y_k solution of (2.1) that J_kY_k satisfies

$$(\partial_t + \mathrm{i}p \cdot k - \Delta_p)J_kY_k + |k|(1 - \mathrm{i})\left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} J_kY_k = |k|J_kF_k + |k|R_kY_k, \tag{3.4}$$

where

$$R_kY_k = 2\mathrm{i}\beta_k\left(\frac{\nu}{|k|}\right)^{\frac12}\nabla_p[(p\cdot\hat k-1)Y_k] - \frac{\nu}{|k|}\alpha_k[\Delta_p,\nabla_p]Y_k - \mathrm{i}\left(\frac{\nu}{|k|}\right)^{\frac12}\beta_k\nabla(p\cdot\hat k)Y_k \tag{3.5}$$

is a remainder term.

Remark 3.2. We insist that through our choice of vector field J_k, the worst term at the right-hand side of (3.5), namely $2\mathrm{i}\beta_k\left(\frac{\nu}{|k|}\right)^{\frac12}\nabla_p[(p\cdot\hat k-1)Y_k]$, vanishes at $p=\hat k$. This will allow us to benefit from the better controls that we have on $\nabla(p\cdot k)Y_k$ or $\nabla(\nabla(p\cdot k)Y_k)$, see the dissipation functional (2.4). Still, this better control will only be achieved away from the *south pole* $p=-\hat k$. Indeed, near the south pole, $\nabla(p\cdot\hat k)Y_k$ vanishes while our term at the right-hand side does not, so that the former cannot control the latter. Namely, we will achieve optimal estimates only for $J_kg_k\chi_k$, where χ_k is zero near the south pole $p=-\hat k$. To obtain control near this south pole, we would need to define another vector field H_k, replacing the relation $\alpha'\approx -2\mathrm{i}\beta$ which was our starting point for J_k by $\alpha'\approx +2\mathrm{i}\beta$. All the estimates we obtain for $J_kY_k\chi_k$, with $\chi_k=0$ near the south pole, would readily extend to $H_kY_kg_k$, with $g_k=0$ near the north pole.

3.1. L^2 estimates. We start with a basic estimate for J_kY_k.

Lemma 3.3. *There exists $M>0$ such that for all $k\neq 0$ it holds that*

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|J_kY_k\|^2 + |k|\frac34\frac{\nu}{|k|}\|\nabla J_kY_k\|^2 + |k|\frac34\left(\frac{\nu}{|k|}\right)^{\frac12}\|J_kY_k\|^2 - |k|\,\mathrm{Re}\langle J_kY_k\chi, J_kF_k\rangle \\ \leq |k|M\left[b_k\|Y_k\|^2 + D_k(Y_k)\right].\end{aligned} \tag{3.6}$$

PROOF. As before, we can take the time rescaling from (2.6) and assume $|k|=1$. We also get rid of the subscript k in the notation when no confusion arises. Performing an L^2 estimate on (3.4), we find

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|JY\|^2 + \nu\|\nabla JY\|^2 + \nu^{\frac12}\|JY\|^2 - \mathrm{Re}\langle JY, JF\rangle \leq \mathrm{Re}\langle JY, RY\rangle.$$

To bound the right-hand side, we integrate by parts the first term in the definition (3.5) of R, and get

$$\begin{aligned}\mathrm{Re}\langle JY, RY\rangle &\lesssim |\beta|\nu^{\frac12}\|\nabla JY\|\|(p\cdot k-1)Y\| + \nu|\alpha|\|JY\|\|\nabla Y\| + |\beta|\nu^{\frac12}\|JY\|\|\nabla(p\cdot k)Y\| \\ &\lesssim \nu^{\frac12}|b|^{\frac12}\|\nabla JY\|\|Y\| + \nu\|JY\|\|\nabla Y\| + \nu^{\frac12}|b|^{\frac12}\|JY\|\|\nabla(p\cdot k)Y\|,\end{aligned}$$

where we used Remark 3.1 for the last bound. The result follows then classically from Young's inequality. □

We now show an improved L^2 bound, when localized away from the *south pole* $p=-\hat k$.

Lemma 3.4. *Let $k\neq 0$, let $\chi_k=\chi_k(p), \tilde\chi_k=\tilde\chi_k(p)$ be two smooth functions, which are zero near $p=-\hat k$, and such that $|\chi_k|+|\nabla\chi_k|\lesssim\tilde\chi_k$. Then, for some $M>0$ independent of k, we have*

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|J_kY_k\chi_k\|^2 + |k|\frac34\frac{\nu}{|k|}\|\nabla J_kY_k\chi_k\|^2 + |k|\frac34\left(\frac{\nu}{|k|}\right)^{\frac12}\|J_kY_k\chi_k\|^2 - |k|\langle J_kY_k\chi_k, J_kF_k\chi_k\rangle \\ \leq |k|M\left[\frac{\nu}{|k|}\|Y\tilde\chi_k\|^2 + D_{\tilde\chi_k,k}(Y_k)\right].\end{aligned}$$

PROOF. As before, we can through (2.6) assume that $|k|=1$. We further omit the subscript k from our notations. The localized L^2 estimate yields

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|JY\chi\|^2 + \nu\|\nabla JY\chi\|^2 + \nu^{\frac12}\|JY\chi\|^2 - \langle JY\chi, JF\chi\rangle \leq 2\nu\|\nabla JY\chi\|\|JY\nabla\chi\| + \mathrm{Re}\langle JY\chi, RY\chi\rangle.$$

Thanks to the definition of J and Remark 3.1, we find that

$$\|JY\nabla\chi\|^2 \lesssim |\alpha|^2\|\nabla Y\nabla\chi\|^2 + \nu^{-1}|\beta|^2\|\nabla(p\cdot k)Y\nabla\chi\|^2 \lesssim \|\nabla Y\tilde\chi\|^2 + \nu^{-1}b\|\nabla(p\cdot k)Y\tilde\chi\|^2.$$

It follows that the first term at the right-hand side satisfies for some $M > 0$:

$$2\nu\|\nabla JY\chi\|\|JY\nabla\chi\| \leq \frac{1}{8}\nu\|\nabla JY\chi\|^2 + M\left(\nu\|\nabla Y\tilde{\chi}\|^2 + b\|\nabla(p\cdot k)Y\tilde{\chi}\|^2\right).$$

For the second term, we integrate by parts the first term at the right-hand side of (3.5), and find

$$\begin{aligned}\mathrm{Re}\langle JY\chi, RY\chi\rangle &\lesssim |\beta|\nu^{\frac{1}{2}}\|\nabla JY\chi\|\|(p\cdot k-1)Y\chi\| + |\beta|\nu^{\frac{1}{2}}\|JY\chi\|\|(p\cdot k-1)Y\nabla\chi\| \\ &\quad + \nu|\alpha|\|JY\chi\|\|\nabla Y\chi\| + |\beta|\nu^{\frac{1}{2}}\|JY\chi\|\|\nabla(p\cdot k)Y\chi\| \\ &\lesssim (\|\nabla JY\chi\| + \|JY\chi\|)\left(b^{\frac{1}{2}}\nu^{\frac{1}{2}}\|\nabla(p\cdot k)Y\tilde{\chi}\| + \nu\|\nabla Y\chi\|\right).\end{aligned}$$

Note that besides Remark 3.1, we have used crucially that $|(p\cdot k-1)| \lesssim |\nabla(p\cdot k)|$ on the support of $\tilde{\chi}$, a property that was missing in the proof of Lemma 3.3. One can conclude using Young's inequality that for some $M > 0$:

$$\mathrm{Re}\langle JY\chi, RY\chi\rangle \lesssim \frac{1}{8}\nu\|\nabla JY\chi\|^2 + \frac{1}{8}\nu\|JY\chi\|^2 + M\left(\nu\|\nabla Y\tilde{\chi}\|^2 + b\|\nabla(p\cdot k)Y\tilde{\chi}\|^2\right).$$

The lemma follows. □

3.2. Hypocoercive estimates. To control JY in the hypocoercivity functional, we first show a non-localized bound, where we do not benefit from the localization and thus lose in terms of scaling in ν.

Lemma 3.5. *Assume the setup of Lemma 2.1. For possibly smaller ν_0, B_0 and a new constant M we find that*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_k(J_kY_k) + \frac{5}{8}|k|\,D_k(J_kY_k) + \frac{3}{4}|k|\left(\frac{\nu}{|k|}\right)^{\frac{1}{2}}E_k(J_kY_k) - |k|\,\mathrm{Re}\,E_k(J_kY_k, JF_k) \\ &\quad \leq |k|M\left(\frac{\nu}{|k|}\right)^{-\frac{1}{2}}\left[D_k(Y_k) + \left(\frac{\nu}{|k|}\right)^{\frac{1}{2}}\frac{c_k^2}{b_k}\|Y_k\|^2\right].\end{aligned}$$

PROOF. By the change of variables (2.6), we can restrict to $|k| = 1$. We omit as before the subscript k. Then, (3.4) and Lemma 2.1 imply (taking into account the extra term $(1-\mathrm{i})\nu^{\frac{1}{2}}JY$)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(JY) + \frac{3}{4}D(JY) + \nu^{\frac{1}{2}}E(JY) - \mathrm{Re}\,E(JY, JF) \leq \mathrm{Re}\,E(RY, JY) + Mc\nu^{\frac{1}{2}}\|JY\|^2.$$

The last term on the right-hand side can directly be absorbed by the third term at the left-hand side, as $c \ll 1$. For $\mathrm{Re}\,E(RY, JY)$, first note that

$$|RY|^2 \lesssim (\nu|\beta|^2 + \nu^2|\alpha|^2)|\nabla Y|^2 + \nu|\beta|^2|\nabla(p\cdot k)Y|^2 \lesssim \nu|\nabla Y|^2 + \nu b|\nabla(p\cdot k)Y|^2, \tag{3.7}$$

where we used Remark 3.1 for the last bound. Then split $\mathrm{Re}\,E_k(RY, JY)$ as

$$\mathrm{Re}\,E_k(RY, JY) = I^0 + I^a + I^b + I^c,$$

where

$$\begin{aligned}I^0 &= \mathrm{Re}\langle RY, JY\rangle, \\ I^a &= a\nu^{\frac{1}{2}}\,\mathrm{Re}\langle\nabla RY, \nabla JY\rangle, \\ I^b &= b\,\mathrm{Re}\langle\mathrm{i}\nabla(p\cdot k)RY, \nabla JY\rangle + b\,\mathrm{Re}\langle\mathrm{i}\nabla(p\cdot k)JY, \nabla RY\rangle, \\ I^c &= c\nu^{-\frac{1}{2}}\,\mathrm{Re}\langle\nabla(p\cdot k)RY, \nabla(p\cdot k)JY\rangle.\end{aligned}$$

Clearly

$$I^0 \leq \|RY\|\,\|JY\|,$$

so that for a constant $\delta > 0$ it holds that

$$I^0 - \delta\nu^{\frac{1}{2}}\|JY\|^2 \lesssim \delta^{-1}\nu^{-\frac{1}{2}}\|RY\|^2 \lesssim \delta^{-1}\nu^{-\frac{1}{2}}\left(\nu\|\nabla Y\|^2 + b\nu\|\nabla(p\cdot k)Y\|^2\right) \lesssim \delta^{-1}\nu^{-\frac{1}{2}}D(Y).$$

For I^a we find after an integration by parts that

$$I^a - \delta a \nu^{\frac{3}{2}} \|\nabla^2 JY\|^2 \lesssim \delta^{-1} a \nu^{-\frac{1}{2}} \|RY\|^2 \lesssim \delta^{-1}\nu^{-\frac{1}{2}}\|RY\|^2 \lesssim \delta^{-1}\nu^{-\frac{1}{2}} D(Y).$$

For I^b we get

$$I^b - \delta \nu^{\frac{1}{2}} c \left[\|\nabla(\nabla(p\cdot k)JY)\|^2 + \|JY\|^2\right] \lesssim \delta^{-1}\frac{b^2}{c}\nu^{-\frac{1}{2}}\|RY\|^2 \lesssim \delta^{-1}\nu^{-\frac{1}{2}}\|RY\|^2 \lesssim \delta^{-1}\nu^{-\frac{1}{2}} D(Y).$$

Finally, we need to bound I^c. We find that

$$\begin{aligned} I^c - \delta b \|\nabla(p\cdot k)JY\|^2 &\lesssim \delta^{-1}\nu^{-1}\frac{c^2}{b}\|\nabla(p\cdot k)RY\|^2 \\ &\lesssim \delta^{-1}\frac{c^2}{b}\left(\|\nabla(p\cdot k)\nabla Y\|^2 + b\|\nabla(p\cdot k)Y\|^2\right) \\ &\lesssim \delta^{-1}\nu^{-\frac{1}{2}}\left(\frac{c^2}{b}\nu^{\frac{1}{2}}\|\nabla(\nabla(p\cdot k)Y)\|^2 + \frac{c^2}{b}\nu^{\frac{1}{2}}\|Y\|^2 + b\|\nabla(p\cdot k)Y\|^2\right) \\ &\lesssim \delta^{-1}\nu^{-\frac{1}{2}}\left(\frac{c^2}{b}\nu^{\frac{1}{2}}\|Y\|^2 + D(Y)\right). \end{aligned}$$

The lemma follows. □

Using a cutoff, we can obtain a sharp estimate.

Lemma 3.6. *Assume the setup of Lemma 2.1 and $\chi_k, \tilde{\chi}_k$ as in Lemma 3.4. For possible smaller ν_0, B_0 and a new constant M we find that*

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_{\chi_k,k}(J_kY_k) + \frac{5}{8}|k|\, D_{\chi_k,k}(J_kY_k) + \frac{3}{4}|k|\left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} E_{\chi_k,k}(J_kY_k) - |k|\,\mathrm{Re}\, E_{\chi_k,k}(J_kY_k, JF_k) \\ &\quad \le |k| M\left[\left(\frac{\nu}{|k|}\right)\frac{c_k^2}{b_k}\|Y_k\tilde{\chi}\|^2 + D_{\tilde{\chi}_k,k}(Y_k) + \left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} c_k\|\nabla(p\cdot\hat{k})J_kY_k\nabla\chi\|^2\right]. \end{aligned}$$

PROOF. Again, we can make the change of variables (2.6) as in the proof of Lemma 2.1, and assume that $|k| = 1$. Starting from Lemma 2.1, we find that (omitting as before the subscript k)

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_\chi(JY) + \frac{3}{4}D_\chi(JY) + \nu^{\frac{1}{2}}E_\chi(JY) - \mathrm{Re}\, E_\chi(JY, JF) \\ &\quad \le \mathrm{Re}\, E_\chi(RY, JY) + Mc\nu^{\frac{1}{2}}\|JY\chi\|^2 \\ &\qquad + M\left[\nu\|JY\nabla\chi\|^2 + \nu^{\frac{3}{2}}a\|\nabla JY\nabla\chi\|^2 + \nu^{\frac{1}{2}}c\|\nabla(p\cdot k)JY\nabla\chi\|^2\right]. \end{aligned} \tag{3.8}$$

We first treat the term $\mathrm{Re}\, E_\chi(RY, JY)$. A preliminary remark is that $|p\cdot k - 1| \sim |\nabla(p\cdot k)|^2$ on the support of $\tilde{\chi}$, from where it is easily deduced that, pointwise in the support of $\tilde{\chi}$:

$$\begin{aligned} |RY|^2 &\lesssim \nu|\beta|^2\left[|\nabla(\nabla(p\cdot k)Y)|^2 + |\nabla(p\cdot k)Y|^2\right] + \nu^2|\alpha|^2|\nabla Y|^2 \\ &\lesssim \nu b\left[|\nabla(\nabla(p\cdot k)Y)|^2 + |\nabla(p\cdot k)Y|^2\right] + \nu^2|\nabla Y|^2. \end{aligned} \tag{3.9}$$

The first term splits as

$$\mathrm{Re}\, E_\chi(RY, JY) = I^0 + I^a + I^b + I^c,$$

where

$$\begin{aligned} I^0 &= \mathrm{Re}\langle RY\chi, JY\chi\rangle, \\ I^a &= a\nu^{\frac{1}{2}}\,\mathrm{Re}\langle\nabla RY\chi, \nabla JY\chi\rangle, \\ I^b &= b\,\mathrm{Re}\langle \mathrm{i}\nabla(p\cdot k)RY\chi, \nabla JY\chi\rangle + b\,\mathrm{Re}\langle\mathrm{i}\nabla(p\cdot k)JY\chi, \nabla RY\chi\rangle, \\ I^c &= c\nu^{-\frac{1}{2}}\,\mathrm{Re}\langle\nabla(p\cdot k)RY\chi, \nabla(p\cdot k)JY\chi\rangle. \end{aligned}$$

For I^0, after integration by parts of the first term in the definition (3.5) of RY, we get

$$|I^0| \leq 2|\beta|\nu^{\frac{1}{2}} \left\{\|\nabla JY\chi\| \, \|(p\cdot k-1)Y\chi\| + 2\|JY\chi\| \, \|(p\cdot k-1)Y\nabla\chi\|\right\}$$
$$+ \nu|\alpha|\|JY\chi\| \, \|[\Delta,\nabla]Y\chi\| + \nu^{\frac{1}{2}}|\beta|\|JY\chi\| \, \|\nabla(p\cdot k)Y\chi\|.$$

Hence we find for a constant $\delta > 0$ that

$$|I^0| - \delta\nu\|\nabla JY\chi\|^2 - \delta\nu^{\frac{1}{2}}\|JY\chi\|^2$$
$$\lesssim \delta^{-1}\left\{|\beta|^2\|\nabla(p\cdot k)Y\chi\|^2 + |\beta|^2\nu^{\frac{1}{2}}\|\nabla(p\cdot k)Y\nabla\chi\|^2 + \nu^{\frac{3}{2}}\|\nabla Y\chi\|^2\right\} \lesssim \delta^{-1}D_{\tilde{\chi}}(Y),$$

taking into account that $|\beta|^2 \sim b$. For I^a note that

$$|I^a| \leq a\nu^{\frac{1}{2}}\left[\|\nabla\nabla JY\chi\| \, \|RY\chi\| + 2\|\nabla JY\chi\| \, \|RY\nabla\chi\|\right].$$

so that

$$|I^a| - \delta\nu^{\frac{3}{2}}a\|\nabla\nabla JY\chi\|^2 - \delta\nu\|\nabla JY\chi\|^2 \lesssim \delta^{-1}\nu^{-\frac{1}{2}}a\|RY\chi\|^2 + \delta^{-1}a^2\|RY\nabla\chi\|^2.$$

Hence, from (3.9), we find (taking into account that $ab \sim c$):

$$|I^a| - \delta\nu^{\frac{3}{2}}a\|\nabla\nabla JY\chi\|^2 - \delta\nu\|\nabla JY\chi\|^2 \lesssim \delta^{-1}D_{\tilde{\chi}}(Y).$$

