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Sergio Morales-Tejera, Victor E. Ambrus, Maxim N. Chernodub. Vortical waves in a quantum fluid with vector, axial and helical charges. II. Dissipative effects. 2024. hal-04528328

HAL Id: hal-04528328

<https://hal.science/hal-04528328>

Preprint submitted on 1 Apr 2024

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Vortical waves in a quantum fluid with vector, axial and helical charges. II. Dissipative effects

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ABSTRACT: In this paper, we consider the effect of interactions on the local, average polarization of a quantum plasma of massless fermion particles characterized by vector, axial, and helical quantum numbers. Due to the helical and axial vortical effects, perturbations in the vector charge in a rotating plasma can lead to chiral and helical charge transfer along the direction of the vorticity vector. At the same time, interactions between the plasma constituents lead to the dissipation of the helical charge through helicity-violating pair annihilation (HVPA) processes and of the axial charge through the axial anomaly. We will discuss separately a QED-like plasma, in which we ignore background electromagnetic fields and thus the axial charge is roughly conserved, as well as a QCD-like plasma, where instanton effects lead to the violation of the axial charge conservation, even in the absence of background chromomagnetic fields. The non-conservation of helicity and chirality leads to a gapping of the Helical, Axial and mixed, Axial-Helical vortical waves that prevents their infrared modes from propagating. On the other hand, usual dissipative effects, such as charge diffusion, lead to significant damping of ultraviolet modes. We end this paper with a discussion of the regimes where these vortical waves may propagate.

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1 Introduction

Anomalous breaking of continuous internal symmetries in chiral fluids leads to the emergence of a distinct class of gapless hydrodynamic modes that appear in nontrivial backgrounds such as classical electromagnetic fields or curved spacetimes [1]. A celebrated example of such modes is produced by the coherent interplay of two anomalous transport effects: the Chiral Magnetic Effect [2–4] and the Chiral Separation Effect [5, 6]. Both these transport phenomena, which appear as a result of the activation of the axial anomaly in the magnetic field background, generate a vector (chiral/axial) current in the regions with a nonvanishing chiral (vector) density. These effects yield a closed system of propagating vector-axial oscillations in currents and densities, known as the Chiral Magnetic Wave. This anomalous hydrodynamic excitation might manifest itself in the quark-gluon plasma and could potentially exhibit distinguishing characteristics that might be observed experimentally [7]. It is worth stressing that the word “gapless”, originating from the condensed matter literature, implies that the excitation has no mass gap in its energy spectrum, making it somewhat similar to the acoustic sound waves.

Vortical backgrounds – represented, for example, by whirlpools in fluids or plasmas – host a similar class of hydrodynamic phenomenon known under the common name of Chiral Vortical Effects [8, 9]. These vortical transport effects generate intertwined oscillations of vector and axial currents, as well as their charges, propagating along the rotation axis [10]. The resulting Chiral Vortical Waves, which are supported by the presence of the chiral anomaly and its mixed chiral-gravitational counterpart [11], were suggested to exist in rotating quark-gluon plasmas that are created in highly-vortical noncentral heavy-ion collisions [10].

The picture described above traditionally represents a chiral fluid as a two-component system that includes only vector and axial degrees of freedom that propagate coherently. However, at this point, it is worth noticing that fermions possess a third type of quantum number associated with the fermion helicity, which is distinct from the vector and axial charges [12]. Helicity and chirality are distinguishable physical properties of fermions that are often inaccurately identified with each other [13, 14]. The presence of this additional third characteristic of the fermion ensemble leads to the associated helical vortical transport phenomena that affect, in particular, the spectrum of the hydrodynamic modes [14].¹

In our companion paper [16], we have been exploring the role of helical degrees of freedom in vortical chiral fluids in an academic dissipationless limit when both axial and helical quantum numbers are conserved. This study has confirmed the presence of the Helical Vortical Wave [14], which corresponds to a gapless hydrodynamic excitation acting mainly in the helical and vector sectors of charge densities. Surprisingly, it also has uncovered yet another gapless wave in the system: the Axial Vortical Wave, which is driven by a purely axial wave accompanied, in the presence of vector (baryon) or helical (spin-polarized) background, by coherent fluctuations in axial and helical/vector charges and their current. At

¹Here, for completeness, we mention that similar, helicity-related nontrivial transport and associated hydrodynamic modes in the magnetic field background has been uncovered in Refs. [15]. In this paper, we concentrate only on rotating systems.

high vector (baryon) densities, where the fluid becomes degenerate, both the Helical and the Axial Vortical Waves merge into a hydrodynamic excitation, the Axial-Helical Vortical Wave. The fascinating feature of the Axial Vortical Wave is its spatial non-reciprocity, which makes this hydrodynamic mode distinguishable from the usual acoustic waves: a pure Axial Vortical Wave propagates only along the direction of the angular velocity while is unable to travel in the opposite direction.

This paper aims to study the fate of the hydrodynamic vortical excitations in realistic plasmas, which are characterized by axial and helical charge relaxation as well as kinetic damping effects. A similar analysis has been extensively applied to determine possible experimentally observable signatures [17, 18] of a similar excitation, the Chiral Magnetic Wave, that emerges in the magnetic field background and incorporates oscillations of the vector and axial charges [19, 20]. In addition to the standard axial charge relaxation effects – related to the fact that the axial charge is not a conserved quantity in quantum field theory – the vector sector is also subjected to relaxation effects caused by the electromagnetic backreaction [20]. The latter phenomenon is supported by a finite electrical conductivity in plasma that leads to an efficient screening of the local vector charge fluctuations carrying nonvanishing electric charge density. As a result, the Chiral Magnetic Wave is essentially “overdamped” in realistic environments such as quark-gluon plasma: the relaxation length of the wave appears to be shorter than the half-period of the wave oscillation [21].² In the vector-chiral sector of rotating plasma, a related analysis has been performed in Ref. [23], where various waves were analyzed, including the longitudinal sound wave, a circularly polarized vortical wave and diffusive modes that involve an oscillating vector charge density.

Here, we include in the analysis the helical charge, which enriches the spectrum of the hydrodynamic excitations. In particular, the Helical Vortical Wave [14] involves the oscillations of the helical and vector charge densities, and, therefore, our analysis gives a higher, conservative bound of the wave propagation length caused by the helical charge relaxation and kinetic dissipation effects. The electromagnetic backreaction of the vector charge density, not considered in this paper, will restrict this length even further. On the other hand, the Axial and Axial-Helical Vortical waves, found in the companion paper [16], are driven essentially by electrically neutral helical and axial charge densities, and therefore, they are not subjected to the backreaction coming from the electromagnetic sector associated with the vector charge fluctuations.

One may argue that the helical degree of freedom is as good (bad) as the chiral charge of massive fermions: neither of them is conserved in the physically relevant theories, such as massive QED or QCD. Chirality is not conserved for massless fermions either, due to instanton-like effects that are present in the QCD medium even in the absence of background chromomagnetic fields. In the relaxation time approximation, the pertinence of these quantum numbers for the dynamics of the system is determined by the relative order of magnitudes of the axial, τ_A , and helical, τ_H , relaxation times, which show how fast these charges dissolve in the system. Similarly to the axial (chiral) charge [24, 25], the relaxation

²A recent work [22] highlights the regimes, in which the Chiral Magnetic Wave can still be observed in a condensed matter setup as an underdamped collective mode in a Weyl semimetal which hosts relativistic fermionic quasiparticles similarly to quark-gluon plasma.

of the helical charge might also be a rather fast process [26]. In addition to the charge relaxation times, the system is also characterized by the kinetic relaxation time τ_R , which encodes how fast fluctuations in the thermodynamic characteristics of the system (such as pressure and energy density) relax towards the thermodynamic equilibrium. In general, all three relaxation times, τ_R , τ_A , and τ_H , are independent of each other. An expression for τ_H was derived in Ref. [14] for a neutral plasma. In this paper, we will extend this derivation for the case of a finite vector imbalance, encoding a difference between the number of particles and anti-particles in the system, which is necessary in order to assess its relevance in the limit of a degenerate (dense) system. A similar calculation for the axial relaxation time is beyond the scope of this work; as such, an analysis must be performed in fully non-perturbative QCD using, e.g., lattice methods.

Below, we concentrate on the theoretical questions related to the very existence of vortical modes, their spectrum, and their lifetimes. The discussion of their experimental signatures will be presented elsewhere.

The structure of the paper is as follows. In Sec. 2, we briefly present the setup for the study of linear perturbations in the quantum plasma with vector/axial/helical degrees of freedom, following the approach in Ref. [16], which we extend to the case when the helical and/or axial charges are not conserved. Our focus is on the case of an unpolarized background plasma, in which both the helical and axial chemical potentials vanish.

In Sec. 3, we take into account the non-conservation of the helical charge stemming from local processes using a relaxation time approximation. Such a scenario is relevant for a massless QED plasma, where both vector and axial charges are conserved due to an unbroken U(1) gauge symmetry and the vector nature of inter-particle interactions, respectively. The Chiral Vortical Wave is then recovered in the limit where the helical charge dissipates instantaneously. An interplay between Chiral and Helical Vortical Waves, as well as a distinct role played by the Axial Vortical Wave, are discussed.

In Sec. 4, we study the spectra of collective excitations when both helical and axial charges are non-conserved. We show that the relaxation of helical and axial charge densities affects the spectrum of the wave in a different manner.

Section 5 discusses in detail the effect of the kinetic dissipation, which appears at the level of corrections to the energy-momentum tensor and the charge currents. In a realistic fluid, this type of dissipation has different roots than the charge relaxation (for example, of axial or helical charge densities), as it occurs due to inter-particle interactions, which drive the system towards thermodynamic equilibrium. As anomalous transport also leads to charge and heat flow in the fluid rest frame, the kinetic dissipation has the effect of shifting the phase velocities by an imaginary quantity, leading to a gradual dissolution of the waves. The corresponding kinematic relaxation times are estimated.

Finally, we briefly summarize in Sec. 6 our findings for the non-trivial modifications of the wave spectrum due to the non-vanishing lifetimes of the helical and axial charges.

In Appendix A, we show that an alternative prescription to incorporate the relaxation of helical charge gives rise to an unphysical instability. Appendix B is devoted to the generalization of the expression given in Ref. [14] for the helicity relaxation time τ_H to the presence of a finite vector imbalance.

2 Hydrodynamic description of waves in V/A/H fluids

In this section, we briefly review the anomalous transport in the presence of rotation. In Subsec. 2.1, we review the constitutive equations for the energy-momentum tensor and the charge currents for a Dirac fluid with vector, axial and helical imbalance, undergoing rigid rotation, as seen in the Landau (or energy) frame. We then summarize the treatment of small perturbations around this rigidly rotating state in Subsec. 2.3.

2.1 Landau frame decomposition for slow rotation

Let us consider a Dirac fluid characterized by the vector (V), axial (A) and helicity (H) quantum numbers undergoing rigid rotation with angular velocity Ω about the z axis. In a region close to the rotation axis, where $\rho\Omega \ll 1$, and considering $\beta\Omega \ll 1$, the energy-momentum tensor and charge currents of this fluid can be approximated as

$$T^{\mu\nu} \simeq E_\beta u^\mu u^\nu - P_\beta \Delta^{\mu\nu} + u^\mu W_\beta^\nu + u^\nu W_\beta^\mu, \quad J_\ell^\mu \simeq Q_\ell u^\mu + V_\beta^\mu, \quad (2.1)$$

where $W_\beta^\mu = \sigma_{\varepsilon;\beta}^\omega \omega^\mu$ and $V^\mu = \sigma_{\ell;\beta}^\omega \omega^\mu$ represent the heat flux and diffusion currents in the fluid rest frame, $\omega^\mu = \frac{1}{2} \varepsilon^{\mu\alpha\beta\gamma} u_\alpha \partial_\beta u_\gamma \simeq \Omega \delta^\mu_z$ is the vorticity four-vector and we neglected terms of quadratic or higher order in Ω . In the above expression, $u^\mu \partial_\mu = \Gamma(\partial_t + \Omega \partial_\varphi)$ is the four-velocity of a rigidly-rotating fluid, which defines the so-called β hydrodynamic frame. The energy density E_β is related to the pressure P through the ultrarelativistic equation of state, $E = 3P$. The charge densities Q_ℓ and the heat $\sigma_{\varepsilon;\beta}^\omega$ and charge $\sigma_{\ell;\beta}^\omega$ vortical conductivities, measured in the β frame, can be obtained from the thermodynamic pressure P via

$$Q_\ell = \frac{\partial P}{\partial \mu_\ell}, \quad \sigma_{\varepsilon;\beta}^\omega = Q_A, \quad \sigma_\ell^\omega = \frac{1}{2} \frac{\partial Q_\ell}{\partial \mu_A}. \quad (2.2)$$

To linear order in Ω , one can employ the Landau hydrodynamic frame, by which the four-velocity u_L^μ satisfies $T^\mu{}_\nu u_L^\nu = E u_L^\mu$, with E being the energy density and $T^{\mu\nu}$ the energy-momentum tensor. The Landau and β -frame velocity are related, to leading order in Ω , through $u_L^\mu = u^\mu + \sigma_\varepsilon^\omega \omega^\mu / (E + P)$. Applying a Lorentz boost along the z axis brings the Landau frame velocity to the form corresponding to rigid rotation, such that the energy-momentum tensor and the charge currents read, to leading order in Ω , as follows:

$$T^{\mu\nu} = E u^\mu u^\nu - P \Delta^{\mu\nu}, \quad J_\ell^\mu = Q_\ell u^\mu + \sigma_\ell^\omega \omega^\mu. \quad (2.3)$$

The new Landau-frame vortical conductivities are related to the ones in the β frame via

$$\sigma_\ell^\omega = \sigma_{\ell;\beta}^\omega - \frac{Q_\ell Q_A}{E + P}, \quad (2.4)$$

where the last term represents the contribution from $\sigma_\varepsilon^\omega = Q_A$.

2.2 Non-conserved charges

Our work is concentrated on hydrodynamic systems that possess non-conserved charges associated with axial and helical degrees of freedom. The non-conservation of axial charge is caused by the axial anomaly in both Abelian (like QED) and non-Abelian (QCD) theories.

In the massless quark limit, the anomaly leads to topologically induced dissipation in QCD due to either inter-vacuum tunnelling or thermal, sphaleron-induced processes related to the breaking of the axial symmetry by the axial anomaly [1, 27–29]. The conservation of axial charge is also spoiled by an eventual non-zero fermionic mass, which, however, provides a relatively small effect for light quarks in heavy-ion collisions. In addition, the conservation of helical charge is also violated, both in QED and QCD, through the so-called helicity-violating pair annihilation (HVPA) processes (see Sec. 5.2 in Ref. [26]).

The non-conservation of a charge Q and the corresponding charge current \mathbf{J} is often implemented in the relaxation time approximation:

$$\nabla_\mu J^\mu \equiv \frac{\partial Q}{\partial t} + \nabla \cdot \mathbf{J} = -\frac{Q}{\tau}, \quad (2.5)$$

where the relaxation rate is given by the appropriate relaxation time scale τ . Equation (2.5) breaks Lorentz symmetry, thus indicating that the charge relaxation occurs only in the presence of matter and/or in thermal background.

It is understood that Eq. (2.5) holds in the vicinity of charge neutrality, where $Q \rightarrow \delta Q$ represents a small charge fluctuation around an otherwise uncharged background state. Large (persistent) departures from $Q = 0$ may be possible only in a far-from-equilibrium system, where the hydrodynamic description considered in this paper may not necessarily hold. Indeed, we are relying on a grand-canonical ensemble description of the system, where a chemical potential μ describes charge imbalance, and such a chemical potential only makes thermodynamic sense if the charge Q is (approximately) conserved. Introducing the charge susceptibility, $\chi = \partial Q / \partial \mu$, Eq. (2.5) can be rewritten as

$$\nabla_\nu J^\nu = -\frac{\chi}{\tau} \mu. \quad (2.6)$$

The above prescription applies straightforwardly to the case when the system supports a single charge, which is approximately conserved. In the presence of more than one charge, there are, however, two different ways how to generalize the charge non-conservation equation (2.5) in the relaxation time approximation, depending on whether we describe the chemical force that drives the charge relaxation as either the charge itself or the corresponding chemical potential. The difference between these approaches arises from the fact that in a system with multiple charges, the condition that a given charge Q_ℓ vanishes does not imply that its associated chemical potential disappears, and vice-versa. Therefore, it is important to discriminate between these two pictures of the charge relaxation as they lead to two physically different outcomes.

In the first approach, one assumes that the system relaxes towards a state with vanishing Q_ℓ charge. The susceptibility becomes a matrix,

$$\chi \delta \mu \rightarrow \chi_{\ell\ell'} \delta \mu_{\ell'} + \chi_{\ell T} \delta T, \quad (2.7)$$

where $\chi_{\ell\ell'} = \partial Q_\ell / \partial \mu_{\ell'}$ and $\chi_{\ell T} = \partial Q_\ell / \partial T$. The variation of the charge density in the right-hand side of Eq. (2.5) should be taken with respect to the variation of all chemical potentials that exist in the system, as well as with respect to the temperature. In our

case of the V , A , H triad, the non-conservation (2.5) for the charge current J_ℓ^μ thus gets generalized to the following form:

$$\nabla_\mu J_\ell^\mu = -\frac{1}{\tau_\ell} \left(\sum_{\ell'=V,A,H} \chi_{\ell\ell'} \delta\mu_{\ell'} + \chi_{\ell T} \delta T \right), \quad (\text{no sum over } \ell = V, A, H). \quad (2.8)$$

The conservation of the vector charge is encoded in the infinite vector relaxation time $\tau_V = \infty$. The other two relaxation times, τ_A and τ_H , are, generally, finite quantities. Notice that despite the vector charge being a conserved quantity, in this prescription, its fluctuations affect the conservation of the other two charges so that the sum over ℓ' in the right-hand-side of Eq. (2.8) runs over all three charges in the V , A , H triad. While the proposal in Eq. (2.8) is in principle viable, detailed analysis within our V , A , H system indicates that it leads to instabilities appearing at the level of the linear modes governing hydrodynamic fluctuations (see Appendix A for a more detailed discussion).

Our preferred approach is to work on the basis of the chemical potential, which implies that the dissipation of the charge is controlled not by the magnitude of the charge density in the right-hand side of Eq. (2.5) but rather by the magnitude of the corresponding chemical potential. One argument supporting this approach is based on the Chiral Magnetic Effect, giving rise to the Chiral Magnetic Wave. Indeed, the non-conservation of the axial charge appears due to the presence of the anomalous triangular diagram with the axial-vector-vector (AVV) vertex corners. The anomalous generation of the vector current (one V-corner) along the magnetic field (another V-corner) is given by the same diagram, whose remaining A-corner directly links the magnitude of the generated current with the axial chemical potential and not the axial charge density. In this sense, the densities serve as the auxiliary thermodynamic characteristics of the system. At the same time, the chemical potentials play a primary role in the grand-canonical description of a system, while charge densities are derived quantities. In addition, the fact that a chemical potential for non-conserved charges is not well-defined, in a thermodynamic sense, quantity suggests that a system with non-conserved charges should be equilibrated to a state where such chemical potential is absent. Taking these considerations into account, we propose that Eq. (2.5) should be generalized as follows:

$$\nabla_\mu J_\ell^\mu = -\frac{\chi_\ell}{\tau_\ell} \delta\mu_\ell, \quad (\text{no sum over } \ell = V, A, H), \quad (2.9)$$

which amounts to taking only the diagonal element in Eq. (2.8).

In summary, while the two approaches are identical in the case of a single-charge system [*c.f.* Eqs. (2.5) and (2.6)], they differ for systems that contain more than one charge. Since the charge dissipation is taken into account within an effective charge-relaxation-time formalism, we are free to choose between any of these prescriptions to proceed further. To this end, we notice that the first approach gives rise to unnatural unstable modes, as we point out in Appendix A. We will, therefore, focus our discussion only on the second approach.

2.3 Linear perturbations

Following Ref. [16], we take small perturbations on top of the rigidly-rotating state. For definiteness, we consider only longitudinal perturbations along the vorticity vector, such that $\delta u^\mu = \delta u \delta_z^\mu$. The detailed derivation of the equations for the linear perturbations is presented in Ref. [16], and for the sake of brevity, we do not repeat it here. Instead, we list the results:

$$DE + (E + P)\theta = 0, \quad (2.10a)$$

$$(E + P)Du^\mu - \nabla^\mu P = 0, \quad (2.10b)$$

$$DQ_V + Q_V\theta + \omega^\mu \partial_\mu \sigma_V^\omega + \sigma_V^\omega \partial_\mu \omega^\mu = 0, \quad (2.10c)$$

$$DQ_A + Q_A\theta + \omega^\mu \partial_\mu \sigma_A^\omega + \sigma_A^\omega \partial_\mu \omega^\mu = -\frac{T^2 \mu_A}{3\tau_A}, \quad (2.10d)$$

$$DQ_H + Q_H\theta + \omega^\mu \partial_\mu \sigma_H^\omega + \sigma_H^\omega \partial_\mu \omega^\mu = -\frac{T^2 \mu_H}{3\tau_H}, \quad (2.10e)$$

where $D = u^\mu \partial_\mu$ represents the comoving derivative, $\nabla^\mu = \Delta^{\mu\nu} \partial_\nu$ is the spatial gradient in the fluid rest frame, while $\theta = \partial_\mu u^\mu$ is the expansion scalar. As we already discussed in Sec. 2.2 at the level of the charge currents, all three currents $J_{V/A/H}^\mu$ would be conserved in a quantum field theory of free (non-interacting) massless fermions. This is certainly true also in the quantum case for the vector current when $\partial_\mu J_V^\mu = 0$. Contrary to Ref. [16], in this paper, we take into account the non-conservation of the axial and helical charges in the so-called relaxation-time approximation.

We now consider the Fourier decomposition

$$\bar{f}(z, t) = f(z, t) + \delta \bar{f}(t, z), \quad \delta \bar{f}(t, z) = \int_{-\infty}^{\infty} dk e^{ikz} \sum_{\omega} e^{-i\omega(k)t} \delta f_{\omega}(k), \quad (2.11)$$

where $\omega(k)$ represent the angular frequencies determined by the system of Eqs. (2.10).³

In order for linear perturbations to be applicable, we consider an unpolarized background state in which $\mu_A = \mu_H = 0$. Then, the right-hand sides of Eqs. (2.10d) and (2.10e) are also linear in perturbations. Overall, considering the Fourier mode characterized by k and $\omega(k)$, Eqs. (2.10a) and (2.10b) reduce to

$$-3\omega \delta P_{\omega} + 4k P_{\omega} \delta u_{\omega} = 0, \quad k \delta P_{\omega} - 4\omega P_{\omega} \delta u_{\omega} = 0, \quad (2.12a)$$

while the equations for the charge currents become

$$\sum_{\ell'=V,A,H} \left(\omega \delta_{\ell,\ell'} - k \Omega \frac{\partial \sigma_{\ell'}^\omega}{\partial Q_{\ell'}} \right) \delta Q_{\ell'} - k \Omega \frac{\partial \sigma_{\ell}^\omega}{\partial P} \delta P - (k Q_{\ell} - 2\omega \Omega \sigma_{\ell}^\omega) \delta u = -\frac{i T^2 \delta \mu_{\ell}}{3\tau_{\ell}}, \quad (2.12b)$$

where we set $\tau_V \rightarrow \infty$ as the vector charge is conserved.

³It is important not to confuse the angular frequency $\omega = \omega(k)$, which determines the time evolution of fluctuations (2.11), with the vorticity ω^μ which enters, for example, the system of equations (2.10e) being associated with the local angular velocity of the vortical fluid.

