On the selection of subaction and measure for perturbed potentials

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Abstract

We prove that when the Aubry set for a Lipschitz continuous potential is a subshift of finite type, then the pressure function converges exponentially fast to its asymptote as the temperature goes to 0. The speed of convergence turns out to be the unique eigenvalue for the matrix whose entries are the *costs* between the different irreducible pieces of the Aubry set.
For a special case of Walter potential we show that pertubation of that potential that go faster to zero than the pressure do not change the selection, nor for the subaction, neither for the limit measure a zero temperature.

Keywords : Ergodic optimization, selection, zero temperature.
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1 Introduction

1.1 Background

In this paper we study the question of *selection in ergodic optimization*. Given a dynamical system (X, T), ergodic theory describes *almost all* orbits, with respect to some/any *invariant probability*. The thermodynamic formalism is a way to select particular invariant measure(s): one fixes some potential $A : X \to \mathbb{R}$ and one considers the measure(s) maximizing the free energy

$$P(\beta \cdot A) := \sup_{\mu \ T-inv} \left\{ h_\mu(T) + \beta \cdot \int A \, d\mu \right\},$$

where $h_\mu(T)$ is the Kolmogorov entropy and β is a real parameter, corresponding to the inverse of the temperature in statistical mechanics. Such a measure is called an equilibrium

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state (for βA). This formalism has been introduced in the Dynamical Systems settings in the 70's, mainly by Sinai, Ruelle and Bowen.

Ergodic optimization is another way to select particular invariant measures. Instead of maximizing the free energy, one simply maximizes the integral of the potential: an invariant (probability) measure μ is A-maximizing if it satisfies

$$m(A) := \sup_{\nu \ T-inv} \int A \, d\nu = \int A \, d\mu.$$

Existence of equilibrium states usually follows from upper-semi-continuity for the entropy, whereas existence of maximizing measures simply follows from continuity on the compact convex set of invariant probabilities.

Furthermore, there is a relation between equilibrium states and maximizing measures: any accumulation point for the equilibrium state $\mu_{\beta A}$ as β goes to $+\infty$ is an A-maximizing measure. It is also known (see [12]) that if (X, T) is uniformly hyperbolic, then generically in the Lipschitz topology for the potential, the family of equilibrium states $\mu_{\beta A}$ converges to the unique A-maximizing measure which is supported on a periodic orbit.

The question of *selection in ergodic theory* deals with the residual case. In the case of existence of several maximizing measures, is there convergence for $\mu_{\beta A}$ (as $\beta \to +$) and/or what makes that some accumulation point[1] is selected instead of another one ?

Despite residual sets may be considered as small, we emphasize that it is usually extremely easy to find A which admits several maximizing measures: pick any 2 disjoints invariant compact subsets K_1 and K_2 and consider the potential $A := -d(. \, , K_1 \cup K_2)$. The question of selection is thus meaningful.

About selection, few is known. If (X, T) is a subshift of finite type and if A is locally constant, then convergence to a ground state as temperature goes to zero holds (see [8, 17]). On the other hand, there are Lipschitz potentials where $\mu_{\beta A}$ does not converge (see [10, 5, 14]). In [18] it is proved that (under certain hyphothesis) flatness of the potential closed to the *Aubry set* determines which pieces may support ground states. On the other hand, in [3] it was proved that the knowledge of the potential in a neighborhood of the Aubry set is not sufficient to determine the result of selection. In [1], all the difficulty of the selection appeared: equilibrium states are usually obtained via an operator[2] and are constructed from its *eigen-function* and *eigen-measure* (see below). Therein, it is shown that the eigen-measure selects one piece, whereas the eigen-function selects the other piece of the Auby set. Then, considering the value of the potential "in the middle" as a parameter, the ground state may change as this parameter changes. A complete description of the limit depending on the parameter was given and the main point was that this selection process was strongly discontinuous. The main reason for that is that the phase diagram results from *Max-Plus* formalism, explaining the discontinuities.

Several properties and relations between equilibrium states and maximizing measures may be understood from convex analysis. The pressure function $\beta \mapsto P(\beta A)$ is convex,

[1] usually called a *ground state* in statistical mechanics.

[2] called the Transfer or the Ruelle-Perron-Frobenius Operator.

hence its slope is increasing. This is related to the fact that accumulation points for $\mu_{\beta A}$ are A-maximizing. Furthermore, the graph of the pressure admits an asymptote as $\beta \to +\infty$ of the form $h_{max} + \beta m(A)$, where h_{max} is the maximal entropy among A-maximizing measures. Some of these properties are re-explained below. We mention [9] for further descriptions of how thermodynamic formalism, including freezing phase transition (that is as $\beta \to +\infty$) is related to convex analysis.

Our first result (see Theorem A) shows that if the Aubry set is a subshift of finite type, then the pressure converges exponentially fast to its asymptote. In [13] an upper-bound in $O(1/\beta^2)$ was proved. Moreover, we show that the exponential rate is the unique eigenvalue of a transition matrix within the Max-Plus formalism, where entries are related to the Peierl's barrier of the potential between the different pieces of the Aubry set.

Our second result (see Theorem B) studies, in the case of Walters potentials, the stability of the selection under small perturbation of the potential. In that case Theorem A holds, and we show that if the perturbation goes faster to zero than how the pressure goes to its asymptote, then the selection does not change.

1.2 Settings and results

In the following $X \subseteq \{0, 1, ..., d\}^{\mathbb{N}}$ is a subshift of finite type given by an aperiodic matrix. X equipped with the metric d defined by

$$d((x_0, x_1, x_2, x_3, ...), (y_0, y_1, y_2, y_3, ...)) = \theta^{\min\{i \,|\, x_i \neq y_i\}},\ \theta \in (0, 1)$$

is a compact space. The shift map $\sigma : X \to X$ is defined by

$$\sigma(x_0, x_1, x_2, x_3, ...) = (x_1, x_2, x_3, x_4, ...).$$

In the special case $d = 2$, the full-2-shift is denoted by X_2. Points in X are also called *infinite words*, hence X_2 is the set of all infinite words with 0 and 1. If $a_0, \ldots, a_n$ are *digits or letters* in $\{0, 1, \ldots, d\}$, the cylinder $[a_0, \ldots, a_n]$ is the set of points $x \in X$ such that $x_i = a_i$ for $0 \leq i \leq n$.

Given a Lipschitz continuous function $A : X \to \mathbb{R}$, we denote by $|A|_\infty = \sup_{x \in X} |A(x)|$ the supremum norm of A and by $Lip(A) = \sup_{x \neq y} \frac{|A(x) - A(y)|}{d(x,y)}$ the Lipschitz constant of A. The Lipschitz norm is $|\,.\,|_\infty + Lip(.)$.