For I^b note that

$$|I^b| \leq b\|RY\chi\|\left[2\|\nabla(\nabla(p\cdot k)JY)\chi\| + \|JY\chi\|\right] + 2b\|RY\nabla\chi\| \, \|\nabla(p\cdot k)JY\chi\|,$$

so that

$$|I^b| - \delta\nu^{\frac{1}{2}}c\|\nabla(\nabla(p\cdot k)JY)\chi\|^2 - \delta\nu^{\frac{1}{2}}\|JY\chi\|^2 - \delta b\|\nabla(p\cdot k)JY\chi\|^2$$
$$\lesssim \delta^{-1}\nu^{-\frac{1}{2}}\frac{b^2}{c}\|RY\chi\|^2 + \delta^{-1}b\|RY\nabla\chi\|^2 \lesssim \delta^{-1}D_{\tilde{\chi}}(Y),$$

where the last inequality comes again from (3.9) (note that $\frac{b^2}{c} \sim a$, $b \sim a^2$, so that the right-hand side is similar to the one for I_a). For I^c, the treatment is more involved. Note that

$$\nabla(p\cdot k)R = 2\nu\nabla_p\left[(p\cdot k-1)(JY - \alpha\nabla Y)\right] - 2\mathrm{i}\beta\nu^{\frac{1}{2}}\nabla\nabla(p\cdot k)(p\cdot k-1)Y$$
$$- \nu\alpha\nabla(p\cdot k)[\Delta,\nabla]Y - \mathrm{i}\nu^{\frac{1}{2}}\beta\nabla(p\cdot k)\nabla(p\cdot k)Y.$$

We remind that on the support of $\tilde{\chi}$, $|p\cdot k-1| \sim |\nabla(p\cdot k)|^2$ so that

$$|(p\cdot k-1)\nabla Y| \lesssim |\nabla(\nabla(p\cdot k)Y)| + |\nabla(p\cdot k)Y|.$$

Hence, we find that

$$|I^c| \lesssim \nu^{\frac{1}{2}}c\|\nabla(\nabla(p\cdot k)JY)\chi\|\left\{\|\nabla(p\cdot k)JY\chi\| + \|\nabla(\nabla(p\cdot k)Y)\chi\| + \|\nabla(p\cdot k)Y\chi\|\right\}$$
$$+ \nu^{\frac{1}{2}}c\|\nabla(p\cdot k)JY\chi\|\left\{\|\nabla(p\cdot k)JY\nabla\chi\| + \|\nabla(\nabla(p\cdot k)Y)\nabla\chi\| + \|\nabla(p\cdot k)Y\nabla\chi\|\right\}$$
$$+ \nu^{-\frac{1}{2}}c\|\nabla(p\cdot k)JY\chi\|\left\{\beta\nu^{\frac{1}{2}}\|\nabla(p\cdot k)Y\chi\| + \nu\|\nabla(\nabla(p\cdot k)Y)\chi\| + \nu\|Y\chi\|\right\}.$$

Hence we can bound I^c as

$$|I^c| - \delta\nu^{\frac{1}{2}}c\|\nabla(\nabla(p\cdot k)JY)\chi\|^2 - \delta b\|\nabla(p\cdot k)JY\chi\|^2$$
$$\lesssim \delta^{-1}\nu^{\frac{1}{2}}c\left\{\|\nabla(p\cdot k)JY\chi\|^2 + \|\nabla(\nabla(p\cdot k)Y)\chi\|^2 + \|\nabla(p\cdot k)Y\chi\|^2\right\}$$
$$+ \delta^{-1}\nu\frac{c^2}{b}\left\{\|\nabla(p\cdot k)JY\nabla\chi\|^2 + \|\nabla(\nabla(p\cdot k)Y)\nabla\chi\|^2 + \|\nabla(p\cdot k)Y\nabla\chi\|^2\right\}$$
$$+ \delta^{-1}\frac{c^2|\beta|^2}{b}\|\nabla(p\cdot k)Y\chi\|^2 + \delta^{-1}\nu\frac{c^2}{b}\left\{\|\nabla(\nabla(p\cdot k)Y)\chi\|^2 + \|Y\chi\|^2\right\}$$
$$\lesssim \delta^{-1}\nu^{\frac{1}{2}}c\|\nabla(p\cdot k)JY\chi\|^2 + \delta^{-1}D_{\tilde{\chi}}(Y) + \delta^{-1}\nu\frac{c^2}{b}\|\nabla(p\cdot k)JY\nabla\chi\|^2 + \delta^{-1}\nu\frac{c^2}{b}\|Y\chi\|^2.$$

The first term at the right-hand side can be absorbed by $D_\chi(JY)$ at the left-hand side As regards the third term, note that by the definition (3.1) we can estimate directly

$$\nu\frac{c^2}{b}\|\nabla(p\cdot k)JY\nabla\chi\|^2 \lesssim \nu\frac{c^2}{b}\big(\|\nabla(\nabla(p\cdot k)Y)\nabla\chi\|^2+\|Y\nabla\chi\|^2\big)+\frac{c^2}{b}|\beta|^2\|\nabla(p\cdot k)Y\nabla\chi\|^2$$
$$\lesssim D_{\tilde\chi}(Y)+\nu\frac{c^2}{b}\|Y\nabla\chi\|^2.$$

The last step of the proof is to control the second to fourth term in the right-hand side of (3.8). The second one can be absorbed by the term $\nu^{\frac12}E_\chi(JY)$ at the left-hand side, as $c\ll 1$. For the next one, we note that by definition (3.1):

$$\nu\|JY\nabla\chi\|^2\lesssim \nu\|\nabla Y\nabla\chi\|^2+b_k\|\nabla(p\cdot k)Y\nabla\chi\|^2\lesssim D_{\tilde\chi}(Y).$$

For the fourth term, again by (3.1):

$$\nu^{\frac32}a\|\nabla JY\nabla\chi\|^2\lesssim \nu^{\frac32}a\|\nabla\nabla Y\nabla\chi\|^2+\nu^{\frac12}ab\|\nabla(\nabla(p\cdot k)Y)\nabla\chi\|^2\lesssim D_{\tilde\chi}(Y).$$

Putting all the above estimates together, we conclude the proof. □

3.3. The role of convection. In this section, we now specialize the discussion to the case in which the forcing term F_k is of the form

$$VY_k=-\mathrm{i}\hat k\sum_\ell v_{k-\ell}Y_\ell, \tag{3.10}$$

for a divergence-free velocity field v. This is precisely the setting in (1.8). As we do not have a strong enough gain of regularization in x, we cannot neglect the gains arising from the divergence-free property. This translates into losing the possibility of working mode-by-mode in k, as all the x-modes are coupled, and instead we work with energy functionals that include all the non-zero x-modes.

Therefore, we look at

$$E_{\chi,s}(Y)=\sum_k|k|^{2s}E_{\chi_k,k}(Y_k) \tag{3.11}$$

for a family of cutoffs $(\chi_k)_{k\neq0}$ (that we can think of as $\chi_k=\chi(p\cdot\hat k)$). The divergence-free cancellations can be seen through the following lemma.

Lemma 3.7. *Consider a divergence free velocity field v, so that $\sum_k k\cdot v_k=0$, and weights $W:\mathbb{R}^3\to\mathbb{R}_+$ and $\tilde W:\mathbb{R}^3\to X$ for some normed space X satisfying*

$$\forall x,y,\quad |y|\geq\frac{|x|}{2}\ \Rightarrow\ W(x)\lesssim W(y),\quad \|\tilde W(x)\|_X\lesssim\|\tilde W(y)\|_X,$$
$$\forall x,y,\quad \|\tilde W(x)-\tilde W(y)\|_X\lesssim\left(\frac{\|\tilde W(x)\|_X}{|x|}+\frac{\|\tilde W(y)\|_X}{|y|}\right)|x-y|.$$

Then for any non-negative sequences $\{H_k\}_{k\in\mathbb{Z}^3},\{G_k\}_{k\in\mathbb{Z}^3}$, we have

$$\sum_{k,\ell}W(k)\|\tilde W(k)-\tilde W(l)\|_X\,|v_{k-\ell}\cdot\ell|\,H_\ell\,G_k$$
$$\lesssim\Big(\sum_k|k|H_k\Big)\Big(\sum_kW(k)\|\tilde W(k)\|_X|v_k|^2\Big)^{\frac12}\Big(\sum_kW(k)\|\tilde W(k)\|_XG_k^2\Big)^{\frac12}$$
$$+\Big(\sum_k|k|G_k\Big)\Big(\sum_kW(k)\|\tilde W(k)\|_X|v_k|^2\Big)^{\frac12}\Big(\sum_kW(k)\|\tilde W(k)\|_XH_k^2\Big)^{\frac12}$$
$$+\Big(\sum_k|k||v_k|\Big)\Big(\sum_kW(k)\|\tilde W(k)\|_XH_k^2\Big)^{\frac12}\Big(\sum_kW(k)\|\tilde W(k)\|_XG_k^2\Big)^{\frac12}.$$

PROOF. We split the sum in three:

$$\sum_{k,\ell} = \sum_{|k|\geq 2|\ell|} + \sum_{|\ell|\geq 2|k|} + \sum_{|\ell|/2\leq|k|\leq 2|\ell|} .$$

We find

$$\begin{aligned}
&\sum_{|k|\geq 2|\ell|} W(k)\|\tilde{W}(k)-\tilde{W}(\ell)\|_X\,|v_{k-\ell}\cdot\ell|\,H_\ell\,G_k \\
&\quad\lesssim \sum_{|k|\geq 2|\ell|} \Big(\sqrt{W(k-\ell)\|\tilde{W}(k-\ell)\|_X}\,|v_{k-\ell}|\Big)\Big(\sqrt{W(k)\|\tilde{W}(k)\|_X}\,G_k\Big)\,|\ell H_\ell| \\
&\quad\lesssim \Big(\sum_k |k|H_k\Big)\Big(\sum_k W(k)\|\tilde{W}(k)\|_X|v_k|^2\Big)^{\frac12}\Big(\sum_k W(k)\|\tilde{W}(k)\|_X G_k^2\Big)^{\frac12}.
\end{aligned}$$

Similarly, as $v_{k-\ell}\cdot\ell = v_{k-\ell}\cdot k$,

$$\begin{aligned}
&\sum_{|\ell|\geq 2|k|} W(k)\|\tilde{W}(k)-\tilde{W}(\ell)\|_X\,|v_{k-\ell}\cdot\ell|\,H_\ell\,G_k \\
&\quad\lesssim \sum_{|\ell|\geq 2|k|} W(\ell)\|\tilde{W}(k)-\tilde{W}(\ell)\|_X\,|v_{k-\ell}\cdot k|\,H_\ell\,G_k \\
&\quad\lesssim \Big(\sum_k |k|G_k\Big)\Big(\sum_k W(k)\|\tilde{W}(k)\|_X|v_k|^2\Big)^{\frac12}\Big(\sum_k W(k)\|\tilde{W}(k)\|_X H_k^2\Big)^{\frac12}.
\end{aligned}$$

Eventually,

$$\begin{aligned}
&\sum_{|\ell|/2\leq|k|\leq 2|\ell|} W(k)\|\tilde{W}(k)-\tilde{W}(\ell)\|_X\,|v_{k-\ell}\cdot\ell|\,H_k\,G_\ell \\
&\quad\lesssim \sum_{|l|/2\leq|k|\leq 2|l|} W(k)\Big(\frac{\|\tilde{W}(k)\|_X}{|k|}+\frac{\|\tilde{W}(\ell)\|_X}{|\ell|}\Big)|k-\ell||v_{k-\ell}\cdot\ell|H_k\,G_\ell \\
&\quad\lesssim \sum_{|\ell|/2\leq|k|\leq 2|\ell|} \sqrt{|W(k)|\|\tilde{W}(k)\|_X}\,\sqrt{|W(\ell)|\|\tilde{W}(\ell)\|_X}\,|k-\ell||v_{k-\ell}|H_k\,G_\ell \\
&\quad\lesssim \Big(\sum_k |k||v_k|\Big)\Big(\sum_k W(k)\|\tilde{W}(k)\|_X H_k^2\Big)^{\frac12}\Big(\sum_k W(k)\|\tilde{W}(k)\|_X G_k^2\Big)^{\frac12}.
\end{aligned}$$

The proof is over. □

Using the cancellation from the divergence-free condition, we control the error in the hypoelliptic estimate by the following bound.

Lemma 3.8. *Assume the setup of Lemma 2.1, and let $(\chi_k)_{k\neq 0}$ a family of smooth functions. Then, the convection operator defined in (3.10) obeys the following estimate, for all $s>\frac52$, for all $\delta>0$:*

$$\begin{aligned}
&\sum_k |k|^{2s+1}\,\mathrm{Re}\,E_{\chi_k,k}(VY_k,Y_k) - \delta\nu^{\frac12}\sum|k|^{2s}\|Y_k\chi_k\|^2 - \delta\nu\sum|k|^{2s}\|\nabla_k Y_k\chi_k\|^2 \\
&\quad - \delta\nu^{\frac12}\sum|k|^{2s}b_k^2\|\nabla(\nabla(p\cdot\hat{k})Y_k)\chi_k\|^2 - (\delta+\|v\|_{H^s}\nu^{-\frac12})\sum|k|^{2s+1}c_k\|\nabla(p\cdot\hat{k})Y_k\chi_k\|^2 \\
&\quad\lesssim \delta^{-1}\nu^{-1}\|v\|_{H^s}^2\sum|k|^{2s}\|Y_k\|^2 + \delta^{-1}\|v\|_{H^s}^2\sum|k|^{2s}\|\nabla Y_k\|^2.
\end{aligned}$$

Remark 3.9. Note that the localisation through a cut-off is lost in the last two terms. This will be the main constraint on the size of v. Note also that the requirement $s>\frac52$ ensures that $\sum_{k\in 2\pi\mathbb{Z}^3}|k||v_k|\lesssim\|v\|_{H^s}$, an inequality that will be used implicitly each time we apply Lemma 3.7.

PROOF OF LEMMA 3.8. We use for short notation V_k instead of VY_k. We decompose

$$\sum_k |k|^{2s+1} \operatorname{Re} E_{\chi_k,k}(V_k, Y_k) = I^0 + I^a + I^b + I^c,$$

where

$$\begin{aligned}
I^0 &= \sum_k |k|^{2s+1} \operatorname{Re}\langle V_k\chi_k, Y_k\chi_k\rangle,\\
I^a &= \nu^{\frac12} \sum_k |k|^{2s-\frac12} a_k \operatorname{Re}\langle \nabla_p V_k\chi_k, \nabla_p Y_k\chi_k\rangle,\\
I^b &= \sum_k |k|^{2s+1} b_k \left[\operatorname{Re}\langle \mathrm{i}\nabla(p\cdot\hat k)V_k\chi_k, \nabla Y_k\chi_k\rangle + \operatorname{Re}\langle \mathrm{i}\nabla(p\cdot\hat k)Y_k\chi_k, \nabla V_k\chi_k\rangle\right],\\
I^c &= \nu^{-\frac12} \sum_k |k|^{2s+\frac32} c_k \operatorname{Re}\langle \nabla(p\cdot\hat k)V_k\chi_k, \nabla(p\cdot\hat k)Y_k\chi_k\rangle.
\end{aligned}$$

Using that v is real, we find from the definition that

$$\begin{aligned}
I^0 &= \sum_{k,\ell} |k|^{2s} \operatorname{Re}\langle \mathrm{i}k v_{k-\ell} Y_\ell\chi_k, Y_k\chi_k\rangle\\
&= \frac12 \sum_{k,\ell} \operatorname{Re}\left\langle \mathrm{i}\left(|k|^{2s}k\chi_k^2 - |\ell|^{2s}\ell\chi_\ell^2\right)\cdot v_{k-\ell}Y_\ell, Y_k\right\rangle \quad = \frac12 \sum_{k,\ell} \operatorname{Re}\left\langle \mathrm{i}\left(|k|^{2s}\chi_k^2 - |\ell|^{2s}\chi_\ell^2\right) v_{k-\ell}\cdot \ell Y_\ell, Y_k\right\rangle,
\end{aligned}$$

where the last equality comes from the divergence-free property: $v_{k-\ell}\cdot k = v_{k-\ell}\cdot \ell$. We can split the difference as

$$|k|^{2s}\chi_k^2 - |\ell|^{2s}\chi_\ell^2 = |k|^{2s}\chi_k(\chi_k - \chi_\ell) + \left(|k|^{2s} - |\ell|^{2s}\right)\chi_k\chi_\ell + |\ell|^{2s}\chi_\ell(\chi_k - \chi_\ell).$$

Hence, using the symmetry in k and ℓ,

$$I^0 \le \sum |k|^{2s}\|\chi_k - \chi_\ell\|_\infty |v_{k-\ell}\cdot\ell| \|Y_\ell\|\,\|Y_k\chi_k\| + \frac12 \sum \left[|k|^{2s} - |\ell|^{2s}\right] |v_{k-\ell}\cdot\ell| \|Y_\ell\chi_\ell\|\,\|Y_k\chi_k\|.$$

We apply Lemma 3.7, taking $W = |k|^{2s}$ and $\tilde W(k) = \chi_k$ for the first sum, and $W = 1$, $\tilde W(k) = |k|^{2s}$ for the second sum. We get

$$I^0 \lesssim \|v\|_{H^s} \left(\sum |k|^{2s}\|Y_k\|^2\right)^{\frac12} \left(\sum |k|^{2s}\|Y_k\chi_k\|^2\right)^{\frac12}.$$

For I^a we find in the same way (that is replacing $\chi_k(p)$, function of p, by $\chi_k(p)a(\nu^1|k|^{\frac12}t)$, function of p and t) that

$$\begin{aligned}
I^a &= \frac{\nu^{\frac12}}{2} \sum \operatorname{Re}\langle \mathrm{i}\left[|k|^{2s-\frac12}a_k\chi_k^2 - |\ell|^{2s-\frac12}a_\ell\chi_\ell^2\right] v_{k-\ell}\cdot\ell\nabla Y_\ell, \nabla Y_k\rangle\\
&\le \nu^{\frac12} \sum |k|^{2s-\frac12}\|a_k^{\frac12}\chi_k - a_\ell^{\frac12}\chi_\ell\|_\infty |v_{k-\ell}\cdot\ell| \|\nabla Y_\ell\|\, a_k^{\frac12}\|\nabla Y\chi_k\|\\
&\quad + \frac{\nu^{\frac12}}{2} \sum \left||k|^{2s-\frac12} - |\ell|^{2s-\frac12}\right| |v_{k-\ell}\cdot\ell| a_\ell^{\frac12}\|\nabla Y_\ell\chi_\ell\|\, a_k^{\frac12}\|\nabla Y\chi_k\|.
\end{aligned}$$

Hence we find that

$$I^a \lesssim \nu^{\frac12}\|v\|_{H^s} \left(\sum |k|^{2s-\frac12}\|\nabla Y_k\|^2\right)^{\frac12} \left(\sum |k|^{2s-\frac12}a_k\|\nabla Y_k\chi_k\|^2\right)^{\frac12}.$$