The perturbations in the energy-momentum sector, described by Eq. (2.12a), are not influenced by perturbations in the charges, δQ_ℓ . The non-trivial solutions of Eqs. (2.12a) correspond to the acoustic modes,

$$\omega_{\text{ac.}}^\pm = \pm c_s k, \quad c_s = 1/\sqrt{3}, \quad (2.13)$$

with c_s being the speed of sound in an ultrarelativistic ideal fluid. For the charge modes, Eqs. (2.12a) enforce $\delta P_\omega = \delta u_\omega = 0$, such that Eq. (2.12b) becomes:

$$\left(\omega \delta Q_\ell - k \Omega \delta \sigma_\ell^\omega \right) \Big|_{\delta P=0} = - \frac{i T^2 \delta \mu_\ell}{3 \tau_\ell}. \quad (2.14)$$

The above equation involves the variations δQ_ℓ and $\delta \sigma_\ell^\omega$ of the charge densities and vortical conductivities at constant pressure. Thinking in terms of the grand canonical ensemble parameters, the constraint $\delta P = 0$, together with the thermodynamic relation $\delta P = s \delta T + \sum_\ell Q_\ell \delta \mu_\ell$ leads to the constraint

$$\delta T = - \sum_\ell \frac{Q_\ell}{s} \delta \mu_\ell. \quad (2.15)$$

Subsequently, Eq. (2.14) can be written in terms of fluctuations in the chemical potentials,

$$\mathbb{M}_{\ell\ell'} \delta \mu_{\ell'} = 0, \quad \mathbb{M}_{\ell\ell'} = \omega T^2 \mathbb{M}_{\ell\ell'}^\omega - k \Omega T \mathbb{M}_{\ell\ell'}^\Omega + \frac{i T^2}{3 \tau_\ell} \delta_{\ell\ell'}, \quad (2.16a)$$

where we introduced the following notations:

$$\mathbb{M}_{\ell\ell'}^\omega = \frac{1}{T^2} \left(\frac{\partial Q_\ell}{\partial \mu_{\ell'}} - \frac{3 Q_\ell Q_{\ell'}}{s T} + \frac{Q_{\ell'} \vec{\mu}}{s T} \cdot \frac{\partial Q_\ell}{\partial \vec{\mu}} \right), \quad (2.16b)$$

$$\mathbb{M}_{\ell\ell'}^\Omega = \frac{1}{T} \left(\frac{\partial \sigma_\ell^\omega}{\partial \mu_{\ell'}} - \frac{2 \sigma_\ell^\omega Q_{\ell'}}{s T} + \frac{Q_{\ell'} \vec{\mu}}{s T} \cdot \frac{\partial \sigma_\ell^\omega}{\partial \vec{\mu}} \right). \quad (2.16c)$$

2.4 Unpolarized background

The pressure corresponding to an ensemble of non-interacting, massless fermions and anti-fermions with polarizations $\lambda = \pm 1/2$ is given by [14, 16]:

$$P = - \frac{T^4}{\pi^2} \sum_{\sigma, \lambda} \text{Li}_4(-e^{\mu_{\sigma, \lambda}/T}), \quad (2.17)$$

where $\text{Li}_s(z) = \sum_{n=1}^{\infty} z^n/n^s$ is the polylogarithm function. In the above, $\mu_{\sigma, \lambda}$ is the effective chemical potential arising from vector, axial and helical imbalance,

$$\mu_{\sigma, \lambda} = \vec{q}_{\sigma, \lambda} \cdot \vec{\mu} = \sigma \mu_V + 2 \lambda \mu_A + 2 \sigma \lambda \mu_H, \quad (2.18)$$

with $\vec{q}_{\sigma, \lambda}$ collecting the vector, axial and helical charges of a fermion ($\sigma = 1$) or antifermion ($\sigma = -1$) with polarization $\lambda = \pm 1/2$, namely

$$q_{\sigma, \lambda}^V = \sigma, \quad q_{\sigma, \lambda}^A = 2 \lambda, \quad q_{\sigma, \lambda}^H = 2 \lambda \sigma. \quad (2.19)$$

The charge densities and vortical conductivities in the β frame read

$$Q_\ell = \frac{\partial P}{\partial \mu_\ell} = -\frac{T^3}{\pi^2} \sum_{\sigma, \lambda} q_{\sigma, \lambda}^\ell \text{Li}_3(-e^{\mu_{\sigma, \lambda}/T}), \quad (2.20)$$

$$\sigma_\ell^\omega = \frac{1}{2} \frac{\partial Q_\ell}{\partial \mu_A} = -\frac{T^2}{2\pi^2} \sum_{\sigma, \lambda} 2\lambda q_{\sigma, \lambda}^\ell \text{Li}_2(-e^{\mu_{\sigma, \lambda}/T}). \quad (2.21)$$

Let us now specialize the above expressions to the case of an unpolarized background state characterized by $\mu_A = \mu_H = 0$. Using the following properties of the polylogarithm function,

$$\text{Li}_4(-e^\alpha) + \text{Li}_4(-e^{-\alpha}) = -\frac{7\pi^4}{360} - \frac{\pi^2 \alpha^2}{12} - \frac{\alpha^4}{24}, \quad (2.22)$$

$$\text{Li}_3(-e^\alpha) - \text{Li}_3(-e^{-\alpha}) = -\frac{\pi^2 \alpha}{6} - \frac{\alpha^3}{6}, \quad (2.23)$$

$$\text{Li}_2(-e^a) + \text{Li}_2(-e^{-a}) = -\frac{\pi^2}{6} - \frac{a^2}{2}, \quad (2.24)$$

we find

$$\begin{aligned} P &= \frac{7\pi^2 T^4}{180} + \frac{\mu_V^2 T^2}{6} + \frac{\mu_V^4}{12\pi^2}, & Q_V &= \frac{\mu_V T^2}{3} + \frac{\mu_V^3}{3\pi^2}, \\ \sigma_H^\omega &= \frac{T^2}{\pi^2} [\text{Li}_2(-e^{-\mu_V/T}) - \text{Li}_2(-e^{\mu_V/T})], & \sigma_A^\omega &= \frac{T^2}{6} + \frac{\mu_V^2}{2\pi^2}, \end{aligned} \quad (2.25)$$

while $Q_A = Q_H = \sigma_V^\omega = 0$. Since $Q_A = 0$, the vortical conductivities in the Landau frame coincide with the corresponding conductivities in the β frame:

$$\sigma_\ell^\omega = \sigma_{\ell; \beta}^\omega - \frac{Q_A Q_\ell}{E + P} = \sigma_{\ell; \beta}^\omega. \quad (2.26)$$

We now divide Eq. (2.16) by T^2 and obtain

$$\left[\omega \mathbb{M}_{\ell\ell'}^\omega - \kappa_\Omega \mathbb{M}_{\ell\ell'}^\Omega + \frac{i}{3\tau_\ell} \delta_{\ell, \ell'} \right] \delta \mu_{\ell'} = 0, \quad (2.27)$$

where we introduced the parameter

$$\kappa_\Omega = \frac{k\Omega}{T}, \quad (2.28)$$

which has a sense of a normalized momentum. Taking into account the following expressions for the derivatives of the charge densities and vortical conductivities with respect to the chemical potentials,

$$\frac{\partial Q_\ell}{\partial \mu_{\ell'}} \Big|_{\mu_A = \mu_H = 0} = 2 \begin{pmatrix} \sigma_A^\omega & 0 & 0 \\ 0 & \sigma_A^\omega & \sigma_H^\omega \\ 0 & \sigma_H^\omega & \sigma_A^\omega \end{pmatrix}, \quad \frac{\partial \sigma_{\ell; \beta}^\omega}{\partial \mu_{\ell'}} = \frac{1}{\pi^2} \begin{pmatrix} 0 & \mu_V & TL \\ \mu_V & 0 & 0 \\ TL & 0 & 0 \end{pmatrix}, \quad (2.29)$$

with $L = 2 \ln \left(2 \cosh \frac{\mu_V}{2T} \right)$, we find for \mathbb{M}_ω and \mathbb{M}_Ω the following expressions [16]:

$$\mathbb{M}_\omega = \frac{2}{T^2} \begin{pmatrix} \sigma_A^\omega - \frac{T^2}{3} \Delta H & 0 & 0 \\ 0 & \sigma_A^\omega & \sigma_H^\omega \\ 0 & \sigma_H^\omega & \sigma_A^\omega \end{pmatrix}, \quad \mathbb{M}_\Omega = \begin{pmatrix} 0 & \frac{1}{H} A & \frac{1}{H} B \\ A & 0 & 0 \\ B & 0 & 0 \end{pmatrix}. \quad (2.30)$$

In the above, $\Delta H = H - 1$ with $H = (e + P)/sT = 1 + \mu_V Q_V/(sT)$, while A and B are defined as

$$A = \frac{\alpha_V}{\pi^2} - \frac{Q_V}{3s}, \quad B = \frac{HL}{\pi^2} - \frac{2Q_V}{sT^2} \sigma_H^\omega. \quad (2.31)$$

For future convenience, we list here the large-temperature behaviour of the above functions when $\alpha_V = \mu_V/T$ is small:

$$\begin{aligned} \sigma_A^\omega &= \frac{T^2}{6} + \frac{\mu_V^2}{2\pi^2}, & \sigma_H^\omega &= \frac{2 \ln 2}{\pi^2} T^2 \alpha_V + O(\alpha_V^3), & H &= 1 + \frac{15\alpha_V^2}{7\pi^2} + O(\alpha_V^4), \\ L &= 2 \ln 2 + O(\alpha_V^2), & A &= \frac{2\alpha_V}{7\pi^2} + O(\alpha_V^3), & B &= \frac{2 \ln 2}{\pi^2} + \frac{7\pi^2 - 120 \ln 2}{28\pi^4} \alpha_V^2 + O(\alpha_V^4). \end{aligned} \quad (2.32)$$

At large chemical potential, when $|\alpha_V| \rightarrow \infty$, we have

$$\begin{aligned} H &= \frac{\alpha_V^2}{\pi^2} \left(1 + \frac{23\pi^2}{15\alpha_V^2} + O(\alpha_V^{-4}) \right), & A &= \frac{2\alpha_V}{3\pi^2} \left(1 - \frac{4\pi^2}{15\alpha_V^2} + O(\alpha_V^{-4}) \right), \\ \frac{4s_V Q_V}{\pi^2 s} &= \frac{4s_V}{\pi^2 \alpha_V} (H - 1) \simeq \frac{4s_V \alpha_V}{\pi^4} \left(1 + \frac{8\pi^2}{15\alpha_V^2} + O(\alpha_V^{-4}) \right). \end{aligned} \quad (2.33a)$$

In what concerns the quantities L , B and σ_H , their large- α_V behaviour receives exponentially-damped corrections of the form $e^{-|\alpha_V|}$, which can be accessed taking into account the sign $s_V = \text{sgn}(\mu_V)$ of μ_V . Then, one obtains trivially:

$$L \simeq |\alpha_V| + 2e^{-|\alpha_V|}, \quad B - s_V A \simeq \frac{2}{\pi^2} \left(H + \frac{2s_V Q_V}{s} \right) e^{-|\alpha_V|}, \quad \sigma_A^\omega - s_V \sigma_H^\omega \simeq \frac{2T^2}{\pi^2} e^{-|\alpha_V|}. \quad (2.33b)$$

In order to find the angular frequencies supported by Eq. (2.30), we must solve the equation $\det \mathbb{M} = 0$. Writing

$$\frac{1}{T^2} \mathbb{M} = \omega \mathbb{M}_\omega - \kappa_\Omega \mathbb{M}_\Omega + \frac{i}{3\tau_A} \mathbb{I}_A + \frac{i}{3\tau_H} \mathbb{I}_H, \quad (2.34)$$

with $\mathbb{I}_{\ell\ell'}^A = \delta_{\ell A} \delta_{\ell' A}$ and $\mathbb{I}_{\ell\ell'}^H = \delta_{\ell H} \delta_{\ell' H}$, one can obtain $\det \left(\frac{1}{T^2} \mathbb{M} \right)$ as

$$\begin{aligned} \det \left(\frac{1}{T^2} \mathbb{M} \right) &= \det(\omega \mathbb{M}_\omega - \kappa_\Omega \mathbb{M}_\Omega) + \frac{i}{3\tau_A} \begin{vmatrix} \frac{2\omega}{T^2} \left(\sigma_A^\omega - \frac{T^2}{3} \Delta H \right) & -\frac{\kappa_\Omega}{H} B \\ -\kappa_\Omega B & \frac{2\omega}{T^2} \sigma_A^\omega \end{vmatrix} \\ &+ \frac{i}{3\tau_H} \begin{vmatrix} \frac{2\omega}{T^2} \left(\sigma_A^\omega - \frac{T^2}{3} \Delta H \right) & -\frac{\kappa_\Omega}{H} A \\ -\kappa_\Omega A & \frac{2\omega}{T^2} \sigma_A^\omega \end{vmatrix} - \frac{2\omega}{9T^2 \tau_A \tau_H} \left(\sigma_A^\omega - \frac{T^2}{3} \Delta H \right) = 0. \end{aligned} \quad (2.35)$$

The first term evaluates to

$$\det(\omega \mathbb{M}_\omega - \kappa_\Omega \mathbb{M}_\Omega) = \left(\frac{2\omega}{T^2}\right)^3 \left(\sigma_A^\omega - \frac{T^2}{3}\Delta H\right) [(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2] - \frac{2\omega\kappa_\Omega^2}{HT^2} [(A^2 + B^2)\sigma_A^\omega - 2AB\sigma_H^\omega]. \quad (2.36)$$

Solving the other 2×2 determinants finally gives

$$\begin{aligned} \det\left(\omega \mathbb{M}_\omega - \kappa_\Omega \mathbb{M}_\Omega + \frac{i}{3\tau_H} \mathbb{I}_H + \frac{i}{3\tau_A} \mathbb{I}_A\right) \\ = \frac{2\omega}{T^2} \left\{ \left(\frac{2\omega}{T^2}\right)^2 \left(\sigma_A^\omega - \frac{T^2}{3}\Delta H\right) [(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2] - \frac{\kappa_\Omega^2}{H} [(A^2 + B^2)\sigma_A^\omega - 2AB\sigma_H^\omega] \right\} \\ + \frac{i}{3} \left[\left(\frac{2\omega}{T^2}\right)^2 \sigma_A^\omega \left(\sigma_A^\omega - \frac{T^2}{3}\Delta H\right) \left(\frac{1}{\tau_H} + \frac{1}{\tau_A}\right) - \frac{\kappa_\Omega^2}{H} \left(\frac{A^2}{\tau_H} + \frac{B^2}{\tau_A}\right) \right] \\ - \frac{2\omega}{9T^2\tau_A\tau_H} \left(\sigma_A^\omega - \frac{T^2}{3}\Delta H\right) = 0. \quad (2.37) \end{aligned}$$

Given a particular solution ω_* to the previous equation, i.e. a collective mode of the system, the perturbations of the chemical potential are related through $\mathbb{M}_{\ell\ell'}\delta\mu_{\ell'} = 0$. Specifically, solving the second and third rows ($\ell = A, H$) gives:

$$\begin{aligned} \delta\mu_A^* &= \frac{\left[\frac{2\omega_*}{T^2}(A\sigma_A^\omega - B\sigma_H^\omega) + \frac{i}{3\tau_H}A\right] \kappa_\Omega \delta\mu_V^*}{\left(\frac{2\omega_*}{T^2}\right)^2 [(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2] + \frac{2i\omega_*}{3T^2}\sigma_A^\omega \left(\frac{1}{\tau_A} + \frac{1}{\tau_H}\right) - \frac{1}{9\tau_A\tau_H}}, \\ \delta\mu_H^* &= \frac{\left[\frac{2\omega_*}{T^2}(B\sigma_A^\omega - A\sigma_H^\omega) + \frac{i}{3\tau_A}B\right] \kappa_\Omega \delta\mu_V^*}{\left(\frac{2\omega_*}{T^2}\right)^2 [(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2] + \frac{2i\omega_*}{3T^2}\sigma_A^\omega \left(\frac{1}{\tau_A} + \frac{1}{\tau_H}\right) - \frac{1}{9\tau_A\tau_H}}. \quad (2.38) \end{aligned}$$

The previous solutions are valid so long as the denominator does not vanish for the particular mode ω_* . Looking ahead, we note that we will encounter the opposite situation in Subsec. 3.2). Then, it is more convenient to solve $\mathbb{M}_{\ell\ell'}\delta\mu_{\ell'} = 0$ for the first and second rows ($\ell = V, A$) and take $\delta\mu_H$ as the reference amplitude. The relations between the amplitudes are given by:

$$\begin{aligned} \delta\mu_V^* &= \frac{\left[\frac{2\omega_*}{T^2}(A\sigma_H^\omega - B\sigma_A^\omega) - \frac{i}{3\tau_A}B\right] \kappa_\Omega \delta\mu_H^*}{A^2\kappa_\Omega^2 - \frac{2\omega_*}{T^2}H\left(\sigma_A - \frac{1}{3}T^2\Delta H\right)\left(\frac{2\omega_*}{T^2}\sigma_A^\omega + \frac{i}{3\tau_A}\right)}, \\ \delta\mu_A^* &= \frac{\left[\left(\frac{2\omega_*}{T^2}\right)^2 H\left(\sigma_A^\omega - \frac{1}{3}T^2\Delta H\right)\sigma_H^\omega - AB\kappa_\Omega^2\right] \delta\mu_H^*}{A^2\kappa_\Omega^2 - \frac{2\omega_*}{T^2}H\left(\sigma_A - \frac{1}{3}T^2\Delta H\right)\left(\frac{2\omega_*}{T^2}\sigma_A^\omega + \frac{i}{3\tau_A}\right)}. \quad (2.39) \end{aligned}$$

We stress that Eqs. (2.38) and (2.39) are equivalent for a given mode ω_* provided the denominators do not vanish.

Before ending this section, we quote from Ref. [16] the solutions for the angular velocities satisfying Eq. (2.37) in the case of conserved axial and helical charges, corresponding to the $\tau_A, \tau_H \rightarrow \infty$ limit:

$$\omega_{\text{im.}} = 0, \quad \omega_h^\pm = \pm \frac{\kappa_\Omega T^2}{2} \sqrt{\frac{\sigma_A^\omega(A^2 + B^2) - 2AB\sigma_H^\omega}{H(\sigma_A^\omega - \frac{T^2}{3}\Delta H)[(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2]}}. \quad (2.40)$$

The mode $\omega_{\text{im.}} = 0$ corresponds to a non-propagating mode for which the vector perturbation does not fluctuate, $\delta\mu_V^{\text{im.}} = 0$. We added the “im.” subscript to refer to this mode, as in the presence of a finite helicity relaxation time, this mode develops a non-vanishing imaginary part. Instead, the axial and helical perturbations are linked through

$$A\delta\mu_A^{\text{im.}} + B\delta\mu_H^{\text{im.}} = 0. \quad (2.41)$$

The other two modes make up the helical vortical wave. Taking $\delta\mu_V^\pm$ as the reference fluctuation amplitude, the axial and helical amplitudes become

$$\delta\mu_A^{h;\pm} = \frac{T^2(A\sigma_A^\omega - B\sigma_H^\omega)\kappa_\Omega}{2\omega_h^\pm[(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2]}\delta\mu_V^{h;\pm}, \quad \delta\mu_H = \frac{T^2(B\sigma_A^\omega - A\sigma_H^\omega)\kappa_\Omega}{2\omega_h^\pm[(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2]}\delta\mu_V^{h;\pm}. \quad (2.42)$$

In Sec. 3, we shall study Eqs. (2.37) and (2.38) for the cases when $0 < \tau_H < \infty$ (the helical charge has a finite relaxation time and thus it is not conserved) and $\tau_A \rightarrow \infty$ (the axial charge is conserved). The case with finite τ_A will be studied in Sec. 4. In both cases, we will take various limits, such as the high temperature or the degenerate gas limit, discussing the effect of charge non-conservation on the wave spectrum uncovered in the case of the VAH plasma with conserved charges, discussed in Ref. [16].

3 Non-conserved helical charge

In this section, we take into account the non-conservation of the helical charge due to local scattering processes. At the same time, we keep not just the vector but also the axial charge conserved, which amounts to letting $\tau_A \rightarrow \infty$. This situation can be encountered, e.g., in a QED (electron-positron) plasma in the absence of external electromagnetic fields, when the axial anomaly vanishes, and the axial current is conserved both at the classical and quantum levels.

As in the previous section, we will consider a plasma with vanishing helical and axial background (mean) chemical potentials. The assumption of vanishing helical chemical potential is justified by the fact that at finite values of τ_H , the violation of the conservation of helical charge prevents introducing the helical chemical as a Lagrange multiplier in the grand canonical ensemble. The assumption of the vanishing μ_A background is not imposed physically but is technically convenient for the simplified analysis. The generalization of our discussion in this section to the case of finite μ_A is relatively straightforward. All in all, Eq. (2.9) becomes

$$\nabla_\mu J_V^\mu = 0, \quad \nabla_\mu J_A^\mu = 0, \quad \nabla_\mu J_H^\mu = -\frac{T^2}{3\tau_H}\delta\mu_H. \quad (3.1)$$

Physically, the non-conservation of helicity in QED arises from helicity-violating pair-annihilation processes of the form $e_R^+ e_L^- \rightarrow e_L^+ e_R^-$. Each such process violates helicity by two units while preserving chirality. Analogous processes are mediated by gluon interchange in QCD. For small helical imbalance, the helical relaxation time arising from such processes was estimated for QCD in the weakly-interacting regime as [14]⁴

$$\tau_H^{\text{QCD}} \simeq \left(\frac{250 \text{ MeV}}{k_B T} \right) \left(\frac{1}{\alpha_{\text{QCD}}} \right) \left(\frac{2}{N_f} \right) \times 4.8 \text{ fm}/c. \quad (3.2)$$

An extension of the above expression to the case of finite background vector chemical potential is provided in Appendix B.

In the case $\tau_A \rightarrow \infty$, the matrix \mathbb{M}/T^2 in Eq. (2.34) reduces to

$$\frac{1}{T^2} \mathbb{M} = \omega \mathbb{M}_\omega - \kappa_\Omega \mathbb{M}_\Omega + \frac{i}{3\tau_H} \mathbb{I}_H, \quad (3.3)$$

while the Eq. (2.37) showing $\det(\mathbb{M}/T^2) = 0$ becomes

$$\begin{aligned} \frac{2\omega}{T^2} \left\{ \left(\frac{2\omega}{T^2} \right)^2 \left(\sigma_A^\omega - \frac{T^2}{3} \Delta H \right) [(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2] - \frac{\kappa_\Omega^2}{H} [(A^2 + B^2) \sigma_A^\omega - 2AB \sigma_H^\omega] \right\} \\ + \frac{i}{3\tau_H} \left[\left(\frac{2\omega}{T^2} \right)^2 \sigma_A^\omega \left(\sigma_A^\omega - \frac{T^2}{3} \Delta H \right) - \frac{\kappa_\Omega^2 A^2}{H} \right] = 0. \end{aligned} \quad (3.4)$$

The mode amplitudes in Eq. (2.38) reduce to

$$\delta\mu_A^* = \frac{\left(A\sigma_A^\omega - B\sigma_H^\omega + \frac{iT^2 A}{6\omega_* \tau_H} \right) \kappa_\Omega \delta\mu_V^*}{\frac{2\omega_*}{T^2} [(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2] + \frac{i}{3\tau_H} \sigma_A^\omega}, \quad \delta\mu_H^* = \frac{(B\sigma_A^\omega - A\sigma_H^\omega) \kappa_\Omega \delta\mu_V^*}{\frac{2\omega_*}{T^2} [(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2] + \frac{i}{3\tau_H} \sigma_A^\omega}. \quad (3.5)$$

As opposed to the idealised case when all charges are conserved, corresponding to the limit $\tau_A, \tau_H \rightarrow \infty$ and summarized in Eqs. (2.40)–(2.42), the determinant

$$\det \left(-\kappa_\Omega \mathbb{M}_\Omega + \frac{i}{3\tau_H} \mathbb{I}_H \right) = -\frac{i}{3H\tau_H} \kappa_\Omega^2 A^2 \quad (3.6)$$

no longer vanishes and the trivial mode $\omega_{\text{im}} = 0$ is no longer a solution of the equation $\det(\mathbb{M}) = 0$. Hence, all three modes are nontrivial. In the following subsections, we will analyze the properties of the angular frequencies ω in the following special cases: the small chemical potential limit (Subsec. 3.1), the small τ_H limit (when the helicity degree of freedom is frozen, see Subsec. 3.2) and the large chemical potential limit (Subsec. 3.3). Subsec. 3.4 presents our general conclusions on the modes spectrum for the considered system.

⁴The numerical value quoted here differs from the one reported in Ref. [14] due to a factor of two error when going from Eq. (160) to Eq. (167) – see Erratum [30] for details.

3.1 Small vector chemical potential limit

Considering the normalized chemical potential $\alpha_V = \mu_V/T$ a small quantity, we seek a power series solution of the energy dispersion ω given as

$$\omega = \omega_0 + \omega_1 \alpha_V + \omega_2 \alpha_V^2 + O(\alpha_V^3). \quad (3.7)$$

We now expand $\det(\mathbb{M})$ in powers of α_V and equate it recursively to zero, repeating the steps order by order with respect to α_V . To find ω_0 , we set $\mu_V = 0$ in Eq. (3.4) and take the leading-order term in the α_V expansion shown in Eq. (2.32), arriving at

$$\frac{\omega_0}{27} \left[\omega_0 \left(\frac{i}{\tau_H} + \omega_0 \right) - k^2 c_h^2 \right] = 0, \quad (3.8)$$

where c_h represents the velocity of the helical vortical wave in a neutral, non-dissipating plasma,

$$c_h = \frac{6 \ln 2}{\pi^2} \frac{\Omega}{T}. \quad (3.9)$$

We identify the mode corresponding to $\omega_0 = 0$ as the axial vortical mode, ω_a , while the helical vortical modes ω_h^\pm have

$$\omega_{h;0}^\pm = -\frac{i}{2\tau_H} \pm k c_h \sqrt{1 - \frac{k_{\text{th}}^2}{k^2}}, \quad k_{\text{th}} = \frac{1}{2\tau_H c_h} = \frac{\pi^2}{12 \ln 2} \frac{T}{\Omega \tau_H}. \quad (3.10)$$

Notice that we work with a dissipative (lossy) medium so that the apparent (in fact, infinite) superluminal group velocity $v_{\text{gr.}} = \partial \text{Re}[\omega(k)]/\partial k$ at the merging point $k = k_{\text{th}}$ does not violate causality as long as the asymptotic causality condition at large momentum, $k \rightarrow \infty$, is fulfilled [31].

The above solution indicates that the HVW propagates only when the wavenumber k exceeds the threshold value k_{th} given by the second relation in Eq. (3.10). At wavenumbers lower than k_{th} , the modes ω_h^\pm are purely dissipative.