The transfer operator L_A is defined by

$$L_A(\psi)(x) = \sum_{\sigma(z) = x} e^{A(z)} \psi(z).$$

We refer to [6] for complete study of this operator. It acts on $\mathcal{C}(X)$ and on the space $\mathcal{C}^{0+1}(X)$ of Lipschitz continuous functions. Its spectral radius λ_A (for $|\quad|_\infty$-norm) is a single dominating eigenvalue on $\mathcal{C}^{0+1}(X)$. It also turns out to be equal to $e^{P(A)}$. The

associated 1-dimensional eigen-space is $span(H_A)$, where H_A is Lipschitz continuous, positive and uniquely determined up to some normalization. The dual operator L_A^* for the $|\quad|_\infty$-norm acts on the set of measures and ν_A is the unique probability satisfying $L_A^*(\nu_A) = \lambda_A \nu_A$. It is referred to as the *eigenmeasure* or the *conformal measure*.

The usual normalization for H_A is $\int H_A \, d\nu_A = 1$, but for Theorem B we shall choose another normalization.

We now recall some notion on ergodic optimization. We refer to [2] for proofs and more details. The set of σ-invariant probabilities is denoted by $\mathcal{M}(\sigma)$. We set

$$m(A) := \max_{\mu \in \mathcal{M}(\sigma)} \int A \, d\mu.$$

Any measure realizing this supremum is called A-maximizing or maximizing for A. We denote by $\mathcal{M}_{max}(A)$ the set of A-maximizing measures.

A *calibrated subaction* for A is a Lipschitz continuous function $V : X \to \mathbb{R}$ satisfying for every $x \in X$

$$\max_{\sigma(z)=x} [A(z) + V(z) - V(x) - m(A)] = 0.$$

It is also known (see [2] and Subsubsection 2.3.1) that any accumulation point for $\frac{1}{\beta} \log H_{\beta A}$ (and for $|\quad|_\infty$) is such a calibrated subaction. Setting $g(z) := A(z) + V(z) - V \circ \sigma(z) - m(A)$, we see that g is a Lipschitz continuous function, cohomoulogous to A (up to an additive constant) and non-positive. It thus satisfies that $m(g) = 0$, $P(\beta g) = P(\beta A) - \beta m(A)$ and the unique equilibrium sate for βA is also the unique equilibrium state for βg. Furthermore a A-maximizing measure is g-maximizing and conversely.

Hence, without loss of generality one shall **assume** in this paper that A is non-positive and satisfies $m(A) = 0$ (except when this assumption is not required for general results on ergodic optimization).

We remind that $S_n(A)$ stands for the Birkhoff sum $A + \ldots + A \circ \sigma^{n-1}$.

Definition 1. *With these notations and assumptions, the* ***Mañé potential*** *associated to A is defined by*

$$S(x,y) := \lim_{\epsilon \to 0} \left[\sup\{ S_n(A)(z),\ \sigma^n(z) = y,\ d(x,z) < \varepsilon \} \right],$$

the ***Aubry set*** *of A is defined by*

$$\Omega := \{ x \in X \,|\, S(x,x) = 0 \}$$

and ***Mather set*** *is*

$$\mathcal{M} = \cup_{\mu \in \mathcal{M}_{\max}(A)} \mathrm{supp}(\mu).$$

We remind that $y \mapsto S(x,y)$ is Lipschitz continuous, whereas $x \mapsto S(x,y)$ is only upper semi-continuous. By definition it is clear that $S(x,y)$ is non-positive.

Remark 2. *The Mather set is non-empty (as it contains the support of any accumulation point for $\mu_{\beta A}$) and is contained in the Aubry set (see [2, Th 3.15]). This shows that the Aubry set is not empty either.*

With these settings our first main result is:

Theorem A. *Let A be a non-positive Lipschitz continuous function satisfying $m(A) = 0$. Suppose that the Aubry set Ω is a subshift of finite type with topological entropy h. Then the limit*

$$\gamma := \lim_{\beta \to \infty} \frac{1}{\beta} \log(P(\beta A) - h) \tag{1}$$

exists.

In the proof of above theorem we show that γ is the unique eigenvalue in max-plus formalism of a matrix which is constructed from an analysis of the Mañé potential between the irreducible components of Ω with maximal entropy (see Prop. 23 below).

We can now deal with the question of selection. The question we are interested in is to know how/if selection of subaction and measure are stable under perturbations. As we will see it is easy to exhibit examples where the selections for the family $\beta A + B$ differ of the selections for the family βA. Therefore, our question is: which conditions on a family of functions B_β do ensure that $\mu_{\beta A + B_\beta}$ selects the same ground state than $\mu_{\beta A}$ and $\frac{1}{\beta} \log(H_{\beta A + B_\beta})$ selects the same subaction than $\frac{1}{\beta} \log(H_{\beta A})$?

It turns out that we can give an answer in a special case of potentials for X_2. For $i \in \{0, 1\}$, let $f_i : [0, 1] \to (-\infty, 0]$ be a Lipshitz function satisfying $f_i^{-1}(0) = \{0\}$ and let $A : \Sigma_2 \to (-\infty, 0]$ be the potential defined by

$$A(x) = \begin{cases} f_0(d(x, 0^\infty)), \text{ if } x \in [0] \\ f_1(d(x, 1^\infty)), \text{ if } x \in [1] \end{cases} .$$

These potentials belong to the family which were introduced by P. Walters in [23]. They represent a class of functions for which complete computations are possible, which allows to make conjecture for more complicated cases. We emphasize that in that case, the Aubry set is the union of two fixed points $\{0^\infty, 1^\infty\}$. It is thus a subshift of finite type with topological entropy equal to 0. Hence, Theorem A holds in that case.

We emphasize that our result gives another proof for the convergence of the Gibbs measures associated to these potentials (see [5]) as the temperature goes to zero, for the case where convergence holds.