For I^b, we find by the definition

$$I^b = \sum \operatorname{Re}\left\langle \mathrm{i}\left[b_k|k|^{2s}\nabla(p\cdot\hat k)\chi_k^2 - b_\ell|\ell|^{2s}\nabla(p\cdot\hat\ell)\chi_\ell^2\right] v_{k-\ell}\cdot\ell Y_\ell, \nabla Y_k\right\rangle,$$

where we split the bracket as

$$\begin{aligned}
&b_k|k|^{2s}\nabla(p\cdot\hat{k})\chi_k^2 - b_\ell|\ell|^{2s}\nabla(p\cdot\hat{\ell})\chi_\ell^2\\
&\quad= |k|^{2s}\nabla(p\cdot\hat{k})b_k\chi_k(\chi_k-\chi_\ell) + \left[|k|^{2s}\nabla(p\cdot\hat{k})b_k - |\ell|^{2s}\nabla(p\cdot\hat{\ell})b_\ell\right]\chi_k\chi_\ell\\
&\qquad+ |\ell|^{2s}\nabla(p\cdot\hat{\ell})b_\ell\chi_\ell(\chi_k-\chi_\ell).
\end{aligned}$$

Hence we find that

$$\begin{aligned}
I^b \le &\sum |k|^{2s}\|\chi_k-\chi_\ell\|_\infty |v_{k-\ell}\cdot\ell|\,\|Y_\ell\|\,b_k\|\nabla(p\cdot\hat{k})\nabla Y_k\chi_k\|\\
&+\sum \||k|^{2s}\nabla(p\cdot\hat{k})b_k - |\ell|^{2s}\nabla(p\cdot\hat{\ell})b_\ell\|_\infty |v_{k-\ell}\cdot\ell|\,\|Y_\ell\chi_\ell\|\,\|\nabla Y_k\chi_k\|\\
&+\sum |k|^{2s}\|\chi_k-\chi_\ell\|_\infty |v_{k-\ell}\cdot\ell|\,\|\nabla Y_\ell\|\,b_k\|\nabla(p\cdot\hat{k})Y_k\chi_k\|.
\end{aligned}$$

This yields that

$$\begin{aligned}
I^b \lesssim &\|v\|_{H^s}\left(\sum_k |k|^{2s}\|Y_k\|^2\right)^{\frac12}\left(\sum_k |k|^{2s}b_k^2\|\nabla(p\cdot\hat{k})\nabla Y_k\chi_k\|^2\right)^{\frac12}\\
&+\|v\|_{H^s}\left(\sum_k |k|^{2s}\|Y_k\chi_k\|^2\right)^{\frac12}\left(\sum_k |k|^{2s}\|\nabla Y_k\chi_k\|^2\right)^{\frac12}\\
&+\|v\|_{H^s}\left(\sum_k |k|^{2s}\|\nabla Y_k\|^2\right)^{\frac12}\left(\sum_k |k|^{2s}b_k^2\|\nabla(p\cdot\hat{k})Y_k\chi_k\|^2\right)^{\frac12}.
\end{aligned}$$

For I^c we find

$$I^c = \frac{\nu^{-\frac12}}{2}\sum \mathrm{Re}\left\langle \mathrm{i}\left[|k|^{2s+\frac12}c_k\nabla(p\cdot\hat{k})\nabla(p\cdot\hat{k})\chi_k^2 - |\ell|^{2s+\frac12}c_\ell\nabla(p\cdot\hat{\ell})\nabla(p\cdot\hat{\ell})\chi_\ell^2\right]v_{k-\ell}\cdot\ell Y_\ell, Y_k\right\rangle.$$

We split the difference as

$$\begin{aligned}
&|k|^{2s+\frac12}c_k\nabla(p\cdot\hat{k})\nabla(p\cdot\hat{k})\chi_k^2 - |\ell|^{2s+\frac12}c_\ell\nabla(p\cdot\hat{\ell})\nabla(p\cdot\hat{\ell})\chi_\ell^2\\
&\quad= |k|^{2s-\frac12}c_k^{\frac12}|k|^{\frac12}\nabla(p\cdot\hat{k})\left[c_k^{\frac12}|k|^{\frac12}\nabla(p\cdot\hat{k})\chi_k - c_\ell^{\frac12}|\ell|^{\frac12}\nabla(p\cdot\hat{\ell})\chi_l\right]\chi_k\\
&\qquad+\left[|k|^{2s-\frac12}-|\ell|^{2s-\frac12}\right]c_k^{\frac12}|k|^{\frac12}\nabla(p\cdot\hat{k})\chi_k c_\ell^{\frac12}|\ell|^{\frac12}\nabla(p\cdot\hat{\ell})\chi_\ell\\
&\qquad+|\ell|^{2s-\frac12}c_\ell^{\frac12}|\ell|^{\frac12}\nabla(p\cdot\hat{\ell})\left[c_k^{\frac12}|k|^{\frac12}\nabla(p\cdot\hat{k})\chi_k - c_\ell^{\frac12}|\ell|^{\frac12}\nabla(p\cdot\hat{\ell})\chi_\ell\right]\chi_\ell.
\end{aligned}$$

Hence we find that

$$\begin{aligned}
I^c \le &\nu^{-\frac12}\sum |k|^{2s-\frac12}\left\|c_k^{\frac12}|k|^{\frac12}\nabla(p\cdot\hat{k}) - c_\ell^{\frac12}|\ell|^{\frac12}\nabla(p\cdot\hat{\ell})\right\|_\infty |v_{k-\ell}\cdot\ell|\|Y_\ell\|c_k^{\frac12}|k|^{\frac12}\|\nabla(p\cdot\hat{k})Y_k\chi_k\|\\
&+\frac{\nu^{-\frac12}}{2}\sum\left||k|^{2s-\frac12}-|\ell|^{2s-\frac12}\right||v_{k-\ell}\cdot\ell|c_\ell^{\frac12}|\ell|^{\frac12}\|\nabla(p\cdot\hat{\ell})Y_\ell\chi_\ell\|c_k^{\frac12}|k|^{\frac12}\|\nabla(p\cdot\hat{k})Y_k\chi_k\|.
\end{aligned}$$

Applying Lemma 3.7, we obtain that

$$\begin{aligned}
I^c \lesssim &\nu^{-\frac12}\|v\|_{H^s}\left(\sum |k|^{2s}\|Y_k\|^2\right)^{\frac12}\left(\sum |k|^{2s+1}c_k\|\nabla(p\cdot\hat{k})Y_k\chi_k\|^2\right)^{\frac12}\\
&+\nu^{-\frac12}\|v\|_{H^s}\left(\sum |k|^{2s+\frac12}c_k\|\nabla(p\cdot\hat{k})Y_k\chi_k\|^2\right).
\end{aligned}$$

Collecting the estimates, we therefore find

$$\begin{aligned}\sum_k |k|^{2s+1} \operatorname{Re} E_{\chi_k,k}(V_k,Y_k) &\lesssim \|v\|_{H^s} \left(\sum |k|^{2s}\|Y_k\|^2\right)^{\frac12} \left(\sum |k|^{2s}\|Y_k\chi_k\|^2\right)^{\frac12} \\ &+ \nu^{\frac12}\|v\|_{H^s} \left(\sum |k|^{2s-\frac12}\|\nabla Y_k\|^2\right)^{\frac12} \left(\sum |k|^{2s-\frac12} a_k\|\nabla Y_k\chi_k\|^2\right)^{\frac12} \\ &+ \|v\|_{H^s} \left(\sum_k |k|^{2s}\|Y_k\|^2\right)^{\frac12} \left(\sum_k |k|^{2s} b_k^2\|\nabla(p\cdot\hat k)\nabla Y_k\chi_k\|^2\right)^{\frac12} \\ &+ \|v\|_{H^s} \left(\sum_k |k|^{2s}\|Y_k\chi_k\|^2\right)^{\frac12} \left(\sum_k |k|^{2s}\|\nabla Y_k\chi_k\|^2\right)^{\frac12} \\ &+ \|v\|_{H^s} \left(\sum_k |k|^{2s}\|\nabla Y_k\|^2\right)^{\frac12} \left(\sum_k |k|^{2s} b_k^2\|\nabla(p\cdot\hat k)Y_k\chi_k\|^2\right)^{\frac12} \\ &+ \nu^{-\frac12}\|v\|_{H^s} \left(\sum_k |k|^{2s}\|Y_k\|^2\right)^{\frac12} \left(\sum_k |k|^{2s+1} c_k\|\nabla(p\cdot\hat k)Y_k\chi_k\|^2\right)^{\frac12} \\ &+ \nu^{-\frac12}\|v\|_{H^s} \left(\sum_k |k|^{2s+\frac12} c_k\|\nabla(p\cdot\hat k)Y_k\chi_k\|^2\right).\end{aligned}$$

Splitting the right-hand side with Young's inequality then gives the claimed control. □

Using the cancellation from the divergence-free condition, we control the error for the energy by the following estimate.

Lemma 3.10. *Let $(\chi_k = \chi_k(p))_{k\neq 0}$ a family of smooth functions. Then, the convection operator defined in (3.10) obeys the following estimate, for all $s > \frac52$:*

$$\sum_k |k|^{2s+1} \operatorname{Re}\langle VY\chi_k, Y\chi_k\rangle \lesssim \|v\|_{H^s} \left(\sum |k|^{2s}\|Y_k\|^2\right)^{\frac12} \left(\sum |k|^{2s}\|Y_k\chi_k\|^2\right)^{\frac12}.$$

PROOF. See the computation for I^0 in the previous lemma. □

3.3.1. *Commutator with the vector fields.* In order to control Jg and JJg, we need to understand the commutator with the convection term, that is we need to understand the influence of

$$SY_k = \sum_\ell J_k(-\mathrm{i}\hat k v_{k-\ell})Y_\ell - \sum_\ell (-\mathrm{i}\hat k v_{k-\ell})J_\ell Y_\ell. \tag{3.12}$$

To control Jg in L^2, we prove a first commutator estimate.

Lemma 3.11. *Let $(\chi_k)_{k\neq 0}$ a family of smooth functions. We find*

$$\begin{aligned}&\sum |k|^{2s+1} \operatorname{Re}\langle SY_k\chi_k, J_kY_k\chi_k\rangle - \delta\nu^{\frac12}\sum_k |k|^{2s+\frac12}\|J_kY_k\chi_k\|^2 \\ &\lesssim \|v\|_{H^s}\left(\sum_k |k|^{2s}\|\nabla Y_k\|^2\right)^{\frac12}\left(\sum_k |k|^{2s}\|J_kY_k\chi_k\|^2\right)^{\frac12} + \delta^{-1}\nu^{-\frac32}\|v\|^2_{H^{s+\frac14}}\sum_k |k|^{2s+\frac12}\|Y_k\|^2.\end{aligned}$$

PROOF. We estimate the α term in SY as

$$\begin{aligned}\sum |k|^{2s} \operatorname{Re}\langle(\alpha_k-\alpha_\ell)(-\mathrm{i}v_{k-\ell}\cdot\ell)\nabla Y_\ell\chi_k, J_kY_k\chi_k\rangle \\ \lesssim \sum |k|^{2s}\|\alpha_k-\alpha_\ell\|_\infty|v_{k-\ell}\cdot\ell|\|\nabla Y_\ell\|\|JY_k\chi_k\| \\ \lesssim \|v\|_{H^s}\left(\sum |k|^{2s}\|\nabla Y_k\|^2\right)^{\frac12}\left(\sum |k|^{2s}\|J_kY_k\chi_k\|^2\right)^{\frac12}.\end{aligned}$$

For the β term we find

$$\begin{aligned}\nu^{-\frac12}\sum |k|^{2s}\langle(\beta_k|k|^{\frac12}\nabla(p\cdot\hat k)-\beta_\ell|\ell|^{\frac12}\nabla(p\cdot\hat\ell)\mathrm{i}v_{k-\ell}\cdot\ell Y_\ell\chi_k, J_kY_k\chi_k\rangle \\ \lesssim \nu^{-\frac12}\sum |k|^{2s}\|\beta_k|k|^{\frac12}\nabla(p\cdot\hat k)-\beta_\ell|\ell|^{\frac12}\nabla(p\cdot\hat\ell)\|_\infty|v_{k-\ell}\cdot\ell|\|Y_\ell\|\|J_kY_k\chi_k\| \\ \lesssim \nu^{-\frac12}\|v\|_{H^{s+\frac14}}\left(|k|^{2s+\frac12}\|Y_k\|^2\right)^{\frac12}\left(|k|^{2s+\frac12}\|J_kY_k\chi_k\|\right)^{\frac12}.\end{aligned}$$

Splitting gives the required estimate. □

Remark 3.12. By a slight modification, the β term can also be bounded by

$$\nu^{-\frac12}\|v\|_{H^{s+\frac12}}\left(|k|^{2s+1}\|Y_k\|^2\right)^{\frac12}\left(|k|^{2s}\|J_kY_k\chi_k\|\right)^{\frac12}.$$

For the hypoelliptic functional, we control the commutator by the following estimate.

Lemma 3.13. *Let $(\chi_k)_{k\neq0}$ a family of smooth functions. Then,*

$$\begin{aligned}\sum |k|^{2s+1}\operatorname{Re}E_{\chi_k,k}(SY_k,J_kY_k)-\delta\left[\sum |k|^{2s+1}D_{\chi_k,k}(J_kY_k)+\nu^{\frac12}\sum |k|^{2s+\frac12}\|J_kY_k\chi_k\|^2\right] \\ \lesssim \delta^{-1}\|v\|^2_{H^{s+\frac12}}\left[\nu^{-1}\sum |k|^{2s+1}\|\nabla Y_k\|^2+\nu^{-2}\sum |k|^{2s+1}\|Y_k\|^2\right].\end{aligned}$$

PROOF. From the definition we find that

$$\begin{aligned}\sum |k|^{2s+1}\operatorname{Re}E_{\chi_k,k}(SY_k,J_kY_k) \\ =\sum |k|^{2s}\operatorname{Re}E_{\chi_k,k}((\alpha_k-\alpha_\ell)(-\mathrm{i}v_{k-\ell}\cdot\ell)\nabla Y_\ell, J_kY_k) \\ +\nu^{-\frac12}\sum |k|^{2s}\operatorname{Re}E_{\chi_k,k}\left(\mathrm{i}(\beta_k|k|^{\frac12}\nabla(p\cdot\hat k)-\beta_\ell|l|^{\frac12}\nabla(p\cdot\hat\ell))(-\mathrm{i}v_{k-\ell}\cdot\ell)Y_\ell, J_kY_k\right).\end{aligned}$$

We split as before the contributions in I^0, I^a, I^b and I^c. For I^0 we find

$$\begin{aligned}I^0=\sum |k|^{2s}\operatorname{Re}\langle(\alpha_k-\alpha_\ell)(-\mathrm{i}v_{k-\ell}\cdot\ell)\nabla Y_\ell\chi_k, J_kY_k\chi_k\rangle \\ +\nu^{-\frac12}\sum |k|^{2s}\operatorname{Re}\langle(\beta_k|k|^{\frac12}\nabla(p\cdot\hat k)-\beta_\ell|\ell|^{\frac12}\nabla(p\cdot\hat\ell))(v_{k-\ell}\cdot\ell)Y_\ell\chi_k, J_kY_k\chi_k\rangle.\end{aligned}$$

The first sum can directly be estimated by Lemma 3.7 as

$$\begin{aligned}\sum |k|^{2s}\operatorname{Re}\langle(\alpha_k-\alpha_\ell)(-\mathrm{i}v_{k-\ell}\cdot\ell)\nabla Y_\ell\chi_k, J_kY_k\chi_k\rangle \\ \lesssim \|v\|_{H^s}\left(\sum |k|^{2s}\|\nabla Y_k\|^2\right)^{\frac12}\left(\sum |k|^{2s}\|J_kY_k\chi_k\|^2\right)^{\frac12}.\end{aligned}$$

For the second sum, we find by Lemma 3.7 that

$$\begin{aligned}\nu^{-\frac12}\sum |k|^{2s}\operatorname{Re}\langle(\beta_k|k|^{\frac12}\nabla(p\cdot\hat k)-\beta_\ell|\ell|^{\frac12}\nabla(p\cdot\hat\ell))(v_{k-\ell}\cdot\ell)Y_\ell\chi_k, J_kY_k\chi_k\rangle \\ \lesssim \nu^{-\frac12}\|v\|_{H^{s+\frac14}}\left(\sum |k|^{2s+\frac12}\|Y_k\|^2\right)^{\frac12}\left(\sum |k|^{2s+\frac12}\|J_kY_k\chi_k\|\right)^{\frac12}.\end{aligned}$$

For I^a, we find

$$\begin{aligned}I^a=\nu^{\frac12}\sum |k|^{2s-\frac12}\operatorname{Re}a_k\langle(\alpha_k-\alpha_\ell)(-\mathrm{i}v_{k-\ell}\cdot\ell)\nabla\nabla Y_\ell\chi_k, \nabla J_kY_k\chi_k\rangle \\ +\sum |k|^{2s-\frac12}\operatorname{Re}a_k\langle\nabla(\mathrm{i}\beta_k|k|^{\frac12}\nabla(p\cdot\hat k)-\mathrm{i}\beta_\ell|\ell|^{\frac12}\nabla(p\cdot\hat\ell))(-\mathrm{i}v_{k-\ell}\cdot\ell)Y_\ell\chi_k, \nabla J_kY_k\chi_k\rangle.\end{aligned}$$

In the first term, we integrate by parts and then find by Lemma 3.7 that

$$I^a \lesssim \nu^{\frac{1}{2}} \|v\|_{H^{s-\frac{1}{4}}} \left(\sum |k|^{2s-\frac{1}{2}} \|\nabla Y_k\|^2\right)^{\frac{1}{2}} \left(\sum |k|^{2s-\frac{1}{2}} a_k^2 \|\nabla\nabla J_k Y_k \chi_k\|^2\right)^{\frac{1}{2}}$$
$$+ \|v\|_{H^s} \left(\sum |k|^{2s} \left(\|\nabla Y_k\|^2 + \|Y_k\|^2\right)\right)^{\frac{1}{2}} \left(\sum |k|^{2s} \|\nabla J_k Y_k \chi_k\|^2\right)^{\frac{1}{2}}.$$