At first order with respect to α_V , Eq. (3.4) reads

$$\frac{\omega_1}{27} \left[\omega_0 \left(\frac{2i}{\tau_H} + 3\omega_0 \right) - k^2 c_h^2 \right] = 0. \quad (3.11)$$

This condition is valid for both the axial and the vector-helical modes. For the axial mode, $\omega_{a;0} = 0$ and the square brackets evaluate to $-k^2 c_h^2 \neq 0$. For the vector-helical modes, $\omega_{h;0}^\pm (i\tau_H^{-1} + \omega_{h;0}^\pm) = k^2 c_h^2$ and the square brackets evaluate to $2k^2 c_h^2 - i\tau_H^{-1} \omega_{h;0}^\pm$, being non-vanishing unless $k = k_{\text{th}}$. We thus conclude that the first-order contributions to the velocities vanish, $\omega_{a;1} = \omega_{h;1}^\pm = 0$, for both the axial and the helical-vortical modes. Thus, the axial mode is of second order with respect to α_V . To find its expression, we expand Eq. (3.4) to second order, while considering $\omega \rightarrow \omega_a = \omega_{a;2} \alpha_V^2$:

$$-\frac{4}{3} \left(\frac{\kappa_\Omega \ln 2}{\pi^2} \right)^2 \left[\omega_{a;2} + \frac{i}{49\tau_H (\ln 2)^2} \right] = 0 \quad \Rightarrow \quad \omega_{a;2} = -\frac{i}{49\tau_H (\ln 2)^2}. \quad (3.12)$$

Thus, the axial mode is purely dissipative. In the limit $\tau_H \rightarrow \infty$ when helicity is conserved, it is clear that $\omega_{a;2} \rightarrow 0$ and indeed, $\omega_a = 0$ at any order. This observation allows us to

identify the mode of Eq. (3.12) with the mode ω_{im} , obtained in the case when all charges are conserved as it was summarized in Eq. (2.40).

The relation between the fluctuations amplitudes can be found by taking the large-temperature expansion of the expressions in Eq. (3.5), using Eqs. (2.32). In the case of the axial mode, we take $\delta\mu_A^a$ as the reference amplitude and obtain the following relations between the fluctuations of the chemical potentials in this mode

$$\delta\mu_V^a \simeq -\frac{i}{k\tau_H c_h} \frac{\alpha_V}{7 \ln 2} \delta\mu_A^a, \quad \delta\mu_H^a \simeq -\frac{\alpha_V}{7 \ln 2} \delta\mu_A^a. \quad (3.13)$$

Hereafter, the approximate relation “ \simeq ” implies that the relation is valid in the leading order in α_V with higher-order corrections omitted.

For the vector-helical modes, we obtain

$$\begin{aligned} \delta\mu_H^{h;\pm} &\simeq \frac{\omega_{h;0}^\pm}{k c_h} \delta\mu_V^{h;\pm} = \left(-\frac{i k_{\text{th}}}{k} \pm c_h \sqrt{k^2 - k_{\text{th}}^2} \right) \delta\mu_V^{h;\pm}, \\ \delta\mu_A^{h;\pm} &\simeq \left(\frac{6\kappa_\Omega}{7\pi^2 \omega_{h;0}^\pm} - \frac{2\omega_{h;0}^\pm}{\kappa_\Omega} \right) \alpha_V \delta\mu_V^\pm \\ &= \left[\frac{i}{\tau_H} \left(\frac{\pi^2}{84(\ln 2)^2} + 1 \right) \pm 2c_h \sqrt{k^2 - k_{\text{th}}^2} \left(\frac{\pi^2}{84(\ln 2)^2} - 1 \right) \right] \frac{\alpha_V}{\kappa_\Omega} \delta\mu_V^\pm. \end{aligned} \quad (3.14)$$

The inverse proportionality between $\delta\mu_V^a$ and $k/k_{\text{th}} = 2c_h \tau_H k$ implied by Eq. (3.13) reveals the breakdown of the small chemical potential expansion, considered in this section, at large wavelengths or small τ_H . This effect appears due to the finite helical relaxation time τ_H , which changes the low- k wave spectrum significantly compared to the case of conserved charges. We will address the $k \rightarrow 0$ limit (or, equivalently, the small τ_H limit) in the following subsection.

Let us now consider as the initial state a harmonic perturbation in the vector charge:

$$\bar{\mu}_V(0, z) = \mu_V + \delta\mu_V \cos(kz), \quad \bar{\mu}_A(0, z) = \bar{\mu}_H(0, z) = 0. \quad (3.15)$$

To leading order in α_V , this setup assumes that the amplitudes of the axial mode vanish, $\delta\mu_\ell^a \simeq 0$. The amplitudes $\delta\mu_V^{h;\pm}$ and $\delta\mu_H^{h;\pm}$ corresponding to the vector-helical modes satisfy

$$\delta\mu_V^{h;\pm} \simeq \frac{\delta\mu_V}{2} \left(1 \pm \frac{i k_{\text{th}}}{\sqrt{k^2 - k_{\text{th}}^2}} \right), \quad \delta\mu_H^{h;\pm} \simeq \pm \frac{k^2 \tau_H c_h \delta\mu_V}{2 \sqrt{k^2 - k_{\text{th}}^2}}, \quad (3.16)$$

while $\delta\mu_A^{h;\pm}$ is proportional to α_V , as shown in Eq. (3.14). The full solution propagates as

$$\begin{aligned} \delta\bar{\mu}_V(t, z) &\simeq \delta\mu_V e^{-t/2\tau_H} \cos(kz) \left[\cos \left(c_h t \sqrt{k^2 - k_{\text{th}}^2} \right) + \frac{\sin \left(c_h t \sqrt{k^2 - k_{\text{th}}^2} \right)}{2\tau_H c_h \sqrt{k^2 - k_{\text{th}}^2}} \right], \\ \delta\bar{\mu}_H(t, z) &\simeq \frac{k^2 \tau_H c_h \delta\mu_V}{\sqrt{k^2 - k_{\text{th}}^2}} e^{-t/2\tau_H} \cos(kz) \sin \left(c_h t \sqrt{k^2 - k_{\text{th}}^2} \right). \end{aligned} \quad (3.17)$$

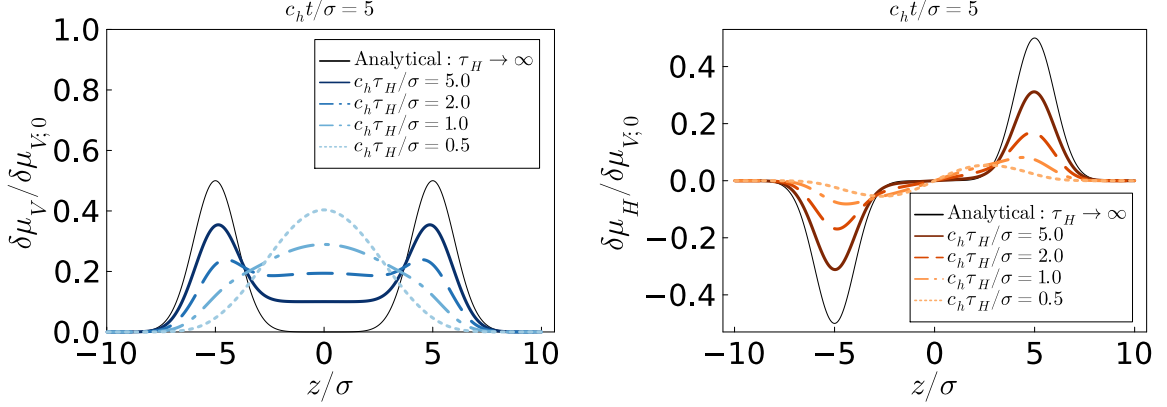


Figure 1: Effect of the helicity relaxation time τ_H on the propagation of the initial Gaussian perturbation of the vector charge, shown in Eq. (3.19), in the case of conserved axial charge ($\tau_A \rightarrow \infty$). The profiles correspond to various values of τ_H and are represented with respect to z/σ , where σ is the Gaussian width. The background fluid is neutral and unpolarized ($\mu_V = \mu_A = \mu_H = 0$), and the temperature and angular velocity are set for definiteness to $T = 300$ MeV and $\Omega = 6.6$ MeV, respectively.

In the infrared, when $k < k_{\text{th}} = 1/2\tau_H c_h$, the square root becomes imaginary, $\sqrt{k^2 - k_{\text{th}}^2} = i\sqrt{k_{\text{th}}^2 - k^2}$, and the trigonometric functions become hyperbolic ones. In the late-time limit, we have

$$\begin{aligned} \delta\bar{\mu}_V \Big|_{k < k_{\text{th}}} &\rightarrow \frac{\delta\mu_V}{2} \left(1 + \frac{k_{\text{th}}}{\sqrt{k_{\text{th}}^2 - k^2}} \right) e^{-\frac{t}{2\tau_H}(1 - \sqrt{1 - k^2/k_{\text{th}}^2})} \cosh(kz), \\ \delta\bar{\mu}_H \Big|_{k < k_{\text{th}}} &\rightarrow \frac{\delta\mu_V k^2 \tau_H c_h}{2\sqrt{k_{\text{th}}^2 - k^2}} e^{-\frac{t}{2\tau_H}(1 - \sqrt{1 - k^2/k_{\text{th}}^2})} \cosh(kz). \end{aligned} \quad (3.18)$$

It can be seen that the modes with $k \rightarrow 0$ suffer negligible damping. When $k > k_{\text{th}} = 1/2\tau_H c_h$, normal propagation resumes, and all modes are damped according to the factor $e^{-t/2\tau_H}$.

Let us now consider an initial Gaussian perturbation of the vector charge:

$$\bar{\mu}_V(0, z) = \mu_V + \delta\mu_V e^{-z^2/2\sigma^2}, \quad \bar{\mu}_A(0, z) = \bar{\mu}_H(0, z) = 0. \quad (3.19)$$

Considering a neutral background state with $\mu_V = 0$, the time evolution of this configuration can be expressed in integral form as

$$\begin{aligned} \bar{\mu}_V(t, z) &\simeq \delta\mu_V e^{-t/2\tau_H} \sigma \sqrt{2\pi} \int_{-\infty}^{\infty} dk e^{-\frac{1}{2}\sigma^2 k^2 + ikz} \\ &\times \left[\cos\left(c_h t \sqrt{k^2 - k_{\text{th}}^2}\right) + \frac{\sin\left(c_h t \sqrt{k^2 - k_{\text{th}}^2}\right)}{2\tau_H c_h \sqrt{k^2 - k_{\text{th}}^2}} \right]. \end{aligned} \quad (3.20)$$

In the limit when the helicity charge is conserved, $\tau_H \rightarrow \infty$, the square brackets on the second line of Eq. (3.20) evaluate to $\cos(c_h kt)$ and the integral with respect to k can be performed analytically (see also Eq. (3.36) of Ref. [16]):

$$\begin{aligned}\lim_{\tau_H \rightarrow \infty} \delta\bar{\mu}_V(t, z) &= \frac{\delta\bar{\mu}_{V;0}}{2} \left[e^{-(z-c_h t)^2/2\sigma^2} + e^{-(z+c_h t)^2/2\sigma^2} \right], \\ \lim_{\tau_H \rightarrow \infty} \delta\bar{\mu}_H(t, z) &= \frac{\delta\bar{\mu}_{V;0}}{2} \left[e^{-(z-c_h t)^2/2\sigma^2} - e^{-(z+c_h t)^2/2\sigma^2} \right].\end{aligned}\quad (3.21)$$

In Fig. 1, we represented $\delta\bar{\mu}_V(t, z)$, $\delta\bar{\mu}_H(t, z)$ and $\delta\bar{\mu}_A(t, z)$ with respect to z/σ , when $c_h t/\sigma = 5$, for the case of a neutral background ($\mu_V = \mu_A = \mu_H = 0$) and temperature $T = 300$ MeV. We considered various values of $c_h \tau_H/\sigma$. It can be seen that, compared to the case when the helicity charge is conserved and $\tau_H \rightarrow \infty$ (shown with the black line), the propagation features are damped as τ_H is decreased. At the lowest value of $c_h \tau_H/\sigma = 0.5$, $\delta\mu_V(t, z)$ simply decays on the time scale given by τ_H .

3.2 Small τ_H limit: Recovering the CVW

Equations (3.13) and (3.14) of the previous subsection show that it is problematic to take the $k, \tau_H \rightarrow 0$ limit from the large-temperature expansion. In order to access the small $\tau_H \kappa_\Omega$ regime, we expand the quantity $w = \tau_H \omega$ and $\delta\mu_\ell$ in powers of τ_H :

$$w = \tau_H \omega = w_0 + w_1 \tau_H + \dots, \quad \delta\mu_\ell = \delta\mu_{\ell;0} + \delta\mu_{\ell;1} \tau_H + \dots \quad (3.22)$$

It is convenient to work with w as in the $\tau_H \rightarrow 0$ limit, we can expect ω to have a τ_H^{-1} leading-order contribution, see e.g. Eq. (3.10). Additionally, from Eq. (3.3) we observe that w is a function of the combination $\kappa_\Omega \tau_H$ and consequently the small τ_H expansion naturally encapsulates the results for small wavenumbers.

Multiplying Eq. (3.4) by τ_H^3 , we get to zeroth order in τ_H :

$$\left(\frac{2w_0}{T^2} \right)^2 \left(\sigma_A^\omega - \frac{T^2}{3} \Delta H \right) \left(\frac{i}{3} \sigma_A^\omega + \frac{2w_0}{T^2} [(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2] \right) = 0, \quad (3.23)$$

which gives as solutions $w_{a;0}^\pm = 0$ as a double root (we will see shortly that these are axial modes, as they involve the axial chemical potential fluctuations), as well as the solution

$$w_{h;0} = -\frac{iT^2 \sigma_A^\omega}{6[(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2]}. \quad (3.24)$$

The above corresponds to a purely dissipative helical mode, with $w_{h;0} = -i + O(\alpha_V^2)$ at large temperature and $w_{h;0} \simeq -\frac{i\pi^2}{24} e^{|\alpha_V|}$ in the degenerate limit.

Going now to the next-to-leading order, the axial modes, characterized by $\tau_H \omega_a = w_{a;1} \tau_H + \dots$, have their leading term given by

$$\frac{i}{3} \left[\left(\frac{2w_{a;1}^\pm}{T^2} \right)^2 \sigma_A^\omega \left(\sigma_A^\omega - \frac{T^2}{3} \Delta H \right) - \frac{A^2}{H} \kappa_\Omega^2 \right] = 0 \Rightarrow w_{a;1}^\pm = \pm \frac{AT^2 \kappa_\Omega}{2\sigma_A^\omega \sqrt{H}} \left(1 - \frac{T^2 \Delta H}{3\sigma_A^\omega} \right)^{-1/2}. \quad (3.25)$$

Using Eqs. (2.32) and (2.33), it can be seen that $w_{a;1}^\pm = \pm 6\alpha_V \kappa_\Omega / (7\pi^2)$ in the large- T limit and $\pm 2\pi \kappa_\Omega / (\alpha_V^2 \sqrt{3})$ in the degenerate limit. We remark that the two propagating modes $w_{a;1}^\pm$ are necessarily different from the ones found in the small chemical potential expansion since those were not propagating at small wavenumber k . Therefore, we can anticipate that at finite but small chemical potential, two modes are propagating for small k . As the wavenumber k increases, these modes will stop propagating, and we will have three purely dissipating modes until we reach yet another critical value of k , where the modes found in the small chemical potential limit start propagating again.

For the helical mode with $\tau_H \omega_h = w_{h;0} + w_{h;1} \tau_H + \dots$, the first-order correction satisfies

$$\frac{4w_{h;0}w_{h;1}}{T^4} \left(\sigma_A^\omega - \frac{T^2}{3} \Delta H \right) \left(\frac{6w_{h;0}}{T^2} [(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2] + \frac{2i}{3} \sigma_A^\omega \right) = 0. \quad (3.26)$$

The term inside the second pair of parentheses evaluates to $-i\sigma_A^\omega/3 \neq 0$, by virtue of Eq. (3.24). Thus, Eq. (3.26) implies that $w_1^h = 0$.

The relation between the fluctuation amplitudes $\delta\mu_\ell$ can be extracted by expanding Eq. (3.5) in powers of τ_H . For the axial modes, we find, to leading order in τ_H ,

$$\begin{aligned} \delta\mu_A^{a;\pm} &\simeq \frac{AT^2 \kappa_\Omega}{2w_{a;1}^\pm \sigma_A^\omega} \delta\mu_V^{a;\pm} = \sqrt{H} \left(1 - \frac{T^2 \Delta H}{3\sigma_A^\omega} \right)^{-1/2} \delta\mu_V^{a;\pm}, \\ \delta\mu_H^{a;\pm} &\simeq -\frac{3i}{\sigma_A^\omega} (B\sigma_A^\omega - A\sigma_H^\omega) \kappa_\Omega \tau_H \delta\mu_V^{a;\pm}. \end{aligned} \quad (3.27)$$

As for the helical mode, the denominator in Eq. (3.5) vanishes at $\omega_* = \omega_{h;0}$ and the above relations are not applicable. Therefore we resort to Eq. (2.39) in the limit $\tau_A \rightarrow \infty$. Expanding the latter for small τ_H gives

$$\delta\mu_V^h \simeq \frac{T^2}{2w_{h;0} H \sigma_A^\omega} \frac{B\sigma_A^\omega - A\sigma_H^\omega}{\sigma_A^\omega - \frac{T^2}{3} \Delta H} \kappa_\Omega \tau_H \delta\mu_H^h, \quad \delta\mu_A^h \simeq -\frac{\sigma_H^\omega}{\sigma_A^\omega} \delta\mu_H^h. \quad (3.28)$$

The Chiral Vortical Wave (CVW) can be recovered as follows. The traditional derivation of the CVW did not include fluctuations of the helical chemical potential. The freezing of the helical degree of freedom is naturally implemented if one requires instantaneous dissipation of helicity fluctuations, i.e. $\tau_H \rightarrow 0$. The angular frequencies $\omega_h = w_h/\tau_H$ and $\omega_a^\pm = w_a^\pm/\tau_H$ of the three modes found at small τ_H can be written in the leading order as:

$$\omega_h = -\frac{iT^2 \sigma_A^\omega}{6\tau_H [(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2]}, \quad \omega_a^\pm = \pm \frac{AT^2}{2\sigma_A^\omega \sqrt{H}} \left(1 - \frac{T^2 \Delta H}{3\sigma_A^\omega} \right)^{-1/2} \frac{k\Omega}{T}. \quad (3.29)$$

Indeed, as $\tau_H \rightarrow 0$, we find that the helical mode dissipates very quickly, removing any fluctuation of helical chemical potential. In contrast, the axial modes give a propagating wave mixing only axial and vector degrees of freedom, with $\delta\mu_H = O(\tau_H)$ [see Eq. (3.27)], i.e. the Chiral Vortical Wave [10]. Note that the CVW is only present at finite (vector) chemical potential μ_V . In agreement with this property, the angular frequency of our axial modes, given by the second relation in Eq. (3.29), does not vanish provided $\mu_V \neq 0$ as it

follows from the definition (2.31) of the coefficient A and the absence of the leading μ_V behaviour in other terms in this expression.

We remark that at finite (non-zero) τ_H , the Chiral Vortical Wave (CVW) is propagating for small wavenumbers, but it can stop propagating at some finite wavenumber. These aspects will be further clarified in Sec. 3.4 (see also Fig. 4).

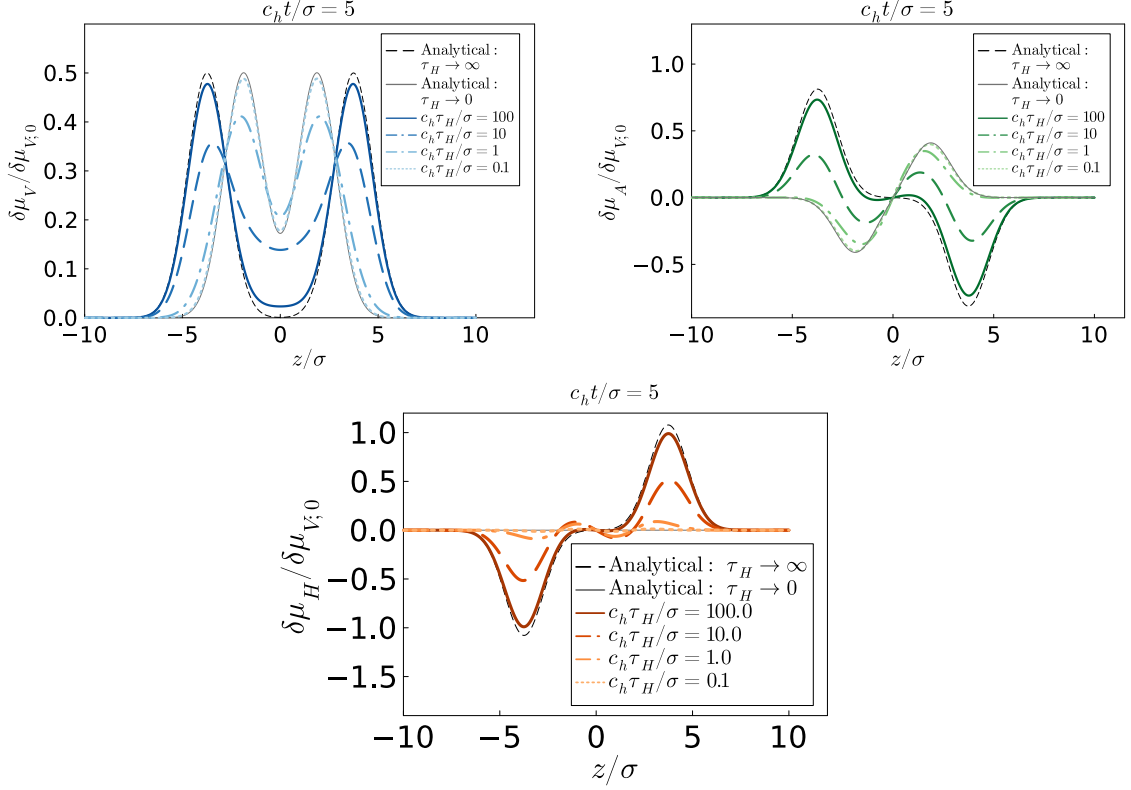


Figure 2: Profiles of the perturbations in the vector, axial and helical chemical potentials at time $c_h t / \sigma = 5$ for the Gaussian initial conditions in Eq. (3.19), set by the initial Gaussian width σ and the velocity c_h of the helical vortical wave in a neutral, non-dissipating plasma (3.9), for various values of the helical relaxation time τ_H . The helicity-preserving limit $\tau_H \rightarrow \infty$ corresponds to the helical vortical wave solutions shown in Eq. (3.21), while the helicity-frozen limit $\tau_H \rightarrow 0$ corresponds to the axial vortical wave described in Eq. (3.32). The parameters of the background state are $\mu_V = 650$ MeV and $T = 300$ MeV, while the rotation parameter is set to $\Omega = 6.6$ MeV.

Let us illustrate the propagation properties of the CVW by considering the initial Gaussian profile for the vector chemical potential perturbation in Eq. (3.19). To leading order, only the axial modes contribute, such that

$$\sum_{\omega=\omega_a^\pm} \delta\mu_V^{a;\pm}(k) = \frac{\sigma}{\sqrt{2\pi}} \delta\mu_V e^{-\sigma^2 k^2/2}. \quad (3.30)$$

Given the relation (3.27) between $\delta\mu_V^{a;\pm}$ and $\delta\mu_A^{a;\pm}$, as well as the constraint $\sum_{\omega=\omega_a^\pm} \delta\mu_A^{a;\pm} =$

0, we find

$$\delta\mu_A^{a;\pm}(k) = \pm \frac{\sigma}{2} \sqrt{\frac{H}{2\pi} \left(1 - \frac{T^2 \Delta H}{3\sigma_A^\omega}\right)} \delta\mu_V e^{-\sigma^2 k^2/2}. \quad (3.31)$$

This leads to the following propagating Gaussian pulse solutions:

$$\begin{aligned} \bar{\mu}_V(t, z) &= \mu_V + \frac{\delta\mu_V}{2} \left[e^{-(z-v_a t)^2/2\sigma^2} + e^{-(z+v_a t)^2/2\sigma^2} \right], \\ \bar{\mu}_A(t, z) &= \frac{\delta\mu_V}{2} \sqrt{\frac{H}{2\pi} \left(1 - \frac{T^2 \Delta H}{3\sigma_A^\omega}\right)} \left[e^{-(z-v_a t)^2/2\sigma^2} - e^{-(z+v_a t)^2/2\sigma^2} \right]. \end{aligned} \quad (3.32)$$

The above solution is confirmed in Fig. 2, where we took $T = 300$ MeV, $\mu_V = 650$ MeV and $\mu_A = \mu_H = 0$ for the background state.

3.3 Large chemical potential limit

The characteristic Eq. (3.4) for generic vector chemical potential is a cubic equation in the following form:

$$a(\alpha_V)\omega^3 + \frac{i}{\tau_H}b(\alpha_V)\omega^2 - \kappa_\Omega^2 c(\alpha_V)\omega - \frac{i}{\tau_H}\kappa_\Omega^2 d(\alpha_V) = 0, \quad (3.33)$$

where the coefficients a , b , c and d are non-negative and can be read off from Eq. (3.4):

$$\begin{aligned} a &= \frac{8}{T^6} \left(\sigma_A^\omega - \frac{T^2}{3} \Delta H \right) [(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2], & b &= \frac{4}{3T^4} \sigma_A^\omega \left(\sigma_A^\omega - \frac{T^2}{3} \Delta H \right), \\ c &= \frac{2}{HT^2} [(A^2 + B^2)\sigma_A^\omega - 2AB\sigma_H^\omega], & d &= \frac{A^2}{3H}. \end{aligned} \quad (3.34)$$

We first notice that Eq. (3.33) always supports a purely imaginary solution, due to the fact that under $\omega \rightarrow i\omega_I$, the characteristic equation turns into a polynomial equation of third order with real coefficients, with at least one real solution. The remaining two modes come in pairs with opposite signs for the real part. In the large chemical potential limit, the coefficients satisfy

$$\lim_{|\alpha_V| \rightarrow \infty} a(\alpha_V) \rightarrow \frac{8\alpha_V^4}{3\pi^6} e^{-|\alpha_V|}, \quad \lim_{|\alpha_V| \rightarrow \infty} b(\alpha_V) \rightarrow \frac{\alpha_V^4}{9\pi^4}, \quad (3.35)$$

$$\lim_{|\alpha_V| \rightarrow \infty} c(\alpha_V) \rightarrow \frac{32}{9\pi^4} e^{-|\alpha_V|}, \quad \lim_{|\alpha_V| \rightarrow \infty} d(\alpha_V) = \frac{4}{27\pi^2}. \quad (3.36)$$

Consequently, as $|\alpha_V| \rightarrow \infty$, the system hosts one purely imaginary mode and two purely propagating modes with real-valued energy dispersion relations, respectively:

$$\omega_{\text{im.}} = -\frac{i\pi^2}{24\tau_H} e^{|\alpha_V|}, \quad \omega_{\pm} = \pm \frac{2\pi\Omega}{\sqrt{3}\alpha_V^2 T} k, \quad (3.37)$$

where we used the definition of κ_Ω in Eq. (2.28). From the above results, we find that as we increase the vector chemical potential, the non-propagating mode $\omega_{\text{im.}}$ dissipates exponentially fast, whereas the phase speed $\omega_{\pm}/k = \pm 2\pi\Omega/\alpha_V^2 \sqrt{3}$ of the other two modes vanishes quadratically, as α_V^{-2} . This result is consistent with Fig. 5.