Theorem B. *Consider a Lipschitz function $A : X_2 \to \mathbb{R}$ satisfying*

$$A(0^\infty) = A(1^\infty) = 0, \ A|_{[01]} = b, \ A|_{[10]} = d, \ A|_{[0^n 1]} = a_n, \ A|_{[1^n 0]} = c_n, \tag{2}$$

for negative numbers $b, d, \{c_n\}_{n \geq 2}$ and $\{a_n\}_{n \geq 2}$. Assume $H_{\beta A}$ is normalized by $H_{\beta A}(0^\infty) = 1$. Then,

1. $\gamma := \lim_{\beta\to+\infty} P(\beta A)$ *exists and it is negative;*

2. *For any* $\delta < \gamma$*, for any family* (B_β) *of Lipschitz functions such that* $|B_\beta|_\infty < e^{\beta\delta}$ *we have*
$$\lim_{\beta\to\infty} \mu_{\beta A+B_\beta} = \lim_{\beta\to\infty} \mu_{\beta A};$$

3. *For any* $\delta < \gamma$ *and any* $c \in \mathbb{R}$*, for any family* (B_β) *such that* $|B_\beta|_\infty < e^{\beta\delta}$ *and* $Lip(B_\beta) < \beta c$ *we have*
$$\lim_{\beta\to\infty} \frac{1}{\beta}\log(H_{\beta A+B_\beta}) = \lim_{\beta\to\infty} \frac{1}{\beta}\log(H_{\beta A}).$$

Remark 3. *We emphasize that the perturbations* B_β *do not need to satisfy similar hypothesis as* A *in* (2)*.*

Remark 4. *Assumptions may be released. We let the reader check that we actually only need* $A(0^\infty) = A(1^\infty) = 0$ *and:*

1. A *non-positiv,*

2. $b + d < 0$*,*

3. $a_2 < 0$*,*

4. $c_2 < 0$*.*

■

For that special case of potentials satisfying (2), existence of γ follows from Theorem A but has also been already proved in [3, 5]. Therein, existence of the limits $\lim_{\beta\to\infty} \mu_{\beta A}$ and $\lim_{\beta\to\infty} \frac{1}{\beta}\log(H_{\beta A})$ were also proved. Here, one novelty is that we give another characterization for the limit $V := \lim_{\beta\to\infty} \frac{1}{\beta}\log(H_{\beta A})$ in terms of max-plus formalism and the main novelty is that we give condition to get stability.

1.3 Plan of the paper

The paper is organized in the following way.

In section 2 we introduce a basic analysis of the problem of selection an some results that will be useful in the proof of Theorem A and in the study of selection of subaction.

In section 3 we prove Theorem A. The proof is divided in subsections in order to facilitate the reading. Using the fact that a subaction is entirely determined by its values on the Aubry set, we obtain a relation between these values on the irreducible component of the Aubry set in the case it is a subshift of finite type. In this relation, the speed of convergence of the pressure turns out to be an eigenvalue for a special matrix.

In section 4 we prove Theorem B. In the first subsection we use Theorem A to identify γ. Proposition 26 is the key point to get selection of the subaction. In Proposition 32 we show the convergences to limit measures for $\mu_{\beta A+B_\beta}$ and $\mu_{\beta A}$. In Section A we give a totally computational example where B_β goes to zero slower than $e^{\beta\cdot\gamma}$ and then there is a change of selections.

1.4 Motivations

An initial motivation for the study of selection for perturbed potentials was given from the results in [19]. In such paper very similar characterizations concerning zero temperature limits for $\mu_{\beta A}$ and $H_{\beta A}$ was getting when supposing $\tilde{X} = M^{\mathbb{N}}$, where M is a compact metric space instead a finite set. In this case the formulation of the Ruelle operator uses a prior probability measure m satisfying $\text{supp}(m) = M$ and then $L_{A,m} : C(\tilde{X}) \to C(\tilde{X})$ is given by $L_{A,m}(\psi)(x) = \int_M e^{A(ax)}\psi(ax)\, dm(a)$.

A natural question concerning this general setting is the dependence of the zero temperature limits with respect to the prior probability measure m which is used to define the operator. The answer is obtained by considering this question in the particular case where $M = \{0, 1, ..., d\}$ and $\tilde{X} = \{0, 1, ..., d\}^{\mathbb{N}}$. For any prior probability $m = (p_0, ..., p_d)$, where $p_i > 0 \, \forall i \in \{0, 1, ..., d\}$, the operator $L_{\beta A,m}$ can be written as

$$L_{\beta A,m}(\psi)(x) = \sum_{a=0}^{d} e^{\beta A(ax)}\psi(ax)p_a = \sum_{a=0}^{d} e^{\beta A(ax)+\log(p_a)}\psi(ax),$$

for each $\beta > 0$. So $L_{\beta A,m}$ is just the usual Ruelle Operator associated to the perturbed potential $\beta A(x) + \log(p_{x_1})$, where $x = (x_1, x_2, x_3, ...)$. It is easy to construct examples (see Example 7 below) where such perturbation will change the results of selection of subaction and measure. Therefore, the conclusion is that selection of subaction and measure in the setting of [19] depends of the prior measure m. In the case of finite alphabet $\{0, 1, ..., d\}$ a natural next step is to study what are the conditions on the perturbation in order to get equal limits.

A second motivation is to inquire how perturbation could enlarge the class of known potentials for which there is convergence. As it is said above, it is known that convergence occurs as soon as the potential is locally constant. In that case the Aubry set is a subshift of finite type. It is thus quite natural to inquire if, for a given potential, one can find some sequence of locally constant perturbations,, refining the initial potential, with all the same Aubry set, and such that convergence for this family finally yields convergence for the initial potential. This is a research program, and one first step is to control how perturbations influence the selection.

2 Auxiliary and preliminary results

2.1 Convexity for the pressure and consequences

Here we do not necessarily assume that $m(A) = 0$ holds.

Definition 5. *For any continuous $B : X \to \mathbb{R}$ we set*

$$P_{A-max}(B) := \sup_{\mu \in \mathcal{M}_{max}(A)} \left\{ \int B \, d\mu + h_\mu(\sigma) \right\}.$$

Any measure that realizes the maximum is called a A-maximizing equilibrium state for B.

Lemma 6. *Let A be a Lipschitz function and let $B_\beta : X \to \mathbb{R}$ be a family of Lipschitz functions converging uniformly to B. Then, any accumulation point for $\mu_{\beta A+B_\beta}$ as $\beta \to +\infty$ is A-maximizing equilibrium state for B.*

Furthermore, if B is Lipschitz continuous, then $\beta \mapsto P(\beta A + B) - \beta m(A)$ is non-increasing and converges to $P_{A-max}(B)$.

Proof. The proof below follows ideas present in [13]. Let μ be any maximizing measure to A and let μ_∞ be an accumulation measure of $\mu_{\beta A+B_\beta}$. From

$$\beta \int A\, d\mu + \int B_\beta\, d\mu + h(\mu) \leq \beta \int A\, d\mu_{\beta A+B_\beta} + \int B_\beta\, d\mu_{\beta A+B_\beta} + h(\mu_{\beta A+B_\beta})$$

we conclude that $\int A\, d\mu \leq \int A\, d\mu_\infty$. This proves that $\mu_\infty \in M_{max}(A)$.