For I^b we find

$$\begin{aligned} I^b = &\sum |k|^{2s} \operatorname{Re}\langle \mathrm{i}(\alpha_k - \alpha_\ell)(-\mathrm{i}v_{k-\ell}\cdot \ell)\nabla Y_\ell \chi_k, b_k \nabla(p\cdot\hat{k})\nabla J_k Y_k \chi_k\rangle \\ &+ \sum |k|^{2s} \operatorname{Re}\langle \nabla(\alpha_k - \alpha_\ell)(-\mathrm{i}v_{k-\ell}\cdot \ell)\nabla Y_\ell \chi_k, \mathrm{i}b_k \nabla(p\cdot\hat{k}) J_k Y_k \chi_k\rangle \\ &+ \nu^{-\frac{1}{2}} \sum |k|^{2s+\frac{1}{2}} \operatorname{Re}\langle \mathrm{i}(\mathrm{i}\beta_k |k|^{\frac{1}{2}} \nabla(p\cdot\hat{k}) - \mathrm{i}\beta_\ell |\ell|^{\frac{1}{2}} \nabla(p\cdot\hat{\ell}))(-\mathrm{i}v_{k-\ell}\cdot\ell) Y_\ell \chi_k, b_k |k|^{-\frac{1}{2}} \nabla(p\cdot\hat{k}) \nabla J_k Y_k \chi_k\rangle \\ &+ \nu^{-\frac{1}{2}} \sum |k|^{2s+\frac{1}{2}} \operatorname{Re}\langle \nabla(\mathrm{i}\beta_k |k|^{\frac{1}{2}} \nabla(p\cdot\hat{k}) - \mathrm{i}\beta_\ell |\ell|^{\frac{1}{2}} \nabla(p\cdot\hat{\ell}))(-\mathrm{i}v_{k-\ell}\cdot\ell) Y_\ell \chi_k, \mathrm{i}\nabla(p\cdot\hat{k}) b_k |k|^{-\frac{1}{2}} J_k Y_k \chi_k\rangle. \end{aligned}$$

so that we find (after integration by parts of the second and fourth terms)

$$I^b \lesssim \|v\|_{H^s} \left(\sum |k|^{2s} \|\nabla Y_k\|^2\right)^{\frac{1}{2}} \left(\sum |k|^{2s} b_k^2 \left(\|\nabla(\nabla(p\cdot\hat{k}) J_k Y_k)\chi_k\|^2 + \|J Y_k \chi_k\|^2\right)\right)^{\frac{1}{2}}$$
$$+ \nu^{-\frac{1}{2}} \|v\|_{H^{s+\frac{1}{2}}} \left(\sum |k|^{2s+1} \|Y_k\|^2\right)^{\frac{1}{2}} \left(\sum |k|^{2s} b_k^2 \left(\|\nabla(\nabla(p\cdot\hat{k}) J Y_k)\chi_k\|^2 + \|J_k Y_k \chi_k\|^2\right)\right)^{\frac{1}{2}}.$$

For I^c we find

$$\begin{aligned} I^c = &\nu^{-\frac{1}{2}} \sum |k|^{2s} \operatorname{Re}\langle \nabla(p\cdot\hat{k})(\alpha_k - \alpha_\ell)(-\mathrm{i}v_{k-\ell}\cdot\ell)\nabla Y_\ell \chi_k, c_k |k|^{\frac{1}{2}} \nabla(p\cdot\hat{k}) J_k Y_k \chi_k\rangle \\ &+ \nu^{-1} \sum |k|^{2s+\frac{1}{2}} \operatorname{Re}\langle \nabla(p\cdot\hat{k})(\beta_k |k|^{\frac{1}{2}} \nabla(p\cdot\hat{k}) - \beta_\ell |\ell|^{\frac{1}{2}} \nabla(p\cdot\hat{\ell}))(-\mathrm{i}v_{k-\ell}\cdot\ell) Y_\ell \chi_k, c_k \nabla(p\cdot\hat{k}) J_k Y_k \chi_k\rangle. \end{aligned}$$

Hence we find that

$$I^c \lesssim \nu^{-\frac{1}{2}} \|v\|_{H^s} \left(\sum |k|^{2s} \|\nabla Y_k\|^2\right)^{\frac{1}{2}} \left(\sum |k|^{2s+1} c_k^2 \|\nabla(p\cdot\hat{k}) J_k Y_k \chi_k\|^2\right)^{\frac{1}{2}}$$
$$+ \nu^{-1} \|v\|_{H^{s+\frac{1}{2}}} \left(\sum |k|^{2s+1} \|Y_k\|^2\right)^{\frac{1}{2}} \left(\sum |k|^{2s+1} c_k^2 \|\nabla(p\cdot\hat{k}) J_k Y_k \chi_k\|^2\right)^{\frac{1}{2}},$$

where the first term comes from the α-term and the second from the β-term.

The inequality of the lemma can then be established as in the proof of Lemma 3.8, through splitting and Young's inequality. □

To control $J_k J_k g$, we need the following estimate.

Lemma 3.14. *Let $(\chi_k)_{k\neq 0}$ a family of smooth functions.*

$$\sum_k |k|^{2s+1} \operatorname{Re}\langle J_k S Y_k \chi_k, J_k J_k Y_k \chi_k\rangle - \delta\nu^{\frac{1}{2}} \sum_k |k|^{2s+\frac{1}{2}} \|J_k J_k Y_k \chi_k\|^2 - \delta\nu \sum_k |k|^{2s} \|\nabla J_k J_k Y_k \chi_k\|^2$$
$$\lesssim \nu^{-\frac{5}{2}} \|v\|^2_{H^{s+\frac{3}{4}}} \sum_k |k|^{2s+\frac{3}{2}} \|Y_k\|^2 + \nu^{-\frac{3}{2}} \|v\|^2_{H^{s+\frac{1}{4}}} \sum_k |k|^{2s+\frac{1}{2}} \|\nabla Y_k\|^2.$$

PROOF. Expanding the definition of $J_k S Y_k$ we find

$$\begin{aligned}|k|J_kSY_k = &\sum_\ell \alpha_k(\alpha_k-\alpha_\ell)(-\mathrm{i}v_{k-\ell}\cdot\ell)\nabla\nabla Y_\ell\\ &+\nu^{-\frac12}\sum_\ell \alpha_k\nabla\Big((\mathrm{i}\beta_k|k|^{\frac12}\nabla(p\cdot\hat k)-\mathrm{i}\beta_\ell|\ell|^{\frac12}\nabla(p\cdot\hat\ell))(-\mathrm{i}v_{k-\ell}\cdot\ell)Y_\ell\Big)\\ &+\nu^{-\frac12}\sum_\ell \mathrm{i}\beta_k|k|^{\frac12}\nabla(p\cdot\hat k)(\alpha_k-\alpha_\ell)(-\mathrm{i}v_{k-\ell}\cdot\ell)\nabla Y_\ell\\ &+\nu^{-1}\sum_\ell \mathrm{i}\beta_k|k|^{\frac12}\nabla(p\cdot\hat k)(\mathrm{i}\beta_k|k|^{\frac12}\nabla(p\cdot\hat k)-\mathrm{i}\beta_\ell|\ell|^{\frac12}\nabla(p\cdot\hat\ell))(-\mathrm{i}v_{k-\ell}\cdot\ell)Y_\ell.\end{aligned}$$

The contribution of the first term can be bounded, after integration by parts, by

$$\begin{aligned}&\sum|k|^{2s}\|\alpha_k-\alpha_\ell\|_\infty|v_{k-\ell}\cdot\ell|\,\|\nabla Y_\ell\|\left(\|\nabla J_kJ_kY_k\chi_k^2\|+2\|J_kJ_kY_k\chi_k|\nabla\chi_k|\|\right)\\ &\lesssim\|v\|_{H^s}\left(\sum_k|k|^{2s}\|\nabla Y_k\|^2\right)^{\frac12}\left(\sum_k|k|^{2s}(\|\nabla J_kJ_kY_k\chi_k\|^2+\|J_kJ_kY_k\chi_k\|^2)\right)^{\frac12}.\end{aligned}$$

The contribution of the second term can be bounded by

$$\begin{aligned}&\sum|k|^{2s}\nu^{-\frac12}\|\beta_k|k|^{\frac12}\nabla(p\cdot\hat k)-\beta_\ell|\ell|^{\frac12}\nabla(p\cdot\hat\ell)\|_\infty\,|v_{k-\ell}\cdot\ell|\,\|\nabla Y_\ell\|\,\|J_kJ_kY_k\chi_k^2\|\\ &\quad+\sum|k|^{2s}\nu^{-\frac12}\|\beta_k|k|^{\frac12}\nabla^2(p\cdot\hat k)-\beta_\ell|\ell|^{\frac12}\nabla^2(p\cdot\hat\ell)\|_\infty\,|v_{k-\ell}\cdot\ell|\,\|Y_\ell\|\,\|J_kJ_kY_k\chi_k^2\|\\ &\lesssim\nu^{-\frac12}\|v\|_{H^{s+\frac14}}\left(\sum_k|k|^{2s+\frac12}(\|\nabla Y_k\|^2+\|Y_k\|^2)\right)^{\frac12}\left(\sum_k|k|^{2s+\frac32}\|J_kJ_kY_k\chi_k\|^2\right)^{\frac12}.\end{aligned}$$

The contribution of the third term can be bounded by

$$\begin{aligned}&\sum|k|^{2s+\frac12}\nu^{-\frac12}\|\alpha_k-\alpha_\ell\|_\infty\,|v_{k-\ell}\cdot\ell|\,\|\nabla Y_\ell\|\,\|J_kJ_kY_k\chi_k^2\|\\ &\lesssim\nu^{-\frac12}\|v\|_{H^{s+\frac14}}\left(\sum_k|k|^{2s+\frac12}\|\nabla Y_k\|^2\right)^{\frac12}\left(\sum_k|k|^{2s+\frac12}\|J_kJ_kY_k\chi_k\|^2\right)^{\frac12}.\end{aligned}$$

The last term can be bounded by

$$\begin{aligned}&\sum|k|^{2s+1}\nu^{-1}\|\beta_k|k|^{\frac12}\nabla(p\cdot\hat k)-\beta_\ell|\ell|^{\frac12}\nabla(p\cdot\hat\ell)\|_\infty\,|v_{k-\ell}\cdot\ell|\,\|Y_\ell\|\,|k|^{-\frac12}\|J_kJ_kY_k\chi_k\|\\ &\lesssim\nu^{-1}\|v\|_{H^{s+\frac34}}\left(\sum_k|k|^{2s+\frac32}\|Y_k\|^2\right)^{\frac12}\left(\sum_k|k|^{2s+\frac12}\|J_kJ_kY_k\chi_k\|^2\right)^{\frac12}.\end{aligned}$$

The result follows from all these bounds and Young's inequality. □

3.3.2. *Application to the density g.* Under suitable assumptions on the velocity field v, we can now obtain controls on g and its vector-fields. Notice that we request something less stringent than (H). We begin from g.

PROPOSITION 3.15. *There exists $\varepsilon_0>0$ with the following property: if $\varepsilon\in(0,\varepsilon_0)$ and*

$$\sup_{t\geq0}\|v(t)\|_{H^s}+\Big(\int_0^{+\infty}\|v(t)\|_{H^s}^2\mathrm{d}t\Big)^{\frac12}\leq\varepsilon\nu^{\frac12}$$

then for any $T>0$ there holds

$$\sup_{0\leq t\leq T}\sum_k|k|^{2s}E_k(g_k)+\nu^{\frac12}\int_0^T\sum_k|k|^{2s+\frac12}a_k\|g_k\|^2+\int_0^T\sum_k|k|^{2s+1}D_k(g_k)\lesssim\|g^{in}\|^2_{H^s_xL^2_p}.$$

As a consequence, (1.11) *holds.*

PROOF. We start from the inequality in Lemma 2.4: we take $F_k = Vg_k$, multiply by $|k|^{2s}$ and sum over k. To bound the term $\sum |k|^{2s} \operatorname{Re} E_k(g_k, Vg_k)$, we apply Lemma 3.8 with $Y_k = g_k$, $\chi = 1$. We find

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_k |k|^{2s} E_k(g_k) + \nu^{\frac12}\sum |k|^{2s+\frac12} a_k \|g_k\|^2 + \sum_k |k|^{2s+1} D_k(g_k)\\
&\quad\lesssim (\delta\nu^{\frac12} + \delta^{-1}\nu^{-1}\|v\|_{H^s}^2)\sum |k|^{2s}\|g_k\|^2 + (\delta\nu + \delta^{-1}\|v\|_{H^s}^2)\sum |k|^{2s}\|\nabla_k g_k\|^2\\
&\qquad + \delta\nu^{\frac12}\sum |k|^{2s} b_k^2 \|\nabla(\nabla(p\cdot\hat k)g_k)\|^2 + (\delta + \|v\|_{H^s}\nu^{-\frac12})\sum |k|^{2s+1} c_k \|\nabla(p\cdot\hat k)g_k\|^2,
\end{aligned}$$

for any $\delta > 0$. We take $\delta = \varepsilon$. If $\varepsilon_0 \in (0,1)$ is small enough, most terms at the right-hand side can be absorbed in the dissipation functional, and we end up with

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_k |k|^{2s} E_k(g_k) + \sum_k |k|^{2s+1} a_k \left(\frac{\nu}{|k|}\right)^{\frac12}\|g_k\|^2 + \sum_k |k|^{2s+1} D_k(g_k)\\
&\qquad\lesssim \left(\varepsilon\nu^{\frac12} + \varepsilon^{-1}\nu^{-1}\|v\|_{H^s}^2\right)\|g\|_{H^s_x L^2_p}^2.
\end{aligned} \tag{3.13}$$

In particular,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|g\|_{H^s_x L^2_p} \lesssim \left(\varepsilon\nu^{\frac12} + \varepsilon^{-1}\nu^{-1}\|v\|_{H^s}^2\right)\|g\|_{H^s_x L^2_p} \tag{3.14}$$

so that our assumptions and Gronwall lemma yield

$$\|g(t)\|_{H^s_x L^2_p} \le M\mathrm{e}^{M\varepsilon\nu^{\frac12}t}\|g^{\mathrm{in}}\|_{H^s_x L^2_p}, \qquad \forall t \ge 0$$

for an absolute constant $M > 0$. If $t \le \nu^{-\frac12}$, injecting this bound in the right-hand side of (3.13) and integrating on $(0,T)$ proves the lemma. If $t \ge \nu^{-\frac12}$, we first integrate (3.13) on $(0,\nu^{-\frac12})$, resulting in

$$\sum_k |k|^{2s} E_k(g_k)\big|_{t=\nu^{-\frac12}} \le \mathrm{e}^{M\varepsilon}\|g^{\mathrm{in}}\|_{H^s_x L^2_p}^2.$$

Using again the interpolation inequality (2.11) with $\sigma = \frac{4\nu a_k}{|k|c_k}$, we deduce that at time $t = \nu^{-\frac12}$ (time for which $\sigma \le 1$ if ν is small enough):

$$\begin{aligned}
E_k(g_k) &\ge \|g_k\|^2 + \frac12 a_k\nu^{\frac12}|k|^{-\frac12}\|\nabla_p g_k\|^2 + \frac12 c_k\nu^{-\frac12}|k|^{\frac12}\|\nabla_p(p\cdot\hat k)g_k\|^2\\
&\ge \left(1 + \frac12(a_k c_k)^{\frac12}\right)\|g_k\|^2 = \left(1 + \frac12(AC)^{\frac12}\right)\|g_k\|^2,
\end{aligned}$$

and eventually

$$\|g(\nu^{-\frac12})\|_{H^s_x L^2_p} \le \lambda\|g^{\mathrm{in}}\|_{H^s_x L^2_p}^2, \qquad \lambda := \frac{\mathrm{e}^{M\varepsilon_0}}{1 + \frac12(AC)^{\frac12}} < 1$$

taking $\varepsilon < \varepsilon_0$ small enough. Iterating this bound gives exactly (1.11) and concludes the proof. □

3.3.3. *Application to the vector fields.* The analogous of Proposition 3.15 for vector fields is contained in Lemma 3.17 (estimate without cut-off) and Lemma 3.18 (estimate with cut-off) below. We start with an L^2 estimate.

Lemma 3.16. *Let $(\chi_k)_{k\neq 0}$ a family of smooth functions with $\chi_k = 0$ near $p = -\hat k$. There exists $\varepsilon_0 > 0$ with the following property: if $\varepsilon \in (0,\varepsilon_0)$ and*

$$\sup_{t\ge 0}\|v(t)\|_{H^s} + \left(\int_0^\infty \|v(t)\|_{H^s}^2 \mathrm{d}t\right)^{\frac12} \le \varepsilon\nu^{\frac34}$$

then for any $T > 0$ there holds

$$\sup_{0\le t\le T}\sum_k |k|^{2s-\frac12}\|J_k g_k\chi_k\|^2 + \int_0^T \nu\sum_k |k|^{2s-\frac12}\|\nabla J_k g_k\chi_k\|^2 + \nu^{\frac12}\sum_k |k|^{2s}\|J_k g_k\chi_k\|^2 \lesssim \|g^{in}\|^2_{H^s_x L^2_p} + \|\nabla g^{in}\|^2_{H^s_x L^2_p}.$$

PROOF. We first establish an estimate without cut-off, starting from Lemma 3.3. We take $Y_k = g_k$, $F_k = VY_k = VJ_k g_k$, multiply by $|k|^{2s-\frac12}$ and sum over k. We find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum |k|^{2s-\frac12}\|J_k g_k\|^2 + \nu\sum |k|^{2s-\frac12}\|\nabla J_k g_k\|^2 + \nu^{\frac12}\sum |k|^{2s}\|J_k g_k\|^2 \\ \lesssim \sum |k|^{2s+\frac12}\langle J_k g_k, VJ_k g_k\rangle + \sum |k|^{2s+\frac12}\langle J_k g_k, Sg_k\rangle + \sum_k |k|^{2s+\frac12} b_k\|g_k\|^2 + \sum_k |k|^{2s+\frac12} D_k(g_k).\end{aligned} \tag{3.15}$$

The first term at the right-hand side is bounded thanks to Lemma 3.10 (applied with $Y = Jg$, $\chi = 1$):

$$\sum_k |k|^{2s+\frac12}\,\mathrm{Re}\langle VJ_k g_k, J_k g_k\rangle \lesssim \|v\|_{H^s}\sum |k|^{2s-\frac12}\|J_k g_k\|^2.$$

It can be absorbed by the left-hand side, as $\|v\|_{H^s} \ll \nu^{\frac12}$. The second term in the right-hand side is estimated thanks to Lemma 3.11 (applied with $Y = g$, $\chi = 1$):

$$\begin{aligned}&\sum |k|^{2s+\frac12}\,\mathrm{Re}\langle SJ_k g_k, J_k g_k\rangle - \delta\nu^{\frac12}\sum_k |k|^{2s}\|J_k g_k\|^2 \\ &\lesssim \|v\|_{H^{s-\frac14}}\left(\sum_k |k|^{2s-\frac12}\|\nabla g_k\|^2\right)^{\frac12}\left(\sum_k |k|^{2s-\frac12}\|J_k g_k\|^2\right)^{\frac12} + \delta^{-1}\nu^{-\frac32}\|v\|^2_{H^s}\sum_k |k|^{2s}\|g_k\|^2 \\ &\lesssim \delta\nu^{\frac12}\sum_k |k|^{2s-\frac12}\|J_k g_k\|^2 + \delta^{-1}\|v\|^2_{H^{s-\frac14}}\nu^{-\frac12}\sum_k |k|^{2s-\frac12}\|\nabla g_k\|^2 + \delta^{-1}\nu^{-\frac32}\|v\|^2_{H^s}\sum_k |k|^{2s}\|g_k\|^2.\end{aligned}$$

We take $\delta = \varepsilon$. For $\varepsilon < \varepsilon_0$ small enough, the first term can be absorbed in the left-hand side, while the second term can be included in $\sum_k |k|^{2s+\frac12} D_k(g_k)$. We end up with

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum |k|^{2s-\frac12}\|J_k g_k\|^2 + \nu\sum |k|^{2s-\frac12}\|\nabla J_k g_k\|^2 + \nu^{\frac12}\sum |k|^{2s}\|J_k g_k\|^2 \\ \lesssim \sum_k |k|^{2s+\frac12} b_k\|g_k\|^2 + \sum_k |k|^{2s+\frac12} D_k(g_k) + \varepsilon^{-1}\nu^{-\frac32}\|v\|^2_{H^s}\|g\|^2_{H^s_x L^2_p}.\end{aligned}$$