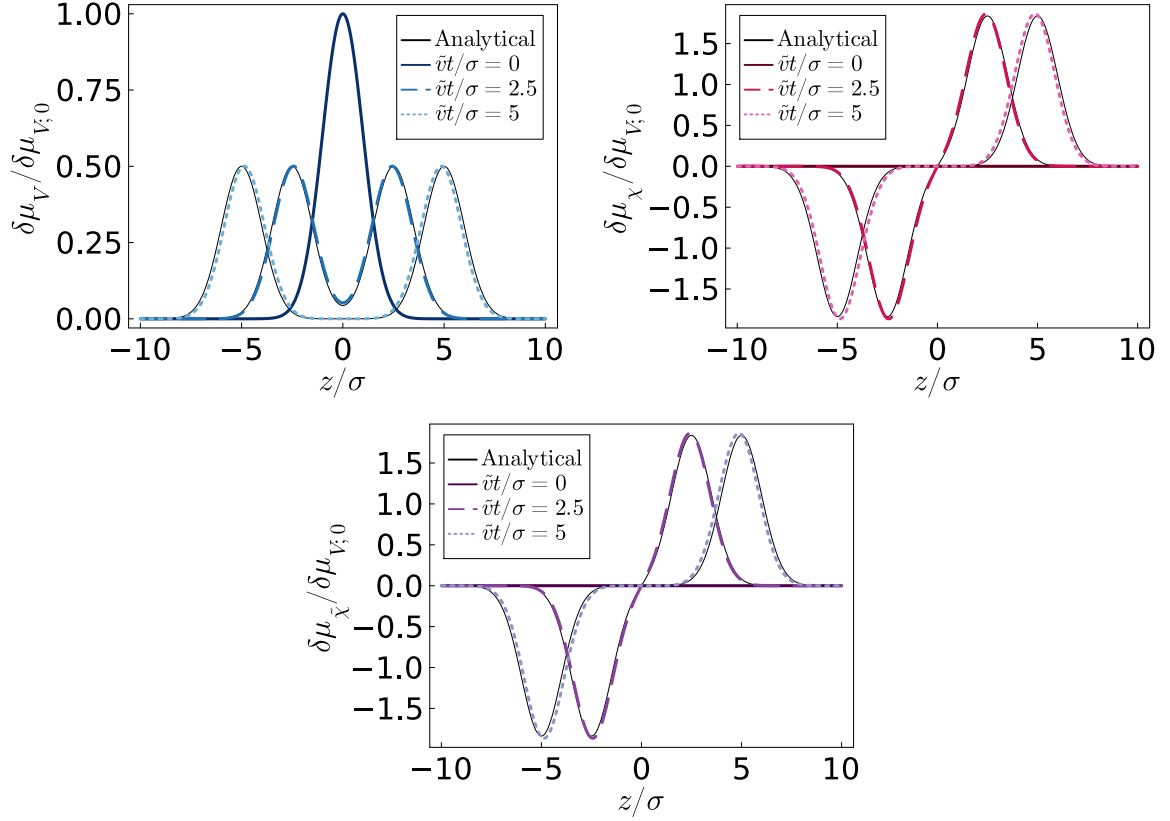


Figure 3: Propagation of an initial Gaussian perturbation in the vector chemical potential, shown in Eq. (3.19), for the degenerate limit discussed in Sec. 3.3. The profiles are represented at three time instances, corresponding to $\tilde{v}t/\sigma = 0, 2.5$ and 5 , where the phase velocity of the propagating solution in the degenerate limit \tilde{v} is given in Eq. (3.41). The background state has $\mu_V = 200$ MeV and $T = 10$ MeV. We took $\Omega = 6.6$ MeV and $\tau_H = 2\sigma/\tilde{v}$. We also represented the analytical solution for the limit $\alpha_V \rightarrow \infty$, shown in Eq. (3.42).

The general solution for $\delta\mu_\chi$ and $\delta\mu_{\tilde{\chi}}$ for arbitrary vector chemical potential and helical relaxation time is given by

$$\begin{aligned}\delta\mu_\chi &= \frac{\kappa_\Omega T^2 (AT^2 \tau_H^{-1} - 6i(A + s_V B)\omega_*(\sigma_A^\omega - s_V \sigma_H^\omega))}{2\omega_* (\sigma_A^\omega T^2 \tau_H^{-1} - 6i\omega_* [(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2])} \delta\mu_V, \\ \delta\mu_{\tilde{\chi}} &= \frac{\kappa_\Omega T^2 (AT^2 \tau_H^{-1} - 6i(A - s_V B)\omega_*(\sigma_A^\omega + s_V \sigma_H^\omega))}{2\omega_* (\sigma_A^\omega T^2 \tau_H^{-1} - 6i\omega_* [(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2])} \delta\mu_V.\end{aligned}\quad (3.38)$$

From the previous expressions, it appears that the degenerate limit $\mu_V \rightarrow \infty$ and the helicity-conservation limit $\tau_H^{-1} \rightarrow 0$ do not commute. Naively, we could take $|\mu_V|/T \rightarrow \infty$ for some finite τ_H^{-1} . In such case, the amplitudes for the propagating modes ω_\pm read

$$\delta\mu_\chi^\pm \simeq \delta\mu_{\tilde{\chi}}^\pm \simeq \frac{\kappa_\Omega T^2}{2\omega^\pm} \frac{A}{\sigma_A^\omega} \delta\mu_V^\pm \simeq \pm \frac{|\alpha_V|}{\sqrt{3}\pi} \delta\mu_V^\pm, \quad (3.39)$$

where the factors of τ_H^{-1} have cancelled out and the result is well defined as $\tau_H^{-1} \rightarrow 0$.

Regarding the exponentially dissipating modes, the denominator of the amplitudes (3.38) vanishes and we resort to the $\tau_A \rightarrow \infty$ limit of (2.39). Taking $\delta\mu_{\tilde{\chi}}$ as the reference perturbation we find:

$$\delta\mu_{\chi}^{\text{im.}} \simeq -\frac{2}{\alpha_V^2} e^{-|\alpha_V|} \delta\mu_{\tilde{\chi}}^{\text{im.}} \quad \delta\mu_V^{\text{im.}} \simeq -\frac{72i\kappa_{\Omega}\tau_H}{\pi^2\alpha_V^2} e^{-2|\alpha_V|} \delta\mu_{\tilde{\chi}}^{\text{im.}} \quad (3.40)$$

In Fig. 3, we show the evolution of the initial Gaussian profile of the vector chemical potential of Eq. (3.19) in the nearly-degenerate case when $\mu_V = 200$ MeV and $T = 10$ MeV. The plots show the profiles of $\delta\bar{\mu}_V(t, z)$, $\delta\bar{\mu}_A(t, z)$ and $\delta\bar{\mu}_H(t, z)$ at initial time and at times satisfying $\tilde{v}t/\sigma = 2.5$ and 5, where

$$\tilde{v} = \frac{|\text{Re}(\omega_{\pm})|}{k} = \frac{2\pi\Omega}{\alpha_V^2 T \sqrt{3}}, \quad (3.41)$$

is the phase velocity of the propagating solution in the degenerate limit [see Eq. (3.37)]. We also represent the analytical solution, which reads

$$\begin{aligned} \delta\bar{\mu}_V(t, z) &= \frac{\delta\mu_V}{2} \left(e^{-(z-\tilde{v}t)/2\sigma^2} + e^{-(z+\tilde{v}t)/2\sigma^2} \right), \\ \delta\bar{\mu}_{\chi}(t, z) &= \frac{|\alpha_V|}{\sqrt{3}\pi} \frac{\delta\mu_V}{2} \left(e^{-(z-\tilde{v}t)/2\sigma^2} - e^{-(z+\tilde{v}t)/2\sigma^2} \right), \end{aligned} \quad (3.42)$$

while $\delta\bar{\mu}_{\tilde{\chi}}(t, z) \simeq \delta\bar{\mu}_{\chi}(t, z)$. As shown in Fig. 3, the above solution provides a good match to the numerical solution in the considered limit at the normalized vector chemical potential $\alpha_V = \mu_V/T = 20$. The amplitude of $\delta\bar{\mu}_{\tilde{\chi}} = \delta\bar{\mu}_A - s_V \delta\bar{\mu}_H$ is finite, contrary to the degenerate limit of the case when the charge is conserved when the amplitude of this mode increases like α_V^4 (see Fig. 4 and Eq. (5.19) from Ref. [16]).

Prior to concluding this section, we address the non-commutativity of the large vector chemical potential ($\mu_V \rightarrow \infty$) and the conserved helicity ($\tau_H \rightarrow 0$) limits. To this end, we need to know the dependence of the helicity relaxation time on the vector chemical potential. This problem is addressed in Appendix B, and a fit to numerical data suggests that the helicity relaxation time diverges in the degenerate limit according to

$$\tau_H^{-1}(\mu_V \gg T) = \frac{\tau_{\text{deg.}}^{-1}}{\alpha_V} \left(1 + O\left(\frac{1}{\alpha_V}\right) \right) e^{-\alpha_V} + O(e^{-2\alpha_V}), \quad (3.43)$$

with $\tau_{\text{deg.}}$ constant. Taking this dependence into account, we can take the degenerate limit and the helicity conservation limit in a correlated manner. While the propagating modes ω_{\pm} remain unaffected, the non-propagating mode $\omega_{\text{im.}}$ becomes, to leading order,

$$\omega_{\text{im.}} = -\frac{i\pi^2}{24\tau_{\text{deg.}}\alpha_V}. \quad (3.44)$$

Therefore, the dissipation is inhibited as we increase the chemical potential, contrary to the naive $\alpha_V \rightarrow \infty$ limit in Eq. (3.37). Finally, taking the limit for the amplitudes (3.40) gives

$$\delta\mu_{\chi}^{\text{im.}} \simeq -\frac{2}{\alpha_V^2} e^{-|\alpha_V|} \delta\mu_{\tilde{\chi}}^{\text{im.}} \quad \delta\mu_V^{\text{im.}} \simeq -\frac{72i\kappa_{\Omega}\tau_{\text{deg.}}}{\pi^2\alpha_V} e^{-|\alpha_V|} \delta\mu_{\tilde{\chi}}^{\text{im.}} \quad (3.45)$$

3.4 Generic properties of spectra

In Subsec. 3.1 we have seen that, at small vector chemical potential, there is a critical wavenumber above which two of the three modes start to propagate. At the same time, for small wavenumbers at finite chemical potential, we showed in Subsec. 3.2 that again, there are two modes that always propagate. Both results are compatible with each other only if, for small chemical potential, the modes that propagate at small k turn into non-propagating modes at some finite k . We will shortly see that this is indeed the case.

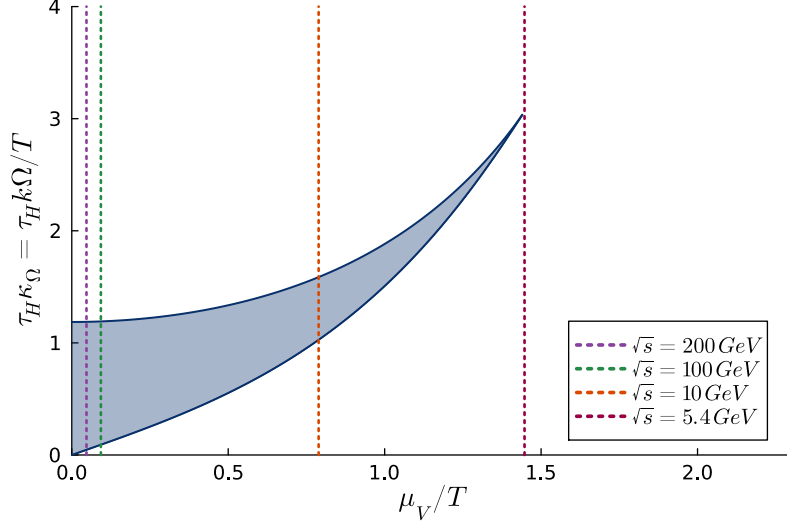


Figure 4: Propagating points κ_Ω^+ (upper black curve) and κ_Ω^- (lower black curve) as functions of the normalized vector chemical potential $\alpha_V = \mu_V/T$ for the unpolarized plasma. In the shaded region, there are no propagating modes. The boundaries $\kappa_\Omega = \kappa_\Omega^\pm$ of the shaded region, defined in Eq. (3.51), join each other at the critical value $\kappa_\Omega^{\text{crit.}}$ given in Eq. (3.56). The vertical lines mark the energy of the collision \sqrt{s} corresponding to the given value of the normalized chemical potential α_V [see a discussion around Eq. (3.57).]

Propagating modes come in pairs with the same imaginary part, whereas non-propagating modes have a vanishing real part. Thus, one is able to compute the critical values k_* at which modes start/stop propagating by looking at purely imaginary degenerate solutions of the characteristic equation, Eq. (3.33). These can be obtained by setting to 0 the derivative with respect to ω of the left-hand side of Eq. (3.33):

$$3a\omega^2 + \frac{2ib}{\tau_H}\omega - \kappa_\Omega^2 c = 0. \quad (3.46)$$

The solution of the above equation reads:

$$\omega_\pm = -\frac{ib}{3a\tau_H} \left(1 \pm \sqrt{1 - \frac{3ac}{b^2}(\tau_H \kappa_\Omega)^2} \right). \quad (3.47)$$

To find the critical wavenumbers k_* , we impose that both Eq. (3.33) and (3.46) hold simultaneously. Multiplying Eq. (3.46) by $\omega/3$ and subtracting the result from Eq. (3.33) leads

to

$$\frac{ib}{3\tau_H}\omega_{\pm}^2 - \frac{2c}{3}\kappa_{\Omega}^2\omega_{\pm} - \frac{i}{\tau_H}\kappa_{\Omega}^2d = 0. \quad (3.48)$$

Multiplying again Eq. (3.46) by $ib/9a\tau_H$ and subtracting the result from the above equation leads to the relation

$$\left(\frac{2b^2}{9a\tau_H^2} - \frac{2c}{3}\kappa_{\Omega}^2\right)\omega_{\pm} = \frac{i}{\tau_H}\left(d - \frac{bc}{9a}\right)\kappa_{\Omega}^2. \quad (3.49)$$

Substituting now the solution in Eq. (3.47) gives the following equation for $\tau_H\kappa_{\Omega}$:

$$4ac^3(\kappa_{\Omega}\tau_H)^6 - (b^2c^2 + 18abcd - 27a^2d^2)(\kappa_{\Omega}\tau_H)^4 + 4b^3d(\tau_H\kappa_{\Omega})^2 = 0. \quad (3.50)$$

The solution $\kappa_{\Omega} = 0$ corresponds to $\omega = 0$ and is thus spurious. The other two solutions for κ_{Ω}^2 read

$$(\kappa_{\Omega}^{\pm})^2 = \frac{1}{8ac^3\tau_H^2} \left[b^2c^2 + 18abcd - 27a^2d^2 \pm \sqrt{bc - ad}(bc - 9ad)^{3/2} \right]. \quad (3.51)$$

In the case of a neutral background, when $\alpha_V = 0$, the coefficients a , b , c and d evaluate to

$$a = b = \frac{1}{27}, \quad c = \frac{1}{3} \left(\frac{2 \ln 2}{\pi^2} \right)^2, \quad d = 0. \quad (3.52)$$

In this case, Eq. (3.51) gives $(\kappa_{\Omega}^{-})^2 = 0$, meaning no infrared modes can propagate. The second solution satisfies

$$\tau_H\kappa_{\Omega}^{+} = \frac{\pi^2}{12 \ln 2}. \quad (3.53)$$

From here, we can recover the threshold wavenumber required for wave propagation, $k_{\text{th}} = \kappa_{\Omega}^{+}T/\Omega = 1/(2c_h\tau_H)$, with $c_h = 6\Omega \ln 2/(\pi^2T)$, as found in Eq. (3.10).

The trajectories of $\kappa_{\Omega}^{\pm} \equiv k^{\pm}\Omega/T$ as functions of $\alpha_V = \mu_V/T$ are shown in Fig. 4. The shaded region has no propagating modes. At zero chemical potential, we recover the known result that the modes do not propagate until the critical wavenumber in Eq. (3.10) is reached, as demonstrated above. As α_V is increased, both κ_{Ω}^{+} and κ_{Ω}^{-} increase, the latter at a faster rate. Close to $\alpha_V = 0$, one can estimate the leading-order increase as:

$$\kappa_{\Omega}^{-} = \frac{\pi^2}{21\tau_H \ln^2 2} \alpha_V + O(\alpha_V^3), \quad \kappa_{\Omega}^{+} = \frac{\pi^2}{12\tau_H \ln 2} + \frac{1.135}{\tau_H} \alpha_V^2 + O(\alpha_V^4). \quad (3.54)$$

The region of no propagation shrinks down to a point when α_V exceeds a threshold value, given by the equation $bc = 9ad$, namely

$$(B^2 - 8A^2)(\sigma_A^{\omega})^2 - 2AB\sigma_A^{\omega}\sigma_H^{\omega} + 9A^2(\sigma_H^{\omega})^2 = 0. \quad (3.55)$$

The above equation is solved when $\alpha_V \simeq 1.4471$, in which case $\kappa_{\Omega} = k\Omega/T$ evaluates to

$$\kappa_{\Omega} \simeq \frac{3.0642}{\tau_H}. \quad (3.56)$$

All three modes merge at this critical value of k for the given chemical potential.

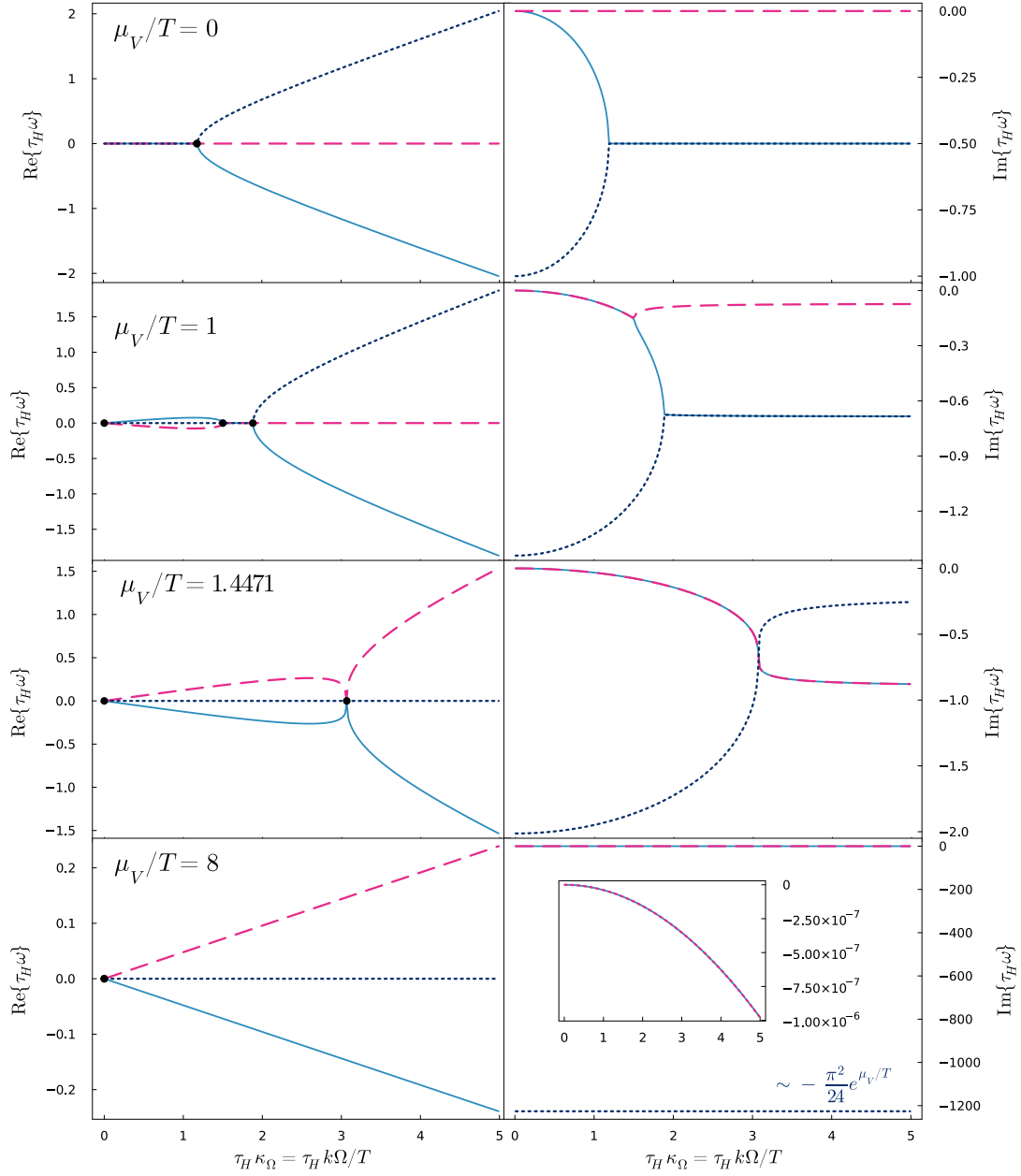


Figure 5: Collective excitations of the charge sector in the unpolarized plasma with dissipating helical charge. Each row shows the mode spectrum for a different value of the dimensionless vector chemical potential, $\alpha_V = \mu_V/T$. The left (right) plots correspond to the real (imaginary) part of the dimensionless angular frequency, $\tau_H \omega$. The black dots mark the points where the wave propagation starts or stops, i.e. when $\text{Re}(\tau_H \omega)$ vanishes identically. Colors are correlated between the left and right plots.

The wave spectrum is shown as the real and imaginary parts of $\tau_H \omega$ on the left and right columns of Fig. 5. The dimensionless chemical potential $\alpha_V = \mu_V/T$ increases from 0 on the first line to 8 on the last line. At $\alpha_V = 0$, $\text{Re}(\tau_H \omega) = 0$ for $\tau_H \kappa_\Omega < \tau_H k_{\text{th}} \Omega / T =$

$\pi^2/12 \ln 2 \simeq 1.18657$. At $\alpha_V = 1$, $\text{Re}(\tau_H \omega)$ is nonvanishing for $0 < \kappa_\Omega < \kappa_\Omega^-$ and when $\kappa_\Omega > \kappa_\Omega^+$, with an intermediate range of finite extent when there is no propagation. At the critical value $\alpha_V = 1.4471$, there is a single point when $\tau_H \kappa_\Omega = 3.0642$ and none of the modes propagate. Otherwise, for $\alpha_V > 1.4471$, all modes are propagating.

As we pointed out in section 3.2, the branch of propagating modes at a smaller wavenumber in Fig. 5 can be thought of as the generalization of the Chiral Vortical Wave previously studied in [10] to fluids possessing an independent helical degree of freedom. On the other hand, the modes propagating at a larger wavenumber in Fig. 5 correspond to the generalization of the Helical Vortical Wave found in [14] for finite vector chemical potential. As the chemical potential is increased, the modes propagate for any wavelength, and distinguishing between CVW and HVW modes becomes problematic.

Figure 4 deserves a further comment. In heavy-ion collisions, the ratio μ_V/T of the plasma is correlated with the energy of the collision \sqrt{s} . We can take the parametrization from [32] for the following freeze-out values:

$$T(\mu_B) = a - b\mu_B^2 - c\mu_B^4, \quad \mu_B(\sqrt{s}) = \frac{d}{1 + f\sqrt{s}}, \quad (3.57)$$

where $\mu_B = 3\mu_V$ and the fitted parameters are

$$\begin{aligned} a &= 0.166(2) \text{ GeV}, & b &= 0.139(16) \text{ GeV}, & c &= 0.053(21) \text{ GeV}, \\ d &= 1.308(28) \text{ GeV}, & f &= 0.273(8) \text{ GeV}^{-1}. \end{aligned} \quad (3.58)$$

We use this data set in Fig. 4 to show different collision energies, which are marked in the figure by vertical lines. It can be seen that at $\sqrt{s} = 5.4$ GeV (or lower energies), the chemical potential is high enough that all wavelengths propagate. As the collision energy is increased, the chemical potential decreases, leading to the development of a non-propagation gap between the infrared ($\kappa_\Omega < \kappa_\Omega^-$) and ultraviolet ($\kappa_\Omega > \kappa_\Omega^+$) propagating modes. The same considerations will apply to Fig. 8.