We have also

$$\int B_\beta\, d\mu + h(\mu) \leq P(\beta A + B_\beta) - \beta \int A\, d\mu$$

$$= \beta(\int A\, d\mu_{\beta A+B_\beta} - \int A\, d\mu) + \int B_\beta\, d\mu_{\beta A+B_\beta} + h(\mu_{\beta A+B_\beta})$$

$$\leq \int B_\beta\, d\mu_{\beta A+B_\beta} + h(\mu_{\beta A+B_\beta}).$$

Using the upper semi-continuity of the entropy, as $\beta \to \infty$ we get

$$\int B\, d\mu + h(\mu) \leq \int B d\mu_\infty + h(\mu_\infty).$$

This yields that μ_∞ has maximal pressure among measures in $M_{max}(A)$.

Let us now assume that B is Lipschitz continuous. We remind that the pressure is defined by

$$P(A) := \max\{h_\mu(\sigma) + \int A\, d\mu\},$$

which immediately yields that $\beta \mapsto P(\beta A + B)$ is convex. It follows from Prop. 4.10 in [22] that

$$\frac{d}{d\beta} P(\beta A + B) = \int A\, d\mu_{\beta A+B}.$$

We can also observe that, by definition of pressure, for any β_0 the line $\beta \to h_{\mu_{\beta_0 A+B}}(\sigma) + \beta \int A\, d\mu_{\beta_0 A+B}$ is below the graph of the pressure function $\beta \to P(\beta A + B)$ and touches it at $\beta = \beta_0$.

As

$$\frac{d}{d\beta}[P(\beta A + B) - \beta m(A)] = \int A\, d\mu_{\beta A+B} - m(A) \leq 0$$

we get that $\beta \to P(\beta A + B) - \beta m(A)$ is a non-increasing function. Let μ_∞ be any accumulation point for $\mu_{\beta A+B}$. As, by definition of pressure,

$$P(\beta A + B) - \beta m(A) - \int B\, d\mu_\infty - h_{\mu_\infty}(\sigma) \geq 0$$

we obtain $P(\beta A+B)-\beta m(A)\geq \int B\,d\mu_\infty + h_{\mu_\infty}(\sigma)$. On the other hand, as the entropy is upper semi-continuous,

$$\lim_{\beta\to+\infty} P(\beta A+B)-\beta m(A)\leq \limsup_{\beta\to+\infty}\int B\,d\mu_{\beta A+B}+h_{\mu_{\beta A+B}}(\sigma)\leq \int B\,d\mu_\infty+h_{\mu_\infty}(\sigma).$$

□

2.1.1 Selection depends on B

The next example shows that if B_β does not go to 0, then selection may (easily) change. It is a consequence of Lemma 6 .

Example 7. *Set A any non-positive Lipschitz continuous function on X_2, negative everywhere except at 0^∞ and 1^∞. Pick B_0 and B_1 Lipschitz continuous functions such that $B_0(0^\infty) > B_0(1^\infty)$ and $B_1(0^\infty) < B_1(1^\infty)$.*

It is easy to check that δ_{0^∞} is the unique A-maximizing equilibrium state for B_0, whereas δ_{1^∞} is the unique A-maximizing equilibrium state for B_1. Hence, $\mu_{\beta A+B_0}$ converges to δ_{0^∞} whereas $\mu_{\beta A+B_1}$ converges to δ_{1^∞}.

2.1.2 Change of selection even if B_β goes to 0

Corollary 8. *Let $A:\{0,1\}^{\mathbb{N}}\to\mathbb{R}$ be a function which is equal to zero at the points 0^∞ and 1^∞ and negative at the other points. Suppose that $\mu_{\beta A}\to\delta_{1^\infty}$. Then there exists a family of potentials B_β which uniformly converge to 0 and such that δ_{0^∞} is an accumulation point for $\mu_{\beta A+B_\beta}$.*

Proof. Set $D(x)=-d(x,0^\infty)$. Then for any integer $n>0$, $\mu_{\beta A+\frac{1}{n}D}$ goes to δ_{0^∞} as β goes to $+\infty$. Therefore, one can find an increasing sequence $(\beta_n)\uparrow+\infty$ such that for any n,

$$\mu_{\beta_n A+\frac{1}{n}D}([0])\geq 1-\frac{1}{n}.$$

Let $s:\mathbb{R}^+\to\mathbb{R}^+$ be a decreasing function satisfying $s(\beta_n)=\frac{1}{n}$. Then, $\mu_{\beta_n A+s(\beta_n)D}$ goes to δ_{0^∞} (as it is the unique invariant measure with support in $[0]$). □

2.2 Kinds of Laplace's methods

We remind the very easy fact:

Lemma 9. *Let $a_1,\dots,a_n$ be real numbers. Then*

$$\lim_{\beta\to+\infty}\frac{1}{\beta}\log\left(\sum_{i=1}^n e^{\beta a_i}\right)=\max_i a_i.$$

Lemma 10 (see [19]). *Let M be any compact metric space. Let $W_\beta : M \to \mathbb{R}$ be a family of continous functions converging uniformly to $W : M \to \mathbb{R}$, as $\beta \to +\infty$. Then, for any finite measure ν with $supp\,(\nu) = M$ we have*

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log \int_M e^{\beta W_\beta(a)} d\nu(a) = \sup_{a\in M} W(a).$$

Proof. Let $m = \sup\{W(a) \,|\, a \in M\}$. We first prove

$$\liminf_{\beta\to+\infty} \frac{1}{\beta} \log \int_M e^{\beta W_\beta(a)} d\nu(a) \geq m.$$

Given $\varepsilon > 0$ let $\bar{a} \in M$ be such that $W(\bar{a}) > m - \varepsilon/2$. There exist β_0 and δ, such that $W_\beta(a) > m - \varepsilon$, for any $a \in B(\bar{a}, \delta) = \{a \,|\, d(a, \bar{a}) < \delta\}$ and $\beta > \beta_0$.