The time integral of the second term at the right-hand side. could be controlled directly thanks to Proposition 3.15. Similarly, the time integral of the last term could be controlled thanks to the assumption on v and (1.11). But the limiting term, due to the absence of cut-off, is $\sum_k |k|^{2s+\frac12} b_k\|g_k\|^2 \lesssim \sum_k |k|^{2s+\frac12} a_k\|g_k\|^2$, where we could not squeeze a factor $\nabla(p\cdot k)$. This forces us to multiply the previous inequality by $\nu^{\frac12}$. We get, with Proposition 3.15 that

$$\begin{aligned}\sup_{0\le t\le T}\nu^{\frac12}\sum |k|^{2s-\frac12}\|J_k g_k\|^2 + \int_0^T \nu^{\frac32}\sum |k|^{2s-\frac12}\|\nabla J_k g_k\|^2 + \int_0^T \nu\sum |k|^{2s}\|J_k g_k\|^2 \\ \lesssim \|g^{in}\|^2_{H^s_x L^2_p} + \nu^{\frac12}\|\nabla_p g^{in}\|^2_{H^s_x L^2_p}.\end{aligned} \tag{3.16}$$

We can then obtain a better bound using the cut-off χ, replacing our starting point (3.6) by the improved Lemma 3.4. After similar manipulations, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum|k|^{2s-\frac12}\|J_kg_k\chi_k\|^2+\nu\sum|k|^{2s-\frac12}\|\nabla J_kg_k\chi_k\|^2+\nu^{\frac12}\sum|k|^{2s}\|J_kg_k\chi_k\|^2\\ \lesssim\|v\|_{H^{s-\frac14}}\Big(\sum|k|^{2s-\frac12}\|J_kg_k\|^2\Big)^{\frac12}\Big(\sum|k|^{2s-\frac12}\|J_kg_k\chi_k\|^2\Big)^{\frac12}\\ +\|v\|_{H^{s-\frac14}}\left(\sum_k|k|^{2s-\frac12}\|\nabla g_k\|^2\right)^{\frac12}\left(\sum_k|k|^{2s-\frac12}\|J_kg_k\chi_k\|^2\right)^{\frac12}\\ +\sum_k|k|^{2s+\frac12}D_k(g_k)+\nu\sum_k|k|^{2s-\frac12}\|g_k\tilde\chi_k\|^2+\varepsilon^{-1}\nu^{-\frac32}\|v\|_{H^s}^2\|g\|^2_{H^s_xL^2_p}\end{aligned}$$

which implies, using the smallness assumption on v:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum|k|^{2s-\frac12}\|J_kg_k\chi_k\|^2+\nu\sum|k|^{2s-\frac12}\|\nabla J_kg_k\chi_k\|^2+\nu^{\frac12}\sum|k|^{2s}\|J_kg_k\chi_k\|^2\\ \lesssim\nu^{-\frac12}\|v\|^2_{H^{s-\frac14}}\sum|k|^{2s-\frac12}\|J_kg_k\|^2+\sum_k|k|^{2s+\frac12}D_k(g_k)+(\nu+\varepsilon^{-1}\nu^{-\frac32}\|v\|^2_{H^s})\|g\|^2_{H^s_xL^2_p}.\end{aligned}$$

Using Proposition 3.15, (3.16) and (1.11) concludes the proof of the lemma. □

For the control of the hypocoercive functional of J_kY, we first show an estimate with a loss in ν, but without localisation. This is

Lemma 3.17. *There exists $\varepsilon_0>0$ with the following property: if $\varepsilon\in(0,\varepsilon_0)$ and*

$$\sup_{t\geq0}\|v(t)\|_{H^s}+\left(\int_0^{+\infty}\|v(t)\|^2_{H^s}\mathrm{d}t\right)^{\frac12}\leq\varepsilon\nu^{\frac34}$$

then for any $T>0$ there holds

$$\nu^{\frac12}\left[\sup_{0\leq t\leq T}\sum_k|k|^{2s-1}E_k(J_kg_k)+\int_0^T\sum_k|k|^{2s}D_k(J_kg_k)+\nu^{\frac12}\sum_k|k|^{2s-\frac12}E_k(J_kg_k)\right]\lesssim\|g^{in},\nabla_pg^{in}\|^2_{H^s_xL^2_p}.$$

PROOF. The proof is very close to proof of (3.16). One starts from Lemma 3.5, that is applied with $Y_k=g_k$, $F_k=Vg_k$. One multiplies by $|k|^{2s-1}$, and sum over k. The nonlinear term

$$\sum|k|^{2s-1}\operatorname{Re}E_k(J_kg_k,J_kVg_k)=\sum|k|^{2s-1}\operatorname{Re}E_k(J_kg_k,VJ_kg_k)+\sum|k|^{2s-1}\operatorname{Re}E_k(J_kg_k,Sg_k)$$

is then treated with Lemma 3.8 and Lemma 3.13. We leave the details to the reader. □

We then get an improved estimate with cut-off.

Lemma 3.18. *Let $(\chi_k)_{k\neq0}$ as in Lemma 3.16. There exists $\varepsilon_0>0$ with the following property: if $\varepsilon\in(0,\varepsilon_0)$ and*

$$\sup_{t\geq0}\|v(t)\|_{H^s}+\Big(\int_0^{+\infty}\|v(t)\|^2_{H^s}\mathrm{d}t\Big)^{\frac12}\leq\varepsilon\nu$$

then for any $T>0$ there holds

$$\sup_{0\leq t\leq T}\sum_k|k|^{2s-1}E_{\chi_k,k}(J_kg_k)+\int_0^T\sum_k|k|^{2s}D_{\chi_k,k}(J_kg_k)+\nu^{\frac12}\int_0^T\sum_k|k|^{2s-\frac12}E_{\chi_k,k}(J_kg_k)\lesssim\|g^{in},\nabla_pg^{in}\|^2_{H^s_xL^2_p}.$$

PROOF. The proof is again very close to the one of Lemma 3.16, once the estimate without cut-off has been established. One needs to combine Lemma 3.6, Lemma 3.17 as well as Lemma 3.8 and Lemma 3.13 to control the convection term. □

The last control that we will need to prove mixing estimates is an $H^{s-\frac{3}{4}}$ estimate on JJg.

Lemma 3.19. *Let χ, χ_k as in Lemma 3.16. There exists $\varepsilon_0 > 0$ with the following property: if $\varepsilon \in (0, \varepsilon_0)$ and*

$$\sup_{t\geq 0} \|v(t)\|_{H^s} + \Big(\int_0^{+\infty} \|v(t)\|_{H^s}^2 \mathrm{d}t\Big)^{\frac{1}{2}} \leq \varepsilon \nu^{\frac{5}{4}}$$

then for any $T > 0$ there holds

$$\sup_{0\leq t\leq T} \sum_k |k|^{2s-\frac{3}{2}} \|J_k J_k g_k \chi_k\|^2 + \int_0^T \nu \sum_k |k|^{2s-\frac{3}{2}} \|\nabla J_k J_k g_k \chi_k\|^2 + \nu^{\frac{1}{2}} \sum_k |k|^{2s-1} \|J_k J_k g_k \chi_k\|^2 \lesssim \|g^{in}, \nabla_p g^{in}, \nabla_p^2 g^{in}\|_{H^s_x L^2_p}^2.$$

PROOF. The starting point is Lemma 3.4 applied with $Y = Jg$, $F = JVg$. We find.

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \sum |k|^{2s-\frac{3}{2}} \|J_k J_k g_k \chi_k\|^2 + \frac{3}{4}\nu \sum |k|^{2s-\frac{3}{2}} \|\nabla J_k J_k g_k \chi_k\|^2 + \frac{3}{4}\nu^{\frac{1}{2}} \sum |k|^{2s-1} \|J_k J_k g \chi_k\|^2$$
$$\lesssim \sum |k|^{2s-\frac{1}{2}} |\langle J_k Y_k \chi, J_k J_k V g_k \chi\rangle| + \nu \sum |k|^{2s-\frac{3}{2}} \|J_k g_k \tilde{\chi}_k\|^2 + \sum |k|^{2s-\frac{1}{2}} D_{\tilde{\chi}_k, k}(J_k g_k).$$

The last two terms are controlled thanks to Lemma 3.18, replacing χ by $\tilde{\chi}$, after integration in time. The convection term is decomposed into

$$\sum |k|^{2s-\frac{1}{2}} \langle J_k J_k g_k \chi, J_k J_k V g_k \chi\rangle = \sum |k|^{2s-\frac{1}{2}} \langle J_k J_k g_k \chi_k, J_k S g_k \chi_k\rangle$$
$$+ \sum |k|^{2s-\frac{1}{2}} \langle J_k J_k g_k \chi_k, S J_k g_k \chi\rangle$$
$$+ \sum |k|^{2s-\frac{1}{2}} \langle J_k J_k g_k \chi_k, V J_k J_k g_k \chi_k\rangle.$$

We use Lemma 3.10 to write

$$\sum_k |k|^{2s-\frac{1}{2}} \operatorname{Re}\langle J_k J_k g_k \chi_k, V J_k J_k g_k \chi_k\rangle$$
$$\lesssim \|v\|_{H^{s-\frac{3}{4}}} \Big(\sum |k|^{2s-\frac{3}{2}} \|J_k J_k g_k\|^2\Big)^{\frac{1}{2}} \Big(\sum |k|^{2s-\frac{3}{2}} \|J_k J_k g_k \chi_k\|^2\Big)^{\frac{1}{2}}$$
$$\lesssim \delta \nu^{\frac{1}{2}} \sum |k|^{2s-\frac{3}{2}} \|J_k J_k g_k \chi_k\|^2 + \delta^{-1} \|v\|_{H^{s-\frac{3}{4}}}^2 \nu^{-\frac{1}{2}} \sum |k|^{2s-\frac{3}{2}} \|J_k J_k g_k\|^2.$$

Replacing the first operator J_k by its definition, we get

$$\|v\|_{H^{s-\frac{3}{4}}}^2 \nu^{-\frac{1}{2}} \sum |k|^{2s-\frac{3}{2}} \|J_k J_k g_k\|^2$$
$$\lesssim \|v\|_{H^{s-\frac{3}{4}}}^2 \nu^{-\frac{3}{2}} \Big(\nu \sum |k|^{2s-\frac{3}{2}} \|\nabla_p J_k g_k\|^2 + \sum |k|^{2s-\frac{1}{2}} b_k \|\nabla (p\cdot \hat{k}) J_k g_k\|^2\Big).$$

The right-hand side can be controlled thanks to Lemma 3.17, after integration in time. The term

$$\sum |k|^{2s-\frac{1}{2}} \langle J_k J_k g_k \chi_k, S J_k g_k \chi_k\rangle$$

can be controlled in an easier way with Lemma 3.11 and Lemma 3.17. Finally, the term

$$\sum |k|^{2s-\frac{1}{2}} \langle J_k J_k g_k \chi_k, J_k S g_k \chi_k\rangle$$

can be controlled thanks to Lemma 3.14. This concludes the proof. □

3.4. Mixing estimates. We now consider integral quantities of the form (1.10), namely

$$\mathsf{V}_k[g] = \int_{\mathbb{S}^2} g_k(p) Z_k(p) \nabla(p \cdot \hat{k}) \mathrm{d}p, \tag{3.17}$$

and establish decay estimates as a consequence of orientation mixing.

PROPOSITION 3.20. *Let $k \neq 0$, $Z_k = Z_k(p)$ a smooth function, and $\chi_k = \chi_k(p)$ a smooth function which is 1 on the support of Z_k. Then, for any $g \in H^1$, we have*

$$|\mathsf{V}_k[g]| \lesssim \frac{\nu^{\frac{1}{2}}}{|k|^{\frac{1}{2}} \beta_k} \left[\|J_k g_k \chi_k\| + \|g_k \chi_k\| \right] \|Z_k\|_{H^1}. \tag{3.18}$$

PROOF. We use the identity $\nabla(p \cdot \hat{k}) g_k = \frac{1}{i} \frac{\nu^{\frac{1}{2}}}{|k|^{\frac{1}{2}} \beta_k} (J_k g_k - \alpha_k \nabla g_k)$, so that

$$\begin{aligned}
|\mathsf{V}_k[g]| &\leq \frac{\nu^{\frac{1}{2}}}{|k|^{\frac{1}{2}} |\beta_k|} \left(\left| \int_{\mathbb{S}^2} J_k g_k Z_k(p) \right| + \left| \int_{\mathbb{S}^2} \alpha_k \nabla g_k Z_k(p) \right| \right) \\
&\leq \frac{\nu^{\frac{1}{2}}}{|k|^{\frac{1}{2}} |\beta_k|} \left(\left| \int_{\mathbb{S}^2} J_k g_k Z_k(p) \right| + \left| \int_{\mathbb{S}^2} \alpha_k g_k \nabla \cdot Z_k(p) \right| \right) \\
&\lesssim \frac{\nu^{\frac{1}{2}}}{|k|^{\frac{1}{2}} |\beta_k|} \left(\int_{\mathbb{S}^2} |J_k g_k \chi_k Z_k(p)| + \int_{\mathbb{S}^2} |g_k \chi_k \nabla \cdot Z_k(p)| \right).
\end{aligned}$$

The result follows from Cauchy-Schwarz inequality. □

When g is the solution of the advection-diffusion equation (1.7), suitable norms of the the functions $\{\mathsf{V}_k\}_{k \in \mathbb{Z}}$ can be estimated in terms of the vector field J as follows.

PROPOSITION 3.21. *Assume*

$$\sup_{t \geq 0} \|v(t)\|_{H^s} + \left(\int_0^\infty \|v(t)\|_{H^s}^2 \mathrm{d}t \right)^{\frac{1}{2}} \leq \varepsilon \nu^{\frac{3}{4}}.$$

Then, for $\varepsilon < \varepsilon_0$ small enough, we find

$$\sum_k |k|^{2s+\frac{1}{2}} |\mathsf{V}_k[g(t)]|^2 \lesssim \frac{\nu}{|\beta(\nu^{\frac{1}{2}} t)|^2} \sum_k |k|^{2s} \left(\|g_k^{in}\|^2 + \|\nabla g_k^{in}\|^2 \right) \left(\sup_\ell \|Z_\ell\|_{H^1}^2 \right).$$

PROOF. For all $k \neq 0$, we introduce χ_k, ψ_k a smooth partition of unity on the sphere ($\chi_k + \psi_k \equiv 1$) such that $\chi_k = 0$ near $p = -\hat{k}$ and $\psi_k = 0$ near $p = \hat{k}$. We can decompose

$$\mathsf{V}_k[g(t)] = \int_{\mathbb{S}^2} g_k(t)(Z_k \chi_k)(p) \nabla(p \cdot \hat{k}) + \int_{\mathbb{S}^2} g_k(t)(Z_k \psi_k)(p) \nabla(p \cdot \hat{k}).$$

It is enough to prove that

$$\sum_k |k|^{2s+\frac{1}{2}} \left| \int_{\mathbb{S}^2} g_k(t)(Z_k \chi_k)(p) \nabla(p \cdot \hat{k}) \right|^2 \lesssim \frac{\nu}{\beta(\nu^{\frac{1}{2}} t)^2} \sum_k |k|^{2s} (\|g_k^{\text{in}}\|^2 + \|\nabla g_k^{\text{in}}\|^2) \left(\sup_\ell \|Z_\ell\|_{H^1}^2 \right).$$

as the other integral could be treated similarly, by exchanging the roles of the north and south poles, see Remark 3.2. We then introduce another family $(\tilde{\chi}_k)_{k \neq 0}$ which is still zero near $p = -\hat{k}$, with $\tilde{\chi}_k = 1$ on the

support of χ_k. It follows from Proposition 3.20 that

$$\begin{aligned}
&\sum_k |k|^{2s+\frac{1}{2}} \Big| \int_{\mathbb{S}^2} g_k(t)(Z_k\chi_k)(p)\nabla(p\cdot\hat{k})\Big|^2 \\
&\quad \lesssim \sum_k |k|^{2s-\frac{1}{2}} \frac{\nu}{|\beta_k|^2} \left[\|J_k g_k(t)\tilde{\chi}_k\|^2 + \|g_k(t)\tilde{\chi}_k\|^2\right] \|Z_k\|_{H^1}^2 \\
&\quad \lesssim \frac{\nu}{|\beta(\nu^{\frac{1}{2}}t)|^2} \sum_k |k|^{2s-\frac{1}{2}} \left[\|J_k g_k(t)\chi_k\|^2 + \|g_k(t)\chi_k\|^2\right] \left(\sup_\ell \|Z_\ell\|_{H^1}^2\right).
\end{aligned}$$

The result then follows from Proposition 3.15 and Lemma 3.16. □

When dealing with JJg, we note the following bound.