4 Non-conserved axial and helical charges

In this section, we study the spectra of collective excitations when both the helical and the axial charges are not conserved. Following the lines of Eq. (2.9), we include the dissipation of axial charge through its chemical potential:

$$\nabla_\mu J_V^\mu = 0, \quad \nabla_\mu J_A^\mu = -\frac{\mu_A T^2}{3\tau_A}, \quad \nabla_\mu J_H^\mu = -\frac{\mu_H T^2}{3\tau_H}. \quad (4.1)$$

The modes supported by Eqs. (4.1) can be obtained by solving the characteristic equation, $\det(T^{-2}\mathbb{M}) = 0$, which reduces to Eq. (2.37). In what follows, we solve this equation in various limits, as follows: the small chemical potential in Sec. 4.1; small axial relaxation times $\tau_A, \tau_H \rightarrow 0$ in Sections 4.2 and 4.3; the large wavenumber limit $k \rightarrow \infty$ in Sec. 4.4; and the large chemical potential limit $|\mu_V| \gg T$ in Sec. 4.5. We summarize our findings in Sec. 4.6

4.1 Small chemical potential

The small chemical potential limit of Eq. (2.37) extends Eq. (3.8) to

$$\frac{1}{27} \left(\frac{i}{\tau_A} + \omega_0 \right) \left[\omega_0 \left(\frac{i}{\tau_H} + \omega_0 \right) - k^2 c_h^2 \right] = 0. \quad (4.2)$$

While the pair of helical modes, $\omega_{h;0}^\pm$, remain unchanged with respect to Eq. (3.10), the axial mode in Eq. (3.12) now picks up a zeroth-order dissipative contribution:

$$\omega_{a;0} = -\frac{i}{\tau_A}, \quad \omega_{h;0}^\pm = -\frac{i}{2\tau_H} \pm kc_h \sqrt{1 - \frac{k_{\text{th}}^2}{k^2}}, \quad (4.3)$$

with $k_{\text{th}} = 1/(2\tau_H c_h)$ given in Eq. (3.10). Compared to the situation encountered in Sec. 3, we see that the axial mode is now damped on a time scale given by τ_A . The propagation properties of the helical vortical wave, uncovered in Eq. (3.10), remain identical as long as $\mu_V/T = 0$.

At first order with respect to $\alpha_V = \mu_V/T$, all three modes receive vanishing contributions,

$$\omega_{a;1} = \omega_{h;1}^\pm = 0, \quad (4.4)$$

while at second order, Eq. (3.12) gets modified to

$$\omega_{a;2} = \frac{3i}{\pi^2 \tau_A (\tau_A - \tau_H - \tau_H \tau_A^2 k^2 c_h^2)} \left[(\tau_A - \tau_H) \left(1 + \frac{12}{49\pi^2} \kappa_\Omega^2 \tau_A^2 \right) + \tau_H \frac{48}{\pi^2} (\ln 2)^2 \left(1 + \frac{3}{28\pi^2} \kappa_\Omega^2 \tau_A^2 \right) \right]. \quad (4.5)$$

Recall that the small chemical potential expansion in the unpolarized QED plasma does not work for small wavenumbers k . Now, the situation is slightly different: the expansion is unreliable whenever the denominator appearing in the expression for $\omega_{a;2}$ vanishes, and the small α_V expansion breaks down. This happens at the “breakdown” wavenumber,

$$k_{\text{br.}} = \frac{1}{c_h \sqrt{\tau_A}} \sqrt{\frac{1}{\tau_H} - \frac{1}{\tau_A}}, \quad \tau_H < \tau_A. \quad (4.6)$$

Note that in the limit where the axial charge is conserved ($\tau_A \rightarrow \infty$), the breakdown wavenumber vanishes, $k_{\text{br.}} \rightarrow 0$. The same happens when $\tau_A = \tau_H$. On the other hand, the value of $k_{\text{br.}}$ reaches a maximum for $\tau_A = 2\tau_H$, giving $k_{\text{br.}} = 1/(2c_h \tau_H)$, which curiously coincides with the threshold wavenumber k_{th} at which the helical wave starts to propagate, derived in Eq. (3.10).

The breakdown of the small chemical potential expansion was already encountered in Sec. 3.1 for conserved axial charge at the level of the fluctuations (c.f. Eq. (3.13)), giving $k_{\text{br.}} = 0$. A closer look at the small wavenumber limit in Sec. 3.2 revealed that the axial modes were propagating in the presence of a finite vector chemical potential μ_V (see Eq. (3.29)). The propagating nature of the axial modes was not captured in the small μ_V expansion of Sec. 3.2. Analogously, at finite τ_A and μ_V , we anticipate that the axial modes may be propagating in the vicinity of $k_{\text{br.}}$. This intuition will be confirmed with the explicit computation of the spectrum in Sec. 4.6 (see Fig. 9).

4.2 Small τ_A limit

We can study the limit where the axial charge dissipates much faster than the helical charge. This is somewhat equivalent to freezing the axial degree of freedom. In this limit, the axial mode dissipates very quickly, according to the dispersion relation:

$$\omega_a \simeq -\frac{iT^2\sigma_A^\omega}{6\tau_A[(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2]}, \quad (4.7)$$

whereas the dispersion relation for the vector and helical fluctuations is

$$\omega_{h;0}^\pm \simeq \frac{T^2}{12\tau_H\sigma_A^\omega} \left(-i \pm \sqrt{\frac{36B^2\sigma_A^\omega}{H(\sigma_A^\omega - \frac{T^2}{3}\Delta H)}\tau_H^2\kappa_\Omega^2 - 1} \right). \quad (4.8)$$

Therefore, the vector and helical fluctuations propagate provided that the normalized wavenumber κ_Ω exceeds some critical value satisfying

$$\tau_H^2\kappa_{\Omega;*}^2 = \frac{H}{36B^2\sigma_A^\omega} \left(\sigma_A^\omega - \frac{T^2}{3}\Delta H \right). \quad (4.9)$$

The corresponding threshold wavenumber ranges from $k_* = T\kappa_{\Omega;*}/(\tau_H\Omega) = 1/(2\tau_H c_h)$ at vanishing chemical potential [recovering Eq. (3.10)] to $\tau_H\kappa_{\Omega;*} = \pi/4\sqrt{3} \simeq 0.45$ at infinite chemical potential. Note that propagation is inhibited for small wavenumbers at any value of the vector chemical potential, contrary to the situation found for conserved axial charge ($\tau_A \rightarrow \infty$) in Sec. 3.4 (see Eq. (3.51) and Fig. 4) where any non-zero chemical potential gave rise to propagating modes in the infrared region with large wavelengths.

In the case of conserved helicity, when $\tau_H \rightarrow \infty$, the helical vortical modes are propagating for any wavelength, and their dispersion relation Eq. (4.8) reduces to

$$\lim_{(\tau_A, \tau_H) \rightarrow (0, \infty)} \omega_{h;0}^\pm = \pm \frac{\kappa_\Omega B T^2}{2} \sqrt{\frac{1}{H\sigma_A^\omega(\sigma_A^\omega - \frac{T^2}{3}\Delta H)}}. \quad (4.10)$$

In this limit, the modes are qualitatively similar to the helical vortical modes found for the unpolarised plasma with conserved charges ($\tau_A, \tau_H \rightarrow \infty$) given in (2.40). The small vector chemical potential limit of both expressions agree quantitatively in the leading order but differ thereafter:

$$\lim_{(\tau_A, \tau_H) \rightarrow (0, \infty)} \omega_{h;0}^\pm(\mu_V \ll T) = \pm c_h k \mp 0.0978 \frac{k\Omega}{T} \alpha_V^2 + O(\alpha_V^4), \quad (4.11)$$

$$\lim_{(\tau_A, \tau_H) \rightarrow (\infty, \infty)} \omega_{h;0}^\pm(\mu_V \ll T) = \pm c_h k \mp 0.0124 \frac{k\Omega}{T} \alpha_V^2 + O(\alpha_V^4), \quad (4.12)$$

where we have used the expansions given in Eq. (2.32) and the definitions for κ_Ω and c_h given in Eqs. (2.28) and (3.9). In the large chemical potential limit, we can employ the relations (2.33) to show that both dispersion relations agree quantitatively up to exponentially small corrections. In Fig. 6, we show the propagation velocity $v_h^\pm \equiv \omega_{h;0}^\pm/k$ for conserved helicity in both cases, i.e. conserved axial charge $\tau_A \rightarrow \infty$ and frozen axial degree of freedom $\tau_A \rightarrow 0$. Notably, the freezing of the axial degree of freedom results in a diminishing of the propagation velocity at finite vector chemical potential as opposed to the case where chirality is present and conserved.

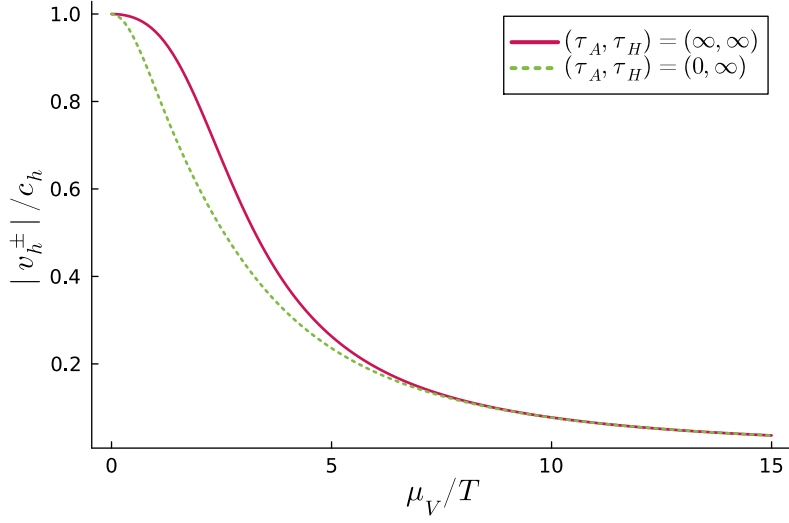


Figure 6: Propagation velocity for the helical vortical wave in the limit of conserved helicity for the cases when chirality is either conserved or frozen.

4.3 Small τ_H limit

Similarly to the previous subsection, we can study the spectra when the fluctuations of the helical degree of freedom dissipate very quickly, implying, in other words, that this degree of freedom is frozen. The analysis in this section extends that of Sec. 3.2 to the case of a finite (non-infinite) axial relaxation time τ_A . As discussed in Sec. 3.2, freezing the helicity degree of freedom opens up the propagation channel for the chiral vortical wave. On the other hand, a finite helical relaxation time inhibits the propagation of the helical vortical wave at large wavelengths. We therefore anticipate that, in a similar manner, a finite axial relaxation time τ_A will inhibit the propagation of the axial vortical wave at large wavelengths.

Multiplying Eq. (2.37) by τ_H^3 and considering that $\tau_H\omega$ is finite when $\tau_H \rightarrow 0$, we find the purely dissipative mode:

$$\omega_h \simeq -\frac{iT^2\sigma_A^\omega}{6\tau_H[(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2]}. \quad (4.13)$$

The angular frequencies for the other two(axial) modes, ω_a^\pm , are finite when the helical mode freezes, $\tau_H \rightarrow 0$, satisfying

$$\omega_a^\pm \simeq \frac{T^2}{12\sigma_A^\omega\tau_A} \left(-i \pm \sqrt{\frac{36A^2\sigma_A^\omega}{H(\sigma_A^\omega - \frac{T^2}{3}\Delta H)}\tau_A^2\kappa_\Omega^2 - 1} \right). \quad (4.14)$$

The above expression generalizes Eq. (3.29), giving the energy dispersion relation for the CVW to the case when the axial charge is not conserved. As already noted in Sec. 3.2, the vector and axial fluctuations do not propagate at vanishing chemical potential. At finite chemical potential and finite (i.e., non-infinite) τ_A , they propagate provided that k exceeds some critical value. Using the large temperature expansion 2.32, we find that the

critical value behaves as $\tau_A \kappa_* = \frac{7\pi^2}{12\alpha_V}$ for small chemical potential. In contrast, it finds a minimum in the limit of infinite chemical potential: $\tau_A \kappa_*^{\min} = k_*^{\min} \tau_A \Omega / T = \pi / 4\sqrt{3} \simeq 0.45$, which coincides with the large chemical potential limit of Eq. (4.9) giving the propagation threshold in the case when $\tau_A \rightarrow 0$. Since the helical and axial degrees of freedom correspond to the same physical quantity at infinite vector chemical potential, it is reasonable that freezing of either the helical or the axial chemical potentials gives the same result in this degenerate limit.

4.4 Large wavenumber limit

As seen in Sec. 4.1, a finite (non-infinite) relaxation time brings significant modifications in the wave spectrum for small wavenumbers k [cf. (4.3)]. We now consider the opposite limit of large wavenumbers (small wavelengths) in order to demonstrate that the relaxation time of the axial and/or helical charges does not affect the wave spectrum in the UV limit. Specifically, we find

$$\omega_h^\pm = \pm k c_h(\alpha_V) + O(k^0), \quad c_h(\alpha_V) = \frac{\Omega T}{2} \sqrt{\frac{\sigma_A^\omega (A^2 + B^2) - 2AB\sigma_H^\omega}{H(\sigma_A^\omega - \frac{T^2}{3}\Delta H)[(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2]}}, \quad (4.15)$$

where $c_h(\alpha_V)$ is the velocity of the helical vortical wave in an unpolarized fluid with vector imbalance $\mu_V = \alpha_V T$, in the absence of helicity- or chirality-violating interactions, as obtained in Eq. (4.20) of Ref. [16]. The axial mode turns out to be purely dissipative:

$$\omega_a = -\frac{iT^2}{6\tau_A\tau_H} \frac{B^2\tau_H + A^2\tau_A}{(A^2 + B^2)\sigma_A^\omega - 2AB\sigma_H^\omega} + O(k^{-1}). \quad (4.16)$$

At large wavenumbers, the purely imaginary phase velocity of the axial mode will settle to a constant negative value, which will depend on the dimensionless vector chemical potential $\alpha_V = \mu_V / T$ and on the axial and helicity relaxation times. These results can be checked in Figure 9.

4.5 Large chemical potential limit

In order to study the large chemical potential limit, we solve the eigenvalue equation Eq. (2.37) perturbatively in $T/|\mu_V|$. Using the expansion in Eq. (2.33), we can compute the energy dispersion relation as a Taylor series in $e^{-|\mu_V|/T}$. Expanding

$$\omega = \omega_{-1} e^{|\mu_V|/T} + \omega_0 + O(e^{-|\mu_V|/T}), \quad (4.17)$$

the characteristic equation Eq. (2.37) becomes, to leading order,

$$\frac{12i\sigma_A \left(\sigma_A - \frac{1}{3}T^2\Delta H\right) \left(\pi^2(\tau_A^{-1} + \tau_H^{-1}) - 24i\omega_{-1}\right) \omega_{-1}^2}{9\pi^2 T^4} e^{2|\mu_V|/T} + O(e^{|\mu_V|/T}) = 0. \quad (4.18)$$

Therefore, we find

$$\omega_{-1}^\pm = 0 \quad (\text{double root}), \quad \omega_{-1}^{\text{im.}} = -\frac{i}{24\pi^2} \left(\frac{1}{\tau_H} + \frac{1}{\tau_A} \right). \quad (4.19)$$

The contributions ω_0^\pm to modes with ω_{-1}^\pm come from the next-to-leading order contribution to Eq. (2.37), namely

$$\tau_A^{-1}\tau_H^{-1}\left(9A^2\kappa_\Omega^2(\tau_A+\tau_H)-2H\omega_0^\pm\left(\sigma_A-\frac{T^2}{3}\Delta H\right)(iT^2+6\sigma_A(\tau_A+\tau_H)\omega_0^\pm)\right)=0. \quad (4.20)$$

Expanding ω_0 in powers of $T/|\mu_V|$ gives, to leading order in $T/|\mu_V|$:

$$\omega_\pm \simeq \frac{-i\pi^2}{6(\tau_A+\tau_H)}\left(1\pm\sqrt{1-\frac{48}{\pi^2}(\tau_A+\tau_H)^2\kappa_\Omega^2}\right)\frac{T^2}{|\mu_V|^2}. \quad (4.21)$$

Note that when both relaxation times are finite, there is no propagation for small wavenumbers. This is in contrast to the result obtained in Subsec. 3.3, where we considered $\tau_A = \infty$, and the analogous modes propagate for all wavelengths. Notably, the \pm modes are sensitive to an emergent effective relaxation time $\tau_{eff} = \tau_A + \tau_H$ which can be traced to the interdependence of the axial and helical degrees of freedom in the degenerate limit. From Eq. (4.21), we extract the critical wavenumber k_* above which these two modes start propagating:

$$k_* = \frac{\kappa_\Omega^* T}{\Omega \tau_H} = \frac{\pi}{4\sqrt{3}} \frac{T}{(\tau_A + \tau_H)\Omega}. \quad (4.22)$$

We now consider the amplitudes of the perturbations in the vector, axial, and helical chemical potentials. As already discussed in Sec. 3.3, it is convenient to discuss the combinations $\delta\mu_\chi = \delta\mu_A + s_V\delta\mu_H$ and $\delta\mu_{\tilde{\chi}} = \delta\mu_A - s_V\delta\mu_H$, where $s_V = \text{sgn } \mu_V$ is the sign of the vector chemical potential, that follow from Eq. (2.38):

$$\begin{aligned} \delta\mu_\chi^* &= \frac{\frac{2\omega_*}{T^2}(A+s_VB)(\sigma_A^\omega - s_V\sigma_H^\omega) + \frac{i}{3}\left(\frac{A}{\tau_H} + \frac{s_VB}{\tau_A}\right)}{\left(\frac{2\omega_*}{T^2}\right)^2[(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2] + \frac{2i\omega_*}{3T^2}\sigma_A^\omega\left(\frac{1}{\tau_A} + \frac{1}{\tau_H}\right) - \frac{1}{9\tau_A\tau_H}}\kappa_\Omega\delta\mu_V^*, \\ \delta\mu_{\tilde{\chi}}^* &= \frac{\frac{2\omega_*}{T^2}(A-s_VB)(\sigma_A^\omega + s_V\sigma_H^\omega) + \frac{i}{3}\left(\frac{A}{\tau_H} - \frac{s_VB}{\tau_A}\right)}{\left(\frac{2\omega_*}{T^2}\right)^2[(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2] + \frac{2i\omega_*}{3T^2}\sigma_A^\omega\left(\frac{1}{\tau_A} + \frac{1}{\tau_H}\right) - \frac{1}{9\tau_A\tau_H}}\kappa_\Omega\delta\mu_V^*. \end{aligned} \quad (4.23)$$

The previous result is exact. As we approach the degenerate limit, Eq. (2.33) shows that $A \rightarrow s_VB$ and $\sigma_A \rightarrow s_V\sigma_H$ up to exponentially small corrections, and the amplitudes for the two propagating modes (4.21) simplify to

$$\begin{aligned} \delta\mu_\chi^\pm &\simeq \frac{AT^2}{4\sigma_A^\omega}\frac{\kappa_\Omega(\tau_A+\tau_H)}{\omega_\pm(\tau_A+\tau_H) + \frac{iT^2}{6\sigma_A^\omega}}\delta\mu_V^\pm \simeq -\frac{\kappa_\Omega(\tau_A+\tau_H)}{3\alpha_V^2\omega_\mp(\tau_A+\tau_H)}\frac{|\mu_V|}{T}\delta\mu_V^\pm, \\ \delta\mu_{\tilde{\chi}}^\pm &\simeq \frac{AT^2}{4\sigma_A^\omega}\frac{\kappa_\Omega(\tau_A-\tau_H)}{\omega_\pm(\tau_A+\tau_H) + \frac{iT^2}{6\sigma_A^\omega}}\delta\mu_V^\pm \simeq -\frac{\kappa_\Omega(\tau_A-\tau_H)}{3\alpha_V^2\omega_\mp(\tau_A+\tau_H)}\frac{|\mu_V|}{T}\delta\mu_V^\pm. \end{aligned} \quad (4.24)$$

As in the case with $\tau_A \rightarrow \infty$ considered in Eq. (3.39), the amplitudes of $\delta\mu_\chi^\pm$ and $\delta\mu_{\tilde{\chi}}^\pm$ are $|\alpha_V| = |\mu_V|/T$ times larger than $\delta\mu_V^\pm$, due to the behaviour $\omega_\mp \sim \alpha_V^{-2}$. Contrary to Eq. (3.39), the degeneracy between $\delta\mu_\chi^\pm$ and $\delta\mu_{\tilde{\chi}}^\pm$ is lifted whenever both τ_A and τ_H are finite (i.e., non-zero and non-infinite).

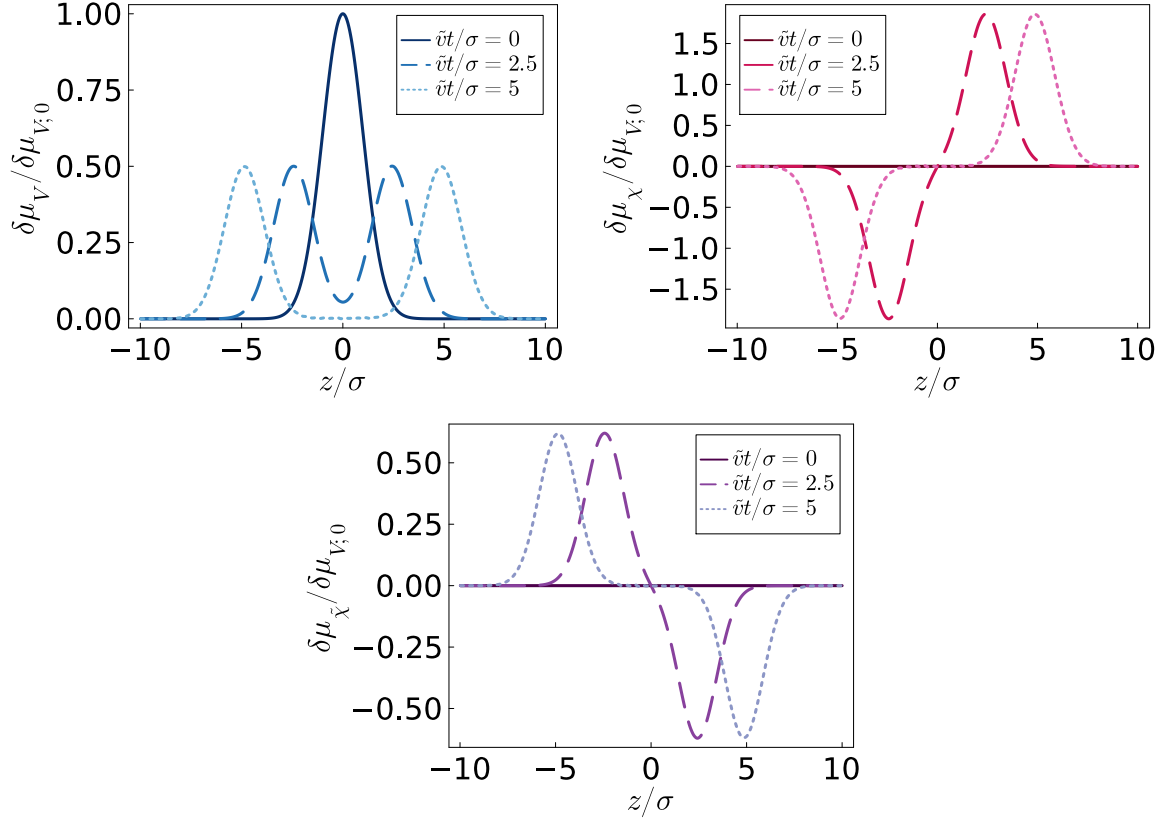


Figure 7: Same as Fig. 3, for the case when the axial charge is not conserved. For definiteness, we took $\tilde{v}\tau_A/\sigma = 1$, such that $\tau_A = \tau_H/2$, where \tilde{v} is defined in Eq. (3.41), with the angular velocities ω_\pm given by Eq. (4.21).

Regarding the exponentially-dissipating modes, the denominators of the amplitudes (4.23) vanish and we take the degenerate limit in Eqs. (2.39). Taking $\delta\mu_{\tilde{\chi}}$ as the reference amplitude we find

$$\delta\mu_\chi^{\text{im.}} \simeq -\frac{2(\tau_A - \tau_H)}{(\tau_A + \tau_H)\alpha_V^2} e^{-|\alpha_V|} \delta\mu_{\tilde{\chi}}^{\text{im.}}, \quad \delta\mu_V^{\text{im.}} \simeq -\frac{72i\kappa_\Omega\tau_A\tau_H}{\pi^2(\tau_A + \tau_H)\alpha_V^2} e^{-2|\alpha_V|} \delta\mu_{\tilde{\chi}}^{\text{im.}} \quad (4.25)$$

We present in Fig. 7 the propagation properties of an initial Gaussian fluctuation in the vector chemical potential, given in Eq. (3.19), similar to the one studied in Fig. 3 (see the end of Sec. 3.3 for details). As opposed to Fig. 3, here we take a finite axial relaxation time, which we set to $\tau_A = \tau_H/2$. While the propagation of $\delta\bar{\mu}_V(t, z)$ and of $\delta\bar{\mu}_\chi(t, z)$ remains unchanged, the amplitude of $\delta\bar{\mu}_{\tilde{\chi}}(t, z)$ is reduced by a factor of about 3 compared to the case when the axial charge is conserved.