Therefore, if $\beta > \beta_0$, we have that

$$\int_M e^{\beta W_\beta(a)} d\nu(a) \geq \int_{B(\bar{a},\delta)} e^{\beta W_\beta(a)} d\nu(a) > \nu\big(B(\bar{a}, \delta)\big) e^{\beta(m-\varepsilon)},$$

and then

$$\frac{1}{\beta} \log \int_X e^{\beta W_\beta(a)} d\nu(a) > \frac{1}{\beta} \log \big(\nu\big(B(\bar{a}, \delta)\big)\big) + m - \varepsilon.$$

We conclude that

$$\liminf_{\beta\to+\infty} \frac{1}{\beta} \log \int_M e^{\beta W_\beta(a)} d\nu(a) \geq m.$$

The converse inequality is obtained in the same way: for a given ε, there exists β_0 such that, $W_\beta(a) < m + \varepsilon$, for any $\beta > \beta_0$ and $a \in M$. □

2.3 Eigenfunctions and calibrated subactions

2.3.1 Calibrated subaction as limit of eigenfunctions

Lemma 11. *Let $B_\beta : X \to \mathbb{R}$ be a family of Lipschitz functions satisfying*

1. *$Lip(B_\beta) \leq \beta c$ for some constant c and any β;*
2. *$\frac{|B_\beta|_\infty}{\beta}$ converges to zero as $\beta \to \infty$.*

There exist accumulation points for $\frac{1}{\beta}\log(H_{\beta A+B_\beta})$ for the norm $|\quad|_\infty$. Any such limit is a calibrated subaction for A.

Proof. It is known that there exists a constant c_1 such that for any Lipschitz function ϕ and any $x, y \in X$

$$e^{-c_1(1+|\phi|_\infty + Lip(\phi))d(x,y)} \leq \frac{H_\phi(x)}{H_\phi(y)} \leq e^{c_1(1+|\phi|_\infty + Lip(\phi))d(x,y)}$$

(see for example [20] pp. 46-51, also see [6, Lemme 1.8]). The main point here is that c_1 is an *universal* constant and does not depend on ϕ. Actually it only depends on the mixing rate in X.

This yields that $\log H_\phi$ is Lipschitz continuous with Lipschitz constant bounded by $c_1(1+|\phi|_\infty+Lip(\phi))$. Applying this to $\phi := \beta A+B_\beta$, we get that the family $\frac{1}{\beta}\log H_{\beta A+B_\beta}$ is equicontinuous with bounded Lipschitz norm.

By Arzelà-Ascoli theorem we conclude that there exists accumulation points for the $|\ \ |_\infty$-norm and they are Lipschitz continuous.

Note the next double-inequality

$$P(\beta A) - |B_\beta|_\infty \leq P(\beta A + B_\beta) \leq P(\beta A) + |B_\beta|_\infty. \tag{3}$$

Now, by definition of the eigenfunction we get that for any $x \in X$:

$$\sum_{\sigma(z)=x} e^{\beta A(z)+\log(H_{\beta A+B_\beta}(z))-\log(H_{\beta A+B_\beta}(x))-P(\beta A+B_\beta)} = 1,$$

holds. Taking $\frac{1}{\beta}\log$, considering any accumulation point $\widetilde{V}$ for $\frac{1}{\beta}\log H_{\beta A+B_\beta}$, Lemma 10 and (3) yield

$$\sup_{\sigma(z)=x}\Big[A(z)+\tilde{V}(z)-\tilde{V}(x)-m(A)\Big]=0.$$

□

2.3.2 Calibrated subactions and the Mañé Potential

We remind that the Aubry set Ω is non-empty as it contains the Mather set. First, we state a technical lemma.

Lemma 12. *Let x and y be in X. Let $\overline{\xi}_n$ be points such that $\overline{\xi}_n \to x$ as n goes to $+\infty$, and for any n there exists k_n with $k_n \to +\infty$ as $n \to +\infty$ such that $\sigma^{k_n}(\overline{\xi}_n) = y$. Then,*

$$S(x,y) \geq \limsup_{n\to+\infty} S_{k_n}(A)(\overline{\xi}_n).$$

Proof. Pick $\varepsilon > 0$. Let N be such that for every $n > N$, $d(x,\overline{\xi}_n) < \varepsilon$. Then

$$S^\varepsilon(x,y) := \sup\{S_k(A)(z),\ \sigma^k(z)=y,\ d(x,z)<\varepsilon\} \geq S_{k_n}(A)(\overline{\xi}_n)$$

holds as soon as $n > N$ holds. This yields, for every ε, $S^\varepsilon(x,y) \geq \limsup_n S_{k_n}(A)(\overline{\xi}_n)$. Hence doing $\varepsilon \downarrow 0$ we get the result. □

Lemma 13. *The potential A is equal to 0 on Ω.*

Proof. By definition x belongs to Ω if and only if $S(x,x)=0$. Hence we get

$$0 \geq A(x) \geq S(x,x) = 0.$$

□

Now we present a very important characterization of calibrated subactions using Mañé Potential and Aubry and Mather sets.

Lemma 14. *For any calibrated subaction V we have*

$$V(x_0) = \sup_{a\in\mathcal{M}}[V(a)+S(a,x_0)] = \sup_{a\in\Omega}[V(a)+S(a,x_0)] \ \forall\ x_0 \in X.$$

Proof. We follow ideas of [4]. Let V be any calibrated subaction. As $\mathcal{M}\subset\Omega$ holds, we get

$$\sup_{a\in\mathcal{M}}[V(a)+S(a,x_0)] \leq \sup_{a\in\Omega}[V(a)+S(a,x_0)].$$

As V is a subaction, $A+V-V\circ\sigma \leq 0$ and then, for any n and any z:

$$S_n(A)(z)+V(z) \leq V\circ\sigma^n(z).$$

This yields for any $a, x_0 \in X$, $S(a,x_0)+V(a)-V(x_0)\leq 0$, hence

$$V(x_0) \geq V(a)+S(a,x_0)\ \forall a, x_0 \in X,$$

holds. This proves that

$$V(x_0) \geq \sup_{a\in\Omega}[V(a)+S(a,x_0)].$$

Now we will prove that $\sup_{a\in\mathcal{M}}[V(a)+S(a,x_0)] \geq V(x_0)$. Set $R_- = A+V-V\circ\sigma$. By definition of calibrated subaction the following holds:

1. $R_- \leq 0$,
2. There exists a sequence of points (x_n) such that for every $n\geq 1$, $\sigma(x_n)=x_{n-1}$ and $R_-(x_n)=0$.