PROPOSITION 3.22. *Let $k \neq 0$; $Z_k = Z_k(p)$ a smooth function, and $\chi_k = \chi_k(p)$ a smooth function which is 1 on the support of Z_k. Then, for all $r = r_k(t)$, the following bound holds*

$$|\mathsf{V}_k[g]| \lesssim \Big(A_{r,k}\|g_k\chi_k\| + B_{r,k}\|J_k g_k\chi_k\| + C_{r,k}\|J_k J_k g_k\chi_k\|\Big)\big(\|Z_k\|_{H^2} + \|Z_k\|_{W^{1,\infty}}\big),$$

where

$$\begin{aligned}
A_{r,k} &:= \frac{\nu^{\frac{1}{2}}}{|k|^{\frac{1}{2}}|\beta_k|}\left(r + r^{-1}\frac{\nu^{\frac{1}{2}}}{|k|^{\frac{1}{2}}|\beta_k|}\right), \\
B_{r,k} &:= \frac{\nu^{\frac{1}{2}}}{|k|^{\frac{1}{2}}|\beta_k|}\left(r + (r^{-1} + |\ln r|^{\frac{1}{2}})\frac{\nu^{\frac{1}{2}}}{|k|^{\frac{1}{2}}|\beta_k|}\right), \\
C_{r,k} &:= \frac{\nu}{|k||\beta_k|^2}|\ln(r)|^{\frac{1}{2}}.
\end{aligned}$$

PROOF. As a preliminary step, we introduce $\chi_r = \chi_r(p)$ a smooth function with $\chi_r = 1$ on an r-neighborhood of $\hat{k}$, and $\chi_r = 0$ outside a $2r$-neighborhood of $\hat{k}$. We then consider, for any smooth G with $G = 0$ near $p = -\hat{k}$ and $\chi_k = 1$ on the support of G, and for any tensor Y_k:

$$\left|\int_{\mathbb{S}^2} Y_k G\right| \leq \left|\int_{\mathbb{S}^2} Y_k\chi_r G\right| + \left|\int_{\mathbb{S}^2} Y_k(1-\chi_r)G\right| =: I_1 + I_2.$$

We find directly by Cauchy-Schwarz

$$I_1 \lesssim \|Y_k\chi_k\|\, r\|G\|_{L^\infty}.$$

Then,

$$\begin{aligned}
I_2 = &\frac{1}{\mathrm{i}\beta_k}\left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} \int_{\mathbb{S}^2} \nabla(p\cdot\hat{k})\cdot J_k Y_k \frac{G}{|\nabla_p(p\cdot\hat{k})|^2}(1-\chi_r) \\
&+ \frac{\alpha_k}{\mathrm{i}\beta_k}\left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} \int_{\mathbb{S}^2} Y_k\nabla_p\cdot\left(\nabla(p\cdot\hat{k})\frac{G}{|\nabla_p(p\cdot\hat{k})|^2}(1-\chi_r)\right),
\end{aligned}$$

from where

$$I_2 \lesssim \frac{1}{|\beta_k|}\left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} \|G\|_{L^\infty}\|J_k Y_k\chi_k\||\ln(r)|^{\frac{1}{2}} + \frac{|\alpha_k|}{|\beta_k|}\left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} \|Y_k\chi_k\|\frac{1}{r}\left(\|G\|_{L^\infty} + \|G\|_{H^1}\right).$$

We conclude that

$$\left|\int_{\mathbb{S}^2} Y_k G\right| \lesssim \left(\left(\frac{1}{|\beta_k|}\left(\frac{\nu}{|k|}\right)^{\frac{1}{2}}\frac{1}{r} + r\right)\|Y_k\chi_k\| + \frac{1}{|\beta_k|}\left(\frac{\nu}{|k|}\right)^{\frac{1}{2}}|\ln(r)|^{\frac{1}{2}}\|J_k Y_k\chi_k\|\right)\left(\|G\|_{L^\infty} + \|G\|_{H^1}\right). \tag{3.19}$$

We then turn to

$$|\mathsf{V}[g_k(t)]| \leq \left| \frac{1}{\mathrm{i}\beta_k} \left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} \int_{\mathbb{S}^2} Z_k J_k g_k \right| + \left| \frac{\alpha_k}{\mathrm{i}\beta_k} \left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} \int_{\mathbb{S}^2} (\nabla_p \cdot Z_k) g_k \right|.$$

Applying the previous formula, we end up with

$$\begin{aligned} |\mathsf{V}[g_k(t)]| \lesssim \frac{1}{|\beta_k|} \left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} \Bigg(\Big(\frac{1}{|\beta_k|} \left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} \frac{1}{r} + r \Big) \|J_k g_k \chi_k\| \\ + \frac{1}{|\beta|} \left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} |\ln(r)|^{\frac{1}{2}} \|J_k J_k g_k \chi_k\| \Bigg) \left(\|Z_k\|_{L^\infty} + \|Z_k\|_{H^1}\right) \\ + \frac{1}{|\beta_k|} \left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} \Bigg(\Big(\frac{1}{|\beta_k|} \left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} \frac{1}{r} + r \Big) \|g_k \chi_k\| \\ + \frac{1}{|\beta_k|} \left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} |\ln(r)|^{\frac{1}{2}} \|J_k g_k \chi_k\| \Bigg) \left(\|\nabla Z_k\|_{L^\infty} + \|\nabla Z_k\|_{H^1}\right). \end{aligned} \tag{3.20}$$

The result follows. □

We can now conclude this linear analysis, proving Theorem 2.

PROOF OF THEOREM 2. For $t \leq 1$, the result follows easily from Proposition 3.21. For $t \geq 1$, we use Proposition 3.22 with the special choice

$$r = r_k(t) = \left(\frac{\nu^{\frac{1}{2}}}{|k|^{\frac{1}{2}} |\beta_k|} \right)^{\frac{1}{2}} \lesssim 1$$

for which

$$A_{r,k}, B_{r,k}, C_{r,k} \lesssim \left(\frac{\nu^{\frac{1}{2}}}{|k|^{\frac{1}{2}} |\beta_k|} \right)^{\frac{3}{2}}.$$

The proof is over. □

4. Nonlinear stability

The proof of the nonlinear stability Theorem 1 relies on two main ingredients: the analysis of a Volterra equation that allows the computation of u from ψ through the relations specified in (1.3), and the use of the linear estimates of Theorem 2, via a proper bootstrapping scheme. These two points will be carried out in Sections 4.2 and 4.3, respectively.

For notational convenience, we will prove Theorem 1 with shifted index $s-1$ instead of s. As u has one more degree of regularity than ψ in x, u will then have H^s regularity in x.

Let $s > \frac{9}{2}$, $T \in (0, \infty]$, and a field v defined on $[0, T)$ and satisfying

$$\sup_{0 \leq t \leq T} \|v(t)\|_{H^s} + \Big(\int_0^T \|v(t)\|_{H^s}^2 \mathrm{d}t \Big)^{\frac{1}{2}} \leq \varepsilon \nu^{\frac{5}{4}}. \tag{H'}$$

We introduce

$$S_v(t, \tau) : L^2 \to L^2, \qquad 0 \leq \tau \leq t < T \tag{4.1}$$

the linear two-parameter process arising as the solution operator of the (non-autonomous) advection-diffusion equation in (1.7), considered on $(0, T)$. Namely, for $0 \leq \tau \leq t < T$ we set $S_v(t, \tau) g^{in} = g_\tau(t)$ where g_τ is the solution on $[\tau, T)$ of

$$\partial_t g_\tau + (v + p) \cdot \nabla_x g_\tau - \nu \Delta_p g_\tau = 0, \qquad g_\tau|_{t=\tau} = g^{in}.$$

The results in Theorem 2 were stated under (H) for the time interval $(0,\infty)$ but extend straightforwardly under (H') to an arbitrary interval (τ,T). In particular, there exist absolute constants $C_0,\varepsilon,\nu_0,\eta_1>0$ such that for all $\nu\le\nu_0$ the condition (H') implies

$$\|(S_v(t,\tau)g^{in})_{\neq 0}\|_{H^s_xL^2_p}\le C_0\mathrm{e}^{-\eta_1\nu^{\frac12}(t-\tau)}\|g^{in}_{\neq 0}\|_{H^s_xL^2_p} \tag{4.2}$$

and

$$\begin{aligned}&\sum_{k\neq 0}|k|^{2s+\frac12-}\left|\mathsf{V}_k[S_v(t,\tau)g^{in}]\right|^2\\ &\quad\le C_0\left(\frac{\nu^{\frac12}}{\min\{1,\nu^{\frac12}(t-\tau)\}}\right)^3\sup_k\left(\|Z_k\|^2_{W^{1,\infty}_p}+\|Z_k\|^2_{H^2_p}\right)\|g^{in}_{\neq 0},\nabla_p g^{in}_{\neq 0},\nabla^2_p g^{in}_{\neq 0}\|^2_{H^s_xL^2_p},\end{aligned} \tag{4.3}$$

for all $t\in[0,T)$. We will now focus on the analysis of the full system (1.3).

4.1. Bootstrap assumptions. Let $\psi^{in}\in\mathcal{H}^{s-1}$. Existence and uniqueness of a local in time solution to (1.3) satisfying

$$\psi\in L^\infty_{loc}([0,T_*),\mathcal{H}^{s-1}),\quad \nabla_p\psi\in L^2_{loc}([0,T_*),\mathcal{H}^{s-1})$$

is standard. Moreover, if T_* is the maximal time of existence, one has

$$\limsup_{t\to T_*}\|u(t)\|_{H^s}=\infty.$$

Let $\eta_0=\frac{\eta_1}{10}$, with η_1 the absolute constant in (4.2). Let δ, δ', δ'' positive constants to be specified later, only depending on γ and ι. For each ν, let $T=T(\nu,\gamma,\iota)>0$ the maximal time upon which the following three bootstrap assumptions hold:

$$\sup_{0\le t\le T}\|u(t)\|_{H^s}\le\delta\nu^{\frac32}\mathrm{e}^{-\eta_0\nu^{\frac12}t}, \tag{BA0}$$

$$\int_0^T\|\nabla_p\psi(t)\|^2_{\mathcal{H}^{s-1}}\mathrm{d}t\le\delta'^2\nu^{\frac12}, \tag{BA1}$$

$$\sup_{0\le t\le T}\|\psi(t)\|_{\mathcal{H}^{s-1}}\le\delta''\nu^{\frac12}. \tag{BA2}$$

For any δ,δ',δ'', the existence of a positive T is guaranteed by the smallness assumption on ψ^{in}, taking δ_0 small enough compared to δ,δ',δ''. Note also that (BA0) implies (H'), for $\delta\le\min\left(\frac{\varepsilon}{\eta_0^{1/4}},1\right)$. The point is to show that there exists ν_0 such that for all $\nu\le\nu_0$, all three bounds are satisfied with improved constants $\delta/2,\delta'/2,\delta''/2$ instead of δ,δ',δ'', and that moreover, on $(0,T)$:

$$\sup_{0\le t<T}\|\psi\|^2_{H^s_xL^2_p}+\nu\int_0^T\|\nabla_p\psi\|^2_{H^s_xL^2_p}\,\mathrm{d}t\lesssim\nu^{-1}\|\psi^{in}\|^2. \tag{4.4}$$

Improvement of the constants will imply that $T=T_*=\infty$, which combined with (4.4) will conclude the proof of Theorem 1.

4.2. Analysis of the Volterra equation. From (1.3), we can deduce a Volterra-like equation for u. On one hand, from the second relation in (1.3), u is obtained from ψ through application of the linear operator U defined by

$$U\psi=\iota\mathrm{St}^{-1}\nabla_x\cdot\int_{\mathbb{S}^2}p\otimes p\,\psi$$

where St is the Stokes operator $-\mathbb{P}\Delta$ on $\mathbb{T}$ (with $\mathbb{P}$ the Leray projector). On the other hand, the solution ψ to the first relation in (1.3) obeys the Duhamel's formula

$$\psi(t)=S_u(t,0)\psi^{in}+\int_0^t S_u(t,\tau)\,F(\tau)\,\mathrm{d}\tau$$

with

$$F(t) = \frac{3\gamma}{4\pi}(p\otimes p) : E(u) - \nabla_p \cdot \Big(\mathbb{P}_{p^\perp}\left[(\gamma E(u) + W(u))p\right]\psi\Big).$$

Applying operator U to both sides of this Duhamel's formula, we get

$$u(t) + \int_0^t K(t,\tau)u(\tau)\mathrm{d}\tau = f(t)$$

where

$$K(t,\tau)u_0 = -\frac{3\gamma\iota}{4\pi}\mathrm{St}^{-1}\nabla_x \cdot \int_{\mathbb{S}^2} p\otimes p\, S_u(t,\tau)(p\otimes p : E(u_0)) \tag{4.5}$$

$$\begin{aligned} f(t) &= \iota \mathrm{St}^{-1}\nabla_x \cdot \int_{\mathbb{S}^2} p\otimes p\, S_u(t,0)\psi^{in} \\ &\quad - \iota \mathrm{St}^{-1}\nabla_x \cdot \int_{\mathbb{S}^2} p\otimes p \int_0^t S_u(t,\tau)\nabla_p \cdot \Big(\mathbb{P}_{p^\perp}\left[(\gamma E(u(\tau)) + W(u(\tau)))p\right]\psi(\tau)\Big)\mathrm{d}\tau. \end{aligned} \tag{4.6}$$

Our first result concerns the integrability properties of the kernel K and only requires the bootstrap assumption (H').

PROPOSITION 4.1. *There exists an absolute constant ν_0 such that for $\nu \le \nu_0$ the condition (H') implies that kernel K satisfies the following estimates*

(a) $\|K(t,\tau)\|_{L(H^s,H^s)} \lesssim 1$ *for all $\tau \le t$ with $t-\tau \le 1$.*

(b) $\|K(t,\tau)\|_{L(H^s,H^s)} \lesssim \mathrm{e}^{-\eta_1\nu^{\frac12}(t-\tau)}$, *for all $\tau \le t$, with η_1 the rate given in the hypocoercive estimate (4.2).*

(c) $\|K(t,\tau)\|_{L(H^s,H^r)} \lesssim \frac{1}{(t-\tau)^{\frac32}} + \nu^{\frac34}$, *for all $\tau < t$, for any $r < s + \frac14$.*

PROOF. We can restrict to the case $\tau = 0$, as all the arguments that we will use are translation invariant. Given $u_0 \in H^s$, We denote by ψ_0 the solution of

$$\partial_t\psi_0 + (u+p)\cdot\nabla_x\psi_0 - \nu\Delta_p\psi_0 = 0, \quad \psi_0|_{t=0} = -\frac{3\gamma\iota}{4\pi}p\otimes p : E(u_0).$$

Note that the initial data is mean-free in x, a property that is propagated through time: $\psi_0 = \psi_{0,\neq 0}$. From the definition of ψ_0,

$$K(t,0)u_0 = \mathrm{St}^{-1}\nabla_x \cdot \int_{\mathbb{S}^2} p\otimes p\,\psi_0$$

By a standard energy estimate on ψ_0 in $\mathcal{H}^{s-1}$, under (H') for u we find

$$\|\psi_0(t)\|_{\mathcal{H}^{s-1}} \lesssim \|\psi_0(0)\|_{\mathcal{H}^{s-1}} \lesssim \|u_0\|_{H^s}, \quad \forall t \le 1$$

Hence,

$$\|K(t,0)u_0\|_{H^s} \lesssim \|\psi_0(t)\|_{\mathcal{H}^{s-1}} \lesssim \|u_0\|_{H^s}, \quad \forall t \le 1$$

and estimate (a) follows. Also, the hypocoercive estimate (4.2), which is valid under (H'), implies for δ small enough that

$$\|\psi_0(t)\|_{\mathcal{H}^{s-1}} \lesssim \mathrm{e}^{-\eta_1\nu^{\frac12}t}\|\psi_0(0)\|_{\mathcal{H}^{s-1}} \lesssim \mathrm{e}^{-\eta_1\nu^{\frac12}t}\|u_0\|_{H^s}, \quad \forall t \ge 0,$$

which implies estimate (b). It remains to prove estimate (c). Expressing the operator $\mathrm{St}^{-1}\nabla_x\cdot$ in Fourier, one checks that

$$\|K(t,0)u_0\|_{H^r}^2 = \sum_{k\neq 0}|k|^{2r-2}\left|\int_{\mathbb{S}^2}(p\cdot\hat{k})P_{\hat{k}^\perp}p\,\psi_{0,k}\right|^2.$$

To a given k, we can associate a cartesian frame $(e_x, e_y, e_z := \hat{k})$ and spherical coordinates (θ, φ) with θ the colatitude and φ the longitude. In particular, $p = \sin\theta\cos\varphi e_x + \sin\theta\sin\varphi e_y + \cos\theta\hat{k}$. We compute

$$p\cdot\hat{k} = \cos\theta, \quad \nabla(p\cdot\hat{k}) = -\sin\theta e_\theta = -\sin\theta(\cos\theta\cos\phi e_x + \cos\theta\sin\phi e_y - \sin\theta\hat{k}),$$
$$P_{\hat{k}^\perp}p = p - (p\cdot\hat{k})\hat{k} = \sin\theta\cos\phi e_x + \sin\theta\sin\phi e_y,$$

so that

$$(p\cdot\hat{k})P_{\hat{k}^\perp}p = -P_{\hat{k}^\perp}\nabla(p\cdot\hat{k}).$$

Hence

$$\|K(t,0)u_0\|_{H^r}^2 = \sum_{k\neq 0}|k|^{2r-2}\left|\int_{\mathbb{S}^2}P_{\hat{k}^\perp}\nabla(p\cdot\hat{k})\,\psi_{0,k}\right|^2.$$

From there, we apply inequality (4.3) (see (1.10) for the definition of V_k), with $Z = Z_k = P_{\hat{k}^\perp}$ and $s = r - \frac{5}{4}+$ to get

$$\begin{aligned}\|K(t,0)u_0\|_{H^r}^2 &\lesssim \left(\frac{\nu^{\frac{1}{2}}}{\min(1,\nu^{\frac{1}{2}}t)}\right)^3\left(\|\psi_0(0)\|^2_{\mathcal{H}^{r-\frac{5}{4}+}} + \|\nabla_p\psi_0(0)\|^2_{\mathcal{H}^{r-\frac{5}{4}+}} + \|\nabla_p^2\psi_0(0)\|^2_{\mathcal{H}^{r-\frac{5}{4}+}}\right)\\ &\lesssim \left(\frac{1}{t^3} + \nu^{\frac{3}{2}}\right)\|u_0\|^2_{H^{r-\frac{1}{4}+}}.\end{aligned}$$

The result follows. □

We can now state a stability estimate for u.

PROPOSITION 4.2. *Assume* (1.4). *Let* $\eta_0 = \frac{\eta_1}{10}$, *with* η_1 *the absolute constant given in* (4.2). *There exists* $\nu_0 > 0$ *depending on* γ *and* ι *such that for* $\nu \leq \nu_0$ *the condition* (H') *implies*

$$\sup_{0\leq t\leq T}\mathrm{e}^{\eta_0\nu^{\frac{1}{2}}t}\|u(t)\|_{H^s} \lesssim \sup_{0\leq t\leq T}\mathrm{e}^{\eta_0\nu^{\frac{1}{2}}t}\|f(t)\|_{H^s}.$$

PROOF. The proof is an adaptation of the reasoning in [8, Section 5.2]. We first extend $K(t,\tau)$ by zero for $t \geq T$ or $\tau > t$, and extend $u(t)$ and $f(t)$ by zero for $t \geq T$. Setting

$$k(t,\tau) = \mathrm{e}^{\eta_0\nu^{\frac{1}{2}}(t-\tau)}K(t,\tau), \qquad \tilde{u}(t) = \mathrm{e}^{\eta_0\nu^{\frac{1}{2}}t}u(t), \qquad \tilde{f}(t) = \mathrm{e}^{\eta_0\nu^{\frac{1}{2}}t}f(t),$$

the Volterra equation is equivalent to

$$\tilde{u}(t) + \int_0^\infty k(t,\tau)\,\tilde{u}(\tau)\,\mathrm{d}\tau = \tilde{f}(\tau) \qquad \text{for all } t\in\mathbb{R}_+ \tag{4.7}$$

and the point is to show that $\sup_{t\geq 0}\|\tilde{u}(t)\|_{H^s} \lesssim \sup_{t\geq 0}\|\tilde{f}(t)\|_{H^s}$. We introduce

$$\mathcal{K}_s := \left\{k:\mathbb{R}_+\times\mathbb{R}_+\to L(H^s,H^s), \quad k(t,\tau) = 0 \text{ for } \tau\leq t, \quad \sup_{t\in\mathbb{R}_+}\int_0^\infty|k(t,\tau)|\,\mathrm{d}\tau < +\infty\right\}$$

equipped with $\|k\|_{\mathcal{K}_s} := \sup_t\int_{\mathbb{R}_+}|k(t,\tau)|\,\mathrm{d}\tau$. This space is an analogue of the space of *Volterra kernels of bounded type* introduced in [10], replacing kernels with values in $\mathbb{C}^n$ by kernels with values in $L(H^s,H^s)$. One can show exactly as in [10] that $\mathcal{K}_s$ is a Banach algebra for the product

$$(k_1\star k_2)(t,\tau) := \int_0^\infty k_1(t,\tau')\,k_2(\tau',\tau)\,\mathrm{d}\tau'.$$

Moreover, as in any Banach algebra, there is a notion of resolvent: we say that $k\in\mathcal{K}_s$ has resolvent $r\in\mathcal{K}_s$ if

$$r + k\star r = r + r\star k = k.$$

One can show that

Lemma 4.3. *(Direct adaptation of* [10, Chapter 9, Lemma 3.4]*) If $k \in \mathcal{K}_s$ has a resolvent $r \in \mathcal{K}_s$ then, for any $\tilde{f} \in L^\infty(\mathbb{R}_+, H^s)$, equation (4.7) has a unique solution $\tilde{u} \in L^\infty(\mathbb{R}_+, H^s)$, given by*

$$\tilde{u}(t) = \tilde{v}(t) - \int_{\mathbb{R}_+} r(t,\tau)\,\tilde{v}(\tau)\,\mathrm{d}\tau.$$

In particular, $\|\tilde{u}\|_{L^\infty(\mathbb{R}_+,H^s)} \leq (1 + \|r\|_{\mathcal{K}_s})\|\tilde{v}\|_{L^\infty(\mathbb{R}_+,H^s)}$.