4.6 Generic properties of spectra

In Sec. 3.4, we have seen that there exists a value of the vector chemical potential which divides the spectrum of excitations into two regions, depending on whether there exist one

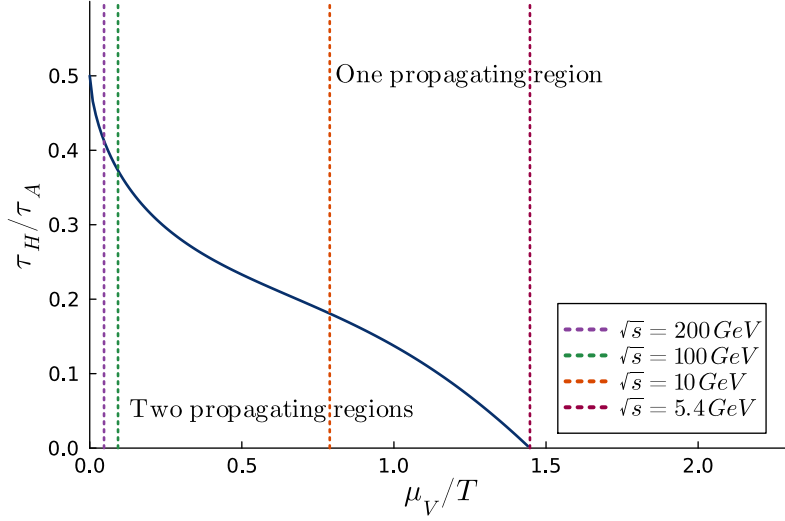


Figure 8: Solution to Eq. (4.31) showing the values of the parameters (blue curve) for which the spectrum of excitations contains a merging point. The curve divides the parameter space into two regions: Above the curve, the spectrum has only one branch of propagating modes; Below the curve, the spectrum displays two disconnected branches of propagating modes.

or two branches of propagating modes (c.f. Fig. 5). The chemical potential separating both regions was found to be $\mu_V = 1.4471 T$, and the wavenumber at which the two propagating branches merge $\tau_H \kappa_\Omega = 3.0642$. In this section, the presence of a finite axial relaxation time τ_A provides an extra parameter, and we expect to find a one-parameter family of such points. We will impose the existence of the merging point in order to find the curve separating both regions in terms of μ_V and τ_H/τ_A .

Similarly to Eq. (3.33), we write the characteristic Eq. (2.37) as

$$a(\alpha_V)\omega^3 + i\left(\frac{1}{\tau_A} + \frac{1}{\tau_H}\right)b(\alpha_V)\omega^2 - \left(c(\alpha_V)\kappa_\Omega^2 + \frac{\varsigma(\alpha_V)}{\tau_A\tau_H}\right)\omega - i\kappa^2\delta(\alpha_V, \tau_A, \tau_H) = 0, \quad (4.26)$$

where a, b and c are already given in (3.3), while ς and δ are defined as

$$\varsigma = \frac{2}{9T^2} \left(\sigma_A^\omega - \frac{1}{3}T^2\Delta H \right), \quad \delta = \frac{A^2\tau_A + B^2\tau_H}{3H\tau_A\tau_H}. \quad (4.27)$$

At the merging point $\omega_{\text{merg.}}$, all three solutions of Eq. (4.6) coincide. Therefore, the first and second derivatives of Eq. (4.6) with respect to ω also vanish:

$$\begin{aligned} 3a\omega_{\text{merg.}}^2 + 2ib\left(\frac{1}{\tau_A} + \frac{1}{\tau_H}\right)\omega_{\text{merg.}} - \left(c(\alpha_V)\kappa_{\text{merg.}}^2 + \frac{\varsigma(\alpha_V)}{\tau_A\tau_H}\right) &= 0, \\ 6a\omega_{\text{merg.}} + 2ib\left(\frac{1}{\tau_A} + \frac{1}{\tau_H}\right) &= 0. \end{aligned} \quad (4.28)$$

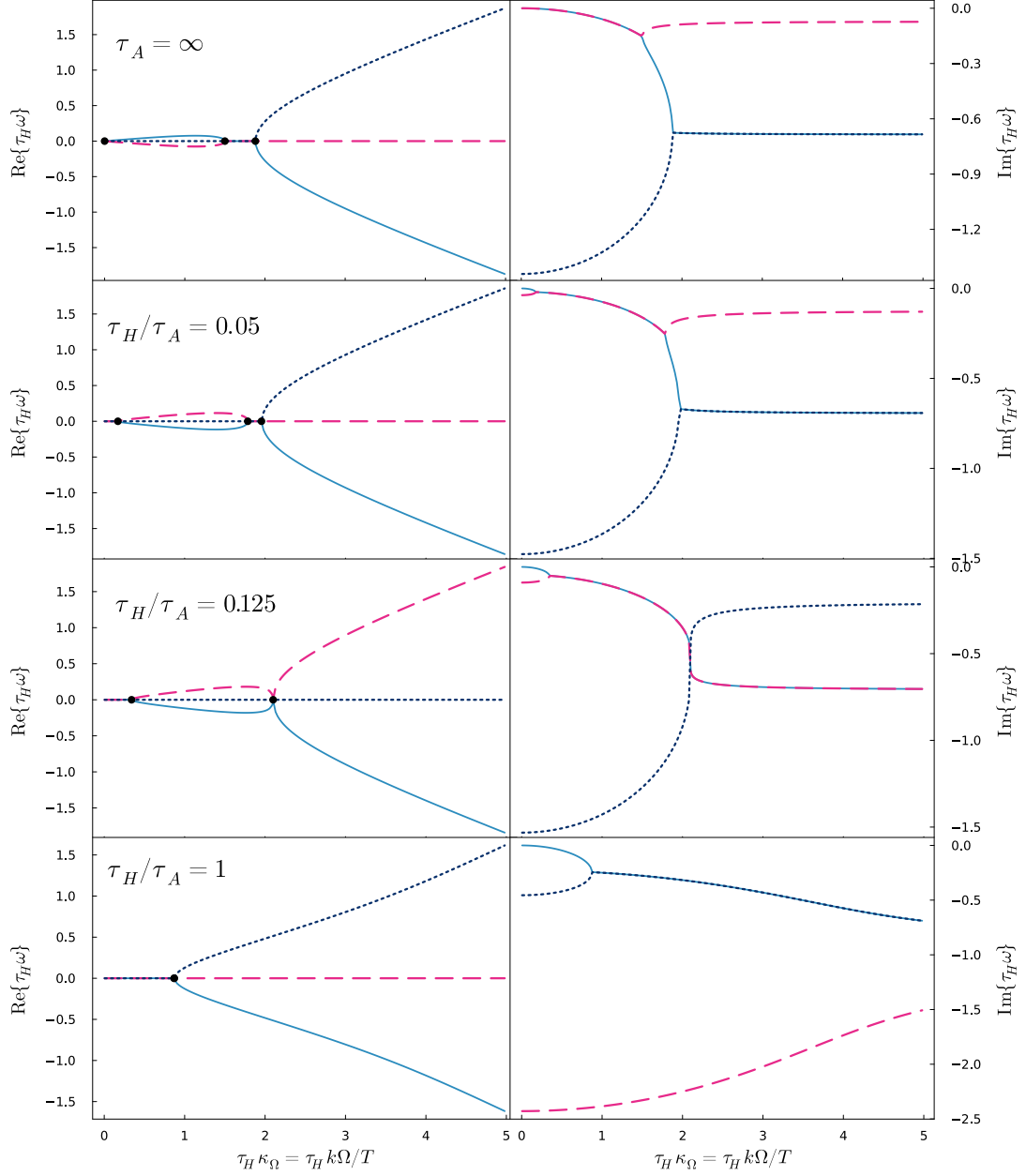


Figure 9: Collective excitations of the charge sector in the unpolarized plasma with dissipating axial and helical charges for $\mu_V/T = 1$. Each row shows the spectrum for a different value of the ratio τ_H/τ_A . The left (right) plots correspond to the real (imaginary) part of the dimensionless angular frequency, $\tau_H \omega$. The lines with identical styles at the left and right plots correspond to the same solutions. The black dots mark the points where propagation starts or stops.

Solving the above equations for $\omega_{\text{merg.}}$ and $\kappa_{\text{merg.}}^2$, gives

$$\omega_{\text{merg.}} = -\frac{2ib}{6a} \left(\frac{1}{\tau_A} + \frac{1}{\tau_H} \right) \quad \kappa_{\text{merg.}}^2 = \frac{b^2}{3ac} \left(\frac{1}{\tau_A} + \frac{1}{\tau_H} \right)^2 - \frac{\varsigma}{c} \frac{1}{\tau_A \tau_H}. \quad (4.29)$$

Replacing $\omega_{\text{merg.}}$ and $\kappa_{\text{merg.}}^2$ into Eq. (4.6) gives the equation in parameter space that determines the existence of the merging point:

$$\frac{b^3}{27a^2} \left(\frac{1}{\tau_A} + \frac{1}{\tau_H} \right)^3 - \frac{b^2\delta}{3ac} \left(\frac{1}{\tau_A} + \frac{1}{\tau_H} \right)^2 + \frac{\delta\zeta}{c} \frac{1}{\tau_A\tau_H} = 0. \quad (4.30)$$

From the definitions in Eqs. (3.3) and (4.6) we can rewrite the previous equation as

$$\frac{(\sigma_A^\omega)^3 \left(\frac{1}{\tau_A} + \frac{1}{\tau_H} \right)^3}{[(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2]} - \frac{9(\sigma_A^\omega)^2 \left(\frac{A^2}{\tau_H} + \frac{B^2}{\tau_A} \right) \left(\frac{1}{\tau_A} + \frac{1}{\tau_H} \right)^2}{[(A^2 + B^2)\sigma_A^\omega - 2AB\sigma_H^\omega]} + \frac{27 [(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2] \left(\frac{A^2}{\tau_H} + \frac{B^2}{\tau_A} \right)}{\tau_A\tau_H [(A^2 + B^2)\sigma_A^\omega - 2AB\sigma_H^\omega]} = 0, \quad (4.31)$$

where we have multiplied by $729 [(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2]$ and divided by $(\sigma_A^\omega - \frac{1}{3}T^2\Delta H)$ to have a more compact expression. Finally, multiplying Eq. (4.31) by τ_H^3 , it is manifest that it depends on the ratio of lifetimes τ_H/τ_A .

The solutions to Eq. (4.31) are shown in Figure 8. The spectra along the blue line feature a point where all three modes coincide, which appear when two branches of propagating modes merge together (see, e.g., Figure 5 in the previous section). We recover the result of the previous section that for a conserved axial charge, i.e., $\tau_H/\tau_A \rightarrow 0$, the merging point exists for $\mu_V \simeq 1.4471 T$. The curve divides the parameter space into two regions: below the curve, there will be two branches of propagating modes, as it happens in the second line of Figure 5; the complementary region contains only one branch of propagating modes, similarly to the last line in the same figure. Furthermore, there is a global maximum at $\mu_V/T = 0$: $\tau_A = 2\tau_H$. Thus, we can assert that for $\tau_A < 2\tau_H$, there will only be one branch of propagating modes, regardless of the value of the vector chemical potential.

In Figure 9, we show the spectrum of excitations for fixed $\mu_V/T = 1$ and varying the ratio of axial and helical lifetimes, which can be understood as varying τ_A for fixed τ_H . As anticipated from Figure 8, two branches of propagating modes exist for sufficiently small τ_H/τ_A . As we increase the ratio, both branches collide, and above a threshold value, there is only one branch of propagating modes. It should also be noted that for a non-conserved axial charge, i.e., finite τ_A , there are no propagating modes for small wavenumber.

The effect of the axial relaxation time on the perturbations of the axial chemical potential μ_A sourced by a Gaussian perturbation in the vector potential μ_V is exemplified in Fig. 10. We choose the same set of the parameters as in Fig. 2 for two different values of the helicity relaxation time, namely $c_h\tau_H/\sigma = 0.1$ and $c_h\tau_H/\sigma = 100$. The first one corresponds to the traditional Chiral Vortical Wave, while the second is closer to the Helical Vortical Wave. As expected, a decrease in the lifetime of the axial charge τ_A results in a damping of the perturbations in the axial chemical potential and the variations of τ_A do not substantially affect the velocity of the axial wave propagation.

5 Dissipative effects

In our considerations so far, we have treated the V/A/H fluid as ideal, ignoring dissipative corrections to the energy-momentum tensor $T^{\mu\nu}$ and the charge currents J_ℓ^μ . In a realistic

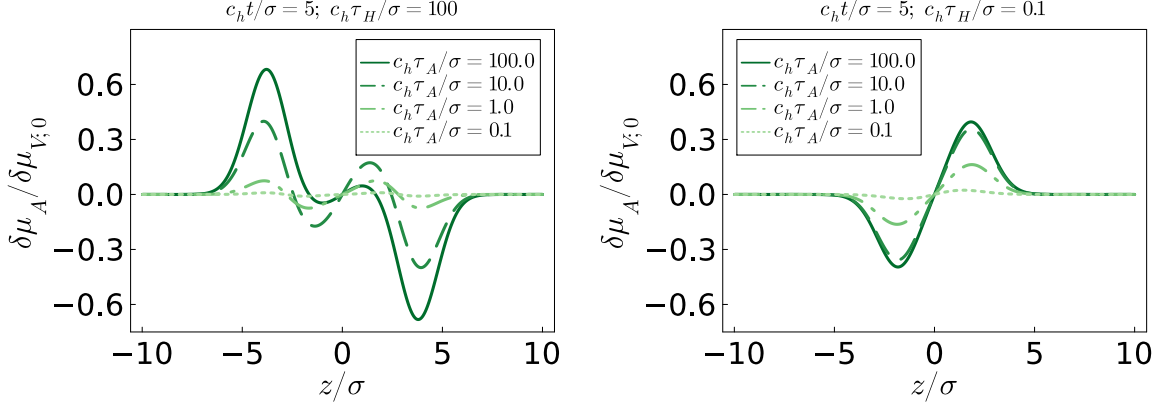


Figure 10: Perturbations in the axial chemical potentials at a fixed time $t = 5$ fm/c as we vary the axial relaxation time for (left) $c_h \tau_H/\sigma = 100$ (right) $c_h \tau_H/\sigma = 0.1$. The system is initialized with a Gaussian perturbation in the vector chemical potential $\delta\mu_V$ on top of a background vector chemical potential $\mu_V = 650$ MeV at temperature $T = 300$ MeV and angular frequency $\Omega = 6.6$ MeV.

fluid, dissipation occurs due to the inter-particle interaction, which drives the system towards thermodynamic equilibrium. Contrary to intuition, the perfect fluid limit is achieved when the interactions are sufficiently strong, taking place on time scales much shorter than the characteristic macroscopic time scale, such that equilibration happens almost instantaneously and the deviation of the fluid from thermal equilibrium is always negligible. Since the relaxation time τ_H arises due to helicity-violating pair annihilation (HVPA) two-to-two interactions, the perfect fluid limit implies also that $\tau_H \rightarrow 0$. The same limit implies also that the axial relaxation time τ_A , describing the dissipation of axial charge, should be taken to 0. In order to provide a more credible assessment of realistic V/A/H fluids, we will consider in this section the effect of dissipation on the spectrum of vortical waves.

5.1 Relaxation-time approximation

Dissipative corrections manifest themselves whenever global thermodynamic equilibrium is perturbed. In the Landau frame, when $T^\mu{}_\nu u^\nu = E u^\mu$, they lead to the modifications [33]

$$T^{\mu\nu} \rightarrow T^{\mu\nu} = T_0^{\mu\nu} + \pi_d^{\mu\nu} - \Pi_d \Delta^{\mu\nu}, \quad J_\ell^\mu \rightarrow J_\ell^\mu = J_{\ell;0}^\mu + V_{\ell;d}^\mu, \quad (5.1)$$

where Π_d , $V_{\ell;d}^\mu$ and $\pi_d^{\mu\nu}$ represent the scalar, vector and tensor dissipative degrees of freedom appearing at the level of $T^{\mu\nu}$ and J_ℓ^μ . By construction, $V_{\ell;d}^\mu$ and $\pi_d^{\mu\nu}$ are orthogonal to the fluid velocity, while $\pi_d^{\mu\nu}$ is taken to be traceless. Since we are dealing with a conformal (massless) fluid with $E = 3P$ and $T^\mu{}_\mu = E - 3P - 3\Pi_d = 0$, the bulk viscous pressure can be neglected, $\Pi_d = 0$. The remaining degrees of freedom contained in $V_{\ell;d}^\mu$ and $\pi_d^{\mu\nu}$ are *a priori* unconstrained and must be specified via constitutive relations.

A relativistic treatment of dissipative effects must be performed in a framework that preserves the hyperbolicity, causality and stability of the hydrodynamic equations [34, 35]. This can be achieved in the so-called Müller-Israel-Stewart hydrodynamics framework [36,

37], where $V_{\ell;d}^\mu$ and $\pi_d^{\mu\nu}$ become dynamical quantities obeying equations that essentially govern their relaxation towards their Navier-Stokes (first-order) asymptotic expressions,

$$V_{\ell;d}^\mu = \sum_{\ell'} \kappa_{\ell,\ell'} \nabla^\mu \alpha_{\ell'}, \quad \pi_d^{\mu\nu} = 2\eta \sigma^{\mu\nu}, \quad (5.2)$$

where η is the shear viscosity, $\kappa_{\ell\ell'}$ is the so-called diffusion matrix [38–40], while $\nabla^\mu = \Delta^{\mu\nu} \partial_\nu$ is the spatial gradient in the fluid rest frame, with $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ being the projector on the hypersurface orthogonal to u^μ . The quantities $V_{\ell;d}^\mu$ and $\pi_d^{\mu\nu}$ can be calculated from field theory via the Kubo formulas [41, 42], or in the frame of kinetic theory [40, 43, 44], once the inter-particle interactions have been specified. Generically, they are inversely proportional to the collision cross-section σ . The same cross-section allows one to define a characteristic relaxation time scale,

$$\tau_R \simeq \frac{T}{P\sigma}, \quad (5.3)$$

where we used P/T instead of the particle number n [45], since the latter is not conserved in a quantum fluid. We thus conclude that both η and $\kappa_{\ell\ell'}$ are proportional to the microscopic timescale τ_R . By similar arguments, the helicity and axial relaxation times τ_H and τ_A should also be proportional to τ_R . One should therefore consider η and $\kappa_{\ell\ell'}$ on an equal footing to τ_H and τ_A .

In MIS hydrodynamics, the Navier-Stokes relations (5.2) represent the asymptotic near-equilibrium, the long-wavelength limit of the dissipative quantities, and are therefore not valid if the fluid is too far from equilibrium or if the macroscopic gradients are too large compared to the microscopic relaxation time scale τ_R . For $k\tau_R \ll 1$, Eq. (5.2) provides a reasonable approximation and second- or higher-order corrections can be neglected. Of course, the framework becomes unreliable for small wavelengths (large k) and we will point out this limitation when relevant in the discussion below.

The evaluation of transport coefficients in theories with vector interactions is a daunting task, and we do not pursue such calculations in what follows [46, 47]. We will consider the qualitative effect of dissipation on the wave spectrum in the frame of kinetic theory under the relaxation-time approximation of Anderson and Witting [48]. Denoting by $f_{\mathbf{p},\lambda}^\sigma$ the number of particles/anti-particles ($\sigma = \pm 1$) with polarization λ , we postulate a kinetic equation of the form

$$p^\mu \partial_\mu f_{\mathbf{p},\lambda}^\sigma = -\frac{E_{\mathbf{p}} \cdot u}{\tau_R} \left(f_{\mathbf{p},\lambda}^\sigma - f_{\mathbf{p},\lambda}^{\text{eq};\sigma} \right), \quad (5.4)$$

where $E_{\mathbf{p}} = p \cdot u$ is the particle energy in the fluid rest frame, u^μ is the fluid four-velocity and

$$f_{\mathbf{p},\lambda}^{\text{eq};\sigma} = \left[\exp \left(\frac{p \cdot u - \mu_{\sigma,\lambda}}{T} \right) + 1 \right]^{-1}, \quad (5.5)$$

is the Fermi-Dirac distribution characterized by the particle four-momentum $p^\mu = (p^0, \mathbf{p})$ with $p^0 = |\mathbf{p}|$ for massless fermions, $\sigma = \pm 1$ distinguishes between particles and anti-particles, and $\lambda = \pm 1/2$ labels the particle helicity.

Due to its classical nature, Eq. (5.4) cannot account for the vortical effects that modify the charge currents J_ℓ^μ in the presence of finite vorticity ω^μ . In order to account for such

effects, one must employ a quantum kinetic equation, such as the one derived from the Wigner equation, leading to a modified equilibrium distribution [49]. Similarly, Eq. (5.4) is unable to account for the non-conservation of the helical and axial charges, thus our current simplified formulation provides access only to the dissipative quantities $V_{\ell;d}^\mu$ and $\pi_d^{\mu\nu}$.

The charge currents J_ℓ^μ and the stress-energy tensor $T^{\mu\nu}$ can be obtained from the distribution functions $f_{\mathbf{p},\lambda}^\sigma$ as follows:

$$J_\ell^\mu = \sum_{\sigma,\lambda} q_{\sigma,\lambda}^\ell \int dP f_{\mathbf{p},\lambda}^\sigma p^\mu, \quad T^{\mu\nu} = \sum_{\sigma,\lambda} \int dP f_{\mathbf{p},\lambda}^\sigma p^\mu p^\nu. \quad (5.6)$$

At equilibrium, when $f_{\mathbf{p},\lambda}^\sigma = f_{\mathbf{p},\lambda}^{\sigma;\text{eq}}$, we have $J_\ell^\mu = Q_\ell u^\mu$ and $T^{\mu\nu} = (E+P)u^\mu u^\nu - Pg^{\mu\nu}$. The conservation equations $\partial_\mu J_\ell^\mu = 0$ and $\partial_\mu T^{\mu\nu} = 0$ imply the Landau matching conditions $u_\mu J_\ell^\mu = Q_\ell$ and $u_\nu T^{\mu\nu} = Eu^\mu$, which we consider to hold in our analysis.

5.2 Chapman-Enskog expansion

We now employ the Chapman-Enskog procedure to derive an approximation for $f_{\mathbf{p},\lambda}^\sigma$, when the fluid is not too far from equilibrium [45] and $\delta f_{\mathbf{p},\lambda}^\sigma = f_{\mathbf{p},\lambda}^\sigma - f_{\mathbf{p},\lambda}^{\sigma;\text{eq}}$ can be considered small and of the order of the relaxation time, τ_R . The deviation from equilibrium $\delta f_{\mathbf{p},\lambda}^\sigma$ can be obtained from Eq. (5.4),

$$\delta f_{\mathbf{p},\lambda}^\sigma = -\frac{\tau_R}{E_{\mathbf{p}}} p^\mu \partial_\mu f_{\mathbf{p},\lambda}^\sigma = -\tau_R \left[\partial_\alpha \left(\frac{p^\alpha}{E_{\mathbf{p}}} f_{\mathbf{p},\lambda}^\sigma \right) + \frac{\partial_\alpha u_\beta}{E_{\mathbf{p}}^2} p^\alpha p^\beta f_{\mathbf{p},\lambda}^\sigma \right]. \quad (5.7)$$

Due to the multiplication with τ_R , the distribution on the RHS of the above equation can be approximated via $f_{\mathbf{p},\lambda}^\sigma \simeq f_{\mathbf{p},\lambda}^{\sigma;\text{eq}}$ and eventually we obtain

$$\begin{aligned} V_{\ell;d}^\mu &= -\tau_R \sum_{\sigma,\lambda} q_{\sigma,\lambda}^\ell \left(\partial_\alpha T_{\sigma,\lambda;1}^{\text{eq};\mu\alpha} + \partial_\alpha u_\beta T_{\sigma,\lambda;2}^{\text{eq};\mu\alpha\beta} \right), \\ \pi_d^{\mu\nu} &= -\tau_R \sum_{\sigma,\lambda} \left(\partial_\alpha T_{\sigma,\lambda;1}^{\text{eq};\mu\nu\alpha} + \partial_\alpha u_\beta T_{\sigma,\lambda;2}^{\text{eq};\mu\nu\alpha\beta} \right), \end{aligned} \quad (5.8)$$

where the following notation was introduced [50]:

$$T_{\sigma,\lambda;n}^{\text{eq};\mu_1\mu_2\cdots\mu_n} = \int \frac{dP}{E_{\mathbf{p}}^n} p_1^\mu p_2^\mu \cdots p_n^\mu f_{\mathbf{p},\lambda}^{\sigma;\text{eq}}. \quad (5.9)$$

In the massless case, when $p^2 = 0$, the integrals appearing in Eq. (5.8) are easily performed:

$$\begin{aligned} \sum_{\sigma,\lambda} q_{\sigma,\lambda}^\ell T_{\sigma,\lambda;1}^{\text{eq};\mu\nu} &= \frac{1}{3} Q_\ell (4u^\mu u^\nu - g^{\mu\nu}), \\ \sum_{\sigma,\lambda} \left(\frac{T_{\sigma,\lambda;1}^{\text{eq};\mu\nu\alpha}}{q_{\sigma,\lambda}^\ell T_{\sigma,\lambda;2}^{\text{eq};\mu\nu\alpha}} \right) &= \frac{1}{3} \left(\frac{E}{Q_\ell} \right) (6u^\mu u^\nu u^\lambda - u^\mu g^{\nu\lambda} - u^\nu g^{\lambda\mu} - u^\lambda g^{\mu\nu}), \\ \sum_{\sigma,\lambda} T_{\sigma,\lambda;2}^{\mu\nu\alpha\beta} &= \frac{1}{15} E [48u^\mu u^\nu u^\lambda u^\kappa - 6(u^\mu u^\nu g^{\lambda\kappa} + u^\mu u^\lambda g^{\nu\kappa} + u^\mu u^\kappa g^{\nu\lambda} + u^\nu u^\lambda g^{\mu\kappa} \\ &\quad + u^\nu u^\kappa g^{\mu\lambda} + u^\lambda u^\kappa g^{\mu\nu}) + g^{\mu\nu} g^{\lambda\kappa} + g^{\mu\lambda} g^{\nu\kappa} + g^{\mu\kappa} g^{\nu\lambda}]. \end{aligned} \quad (5.10)$$

Substituting the above results into Eq. (5.8), we obtain

$$\begin{aligned} V_{\ell;d}^\mu &= -\tau_R \left[u^\mu \partial_\lambda (Q_\ell u^\lambda) - \frac{1}{3} \nabla^\mu Q_\ell + Q_\ell D u^\mu \right], \\ \pi_d^{\mu\nu} &= -\tau_R \left[(4u^\mu u^\nu - g^{\mu\nu}) DP - u^\mu \nabla^\nu P - u^\nu \nabla^\mu P + \frac{24P}{5} (u^\mu u^\nu \theta + u^\mu D u^\nu + u^\nu D u^\mu) \right. \\ &\quad \left. - \frac{4P}{5} (g^{\mu\nu} \theta - \partial^\mu u^\nu - \partial^\nu u^\mu) \right], \end{aligned} \quad (5.11)$$

with $Df = u^\mu \partial_\mu f$ being the comoving derivative and $\nabla^\mu f = \Delta^{\mu\nu} \partial_\nu f$ the derivative in the fluid's rest frame.