Let us consider the sequence of probabilities $\nu_n = \frac{1}{n}\sum_{i=1}^{n}\delta_{x_i}$, $n\geq 1$. Let ν be an accumulation point for the weak* topology as $n\to\infty$, say $\nu := \lim_{n_j\to\infty}\nu_{n_j}$. As for any continuous function $f: X\to\mathbb{R}$ we have

$$\begin{aligned}\int f\circ\sigma\, d\nu &= \lim_{n_j}\frac{1}{n_j}\sum_{i=1}^{n_j} f(\sigma(x_i)) \overset{\sigma(x_i)=x_{i-1}}{=} \lim_{n_j}\frac{1}{n_j}\sum_{i=1}^{n_j} f(x_{i-1}) \\ &= \lim_{n_j}\frac{1}{n_j}[f(x_0)-f(x_{n_j})+\sum_{i=1}^{n_j} f(x_i)] = \lim_{n_j}\frac{1}{n_j}[\sum_{i=1}^{n_j} f(x_i)] = \int f\, d\nu,\end{aligned}$$

then ν is $\sigma-$invariant.

By construction of R_- and ν_n we have $\int A\, d\nu_n = \int R_-\, d\nu_n = 0$, thus $\int A\, d\nu = 0$. Therefore ν is A-maximizing.

Now we fix any point $a \in \mathrm{supp}(\nu)$. Note that a belongs to $\mathcal{M} \subset \Omega$. By definition of support we have $\nu(\{z | d(z,a) \leq \epsilon\}) > 0$ for any $\epsilon > 0$. It follows that, for arbitrarily large n we have also $\nu_n(\{z : d(z,a) \leq \epsilon\}) > 0$, which proves that a is an accumulation point of the sequence (x_n). By considering a subsequence (x_{n_k}) satisfying $x_{n_k} \to a$ we get

$$\begin{aligned} S(a, x_0) &\geq \lim_{n_k \to \infty} [S_{n_k}(A)(x_{n_k})] \\ &= \lim_{n_k \to \infty} [S_{n_k}(R_-)(x_{n_k}) - V(x_{n_k}) + V(x_0)] \\ &= V(x_0) - V(a). \end{aligned}$$

This proves

$$V(x_0) \leq \sup_{a \in \mathcal{M}} [V(a) + S(a, x_0)].$$

□

3 Proof of Theorem A

We remind that A is assumed to be Lipschitz continuous, non-positive and satisfying $m(A) = 0$. The Aubry set is defined by

$$\Omega := \{x \in X,\ S(x,x) = 0\},$$

where $S(.,.)$ is the Mañé potential (see Def. 1).

3.1 Technical results for the irreducible pieces of Ω

Let us consider a decomposition of the Aubry set in irreductible sub-shifts of finite type

$$\Omega = \Sigma_1 \cup ... \cup \Sigma_k \cup \Sigma_{k+1} \cup ... \cup \Sigma_L$$

where $\Sigma_i \cap \Sigma_j = \emptyset$ for $i \neq j$, $h_{top}(\Sigma_i) = h$ if $i \in \{1, ..., k\}$ and $h_{top}(\Sigma_i) < h$ if $i \in \{k+1, ..., L\}$.

It follows that $A = 0$ over the sets Σ_i where $i \in \{1, ..., L\}$.

Lemma 15. *For each $i \in \{1, ..., L\}$, the set $\sigma^{-1}(\Sigma_i) \setminus \Sigma_i$ is compact.*

Proof. As X is compact, we only need to prove that $\sigma^{-1}(\Sigma_i) \setminus \Sigma_i$ is closed. Let (y_n) be a sequence in $\sigma^{-1}(\Sigma_i) \setminus \Sigma_i$ converging to y in X. We will prove that $y \in \sigma^{-1}(\Sigma_i) \setminus \Sigma_i$.

As Σ_i is a sub-shift of finite type we can find some positive integer l and a collection of cylinders of length l, say $\mathcal{A}$, coinciding with the admissible words (of length l) for Σ_i. That is that x belongs to Σ_i if and only if for every $n \geq 0$, $\sigma^n(x)$ belongs to one of the cylinders in $\mathcal{A}$.

By continuity, $\sigma(y_n)$ converges to $\sigma(y)$. Furthermore, $\sigma(y_n)$ belongs to the closed set Σ_i, by hypothesis, and thus $\sigma(y)$ also belongs to Σ_i.

As $y_n \to y$, we can assume that all the y_n belong to the same l-cylinder than y. As $\sigma(y_n) \in \Sigma_i$ and $y_n \notin \Sigma_i$ this cylinder does not belong to the collection $\mathcal{A}$ and so $y \notin \Sigma_i$. Therefore, $y \in \sigma^{-1}(\Sigma_i) \setminus \Sigma_i$ □

3.2 Another characterization for the Σ_i's

We recall that the Aubry set Ω is defined as the set of points x satisfying $S(x,x) = 0$, where the function S is the Mañé potential.

We define an equivalence relation by

$$x \sim y \iff S(x,y) + S(y,x) = 0.$$

Our main result for this section (Prop. 17) is that equivalences classes for that relation coincide with the decomposition of Ω in irreducible components.

Let us first recall a classical result for the Mañé potential:

Lemma 16. *Let S be the Mañé potential. We have:*

1. *$S(x,y) + S(y,z) \leq S(x,z)$ for every x, y, z in X;*
2. *for any fixed $x \in X$, the map $y \mapsto S(x,y)$ is Lipschitz continuous;*
3. *if $x \sim y$, then $S(x,.) = S(y,.)$;*
4. *if $\sigma(x) = y$, then $S(x,y) = A(x)$.*

Proof. For items 1 and 2 see [2].

In order to prove 3, just observe that $S(x,y) + S(y,x) = 0$ and $S(.,.) \leq 0$ yields

$$S(x,y) = S(y,x) = 0.$$

Hence, for any $z \in X$,

$$S(x,z) \overset{1.}{\geq} S(x,y) + S(y,z) = S(y,z) \overset{1.}{\geq} S(y,x) + S(x,z) = S(x,z).$$

The proof of item 4 has two steps. First, note that if z is such that $\sigma^n(z) = y$ and $d(x,z) < \varepsilon$, then

$$S_n A(z) \leq A(z) \leq A(x) + Cd(x,z) \leq A(x) + C\varepsilon.$$

This yields for any positive ε:

$$S^\varepsilon(x,y) = \sup\{S_n(A)(z),\ \sigma^n(z) = y,\ d(x,z) < \varepsilon\} \leq A(x) + C\varepsilon.$$

Doing $\varepsilon \downarrow 0$ we get $S(x,y) \leq A(x)$.

On the other hand, as $\sigma(x) = y$,

$$\sup\{S_n(A)(z),\ \sigma^n(z) = y,\ d(x,z) < \varepsilon\} \geq A(x),$$

which yields the other inequality. □

Proposition 17. *The irreducible components Σ_i, $i \in \{1, ..., L\}$, of the Aubry set as sub-shift of finite type coincide with the classes of equivalence, which are defined by the equation*

$$S(x, y) + S(y, x) = 0.$$

Proof. Let $x = (x_1, x_2, x_3, ...)$ and $y = (y_1, y_2, y_3, ...)$ be points of X.