Moreover, the set of kernels having a resolvent is open, which can be proved through a Von Neumann series argument. Namely,

PROPOSITION 4.4. *(Direct adaptation of* [10, Chapter 9, Theorem 3.9]*)*
If $k = k_1 + k_2$ is the sum of two elements of $\mathcal{K}_s$, if k_1 has a resolvent r_1 and if

$$\|k_2\|_{\mathcal{K}_s} < \frac{1}{1 + \|r_1\|_{\mathcal{K}_s}}$$

then k has a resolvent r, given by $r = \sum_{n=0}^{+\infty}(-1)^n\big((k_2 - r_1 \star k_2)\star\big)^n(k - r_1 \star k)$.

Following [8], we decompose

$$k(t,\tau) \;= K(t,\tau) + \big(\mathrm{e}^{\eta_0 \nu^{\frac{1}{2}}(t-\tau)} - 1\big)K(t,\tau) =: k_1(t,\tau) + k_2(t,\tau).$$

It is easily seen that $k \in \mathcal{K}_s$. Moreover, we shall prove below:

Lemma 4.5. *Assume (1.4). There exists $\nu_0 > 0$ depending on γ and ι such that: for all $\nu \leq \nu_0$, if (H') holds, then the kernel $(t,\tau) \mapsto K(t,\tau)$ has a resolvent $R = R(t,\tau)$, and*

$$\|K(t,\tau)\|_{\mathcal{K}_s} \lesssim 1, \quad \|R(t,\tau)\|_{\mathcal{K}_s} \lesssim 1.$$

Assuming for the moment that this lemma is satisfied, we now prove that $\|k_2\|_{\mathcal{K}_S} \to 0$ as $\nu \to 0$. Let $a \in (0,1)$ to be specified, $b := \frac{1}{8\eta_0}$, and decompose:

$$\begin{aligned}
\|k_2\|_{\mathcal{K}_S} &\leq \sup_{t\in\mathbb{R}_+} \int_0^\infty \big(\mathrm{e}^{\eta_0\nu^{\frac{1}{2}}\tau} - 1\big)\|K(t,t-\tau)\|_{L(H^s,H^s)}\mathrm{d}\tau \\
&\leq \sup_{t\in\mathbb{R}_+} \Big(\int_0^{a\nu^{-\frac{1}{2}}} + \int_{a\nu^{-\frac{1}{2}}}^{b|\ln\nu|\nu^{-\frac{1}{2}}} + \int_{b|\ln\nu|\nu^{-\frac{1}{2}}}^\infty\Big)\big(\mathrm{e}^{\eta_0\nu^{\frac{1}{2}}\tau} - 1\big)\|K(t,t-\tau)\|_{L(H^s,H^s)}\mathrm{d}\tau \\
&=: \sup_{t\in\mathbb{R}_+} I_1(t) + I_2(t) + I_3(t).
\end{aligned}$$

Let $\kappa > 0$. Using the bound $\|K(t,t-\tau)\|_{L(H^s,H^s)} \lesssim \langle\tau\rangle^{-\frac{3}{2}}$ for $\tau \leq a\nu^{-\frac{1}{2}}$, see Proposition 4.1, we find

$$I_1(t) \lesssim (\mathrm{e}^{\eta_0 a} - 1) \leq \kappa \quad \text{for } a \text{ small enough.}$$

This a being fixed, using the bound $\|K(t,t-\tau)\|_{L(H^s,H^s)} \lesssim \nu^{\frac{3}{4}}$ for $\tau \in [a\nu^{-\frac{1}{2}}, b\nu^{-\frac{1}{2}}|\ln\nu|]$, *cf.* again Proposition 4.1, we get

$$I_2(t) \lesssim \nu^{-\eta_0 b}\nu^{1/4} \lesssim \nu^{1/4-1/8} \leq \kappa,$$

for ν small enough. Eventually, using the second inequality in Proposition 4.1, we get

$$I_3(t) \lesssim \int_{b|\ln\nu|\nu^{-\frac{1}{2}}}^{+\infty} \mathrm{e}^{(\eta_0-\eta_1)\nu^{\frac{1}{2}}\tau}\mathrm{d}\tau \lesssim \nu^{-\frac{1}{2}+(\eta_1-\eta_0)b} = \nu^{\frac{1}{8}(\frac{\eta_1}{\eta_0}-5)} \leq \kappa,$$

for ν small enough. Hence, $\|k_2\|_{\mathcal{K}_S}$ goes to zero with ν. By Lemma 4.4 and Lemma 4.5 (still to be proved), we deduce that the kernel k has a resolvent r, with $\|r\|_{\mathcal{K}_s} \lesssim 1$. Proposition 4.2 is then a direct consequence of Lemma 4.3. The only missing step is the proof of Lemma 4.5, which is done below. □

PROOF OF LEMMA 4.5. We remind for $0 \le \tau \le t < T$ the formula

$$K(t,\tau)u_0 := \mathrm{St}^{-1}\nabla_x \cdot \int_{\mathbb{S}^2} p \otimes p\, \psi_\tau,$$

where $\psi_0 = \psi_0(t)$ is the solution of

$$\partial_t \psi_\tau + (u+p)\cdot\nabla_x \psi_\tau - \nu\Delta_p \psi_\tau = 0, \quad t \ge \tau, \quad \psi_\tau|_{t=\tau} = -\frac{3\gamma\iota}{4\pi} p\otimes p : E(u_0).$$

The first estimate of the lemma follows from the estimates of Proposition 4.1 (see the treatment of the kernel k_2 for very close computations). To show that K has a resolvent with norm $O(1)$, we shall again rely on Lemma 4.4, seeing K as a perturbation of the kernel $\bar{K}(t,\tau) = \bar{K}_0(t,\tau)1_{t<T}$, where

$$\text{for all } \tau > t, \quad \bar{K}_0(t,\tau) = 0, \quad \text{while for } \tau \le t, \quad \bar{K}_0(t,\tau)u_0 := \mathrm{St}^{-1}\nabla_x \cdot \int_{\mathbb{S}^2} p\otimes p\, \bar{\psi}_\tau(t) \tag{4.8}$$

where this time $\bar{\psi}_\tau$ is the solution (mean-free in x) of

$$\partial_t \bar{\psi}_\tau + p\cdot\nabla_x \bar{\psi}_\tau - \nu\Delta_p \bar{\psi}_\tau = 0, \quad t \ge \tau, \quad \bar{\psi}_\tau|_{t=\tau} = -\frac{3\gamma\iota}{4\pi} p\otimes p : E(u_0). \tag{4.9}$$

The kernel $\bar{K}_0$, corresponding to the case $u = 0$, was analyzed Fourier mode by Fourier mode in [8]. We remind some elements of this analysis in Appendix A. This analysis shows in particular that there exists $\nu_0 > 0$ depending on γ and ι such that for $\nu \le \nu_0$ the kernel $\bar{K}_0$ has a resolvent $\bar{R}_0$ satisfying

$$\|\bar{R}_0(t,\tau)\|_{\mathcal{K}_s} \lesssim 1. \tag{4.10}$$

It implies directly that $\bar{K}(t,\tau) = \bar{K}_0(t,\tau)1_{t<T}$ has for resolvent $\bar{R}(t,\tau) = \bar{R}_0(t,\tau)1_{t<T}$ whose norm in $\mathcal{K}_s$ satisfies the same bound. By Lemma 4.4, it is then enough to show that under (H')

$$\lim_{\nu\to 0} \|(K - \bar{K})\|_{\mathcal{K}^s} = 0 \tag{4.11}$$

Let $\kappa > 0$. We decompose, for some large $\tilde{T}$ the difference as

$$\begin{aligned}\|(K-\bar{K})\|_{\mathcal{K}^s} \le &\int_0^{\tilde{T}} \|(K-\bar{K})(t,t-\tau)\|_{L(H^s,H^s)}\,\mathrm{d}\tau \\ &+ \int_{\tilde{T}}^{\infty} \Big(\|K(t,t-\tau)\|_{L(H^s,H^s)} + \|\bar{K}(t,t-\tau)\|_{L(H^s,H^s)}\Big)\,\mathrm{d}\tau.\end{aligned}$$

For the second part, we use the estimates of Proposition 4.1, which are also valid for $\bar{K}$, as $u = 0$ satisfies (H'): introducing $c = \frac{1}{\eta_1}$

$$\begin{aligned}&\int_{\tilde{T}}^{\infty} \Big(\|K(t,t-\tau)\|_{L(H^s,H^s)} + \|\bar{K}(t,t-\tau)\|_{L(H^s,H^s)}\Big)\mathrm{d}\tau \\ &= \Big(\int_{\tilde{T}}^{\nu^{-\frac12}} + \int_{\nu^{-\frac12}}^{c\nu^{-\frac12}|\ln\nu|} + \int_{c\nu^{-\frac12}|\ln\nu|}^{+\infty}\Big)\Big(\|K(t,t-\tau)\|_{L(H^s,H^s)} + \|\bar{K}(t,t-\tau)\|_{L(H^s,H^s)}\Big)\mathrm{d}\tau \\ &\lesssim \int_{\tilde{T}}^{\nu^{-\frac12}} \frac{1}{\langle\tau\rangle^{\frac32}}\mathrm{d}\tau + \int_{\nu^{-\frac12}}^{c\nu^{-\frac12}|\ln\nu|} \nu^{\frac34}\mathrm{d}\tau + \int_{c\nu^{-\frac12}|\ln\nu|}^{+\infty} \mathrm{e}^{-\eta_1\nu^{\frac12}\tau}\mathrm{d}\tau \\ &\lesssim \int_{\tilde{T}}^{\infty} \frac{1}{\langle\tau\rangle^{\frac32}}\mathrm{d}\tau + \nu^{1/4}|\ln\nu| + \nu^{-\frac12+\eta_1 c} \le \kappa,\end{aligned}$$

for $\tilde{T}$ large enough (depending on γ and ι) and ν small enough. This time $\tilde{T}$ being fixed, we turn to the first term. We claim that for ν small enough, if (H') holds, for all $\tau \le \tilde{T}$

$$\|(K-\bar{K})(t,t-\tau)\|_{L(H^s,H^{s-1})} \lesssim \nu^{\frac54}. \tag{4.12}$$

Let us assume temporarily that (4.12) holds. We also have, by the third inequality in Proposition 4.1

$$\|K(t,t-\tau)\|_{L(H^s,H^{s+s'})}+\|\bar{K}(t,t-\tau)\|_{L(H^s,H^{s+s'})}\lesssim\frac{1}{\tau^{\frac{3}{2}}} \tag{4.13}$$

for $s'=\frac{1}{8}$ (any $s'\in(0,\frac{1}{4})$ would do). By interpolation of (4.12) and (4.13) , we get

$$\|(K-\bar{K})(t,t-\tau)\|_{L(H^s,H^s)} \tag{4.14}$$

$$\leq\|(K-\bar{K})(t,t-\tau)\|^{1-\theta}_{L(H^s,H^{s-1})}\Big(\|K(t,t-\tau)\|_{L(H^s,H^{s+s'})}+\|\bar{K}(t,t-\tau)\|_{L(H^s,H^{s+s'})}\Big)^{\theta}$$

$$\lesssim\nu^{\frac{5}{4}(1-\theta)}\tau^{-\frac{3}{2}\theta}, \tag{4.15}$$

with θ such that $\theta s'-(1-\theta)=0$, that is $\theta=\frac{1}{1+s'}=\frac{8}{9}$. We deduce from this estimate and the estimate

$$\|(K-\bar{K})(t,t-\tau)\|_{L(H^s,H^s)}\leq\|K(t,t-\tau)\|_{L(H^s,H^s)}+\|\bar{K}(t,t-\tau)\|_{L(H^s,H^s)}\lesssim 1$$

that

$$\int_0^{\tilde{T}}\|(K-\bar{K})(t,t-\tau)\|_{L(H^s,H^s)}\,\mathrm{d}\tau$$

$$=\int_0^{\kappa}\|(K-\bar{K})(t,t-\tau)\|_{L(H^s,H^s)}\,\mathrm{d}\tau+\int_{\kappa}^{\tilde{T}}\|(K-\bar{K})(t,t-\tau)\|_{L(H^s,H^s)}\,\mathrm{d}\tau$$

$$\leq C\kappa+C'\nu^{\frac{5}{4}(1-\theta)}\int_{\kappa}^{+\infty}\tau^{-\frac{3}{2}\theta}\leq C\kappa+C''\nu^{\frac{5}{4}(1-\theta)}\kappa^{-\frac{1}{3}}\leq(C+1)\kappa,$$

for ν small enough. As κ is arbitrary, this proves (4.11). The final step is to establish (4.12). From the definition of the kernel, we find for $t\leq T$ (otherwise all quantities are zero):

$$\|(K-\bar{K})(t,\tau)u_0\|_{H^{s-1}}\leq\|(\psi_0-\bar{\psi}_0)(t)\|_{\mathcal{H}^{s-2}}.$$

The function $\psi:=\psi_\tau-\bar{\psi}_\tau$ satisfies

$$\partial_t\psi+(u+p)\cdot\nabla_x\psi-\nu\Delta_p\psi=-u\cdot\nabla_x\bar{\psi}_\tau,\quad t\geq\tau,\quad \psi|_{t=\tau}=0.$$

A standard estimate yields (for $s-2>\frac{5}{2}$),

$$\partial_t\|\psi\|^2_{\mathcal{H}^{s-2}}\lesssim\|u\|_{\mathcal{H}^{s-2}}\|\psi\|^2_{\mathcal{H}^{s-2}}+\|u\|_{\mathcal{H}^{s-2}}\|\nabla_x\bar{\psi}_\tau\|_{\mathcal{H}^{s-2}}\|\psi\|_{\mathcal{H}^{s-2}}.$$

For $\tau\leq t$ with $t-\tau\leq\tilde{T}$ (where we remind that $\tilde{T}$ is fixed) this implies

$$\|\psi(t)\|_{\mathcal{H}^{s-2}}\lesssim\int_\tau^t\|\bar{\psi}_\tau(t')\|_{\mathcal{H}^{s-1}}\|u(t')\|_{\mathcal{H}^{s-2}}\mathrm{d}t'\lesssim\sup_{\tau\leq t'\leq t}\|\bar{\psi}_\tau(t')\|_{\mathcal{H}^{s-1}}\nu^{\frac{5}{4}}\lesssim\|u_0\|_{H^s}\nu^{\frac{5}{4}},$$

where the second inequality is coming from (H') and from the standard Sobolev estimate $\sup_{\tau\leq t'\leq t}\|\bar{\psi}_\tau(t')\|_{\mathcal{H}^{s-1}}\lesssim\|\bar{\psi}_\tau(\tau)\|_{\mathcal{H}^{s-1}}=\|\frac{3\gamma\iota}{4\pi}p\otimes p:E(u_0)\|_{\mathcal{H}^{s-1}}$. Estimate (4.12) follows, and the proof of the lemma is concluded. □

4.3. Improvement of the bootstrap assumptions. The goal of this section is to prove the following:

PROPOSITION 4.6. *Assume* (1.4). *Let* $\eta_0=\frac{\eta_1}{10}$, *with* η_1 *the absolute constant given in* (4.2). *There exists* $\delta,\delta',\delta'',\nu_0>0$ *depending on* γ *and* ι *such that for* $\nu\leq\nu_0$ *the assumptions* (BA0)-(BA1)-(BA2) *imply*

$$\sup_{0\leq t\leq T}\|u(t)\|_{H^s}\lesssim\|\psi_{in}\|_{\mathcal{H}^{s-1}}\mathrm{e}^{-\eta_0\nu^{\frac{1}{2}}t}, \tag{4.16}$$

$$\nu\int_0^T\|\nabla_p\psi(t)\|^2_{\mathcal{H}^{s-1}}\,\mathrm{d}t\lesssim\|\psi_{in}\|^2_{\mathcal{H}^{s-1}}\nu^{-1}+\|\psi_{in}\|^3_{\mathcal{H}^{s-1}}\nu^{-\frac{3}{2}}+\|\psi_{in}\|^4_{\mathcal{H}^{s-1}}\nu^{-\frac{5}{2}}, \tag{4.17}$$

$$\sup_{0\leq t\leq T}\|\psi(t)\|_{\mathcal{H}^{s-1}}\lesssim\|\psi^{in}\|_{\mathcal{H}^{s-1}}\nu^{-\frac{1}{2}}. \tag{4.18}$$

In particular, there exists δ_0, depending on γ and ι, such that for

$$\|\psi_{in}\|_{\mathcal{H}^{s-1}} \leq \delta_0 \nu^{\frac{3}{2}}$$

(BA0)-(BA1)-(BA2) *can be improved, with $\delta/2, \delta'/2, \delta''/2$ replacing $\delta, \delta', \delta''$, and also such that* (4.4) *holds.*

As explained in Paragraph 4.1, the last part of the proposition implies Theorem 1.