The relations in Eq. (5.11) can be simplified by taking into account the conservation relations $\partial_\mu J_\ell^\mu = 0$ and $\partial_\mu T^{\mu\nu} = 0$, applied to zeroth order in τ_R , when $J_\ell^\mu \simeq Q_\ell u^\mu$ and $T^{\mu\nu} \simeq (E + P)u^\mu u^\nu - P g^{\mu\nu}$:

$$DQ_\ell + Q_\ell \theta = O(\tau_R), \quad DE + (E + P)\theta = O(\tau_R), \quad (E + P)Du^\mu - \nabla^\mu P = O(\tau_R). \quad (5.12)$$

Using the above relations into Eq. (5.11) gives

$$V_{\ell;d}^\mu = \tau_R \left(\frac{1}{3} \nabla^\mu Q_\ell - \frac{Q_\ell \nabla^\mu P}{E + P} \right), \quad \pi_d^{\mu\nu} = 2\eta \sigma^{\mu\nu}, \quad (5.13)$$

where $\sigma^{\mu\nu} = (\frac{1}{2} \Delta^{\mu\lambda} \Delta^{\nu\kappa} + \frac{1}{2} \Delta^{\nu\lambda} \Delta^{\mu\kappa} - \frac{1}{3} \Delta^{\mu\nu} \Delta^{\lambda\kappa}) \partial_\lambda u_\kappa$ is the shear stress tensor, and we identified the shear viscosity as

$$\eta = \frac{4}{5} \tau_R P. \quad (5.14)$$

While Eq. (5.13) is sufficient for our analysis, we present below the diffusion matrix for completeness. In the massless limit, the quantities Q_ℓ/T^3 and P/T^4 depend only on the ratios $\alpha_{\ell'} = \mu_{\ell'}/T$. Noting that $E + P = 4P$, it can be seen that the diffusion flux depends only on the derivatives of $\alpha_{\ell'}$. Employing $\partial P / \partial \alpha_{\ell'} = Q_{\ell'} T$, we get

$$V_{\ell;d}^\mu = \sum_{\ell'} \kappa_{\ell\ell'} \nabla^\mu \alpha_{\ell'}, \quad \kappa_{\ell\ell'} = \tau_R \left(\frac{1}{3} \frac{\partial Q_\ell}{\partial \alpha_{\ell'}} - \frac{Q_\ell Q_{\ell'} T}{E + P} \right). \quad (5.15)$$

5.3 Dispersion relations

We now add the dissipative contributions $V_{\ell;d}^\mu$ and $\pi_d^{\mu\nu}$ to the Landau-frame expressions for the charge currents and energy-momentum tensor given in Eq. (2.3):

$$\begin{aligned} J_\ell^\mu &= Q_\ell u^\mu + \sigma_\ell^\omega \omega^\mu + V_{\ell;d}^\mu, \\ T^{\mu\nu} &= (E + P) u^\mu u^\nu - P g^{\mu\nu} + \pi_d^{\mu\nu}. \end{aligned} \quad (5.16)$$

The dissipative quantities appearing above modify the conservation Eqs. (2.10) to

$$\begin{aligned} DQ_\ell + Q_\ell \theta + \omega^\mu \partial_\mu \sigma_\ell^\omega + \sigma_\ell^\omega \partial_\mu \omega^\mu + \partial_\mu V_{\ell;d}^\mu &= 0, \\ DE + (E + P)\theta + u_\mu \partial_\nu \pi_d^{\mu\nu} &= 0, \\ (E + P)Du^\mu - \nabla^\mu P + \Delta^\mu{}_\lambda \partial_\nu \pi_d^{\lambda\nu} &= 0, \end{aligned} \quad (5.17)$$

that can be cast with the help of the relations (5.15) into the following form in the rotating fluid:

$$\begin{aligned} \pi^{\mu\nu} &= \frac{2i\eta k}{3} \begin{pmatrix} 0 & -y\Omega & x\Omega & 0 \\ -y\Omega & 1 & 0 & -\frac{3}{2}yv\Omega \\ x\Omega & 0 & 1 & \frac{3}{2}xv\Omega \\ 0 & -\frac{3}{2}yv\Omega & \frac{3}{2}xv\Omega & -2 \end{pmatrix} e^{-i(\omega t - kz)} \delta u, \\ V_\ell^\mu &= -i\tau_R \begin{pmatrix} 0 \\ y\Omega\omega \\ -x\Omega\omega \\ k \end{pmatrix} e^{-i(\omega t - kz)} \left(\frac{1}{3} \delta Q_\ell - \frac{Q_\ell \delta P}{E + P} \right). \end{aligned} \quad (5.18)$$

Furthermore, taking into account that $u_\mu \partial_\nu \pi^{\mu\nu} = -\pi^{\mu\nu} \nabla_\nu u_\mu$ is of second order in perturbations, while $\Delta^\mu{}_\lambda \partial_\nu \pi^{\lambda\nu} = \frac{4\eta k^2}{3} \delta u(0, 0, 0, 1)^\mu e^{-i(\omega t - kz)}$, it is clear that the energy-momentum sector remains decoupled from the charge currents sector, since it involves only the amplitudes of perturbations in the velocity δu and the pressure δP :

$$\begin{pmatrix} -3\omega & 4Pk \\ k & -4P\omega - \frac{4}{3}i\eta k^2 \end{pmatrix} \begin{pmatrix} \delta P \\ \delta u \end{pmatrix} = 0. \quad (5.19)$$

The above equation reveals that the effect of the dissipating shear is to shift the angular frequencies of the acoustic modes by an imaginary quantity, such that the non-dissipative solution of Eq. (2.13) becomes

$$\omega_{R;\text{ac.}}^\pm = \pm k c_s(\eta) - \frac{ik^2\eta}{6P}, \quad c_s(\eta) = \frac{1}{\sqrt{3}} \sqrt{1 - \frac{k^2\eta^2}{12P^2}}. \quad (5.20)$$

In other words, one of the effects of the presence of a finite kinetic relaxation time τ_R is to set the attenuation of the acoustic waves due to the nonvanishing kinetic shear viscosity (5.14). In addition, Eqs. (5.14) and (5.20) reveal that the speed of sound in the dissipative fluid is slower than the one in its non-dissipative limit (2.13). The speed of the sound wave depends on the momentum k , which has an ultraviolet bound, implying that when $k\tau_R > 5\sqrt{3}/2$, the acoustic wave stops propagating. The latter effect is a feature of the first-order theory, as the inclusion of second-order terms may alter this behaviour [51].

What are the effects of a finite kinetic relaxation time for the propagation of perturbations in the charge sector? In the case of charge (non-)conservation, Eq. (2.12b) is modified to:

$$\left[\left(\omega + \frac{ik^2\tau_R}{3} \right) \delta_{\ell,\ell'} - \Omega \frac{\partial \sigma_\ell^\omega}{\partial Q_{\ell'}} \right] \delta Q_{\ell'} - \left(\frac{ik^2\tau_R Q_\ell}{4P} + k\Omega \frac{\partial \sigma_\ell^\omega}{\partial P} \right) \delta P - (kQ_\ell - 2\omega\Omega \sigma_\ell^\omega) \delta u = -\frac{iT^2 \delta \mu_\ell}{3\tau_\ell}. \quad (5.21)$$

As in the case of the perfect fluid, the charge modes have vanishing pressure and velocity fluctuations, $\delta P = \delta u = 0$, such that the last two terms above cancel. This leads to

$$\left[\left(\omega + \frac{ik^2\tau_R}{3} \right) \delta Q_\ell - k\Omega \delta \sigma_\ell^\omega \right] \Big|_{\delta P=0} = -\frac{iT^2 \delta \mu_\ell}{3\tau_\ell}. \quad (5.22)$$

Comparing the above relations with Eq. (2.14), we see that, as in the case of the dissipative shear stress, the charge diffusion has the effect of shifting the angular frequencies by an imaginary quantity. Denoting by ω_R the energy dispersion relation in the presence of dissipation, we have

$$\omega_R = \omega - \frac{ik^2\tau_R}{3}, \quad (5.23)$$

where ω is the mode angular frequency.

5.4 Estimation of damping due to kinetic effects

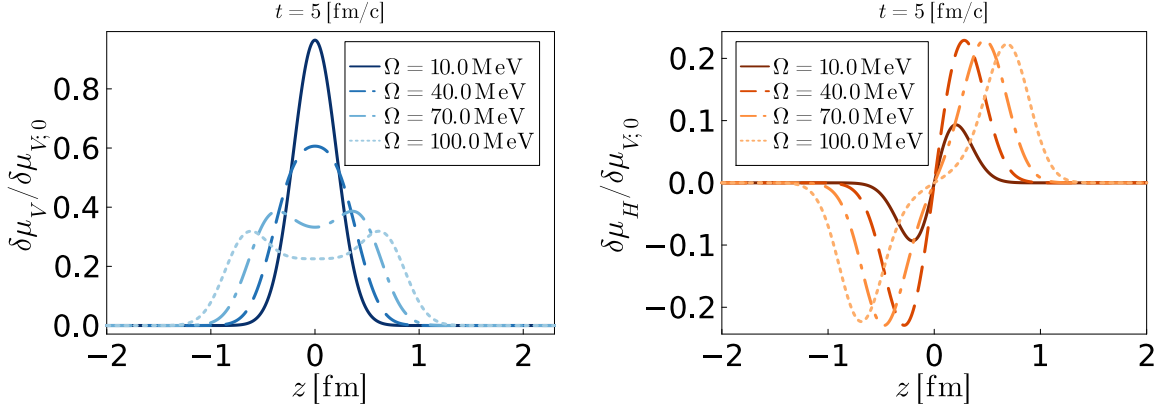


Figure 11: Perturbations in the chemical potentials at a fixed time $t = 5$ fm/c with finite helical ($\tau_H = 2.7$ fm/c) and axial ($\tau_A = 0.25$ fm/c) relaxation times as we increase the angular frequency Ω . The system is initialized with a Gaussian perturbation in the vector chemical potential $\delta\mu_V$ of width $\sigma = 0.2$ fm on top of a neutral unpolarized background $\mu_V = \mu_A = \mu_H = 0$ at temperature $T = 300$ MeV.

We now estimate the effect of damping in the setup of the quark-gluon plasma formed in heavy-ion collisions. As usual, we consider $T = 300$ MeV and $\mu_V = 0$. For definiteness, we consider a QGP with $N_f = 3$ flavours, for which the helicity relaxation time evaluates to $\tau_H = 2.7$ fm/c, according to Eq. (3.2). For the axial chemical potential, we take the estimate $\tau_A \simeq 0.25$ fm/c, obtained from first-principle simulations in Ref. [25].

Ignoring for the time being the kinetic dissipation (i.e., $\tau_R \rightarrow 0$), we estimate the propagation properties of the vortical waves discussed in the previous sections at these realistic values of the relaxation times. In the case of an ideal plasma, where the axial and helical relaxation times tend to infinity, the helical vortical wave propagates with the velocity c_h given in Eq. (3.9). Thus, over a time period equal to τ_H , the helical vortical wave propagates a distance

$$z = c_h\tau_H \simeq 0.38 \left(\frac{\Omega}{100 \text{ MeV}} \right) \text{ fm}. \quad (5.24)$$

The above estimate implies that the wave travels a noticeable distance when Ω is of the order of 10^2 MeV. At lower values of Ω , the dissipative effects will lead to the damping of

the fluctuations in polarization with no observable propagation. Moreover, the expected lifetime of the QGP fireball is of the order of 10 fm/c, after which the system undergoes the phase transition towards hadronic degrees of freedom, and our considerations related to vortical effects in a conformal plasma cease to apply.

To illustrate the effect of the rotation parameter Ω on the propagation properties of the vortical waves, we considered a system initialized with a Gaussian perturbation in the vector chemical potential $\delta\mu_V$ of width $\sigma = 0.2$ fm. Figure 11 shows the vector and helical perturbations at a time $t = 5$ fm/c after initialization. We do not show the axial perturbation, as in the neutral plasma, it is not excited. It can be seen that at small angular frequency Ω , the dissipative effects dominate the evolution of the perturbations while increasing Ω enhances the propagating features of the waves.

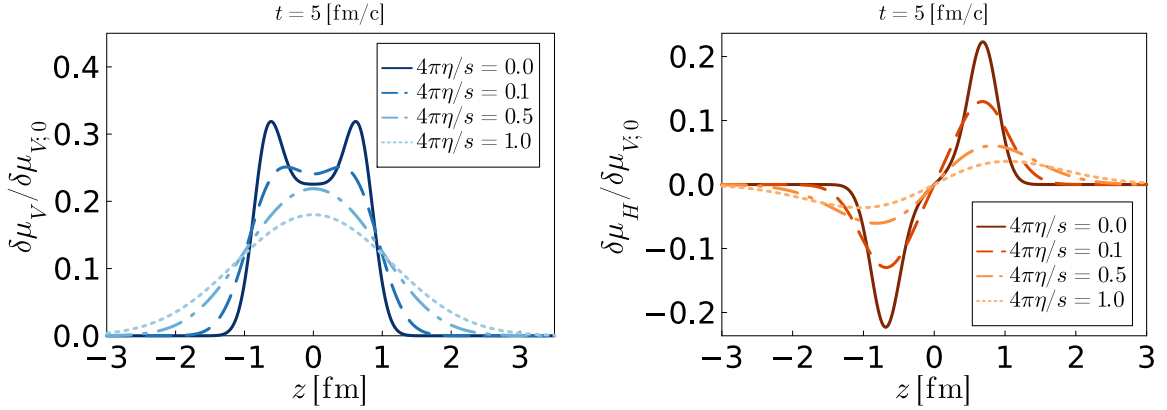


Figure 12: Perturbations in the chemical potentials at a fixed time $t = 5$ fm/c as we increase the damping due to kinetic effects, similar to the one of Fig. 11. We choose $\sigma = 0.2$ fm for the width of initial Gaussian perturbations in all the chemical potentials $\delta\mu_\ell$ on top of a neutral background $\mu_V = 0$ MeV at temperature $T = 300$ MeV and the angular frequency $\Omega = 100$ MeV for the finite helical ($\tau_H = 2.7$ fm/c) and axial ($\tau_A = 0.25$ fm/c) relaxation times.

We now estimate the effects of kinetic dissipation, governed by the kinetic relaxation time τ_R . To estimate τ_R , we consider that the ratio between the shear viscosity $\eta = \frac{4}{5}\tau_R P$ and the entropy density $s = (E + P - \vec{\mu} \cdot \vec{Q})/T$ is constant. According to Ref. [52], the ratio η/s should be bounded from below by $1/4\pi$ for strongly-interacting theories. Bayesian estimates based on matching hydrodynamics simulations to experimental data from heavy-ion collisions give $4\pi\eta/s \lesssim 1.5$ around $T = 150$ MeV [53], thus confirming this conjecture. Taking $4\pi\eta/s = 1$ for definiteness and considering the choice of parameters discussed above ($T = 300$ MeV, $\mu_V = 0$), we obtain

$$\tau_R = \frac{5\eta}{sT} \left(1 - \frac{\mu_\ell Q_\ell}{E + P} \right) \simeq 0.26 \text{ fm/c}, \quad (5.25)$$

which is very similar to the axial relaxation time τ_A . Contrary to the relaxation times τ_A and τ_H , which govern the dissipation of long-wavelength excitations, τ_R becomes important

in the ultraviolet limit, i.e. at short wavelengths, thus leading to the smearing of the sharp features of propagating waves. The particular form of the correction to the dispersion relation due to kinetic dissipation, implied by Eq. (5.23), indicates that the lifetime of a mode with wavenumber k is $\tau_d = 3/(k^2\tau_R)$. Considering now the smallest wavenumber that supports the propagation of the helical vortical wave, given in Eq. (3.10), we estimate τ_d as

$$\tau_d = \frac{12\tau_H^2 c_h^2}{\tau_R} \simeq 6.64 \left(\frac{\Omega}{100 \text{ MeV}} \right)^2 \text{ fm}/c. \quad (5.26)$$

Remarkably, the lifetime due to kinetic dissipation depends quadratically on the angular velocity Ω and becomes sizeable also around $\Omega \simeq 100 \text{ MeV}$. For this purpose, we will discuss the effect of varying η/s (or equivalently, τ_R) at this particular value of Ω .

Considering the configuration and parameters shown in Fig. 11, we now vary η/s while keeping $\Omega = 100 \text{ MeV}$ fixed. The case $\tau_R = 0$ is already depicted in Fig. 11. Figure 12 illustrate the effect of the kinetic dissipation as η/s from 0 to $1/(4\pi)$. The figure confirms that kinetic dissipation damps short wavelength perturbations, smoothing the propagating peaks and progressively erasing the propagating features of the waves as we increase the η/s ratio.

Before ending this section, we come back to our initial considerations and take into account realistic values Ω . As pointed out in Ref. [54], the angular velocity is estimated at $\Omega \simeq 6.6 \text{ MeV}$ for the fireball created in non-central heavy-ion collisions. At this value of Ω , the propagation speed of the helical vortical wave becomes a mere

$$c_h \Big|_{\Omega=6.6 \text{ MeV}} = \frac{6\Omega \ln 2}{\pi^2 T} \simeq 0.0093 c. \quad (5.27)$$

Over the lifetime $\tau_{QGP} \simeq 10 \text{ fm}/c$ of the QGP fireball, the HVW transports perturbations over a tiny distance of $c_h \tau_{QGP} \simeq 0.093 \text{ fm}$. The threshold wavenumber over which perturbations can propagate is set by the helical relaxation time (3.10): $k_{\text{th}} = 1/(2\tau_H c_h) \simeq 20 \text{ fm}^{-1}$. At this wavenumber, the lifetime due to kinetic dissipation is decreased to $\tau_d \simeq 0.029 \text{ fm}/c$, which gives an infinitesimal value for the distance over which the wave can propagate: $c_h \tau_d \simeq 2.7 \times 10^{-4} \text{ fm}$. Clearly, in such conditions, no signs of propagation due to the vortical effects can be seen: at small wavelengths, restriction of the wave propagation appears as a result of kinetic dissipation, while at large wavelengths, the wave overdamping occurs due to the non-conservation of polarization-related charges.

6 Summary and Conclusions

In our paper, we further developed an alternative approach to chiral fermionic systems, put forward in Refs. [13, 14, 16], which extends the traditional premise that chiral fluids are fully described by the pair of vector and axial local charges. In the previous works, we have demonstrated that the incorporation of the helical degree of freedom of non-interacting relativistic fermions – which is a conserved degree of freedom related but not equal to the axial charge in general – enriches the hydrodynamic spectrum of the system via interplay for Axial and Helical Vortical Effects visualized in an intuitively appealing pictorial form in Fig. 1 of Ref. [16].

Helical and Axial Waves. In Ref. [14], we uncovered a new type of gapless mode, the Helical Vortical Wave, which represents a coherent propagating excitation of vector and helical charge densities and their currents. We also found the Axial Vortical Wave [16], which is driven by fluctuations in the axial charge density that propagates in the background of a large axial chemical potential μ_A . Moreover, a background with non-zero μ_A induces non-reciprocity in the spectrum, namely that waves travel with different velocities along and opposite to the direction of the angular velocity. Contrary to the Chiral Vortical Wave [10], which operates in vector and axial sectors, the Axial and Helical Vortical Waves emerge in an electrically neutral regime characterized by a vanishing vector chemical potential. In the opposite, high-density limit, these waves merge and form a mixed Axial-Vortical Wave [16]. In this work, we question whether these hydrodynamic modes could withstand realistic conditions that appear in the interacting vortical quark-gluon plasmas.

Relaxation time approximation. In our analysis, we used the relaxation time approximation in which interactions can phenomenologically be described in a relaxation time approximation via the introduction of the helical, τ_H , and axial, τ_A , and kinetic, τ_R , relaxation times. The first two quantities control the time scale of the non-conservation of the corresponding charges, while the third one determines the dissipative evolution of occupation numbers out of the thermal equilibrium. The relaxation time of the vector charge is taken to be infinite as the electric charge is a conserved quantity. This property allowed us to study finite-density systems at non-zero vector chemical potential. Accordingly, in this paper, we do not consider the linear hydrodynamic waves at a finite background axial density since this case is not consistent with a finite axial relaxation time.

Helical charge relaxation. We first analyzed the genuine effects of finite helical relaxation time τ_H in the limit when other relaxation effects are absent, $\tau_A = \tau_R = \infty$ (Section 3). In the dilute plasma regime, where the density of the vector chemical potential is small compared to temperature, $\mu_V \ll T$, and the other chemical potentials are zero, $\mu_A = \mu_H = 0$, we get an expected effect: in addition to the damping due to the finite τ_H , the Helical Vortical Wave evolves to a purely diffusive mode when the half-period of the wave exceeds the helical relaxation time τ_H . In other words, the helical relaxation cuts off the low-frequency “helical” modes, corresponding predominantly to the helical-vortical charge fluctuations of the Helical Vortical Wave. The “axial” mode, a leftover of the non-reciprocal Axial Vortical Wave branch, is purely diffusive in this regime.

In the dense plasma, at $\mu_V \gtrsim T$, the magnitude of the helical relaxation time determines the qualitative properties of the hydrodynamic spectrum. For slowly relaxing helical plasma with a large τ_H , the oscillations of helical and axial charge densities – which are indistinguishable in this limit⁵ – and the vector charge oscillations merge into a common hydrodynamic excitation, the Axial-Helical Vortical Wave.

The Chiral Vortical Wave emerges in the vanishing- τ_H limit where the helical degree of freedom disappears from the spectrum (gets frozen). It is worth stressing that the Chiral

⁵The helical and axial charges of an ensemble of fermions carrying the same vector charge (i.e. either fermions or anti-fermions) are identical up to a sign since the same statement is valid for the helical and axial charges of a single given particle (or anti-particle).

Vortical Wave and the mixed Axial-Helical Vortical Wave, both existing in the high-density limit at a finite τ_H , are different excitations, as is reflected in their different propagation velocities and the chemical content of these waves.

Notice also that for a degenerate ($\mu_V \gg T$) axially conserved ($\tau_A \rightarrow \infty$) chiral fluid, the limit of the fast helical relaxation, $\tau_H \rightarrow 0$, is accompanied by the redistribution of the degrees of freedom. At large but finite τ_H (approximately conserved helicity), the spectrum is represented by two non-reciprocal modes of the Axial-Helical Wave that mixes almost indistinguishable helical-axial oscillations with fluctuations in the vector charge density. However, at small τ_H , these modes get transformed to a non-propagating helical mode (dissipation of all charges driven by the helical charge relaxation) and two bi-directionally propagating axial-vector modes (the Chiral Vortical Wave).

At a finite (non-zero and non-infinite) helical relaxation time, the interplay of the axial and helical modes in dense fermionic matter becomes rather non-trivial, as summarized in Fig. 4. For the chemical potential below a certain limiting value, the wave spectrum contains three regions: (i) two propagating waves with a small wavenumber (a large wavelength) are the generalization of the Chiral Vortical Wave (axial-vector oscillations) that includes an accompanying helical mode; (ii) two propagating waves with a large wavenumber which is essentially a generalization of the Helical Vortical Wave (helical-vector oscillations) to the finite-density case; (iii) an intermediate region with a moderate wavelength in which both mentioned modes are diffusive. Thus, we have two branches of hydrodynamic excitations. When the vector chemical potential exceeds the mentioned critical value, the diffusive region (iii) disappears, and the (i) CVE and (ii) HVW domains merge, constituting a single branch of solutions. In all mentioned regions, these two helical modes are accompanied by a single diffusive axial non-propagating mode, a reminder of the Axial Vortical Wave.

Simultaneous helical and axial charge relaxations. The accounting for a finite axial relaxation time μ_A makes the whole picture more realistic and, somewhat expectedly, more intricate. While we refer an interested reader to Section 5 for more details, here we describe the main features of the hydrodynamic waves with finite helical and axial relaxation times.

For a non-conserved axial charge, there are no propagating modes for small wavenumbers (large wavelengths). This property contrasts with the picture at the conserved axial charge (and non-conserved helical charge), where two modes propagate for all wavelengths at high vector chemical potential. At the same time, the relaxation time of the axial and/or helical charges does not affect qualitatively the wave spectrum in the large wavenumber limit compared to the previously studied case of the conserved axial charge.

If the helical charge dissipates rapidly (faster than the helical charge), then a single non-propagating helical mode dissipates rather quickly, while the two axial modes propagate at small wavevectors and are purely diffusive otherwise. In the case of conserved helicity, $\tau_H \rightarrow \infty$ (and at finite τ_A), two non-dissipative, for all wavelengths, helical modes co-exist along with the purely dissipative axial mode.

The ratio of the relaxation times, τ_H/τ_A , determines whether we have two branches of thermodynamic excitations (large-wavenumber Helical and low-wavenumber Chiral Vortical

Waves separated by a non-propagating domain of wavelengths) or a single branch that merges these excitations. The corresponding diagram is shown in Fig. 8.