Step one: Suppose $x, y \in \Sigma_i$. We show that $S(x, y) = 0$.

As Σ_i is irreducible, for each n there exists a word $z_1^n, ..., z_{N(n)}^n$ such that

$$\overline{\xi}_n := (x_1, ..., x_n, z_1^n, ..., z_{N(n)}^n, y_1, y_2, y_3, ...) \in \Sigma_i.$$

As $\Sigma_i \subset \Omega$ we get $S_{n+N(n)}(A)(\overline{\xi}_n) = 0$, $\lim_{n\to+\infty} \overline{\xi}_n = x$, and $\sigma^{n+N_n}(\overline{\xi}_n) = y$. Then Lemma 12 yields

$$0 \geq S(x, y) \geq \lim_{n\to+\infty} S_{n+N(n)}(A)(\overline{\xi}_n) = 0.$$

This shows that $S(x, y) = 0$ holds. Similarly, exchanging the roles for x and y we get $S(y, x) = 0$.

Step two: Suppose $S(x, y) + S(y, x) = 0$. Item 1 in Lemma 16 yields $S(x, x) = S(y, y) = 0$, hence $x, y \in \Omega$.

Let us suppose by contradiction that $y \in \Sigma_i$ and $x \notin \Sigma_i$. As $S(x, y) = 0$ there exists a sequence of points $\overline{\xi}_n := (x_{N(n)}^n, ..., x_1^n, y_1, y_2, y_3, ...), n \in \mathbb{N}$, which converges to x, such that $S_{N(n)}(A)(\overline{\xi}_n) > -\frac{1}{n}$.

As Σ_i is closed and $x \notin \Sigma_i$ we have that $\overline{\xi}_n \notin \Sigma_i$ for n large enough. For simplicity we suppose that for every n, $\overline{\xi}_n \notin \Sigma_i$.

As $y \in \Sigma_i$ and Σ_i is invariant, we can consider $M(n) \leq N(n)$ such that

$$\overline{z}_n := \sigma^{N(n)-M(n)}(\overline{\xi}_n) \notin \Sigma_i, \ \sigma(\overline{z}_n) := \sigma^{N(n)-M(n)+1}(\overline{\xi}_n) \in \Sigma_i.$$

Then $\overline{z}_n \in \sigma^{-1}(\Sigma_i) \setminus \Sigma_i$ which is compact, therefore there exists a converging subsequence for $(\overline{z}_n)$. Again, for simplicity we assume that $\overline{z}_n \to z$ holds. Note that $z \in \sigma^{-1}(\Sigma_i) \setminus \Sigma_i$ holds. Furthermore we have

$$0 \geq S_{M(n)}(A)(\overline{z}_n) \geq S_{N(n)}(A)(\overline{\xi}_n) > -\frac{1}{n}.$$

Then, Lemma 12 yields $0 \geq S(z, y) \geq 0$, hence $S(z, y) = 0$.

As $\overline{\xi}_n$ converges to x and $\sigma^{N(n)-M(n)}(\overline{\xi}_n) = \overline{z}_n$ we get for any positive ε and for any sufficiently big n,

$$S^{\varepsilon}(x, \overline{z}_n) \geq S_{N(n)-M(n)}(A)(\overline{\xi}_n) > -\frac{1}{n}.$$

Then, Lemma 12 and Lipschitz continuity in the second variable for $S(x, .)$ yield

$$0 \geq S(x, z) = \lim_n S(x, \overline{z}_n) = 0.$$

Finally, $S(z, z) \geq S(z, y) + S(y, x) + S(x, z) = 0$ shows that z belongs to some Σ_k. So does $\sigma(z)$, hence $k = i$ and this produces a contradiction.

□

Lemma 18. *If V is a calibrated subaction for A then V is constant on each set Σ_i.*

Proof. Let $x, y \in \Sigma_i$ and V be a calibrated subaction. As $S(x,y) = S(y,x) = 0$, by applying Lemma 14 we have

$$V(x) = \sup_{a\in\Omega}[S(a,x) + V(a)] \geq S(y,x) + V(y) = V(y).$$

and

$$V(y) = \sup_{a\in\Omega}[S(a,y) + V(a)] \geq S(x,y) + V(x) = V(x).$$

□

3.3 A Max-Plus formula for the calibrated subaction V

From now on, let us denote $S_i(.) := S(x,.)$, where x is any point of Σ_i (see Lemma 16 item 3.). If Σ_i and Σ_j are two irreducible components of Ω we set

$$a_{ij} := \sup_{x\in\Sigma_i} \sup_{y\in\sigma^{-1}(x), y\notin\Sigma_i} A(y) + S_j(y) = \sup_{y\in\sigma^{-1}(\Sigma_i)\setminus\Sigma_i} A(y) + S_j(y).$$

We remember that we are supposing in this section that

$$\Omega = \Sigma_1 \cup ... \cup \Sigma_k \cup \Sigma_{k+1} \cup ... \cup \Sigma_L$$

where $\Sigma_i \cap \Sigma_j = \emptyset$ for $i \neq j$, $h_{top}(\Sigma_i) = h$ if $i \in \{1, ..., k\}$ and $h_{top}(\Sigma_i) < h$ if $i \in \{k+1, ..., L\}$.

Lemma 19. *For every $i, j, l \in \{1, ..., L\}$ the next holds:*

1. $-\infty < a_{ij} \leq 0$;
2. $a_{ii} < 0$;
3. $a_{li} + a_{ij} \leq a_{lj}$.

Proof. First, A is non-positive, and thus so is S. Second, Lemma 16 yields each S_j is Lipschitz continuous, hence bounded on the compact set $\sigma^{-1}(\Sigma_i) \setminus \Sigma_i$ (see Lemma 15). This also holds for A, and thus each a_{ij} is a real number. This finishes to prove 1.

Proof of 2.
As $y \mapsto A(y)+S_i(y)$ is continuous on the compact set $\sigma^{-1}(\Sigma_i)\setminus\Sigma_i$, it attains its maximum.
If $y \in \sigma^{-1}(\Sigma_i) \setminus \Sigma_i$ is such that

$$a_{ii} = A(y) + S_i(y),$$

then, setting $x := \sigma(y)$, we have $x \in \Sigma_i$ and $a_{ii} = A(y) + S(x,y)$. By item 4 in Lemma 16, $S(y,x) = A(y)$. This yields,

$$a_{ii} = A(y) + S(x,y) = S(y,x) + S(x,y) < 0,$$

since $y \notin \Sigma_i$ (see Prop 17).