4.3.1. *Bound on u.* In all what follows, we take $\delta \leq \min\left(\frac{\varepsilon}{\eta_0^{1/4}}, 1\right)$, so that (BA0) implies (H'), and results of Paragraph 4.2 can be applied. The starting point is Proposition 4.2, which says that

$$\sup_{0\leq t<T} \mathrm{e}^{\eta_0 \nu^{\frac{1}{2}} t} \|u(t)\|_{H^s} \lesssim \sup_{0\leq t<T} \mathrm{e}^{\eta_0 \nu^{\frac{1}{2}} t} \|f(t)\|_{H^s},$$

with the source term

$$\begin{aligned} f(t) &= \iota \mathrm{St}^{-1} \nabla_x \cdot \int_{\mathbb{S}^2} p \otimes p \, S_u(t,0)\psi_{in} \\ &- \iota \mathrm{St}^{-1} \nabla_x \cdot \int_{\mathbb{S}^2} p \otimes p \int_0^t S_u(t,\tau) \nabla_p \cdot \Big(\mathbb{P}_{p^\perp} \left[(\gamma E(u(\tau)) + W(u(\tau)))p \right] \psi(\tau) \Big) \mathrm{d}\tau \\ &=: f_{init}(t) + f_{NL}(t). \end{aligned}$$

We remind that $S_u(t,\tau)$ is the solution operator of $L = -(u+p)\cdot\nabla_x u + \nu \Delta_p$. The first term is estimated as

$$\|f_{init}(t)\|_{H^s} \lesssim \|(S_u(t,0)\psi_{init})_{\neq 0}\|_{\mathcal{H}^{s-1}} \lesssim \mathrm{e}^{-\eta_1 \nu^{\frac{1}{2}} t} \|\psi_{init}\|_{\mathcal{H}^{s-1}},$$

where the last inequality comes from (4.2). Similarly,

$$\begin{aligned} \|f_{NL}(t)\|_{H^s} &\lesssim \int_0^t \Big\| \Big(S(t,\tau) \nabla_p \cdot \Big(\mathbb{P}_{p^\perp} \left[(\gamma E(u(\tau)) + W(u(\tau)))p \right] \psi(\tau) \Big) \Big)_{\neq 0} \Big\|_{\mathcal{H}^{s-1}} \mathrm{d}\tau \\ &\lesssim \int_0^t \mathrm{e}^{-\eta_1 \nu^{\frac{1}{2}}(t-\tau)} \Big(\|\nabla_x u(\tau) \otimes \nabla_p \psi(\tau)\|_{\mathcal{H}^{s-1}} + \|\nabla_x u(\tau) \otimes \psi(\tau)\|_{\mathcal{H}^{s-1}} \Big) \mathrm{d}\tau \\ &\lesssim \int_0^t \mathrm{e}^{-\eta_1 \nu^{\frac{1}{2}}(t-\tau)} \|u(\tau)\|_{H^s} \Big(\|\nabla_p \psi(\tau)\|_{\mathcal{H}^{s-1}} + \|\psi(\tau)\|_{\mathcal{H}^{s-1}} \Big) \mathrm{d}\tau \\ &\lesssim \Big(\sup_{0\leq\tau<T} \mathrm{e}^{\eta_0 \nu^{\frac{1}{2}} \tau} \|u(\tau)\|_{H^s} \Big) \int_0^t \mathrm{e}^{-\eta_1 \nu^{\frac{1}{2}}(t-\tau)} \mathrm{e}^{-\eta_0 \nu^{\frac{1}{2}} \tau} \Big(\|\nabla_p \psi(\tau)\|_{\mathcal{H}^{s-1}} + \|\psi(\tau)\|_{\mathcal{H}^{s-1}} \Big) \mathrm{d}\tau. \end{aligned}$$

Hence,

$$\mathrm{e}^{\eta_0 \nu^{\frac{1}{2}} t} \|f_{NL}(t)\|_{H^s} \lesssim \Big(\sup_{0\leq\tau<T} \mathrm{e}^{\eta_0 \nu^{\frac{1}{2}} \tau} \|u(\tau)\|_{H^s} \Big) \int_0^t \mathrm{e}^{-(\eta_1-\eta_0)\nu^{\frac{1}{2}}(t-\tau)} \Big(\|\nabla_p \psi(\tau)\|_{\mathcal{H}^{s-1}} + \|\psi(\tau)\|_{\mathcal{H}^{s-1}} \Big) \mathrm{d}\tau,$$

where the second factor at the right-hand side is a convolution, resulting in

$$\begin{aligned} \mathrm{e}^{\eta_0 \nu^{\frac{1}{2}} t} \|f_{NL}(t)\|_{H^s} &\lesssim \Big(\sup_{0\leq\tau<T} \mathrm{e}^{\eta_0 \nu^{\frac{1}{2}} \tau} \|u(\tau)\|_{H^s} \Big) \Big(\|\mathrm{e}^{-(\eta_1-\eta_0)\nu^{\frac{1}{2}}}\|_{L^2(\mathbb{R}_+)} \|\|\nabla_p \psi(\cdot)\|_{\mathcal{H}^{s-1}}\|_{L^2(0,T)} \\ &\qquad + \|\mathrm{e}^{-(\eta_1-\eta_0)\nu^{\frac{1}{2}}\cdot}\|_{L^1(\mathbb{R}_+)} \|\|\psi(\cdot)\|_{\mathcal{H}^{s-1}}\|_{L^\infty(0,T)} \Big) \\ &\lesssim \Big(\sup_{0\leq\tau<T} \mathrm{e}^{\eta_0 \nu^{\frac{1}{2}} \tau} \|u(\tau)\|_{H^s} \Big) \Big(\nu^{-1/4} \|\|\nabla_p \psi(\cdot)\|_{\mathcal{H}^{s-1}}\|_{L^2(0,T)} + \nu^{-\frac{1}{2}} \|\|\psi(\cdot)\|_{\mathcal{H}^{s-1}}\|_{L^\infty(0,T)} \Big) \\ &\lesssim \Big(\sup_{0\leq\tau<T} \mathrm{e}^{\eta_0 \nu^{\frac{1}{2}} \tau} \|u(\tau)\|_{H^s} \Big) (\delta' + \delta''), \end{aligned}$$

where the last inequality comes from (BA1)-(BA2). Hence,

$$\sup_{0\leq\tau<T} \mathrm{e}^{\eta_0 \nu^{\frac{1}{2}} \tau} \|u(t)\|_{H^s} \lesssim \sup_{t\geq 0} \mathrm{e}^{(\eta_0-\eta_1)\nu^{\frac{1}{2}} t} \|\psi_{init}\|_{\mathcal{H}^{s-1}} + (\delta' + \delta'') \Big(\sup_{\tau\geq 0} \mathrm{e}^{\eta_0 \nu^{\frac{1}{2}} \tau} \|u(\tau)\|_{H^s} \Big).$$

For δ', δ'' small enough (with a threshold depending on γ and ι), we can absorb the second term at the right-hand side, which implies the first bound of the proposition.

4.3.2. *Bound on* ψ. We come back to the equation (1.3). Performing standard $\mathcal{H}^{s-1}$ Sobolev estimates on the equation, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|^2_{\mathcal{H}^{s-1}} + \nu\|\nabla_p\psi\|^2_{\mathcal{H}^{s-1}} \lesssim \|u\|_{H^s}\|\psi\|_{\mathcal{H}^{s-1}} + \|u\|_{H^{s-1}}\|\psi\|^2_{\mathcal{H}^{s-1}} + \|u\|_{H^s}\left(\|\psi\|_{\mathcal{H}^{s-1}} + \|\nabla_p\psi\|_{\mathcal{H}^{s-1}}\right)\|\psi\|_{\mathcal{H}^{s-1}}.$$

At the right-hand side, the first term corresponds to the contribution of the linear term $-\frac{3\Gamma}{4\pi}(p\otimes p) : E(u)$. The second term corresponds to the contribution of the transport term $(u+p)\cdot\nabla_x\psi$, while the third one corresponds to the contribution of $\nabla_p\cdot\left(\mathbb{P}_{p^\perp}\left[(\gamma E(u) + W(u))p\right]\psi\right)$. It implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|^2_{\mathcal{H}^{s-1}} + \frac{\nu}{2}\|\nabla_p\psi\|^2_{\mathcal{H}^{s-1}} \lesssim \|u\|_{H^s}\|\psi\|_{\mathcal{H}^{s-1}} + \|u\|_{H^s}\|\psi\|^2_{\mathcal{H}^{s-1}} + \nu^{-1}\|u\|^2_{H^s}\|\psi\|^2_{\mathcal{H}^{s-1}} \tag{4.19}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_{\mathcal{H}^{s-1}} \lesssim \|u\|_{H^s} + (\|u\|_{H^s} + \nu^{-1}\|u\|^2_{H^s})\|\psi\|_{\mathcal{H}^{s-1}}.$$

The Gronwall lemma together with the bound (4.16) that we have just established yield

$$\begin{aligned}\|\psi(t)\|_{\mathcal{H}^{s-1}} &\lesssim \|\psi_{in}\|_{\mathcal{H}^{s-1}}\nu^{-\frac{1}{2}}\exp\left(\int_0^t(\|u\|_{H^s} + \nu^{-1}\|u\|^2_{H^s})\mathrm{d}\tau\right)\\ &\lesssim \|\psi_{in}\|_{\mathcal{H}^{s-1}}\nu^{-\frac{1}{2}}\exp\left(\int_0^t(\delta\nu^{\frac{3}{2}}\mathrm{e}^{-\eta_0\nu^{\frac{1}{2}}\tau} + \delta^2\nu^2\mathrm{e}^{-2\eta_0\nu^{\frac{1}{2}}\tau}\mathrm{d}\tau\right)\\ &\lesssim \|\psi^{in}\|_{\mathcal{H}^{s-1}}\nu^{-\frac{1}{2}},\end{aligned}$$

for ν small enough. Back to (4.19), integrating from 0 to t and using (4.16), we find

$$\begin{aligned}\frac{\nu}{2}\int_0^t\|\nabla_p\psi\|^2_{\mathcal{H}^{s-1}} &\lesssim \|\psi_{in}\|^2_{\mathcal{H}^{s-1}} + \int_0^t\left(\|u\|_{H^s}\|\psi\|_{\mathcal{H}^{s-1}} + \|u\|_{H^s}\|\psi\|^2_{\mathcal{H}^{s-1}} + \nu^{-1}\|u\|^2_{H^s}\|\psi\|^2_{\mathcal{H}^{s-1}}\right)\mathrm{d}\tau\\ &\lesssim \|\psi_{in}\|^2_{\mathcal{H}^{s-1}} + \|\psi^{in}\|^2_{\mathcal{H}^{s-1}}\nu^{-1} + \|\psi_{in}\|^3_{\mathcal{H}^{s-1}}\nu^{-\frac{3}{2}} + \|\psi_{in}\|^4_{\mathcal{H}^{s-1}}\nu^{-\frac{5}{2}}\\ &\lesssim \|\psi^{in}\|^2_{\mathcal{H}^{s-1}}\nu^{-1} + \|\psi_{in}\|^3_{\mathcal{H}^{s-1}}\nu^{-\frac{3}{2}} + \|\psi_{in}\|^4_{\mathcal{H}^{s-1}}\nu^{-\frac{5}{2}}.\end{aligned}$$

Appendix A. Resolvent estimates in the linear setting

The goal of this appendix is to show that the kernel $\bar{K}_0$ defined in (4.8) has a resolvent $\bar{R}_0$ satisfying $\|\bar{R}_0\|_{\mathcal{K}_s} \lesssim 1$. As the equation in (4.9) is autonomous in time, we have $\bar{K}_0(t,\tau) = K_0(t-\tau)1_{\tau\le t}$, with

$$K_0(t)u_0 := \mathrm{St}^{-1}\nabla_x\cdot\int_{\mathbb{S}^2} p\otimes p\,\bar{\psi}_0(t).$$

Accordingly, we look for a resolvent under the form $\bar{R}_0(t,\tau) = R_0(t-\tau)1_{\tau\le t}$, where R_0 is the resolvent of K_0 for the usual convolution product

$$R_0 + K_0\star R_0 = R_0 + R_0\star K_0 = K_0, \quad f\star g(t) = \int_0^t f(\tau)\,g(t-\tau)\,\mathrm{d}\tau.$$

In this case, $\|\bar{R}_0\|_{\mathcal{K}_s} = \|R_0\|_{L^1_t(\mathbb{R}_+, L(H^s,H^s))}$. Hence, we want to show

$$\|R_0\|_{L^1_t(\mathbb{R}_+, L(H^s,H^s))} \lesssim 1. \tag{A.1}$$

The properties of K_0 have been studied in [8]. More precisely, we performed a mode by mode Fourier analysis in x, with the study for an arbitrary $k\in 2\pi\mathbb{Z}^3_*$ of

$$\partial_t\hat{\psi}_k + \mathrm{i}p\cdot k\hat{\psi}_k - \nu\Delta_p\hat{\psi}_k = 0, \quad t\ge 0, \quad \hat{\psi}_k|_{t=0} = -\frac{3\gamma\iota}{4\pi}p\otimes p : \frac{k\otimes u_0 + u_0\otimes k}{2}$$

and

$$\hat{K}_k(t) = \frac{\mathrm{i}}{|k|^2}\int_{\mathbb{S}^2} p\cdot kP_{k^\perp}p\,\hat{\psi}_k(t) \in L(\mathbb{C}^3,\mathbb{C}^3).$$

The analysis in [8] showed in particular that under the spectral condition $\frac{\gamma|\iota|}{|k|} < \Gamma_c$, the $\hat{K}_k$ has a unique resolvent $\hat{R}_k$, satisfying $\|\hat{R}_k\|_{L^1(\mathbb{R}_+)} \le C_k$ where C_k is independent of ν, depends on γ, ι and possibly on k (the dependence with respect to k was not examined in details in [8], see below). Note that for $u = \sum_{k\in 2\pi\mathbb{Z}^3_*}\hat{u}_k\mathrm{e}^{\mathrm{i}k\cdot x}$, one has

$$K_0(t)u = \sum_{k\in 2\pi\mathbb{Z}^3_*}\hat{K}_k(t)\hat{u}_k\mathrm{e}^{\mathrm{i}k\cdot x}$$

and, at least formally,

$$R_0(t)u = \sum_{k\in 2\pi\mathbb{Z}^3_*}\hat{R}_k(t)\hat{u}_k\mathrm{e}^{\mathrm{i}k\cdot x}.$$

The remaining step is to show convergence of the series defining R_0, and to get a bound independent of ν. This requires more accurate bounds on the $\hat{R}_k$ for $k\in 2\pi\mathbb{Z}^3_*$. We shall distinguish between low and high frequencies: for some cut-off frequency N to be fixed later, we write

$$R_0^\flat(t)u = \sum_{|k|\le N}\hat{R}_k(t)\hat{u}_k\mathrm{e}^{\mathrm{i}k\cdot x},\quad R_0^\sharp(t)u = \sum_{|k|>N}\hat{R}_k(t)\hat{u}_k\mathrm{e}^{\mathrm{i}k\cdot x}.$$

We will prove that there exists N, depending only on γ and ι such that

$$\sup_{|k|>N}|\hat{R}_k(t)| \lesssim \frac{1}{(1+t)^{\frac{3}{2}}},\quad \forall t\ge 0. \tag{A.2}$$

As $\|R_0^\sharp(t)u\|_{H^s} \le \sup_{|k|>N}|\hat{R}_k(t)|\|u\|_{H^s}$, one has easily $\|R_0^\sharp\|_{L^1_t(\mathbb{R}_+,L(H^s,H^s))}\lesssim 1$. This N being fixed, $R_0^\flat$ is made of a finite number of terms, for which the bound $\|\hat{R}_k\|_{L^1(\mathbb{R}_+)}\le C_k$ allows to conclude that $\|R_0^\flat\|_{L^1_t(\mathbb{R}_+,L(H^s,H^s))}\lesssim 1$, which in turn yields (A.1).

To prove (A.2), let

$$t' = |k|t,\quad k' = \frac{k}{|k|}, \nu' = \frac{\nu}{|k|},\quad \hat{\psi}'_{k'}(t') = \frac{1}{|k|}\hat{\psi}_k(t)$$

so that

$$\partial_{t'}\hat{\psi}'_{k'} + \mathrm{i}p\cdot k'\hat{\psi}'_{k'} - \nu'\Delta_p\hat{\psi}'_{k'} = 0,\quad t\ge 0,\quad \hat{\psi}'_{k'} = -\frac{3\gamma\iota}{4\pi}p\otimes p : \frac{k'\otimes u_0 + u_0\otimes k'}{2}$$

We further introduce

$$\hat{K}_{k'}(t')u_0 = \mathrm{i}\int_{\mathbb{S}^2} p\cdot k'P_{(k')^\perp}p\,\hat{\psi}'_{k'}(t)$$

It is an easy verification that

$$\hat{K}_k(t) = \hat{K}_{k'}\big(|k|t\big).$$

The analysis of [8] was actually focused on the normalized kernels $\hat{K}_{k'}$ (normalized because $|k'| = 1$). There, it was shown that there exists ν_0, C_0, η_1, m depending on γ, ι such that for $\nu'\le\nu_0$ following inequalities hold

$$|\hat{K}_{k'}(t')| \le C_0\frac{\ln(2+t')}{(1+t')^2}\quad \forall t'\le(\nu')^{-\frac{1}{2}},$$

$$|\hat{K}_{k'}(t')| \le C_0\nu'|\ln(\nu')|^m\quad \forall t'\in[\nu'^{-\frac{1}{2}}, c|\ln(\nu')|(\nu')^{-\frac{1}{2}}],$$

$$|\hat{K}_{k'}(t')| \le C_0 e^{-\eta_1(\nu')^{\frac{1}{2}}t'}\quad \forall t'\ge c|\ln(\nu')|(\nu')^{-\frac{1}{2}}.$$

Taking c large enough compared to η_1, these estimates imply

$$|\hat{K}_{k'}(t')| \lesssim \frac{1}{(1+t')^{\frac{3}{2}}}$$

(any power less than 2 would do). Hence,

$$|\hat{K}_k(t)| \lesssim \frac{1}{(1+|k|t)^{\frac{3}{2}}}.$$

This implies

$$\|\hat{K}_k\|_{L^1(\mathbb{R}_+)} \lesssim \frac{1}{|k|} \lesssim \frac{1}{N}.$$

For N large enough, we have in particular that $\|\hat{K}_k\|_{L^1(\mathbb{R}_+)} < 1$, and we know in this case that the resolvent is given explicitly by the Neumann series

$$\hat{R}_k = \sum_{j\geq 0} (-1)^j (\hat{K}_k \star)^j \hat{K}_k.$$

It is moreover straightforward to show that if

$$|f(t)| \lesssim \frac{C_f}{(1+|k|t)^{\frac{3}{2}}}, \quad |\hat{K}_k(t)| \leq \frac{C_K}{(1+|k|t)^{\frac{3}{2}}}$$

then for some absolute constant C_0

$$|\hat{K}_k \star f(t)| \leq \frac{C_0 C_f C_K}{|k|(1+|k|t)^{\frac{3}{2}}}.$$

By induction we get

$$|(\hat{K}_k \star)^j \hat{K}_k(t)| \leq \Big(\frac{C_0 C_K}{|k|}\Big)^j \frac{C_K}{(1+|k|t)^{\frac{3}{2}}}.$$

Eventually, for N large enough so that $\frac{C_0 C_K}{N} < 1$, we find

$$|R_k(t)| \lesssim \sum_{j\geq 0} \Big(\frac{C_0 C_K}{|k|}\Big)^j \frac{1}{(1+|k|t)^{\frac{3}{2}}} \lesssim \frac{1}{(1+|k|t)^{\frac{3}{2}}}.$$

The result (A.2) follows.

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DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON, LONDON, SW7 2AZ, UK
Email address: m.coti-zelati@imperial.ac.uk

UNIVERSITÉ PARIS CITÉ AND SORBONNE UNIVERSITÉ, CNRS, INSTITUT DE MATHÉMATIQUES DE JUSSIEU-PARIS RIVE GAUCHE (IMJ-PRG), F-75013, PARIS, FRANCE
Email address: helge.dietert@imj-prg.fr

UNIVERSITÉ PARIS CITÉ AND SORBONNE UNIVERSITÉ, CNRS, INSTITUT DE MATHÉMATIQUES DE JUSSIEU-PARIS RIVE GAUCHE (IMJ-PRG), F-75013, PARIS, FRANCE
Email address: david.gerard-varet@u-paris.fr