Kinetic dissipation. We also considered the effects of kinetic dissipation produced by the corrections to the energy-momentum tensor and charge currents due to interactions in the fluid. In our work, these corrections are also taken into account in the relaxation time approximation governed by the kinetic relaxation time τ_R within the relativistically consistent framework of Müller-Israel-Stewart hydrodynamics. The presence of the kinetic dissipation leads to the damping of the high-wavenumber modes. This property, together with the fact that for a non-conserved axial charge, there are no propagating modes for small wavenumbers, poses a stringent restriction to the existence of the vortical waves in realistic plasmas.

Consequences for quark-gluon plasma. For the environment of the quark-gluon plasma produced in the current experiments in relativistic heavy-ion collisions, the estimations made in Section 5 do not favour the observation of the Helical Vortical Waves. It appears that the angular velocity of the plasma is too small, so the wave propagates too slowly (about 1% of the speed of light), implying that during the lifetime of the plasma fireball, the wave advances over a phenomenologically irrelevant distance of a small fraction of a fermi. However, it appears that even this estimation is too optimistic because of the damping effects. The helical relaxation time sets an upper bound of the wavelength of the propagating wave because when the wavelengths are too large, wave propagation gets restricted as a result of the non-conservation of the helical charge. However, even this wavelength appears to be too small as the kinetic dissipation effects – that increase with the decrease of the wavelength – appear to be too strong so that they damp the wave efficiently. Therefore, we conclude that helical vortical waves cannot propagate in the realistic environment of quark-gluon plasmas.

It is worth mentioning that we do not consider the effect of finite electric conductivity that promotes a diffusive electromagnetic backreaction of an oscillating electric charge density generated by the waves. This electromagnetic effect can be relevant for all hydrodynamic excitations that involve oscillations of the vector chemical potential. An analysis of the electromagnetic backreaction has been performed for the Chiral Magnetic Wave in the background magnetic field in Ref. [21] and for the longitudinal sound wave, a circularly polarized vortical wave and diffusive modes, among others, in a vortical background in Ref. [23]. It appears that these waves are overdamped in quark-gluon plasma, which aligns well with our conclusions made for the helical and axial hydrodynamic excitations. Moreover, a finite electric conductivity will further limit the propagation of helical and axial waves as they also contain a vector component. Therefore, our theoretical analysis gives an upper bound on the dissolution time of these modes. On the other hand, the electromagnetic effects are irrelevant to the charge-neutral Axial Vortical excitations, which will be discussed in a separate publication.

Acknowledgments. This work is supported by the European Union - NextGenerationEU through grant No. 760079/23.05.2023, funded by the Romanian ministry of re-

search, innovation and digitalization through Romania's National Recovery and Resilience Plan, call no. PNRR-III-C9-2022-I8.

A Dissipation basis: charge vs. chemical potential

In section 3, we discussed the collective excitations of the unpolarized plasma when the non-conservation of helical charge is included through $\dot{Q}_H \propto -\mu_H/\tau_H$. In this Appendix, we review the results for the inclusion of dissipation as $\dot{Q}_H \propto -Q_H/\tau_H$ and show that it unavoidably gives rise to instability.

In order to see that the two approaches are inequivalent, let us consider the variation of Q_H with respect to small perturbations in T and $\mu_{V/A/H}$. For simplicity, we restrict ourselves to the large temperature expression for Q_H . From Eq. (2.20) with $\ell = H$ (see also Eq.(3.5b) in Ref. [16]), we find that

$$Q_H = \frac{\mu_H T^2}{3} + \frac{4T \ln 2}{\pi^2} \mu_A \mu_V + \frac{\mu_H (3\mu_V^2 + 3\mu_A^2 + \mu_H^2)}{3\pi^2} + O(T^{-1}) \quad (\text{A.1})$$

The fluctuations of the helicity charge δQ_H receive non-trivial contributions from fluctuations in both the helical and the axial charges, even in the case of an unpolarized (but not necessarily neutral, $\mu_V \neq 0$) background,

$$\delta Q_H \Big|_{\mu_A = \mu_H = 0} = \left(\frac{T^2}{3} + \frac{\mu_V^2}{\pi^2} \right) \delta \mu_H + \frac{4 \ln 2}{\pi^2} \mu_V T \delta \mu_A + O(T^{-1}). \quad (\text{A.2})$$

Therefore, in the prescription for relaxation given by $\nabla_\mu J_H^\mu = -Q_H/\tau_H$ there is an extra term coming from the crossed susceptibility $\chi_{HA} = \partial Q_H / \partial \mu_A$ which is responsible for a relaxation towards a state with $Q_H = 0$, which does not necessarily enforce $\mu_H = 0$. On the contrary, the prescription used in the main text $\nabla_\mu J_H^\mu = -T^2 \mu_H / 3\tau_H$ does lead to a relaxation towards $\mu_H = 0$. We proceed now to discuss the consequences of introducing dissipation on the basis of charge.

The (non)-conservation equations for the charge sector become

$$\nabla_\mu J_V^\mu = 0, \quad \nabla_\mu J_A^\mu = 0, \quad \nabla_\mu J_H^\mu = -\frac{Q_H}{\tau_H}. \quad (\text{A.3})$$

The matrix \mathbb{M} is now slightly different from that of section 3. In particular:

$$\frac{k}{T^2} \mathbb{M} = \omega \mathbb{M}_\omega - \kappa_\Omega \mathbb{M}_\Omega + \frac{2i\sigma_A^\omega}{\tau_H T^2} \mathbb{I}_H + \frac{2i\sigma_H^\omega}{\tau_H T^2} (\delta_{\ell H} \delta_{\ell' A}). \quad (\text{A.4})$$

In this case, the matrix \mathbb{M} in Eq. (3.3) changes to

$$\frac{1}{T^2} \mathbb{M}_{\ell\ell'} = \omega \mathbb{M}_\omega - \kappa_\Omega \mathbb{M}_\Omega + \frac{2i\sigma_A^\omega}{\tau_H T^2} \mathbb{I}_H + \frac{2i\sigma_H^\omega}{\tau_H T^2} (\delta_{\ell H} \delta_{\ell' A}). \quad (\text{A.5})$$

Its determinant can be evaluated as follows:

$$\det \left(\frac{1}{T^2} \mathbb{M} \right) = \det \left(\omega \mathbb{M}_\omega - \kappa_\Omega \mathbb{M}_\Omega + \frac{2i\sigma_A^\omega}{\tau_H T^2} \mathbb{I}_H \right) - \frac{2i\sigma_H^\omega}{\tau_H T^2} \left| \begin{array}{cc} \frac{2\omega}{T^2} \left(\sigma_A^\omega - \frac{T^2}{3} \Delta H \right) & -\frac{\kappa_\Omega}{H} B \\ -\kappa_\Omega A & \frac{2\omega}{T^2} \sigma_H^\omega \end{array} \right|, \quad (\text{A.6})$$

where the first term can be obtained by substituting $\tau_H \rightarrow \tau_H \times T^2/(6\sigma_A^\omega)$ in Eq. (3.4).

Accordingly, the master equation $\det(\mathbb{M}) = 0$ gets modified to

$$\begin{aligned} & \frac{2\omega}{T^2} \left\{ \left(\frac{2\omega}{T^2} \right)^2 \left(\sigma_A^\omega - \frac{T^2}{3} \Delta H \right) [(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2] - \frac{\kappa_\Omega^2}{H} [(A^2 + B^2)\sigma_A^\omega - 2AB\sigma_H^\omega] \right\} \\ & + \frac{i}{3\tau_H} \left[\left(\frac{2\omega}{T^2} \right)^2 \frac{6}{T^2} \left(\sigma_A^\omega - \frac{T^2}{3} \Delta H \right) [(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2] - \frac{\kappa_\Omega^2 A}{H} \frac{6}{T^2} (A\sigma_A^\omega - B\sigma_H^\omega) \right] = 0. \end{aligned} \quad (\text{A.7})$$

We can solve the previous equation perturbatively in κ_Ω . Let us expand ω in powers of κ_Ω :

$$\omega = \omega_0 + \omega_1 \kappa_\Omega + \dots \quad (\text{A.8})$$

The master equation to zeroth order in κ_Ω reads:

$$\left\{ \left(\frac{2\omega_0}{T^2} \right)^2 \left(\sigma_A^\omega - \frac{T^2}{3} \Delta H \right) [(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2] \right\} \left(\frac{2\omega_0}{T^2} + \frac{2i}{\tau_H T^2} \right) = 0. \quad (\text{A.9})$$

There is a non-trivial and purely dissipating solution $\omega_0 = -i/\tau_H$ and a double trivial root $\omega_0 = 0$. We now compute the corrections to the trivial root from the next to the leading order for the master equation:

$$\frac{2i}{\tau_H T^2} \left[\left(\frac{2\tau_H \omega_1}{T^2} \right)^2 \left(\sigma_A^\omega - \frac{T^2}{3} \Delta H \right) [(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2] - \frac{A}{H} (A\sigma_A^\omega - B\sigma_H^\omega) \right] \kappa_\Omega^2 = 0 \quad (\text{A.10})$$

Therefore, we find

$$\omega_1^\pm = \pm \frac{T^2}{2\tau_H} \sqrt{\frac{A(A\sigma_A^\omega - B\sigma_H^\omega)}{H \left(\sigma_A^\omega - \frac{T^2}{3} \Delta H \right) [(\sigma_A^\omega)^2 - (\sigma_H^\omega)^2]}}. \quad (\text{A.11})$$

The quantity $A(A\sigma_A^\omega - B\sigma_H^\omega)$ is negative semi-definite, whereas the remaining factor inside the square root is positive semi-definite. Specifically, from the definitions, we find $\sigma_A^\omega > \sigma_H^\omega$ and

$$H = \frac{15\alpha_V^4 + 30\pi^2\alpha_V^2 + 7\pi^4}{15\alpha_V^2 + 7\pi^2} > 0, \quad \frac{1}{T^2} \left(\sigma_A^\omega - \frac{T^2}{3} \Delta H \right) = \frac{15\alpha_V^4 + 6\alpha_V^2\pi^2 + 7\pi^4}{90\alpha_V^2 + 42\pi^2} > 0, \quad (\text{A.12})$$

where we have written $\alpha_V = \mu_V/T$. The behaviour of $A(A\sigma_A^\omega - B\sigma_H^\omega)$ at small chemical potential is given by

$$\frac{1}{T^2} A(A\sigma_A^\omega - B\sigma_H^\omega) \simeq -0.000432\alpha_V^2 < 0 \quad (\text{A.13})$$

As a result, ω_1^\pm come in complex conjugate pairs, and there is one unstable mode at finite vector chemical potential. While the previous result is derived for small chemical potential, it can be verified (see Fig. 13) that $A(A\sigma_A^\omega - B\sigma_H^\omega)$ is indeed negative definite for any non-zero chemical potential, and thus, the instability always appears for small wavenumber.

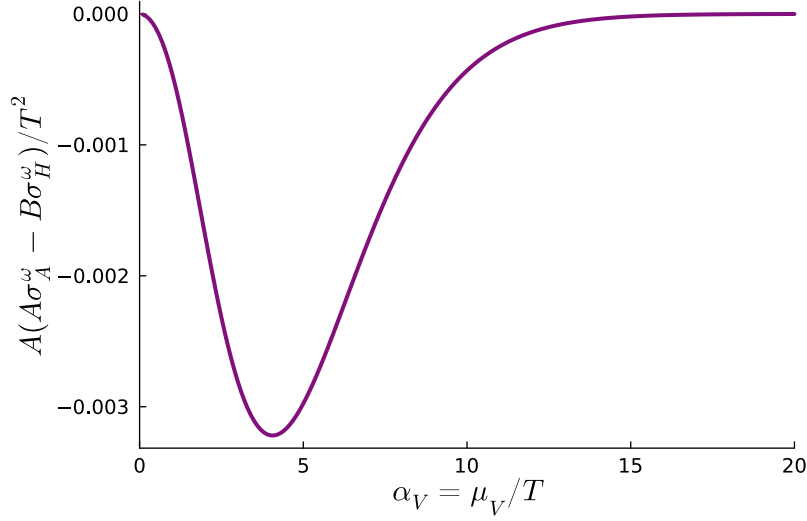


Figure 13: Explicit demonstration that the left hand side of Eq. (A.13) is negative definite and therefore there is always one unstable mode at small wavenumber.

B Helicity relaxation time at finite chemical potential

In this section of the appendix, we extend the calculation of the helicity relaxation time τ_H presented Ref. [14] to the case of an unpolarized, charged plasma, i.e. at finite vector chemical potential $\mu_V = \alpha_V T$. We start from Eqs. (101), (102) and (104) of Ref. [14]:

$$\begin{aligned} \frac{dQ_H}{dt} &= g \sum_{\lambda, \sigma} 2\sigma \lambda \int dP C_{\text{HVPA}}[f], \\ C_{\text{HVPA}}[f] &= \int dK dP' dK' \delta^4(p + k - p' - k') s (2\pi)^6 \frac{g\alpha_{\text{QCD}}^2}{18E_{\text{cm}}^2} (1 - \cos \theta_{\text{cm}}) \\ &\quad \times [f_{\mathbf{p}', -\lambda}^\sigma f_{\mathbf{k}', \lambda}^{-\sigma} \tilde{f}_{\mathbf{p}, \lambda}^\sigma \tilde{f}_{\mathbf{k}, -\lambda}^{-\sigma} - f_{\mathbf{p}, \lambda}^\sigma f_{\mathbf{k}, -\lambda}^{-\sigma} \tilde{f}_{\mathbf{p}', -\lambda}^\sigma \tilde{f}_{\mathbf{k}', \lambda}^{-\sigma}], \end{aligned} \quad (\text{B.1})$$

with $s = E_{\text{cm}}^2 = (p + k)^2 = (p' + k')^2$ being the center-of-mass energy and θ_{cm} the angle between \mathbf{p}' and \mathbf{p} , measured in the center of mass frame, satisfying

$$1 - \cos \theta_{\text{cm}} = \frac{4p \cdot p'}{E_{\text{cm}}^2}. \quad (\text{B.2})$$

We now consider that the plasma is near thermal equilibrium at finite T and μ_V . A small helicity imbalance can be modelled via the Fermi-Dirac distribution,

$$f_{\mathbf{p}, \lambda}^\sigma \simeq f_{\mathbf{p}, \lambda}^{\text{eq}; \sigma} \simeq f_{0\mathbf{p}}^\sigma + 2\lambda \sigma \beta \mu_H f_{0\mathbf{p}}^\sigma \tilde{f}_{0\mathbf{p}}^\sigma, \quad (\text{B.3})$$

where $f_{0\mathbf{p}}^\sigma = [e^{\beta E_{\mathbf{p}} - \sigma \alpha_V} + 1]^{-1}$ is the equilibrium distribution for a charged, unpolarized fluid, thus extending Eq. (105) of Ref. [14] for the neutral fluid to the case of a charged fluid. Note that in Eq. (107) of Ref. [14], Eq. (B.1) was replaced by $dQ_H/dt = -Q_H/\tau_H$, which is valid under the assumption of a neutral background with a small helicity imbalance, i.e.

$\mu_V = \mu_A = 0$ and $|\mu_H| \ll T$, when $Q_H \simeq \mu_H T^2/3$. In this work, we consider the case when μ_V is arbitrary, such that instead Eq. (B.1) becomes

$$\frac{dQ_H}{dt} = -\frac{\mu_H T^2}{3\tau_H}, \quad (\text{B.4})$$

with

$$\begin{aligned} \tau_H^{-1} = & \frac{8}{3}(2\pi)^6 g \alpha_{\text{QCD}}^2 \beta^3 \int dP dK dP' dK' (1 - \cos \theta_{\text{cm}})^2 \delta^4(p + k - p' - k') \\ & \times \frac{1}{4} \sum_{\sigma=\pm 1} f_{0\mathbf{p}}^\sigma f_{0\mathbf{k}}^{-\sigma} \tilde{f}_{0\mathbf{p}'}^\sigma \tilde{f}_{0\mathbf{k}'}^{-\sigma} (\tilde{f}_{0\mathbf{p}}^\sigma + \tilde{f}_{0\mathbf{k}}^{-\sigma} + f_{0\mathbf{p}'}^\sigma + f_{0\mathbf{k}'}^{-\sigma}). \end{aligned} \quad (\text{B.5})$$

Noting that, under the conservation of total four-momentum, $f_{0\mathbf{p}}^\sigma f_{0\mathbf{k}}^{-\sigma} \tilde{f}_{0\mathbf{p}'}^\sigma \tilde{f}_{0\mathbf{k}'}^{-\sigma} (f_{0\mathbf{p}'}^\sigma + f_{0\mathbf{k}'}^{-\sigma}) = f_{0\mathbf{p}}^\sigma f_{0\mathbf{k}}^{-\sigma} \tilde{f}_{0\mathbf{p}'}^\sigma \tilde{f}_{0\mathbf{k}'}^{-\sigma} (f_{0\mathbf{p}}^\sigma + f_{0\mathbf{k}}^{-\sigma})$, the expression between the parentheses above evaluates to

$$\tilde{f}_{0\mathbf{p}}^\sigma + \tilde{f}_{0\mathbf{k}}^{-\sigma} + f_{0\mathbf{p}'}^\sigma + f_{0\mathbf{k}'}^{-\sigma} \rightarrow \tilde{f}_{0\mathbf{p}}^\sigma + \tilde{f}_{0\mathbf{k}}^{-\sigma} + f_{0\mathbf{p}}^\sigma + f_{0\mathbf{k}}^{-\sigma} = 2, \quad (\text{B.6})$$

where we used the property $\tilde{f}_{0\mathbf{p}}^\sigma = 1 - f_{0\mathbf{p}}^\sigma$. Writing now as in Eq. (159),

$$\tau_H^{-1} = \frac{8}{3}(2\pi)^6 g \alpha_{\text{QCD}}^2 \beta^3 \int dP dK \frac{1}{2} \sum_{\sigma} f_{0\mathbf{p}}^\sigma f_{0\mathbf{k}}^{-\sigma} I_{\Omega}^{\sigma}, \quad (\text{B.7})$$

with

$$I_{\Omega}^{\sigma} = \int dP' dK' (1 - \cos \theta_{\text{cm}})^2 \delta^4(p + k - p' - k') \tilde{f}_{0\mathbf{p}'}^\sigma \tilde{f}_{0\mathbf{k}'}^{-\sigma}, \quad (\text{B.8})$$

the k' and p' outgoing momenta can be boosted in the center of mass frame, where the delta function reads $\delta(E_{\text{cm}} - p'_0 - k'_0) \delta^3(-\mathbf{p}' - \mathbf{k}')$. The dK' integral can be performed automatically, leading to

$$I_{\Omega}^{\sigma} = \frac{1}{(2\pi)^6} \int_0^\infty dp' \int d\Omega_{\mathbf{p}'} \delta(E_{\text{cm}} - 2p'_0) (1 - \cos \theta_{\text{cm}})^2 \tilde{f}_{0\mathbf{p}'}^\sigma \tilde{f}_{0\mathbf{k}'}^{-\sigma} \quad (\text{B.9})$$

The integration with respect to p' yields a factor of $1/2$ (see Ref. [30]), leading to:

$$I_{\Omega}^{\sigma} = \frac{1}{2(2\pi)^5} \int dx \tilde{f}_{0\mathbf{p}'}^\sigma \tilde{f}_{0\mathbf{k}'}^{-\sigma} \int_0^{2\pi} \frac{d\varphi_{\mathbf{p}'}}{2\pi} (1 - \cos \theta_{\text{cm}})^2. \quad (\text{B.10})$$

In the above, $x = \cos \theta_{\mathbf{p}'}$ represents the angle between \mathbf{p}' and $\mathbf{p} + \mathbf{k}$, while $\varphi_{\mathbf{p}'}$ measures an angle in the plane perpendicular to $\mathbf{p} + \mathbf{k}$.

The integral with respect to $\varphi_{\mathbf{p}'}$ can be performed using the explicit expression for $1 - \cos \theta_{\text{cm}}$,

$$1 - \cos \theta_{\text{cm}} = \frac{4\mathbf{p} \cdot \mathbf{p}'}{E_{\text{cm}}^2}. \quad (\text{B.11})$$

After boosting p' to the center of mass frame, p^μ is replaced by $p_{\text{cm}}^\mu = (E_{\text{cm}}/2, \mathbf{p}_{\text{cm}})$, with

$$\mathbf{p}_{\text{cm}} = \mathbf{p} - \frac{p^0 + E_{\text{cm}}/2}{p^0 + k^0 + E_{\text{cm}}} (\mathbf{p} + \mathbf{k}). \quad (\text{B.12})$$

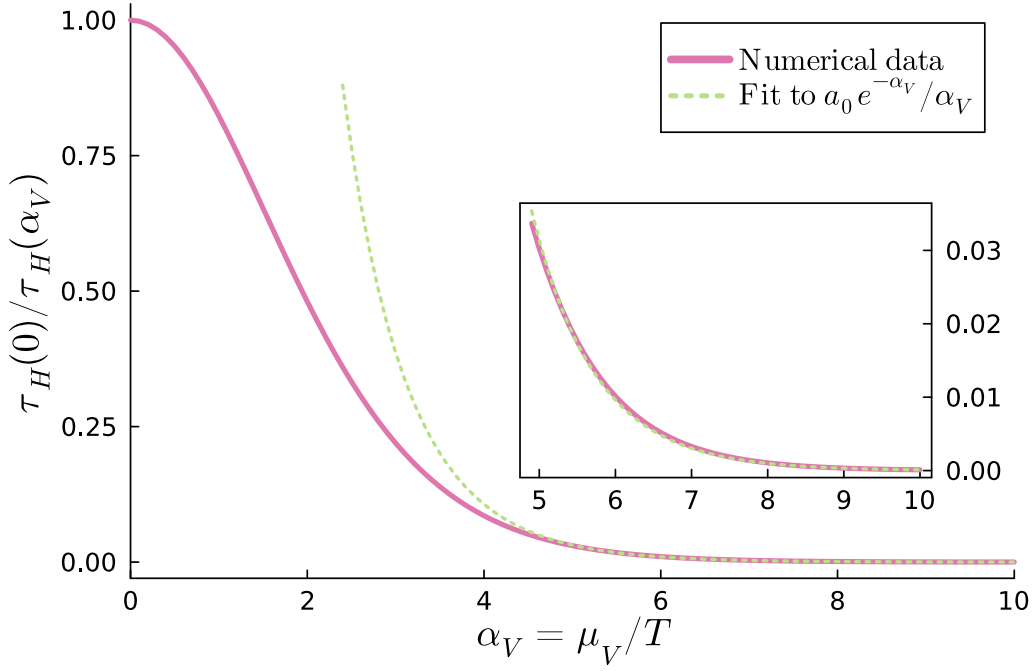


Figure 14: Ratio $\tau_H(0)/\tau_H(\alpha_V)$ between the helicity relaxation time in a neutral plasma and at dimensionless vector chemical potential $\alpha_V = \mu_V/T$, in an unpolarized plasma.

Splitting $\mathbf{p}_{\text{cm}} = \mathbf{p}_{\parallel} + \mathbf{p}_{\perp}$, where \mathbf{p}_{\parallel} is oriented along $\mathbf{k} + \mathbf{p}$ and \mathbf{p}_{\perp} is perpendicular to this vector, we have

$$p_{\parallel} = \frac{\mathbf{p}_{\text{cm}} \cdot (\mathbf{p} + \mathbf{k})}{|\mathbf{p} + \mathbf{k}|} = \frac{E_{\text{cm}}(p^0 - k^0)}{2|\mathbf{p} + \mathbf{k}|}, \quad (\text{B.13})$$

same as Eq. (168) in Ref. [30]. The calculation proceeds further exactly as shown in Ref. [14], leading to

$$\tau_H = \frac{6\pi^3\beta}{g\alpha_{\text{QCD}}^2\mathcal{I}}, \quad (\text{B.14})$$

with

$$\begin{aligned} \mathcal{I} = & \int_0^\infty dz z^3 \int_{-1}^1 dx \int_{-1}^1 d\delta \int_{|\delta|}^1 d\xi \left[\frac{3-x^2}{2}\xi - 2x\delta + \frac{3x^2-1}{2\xi}\delta^2 \right] \\ & \times \frac{1}{2} \sum_{\sigma=\pm 1} [e^{\frac{z}{2}(1+\delta)-\sigma\alpha_V} + 1]^{-1} [e^{\frac{z}{2}(1-\delta)+\sigma\alpha_V} + 1]^{-1} [e^{-\frac{z}{2}(1+x\xi)+\sigma\alpha_V} + 1]^{-1} [e^{-\frac{z}{2}(1-x\xi)-\sigma\alpha_V} + 1]^{-1}. \end{aligned} \quad (\text{B.15})$$

At vanishing chemical potential, $\mathcal{I} = 5.09434$ [30] and the helicity relaxation time in the neutral plasma reads

$$\tau_H = 0.196 \times \frac{6\pi^3\beta}{g\alpha_{\text{QCD}}^2} \simeq \left(\frac{250 \text{ MeV}}{k_B T} \right) \left(\frac{1}{\alpha_{\text{QCD}}} \right)^2 \left(\frac{2}{N_f} \right) \times 4.804 \text{ fm}/c. \quad (\text{B.16})$$

As the vector chemical potential is increased, the relaxation time also increases, as shown in Fig. 14. The large chemical potential behaviour of the relaxation time can be well fitted by the following function:

$$\tau_H^{-1}(\alpha_V \gg 1) = c_0 \frac{e^{-\alpha_V}}{\alpha_V} \quad (\text{B.17})$$

where the best fit was found to be given by $c_0 = 118.6$.

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