Proof of 3.
If $i = l$ we just need to apply item 2. Suppose then $i \neq l$ and let $y_i \in \sigma^{-1}(\Sigma_i) \setminus \Sigma_i$ and $y_l \in \sigma^{-1}(\Sigma_l) \setminus \Sigma_l$ be such that

$$a_{ij} = A(y_i) + S_j(y_i), \quad a_{li} = A(y_l) + S_i(y_l).$$

Set $x_i := \sigma(y_i) \in \Sigma_i$ and note that $S_i(y_l) = S(x_i, y_l)$ holds. Again Lemma 16 yields $S(y_i, x_i) = A(y_i)$. Hence we get

$$\begin{aligned} a_{li} + a_{ij} &= A(y_l) + S_i(y_l) + A(y_i) + S_j(y_i) \\ &= A(y_l) + S(x_i, y_l) + S(y_i, x_i) + S_j(y_i) \\ &\leq A(y_l) + S_j(y_l) \leq a_{lj}. \end{aligned}$$

□

Remark 20. *If V is a calibrated subaction, then it is constant on each set Σ_i (by Lemma 18). It thus makes sense to set $V(\Sigma_i)$ and we will use such notation from now on in the present section.*

Proposition 21. *Any accumulation point for $\frac{1}{\beta}\log(P(\beta A) - h)$ belongs to $]-\infty, 0]$. Furthermore, if $\lim_{\beta_j \to \infty} \frac{1}{\beta_j}\log(P(\beta_j A) - h) = \gamma$ and $\lim_{\beta_j \to \infty} \frac{1}{\beta_j}\log(H_{\beta_j A}) = V$, then,*

$$\gamma + V(\Sigma_i) = \max_{j \in \{1,...,L\}} V(\Sigma_j) + a_{ij}, \ \ \forall\, i \in \{1, ..., k\} \tag{4}$$

and

$$V(\Sigma_i) = \max_{j \in \{1,...,L\}} V(\Sigma_j) + a_{ij}, \ \ \forall\, i \in \{k+1, ..., L\}. \tag{5}$$

Proof. As it is said above (see Lemma 6), $\beta \mapsto P(\beta A) - h$ is non-increasing and goes to 0 as $\beta \to +\infty$. Hence, $\beta \mapsto \frac{1}{\beta}\log(P(\beta A) - h)$ is also non-increasing and it is negative for sufficiently large β. This shows that any accumulation point for $\frac{1}{\beta}\log(P(\beta A) - h)$ is in $[-\infty, 0]$.

Because V is Lipschitz continuous on Ω, thus bounded, and beause all the a_{ij} are real numbers, Equation (4) shows that γ is a real number, hence $\gamma > -\infty$.

Let us now consider the Ruelle operator for βA, given by

$$L_{\beta A}(\varphi)(x) = \sum_{y \in \sigma^{-1}(x)} e^{\beta A(y)} \varphi(y).$$

For each $i \in \{1, ..., L\}$, we can also consider the Ruelle operator for the zero potential acting over $\mathcal{C}(\Sigma_i)$:

$$\mathcal{L}_i(\varphi)(x) = \sum_{y \in \sigma^{-1}(x),\, y \in \Sigma_i} \varphi(y)$$

where $x \in \Sigma_i$. As we suppose that Σ_i is an irreducible sub-shift of finite type, there exists an eigen-measure ν_i with $\text{supp}\,(\nu_i) = \Sigma_i$ such that

$$e^{h_i} \int \varphi(x) d\nu_i(x) = \int \mathcal{L}_i(\varphi)(x)\, d\nu_i(x),$$

where h_i is the topological entropy of Σ_i, which corresponds to the pressure of the zero function on Σ_i.

We start by supposing $i \in \{1, ..., k\}$. Given $x \in \Sigma_i$ we have

$$e^{P(\beta A)} H_{\beta A}(x) = \sum_{y \in \sigma^{-1}(x),\, y \in \Sigma_i} e^{\beta A(y)} H_{\beta A}(y) + \sum_{y \in \sigma^{-1}(x),\, y \notin \Sigma_i} e^{\beta A(y)} H_{\beta A}(y)$$

and using that $A = 0$ in Σ_i we get

$$e^{P(\beta A)} H_{\beta A}(x) = \mathcal{L}_i(H_{\beta A})(x) + \sum_{y \in \sigma^{-1}(x),\, y \notin \Sigma_i} e^{\beta A(y)} H_{\beta A}(y).$$

Integrating both sides with respect to the eigenmeasure ν_i of $\mathcal{L}_i$ we have

$$e^{P(\beta A)} \int H_{\beta A}(x)\, d\nu_i(x) = \int \mathcal{L}_i(H_{\beta A})(x)\, d\nu_i(x) + \int \sum_{y \in \sigma^{-1}(x),\, y \notin \Sigma_i} e^{\beta A(y)} H_{\beta A}(y)\, d\nu_i(x)$$

i.e.

$$e^{P(\beta A)} \int H_{\beta A}(x)\, d\nu_i(x) = e^{h} \int H_{\beta A}(x)\, d\nu_i(x) + \int \sum_{y \in \sigma^{-1}(x),\, y \notin \Sigma_i} e^{\beta A(y)} H_{\beta A}(y)\, d\nu_i(x).$$

Then

$$(e^{P(\beta A) - h} - 1) e^{h} \int H_{\beta A}(x)\, d\nu_i(x) = \int \sum_{y \in \sigma^{-1}(x),\, y \notin \Sigma_i} e^{\beta A(y)} H_{\beta A}(y)\, d\nu_i(x). \tag{6}$$

Taking $\lim_{\beta_j \to +\infty} 1/\beta_j \log(\cdot)$ in both sides of (6) we get

$$\gamma + 0 + V(\Sigma_i) = \sup_{x \in \Sigma_i} \sup_{y \in \sigma^{-1}(x), y \notin \Sigma_i} A(y) + V(y), \tag{7}$$

where for both integrals we use Lemma 10 and that ν_i has full support in Σ_i.

Now, it follows from Lemmas 14 and 18 that

$$V(x) = \max_{j \in \{1, ..., L\}} V(\Sigma_j) + S_j(x)$$

holds where S_j is the Mañé potential with respect to Σ_j. Therefore, Equation (7) can be rewritten as

$$\begin{aligned}\gamma + V(\Sigma_i) &= \sup_{x\in\Sigma_i}\sup_{y\in\sigma^{-1}(x),y\notin\Sigma_i} A(y) + \left(\max_{j\in\{1,\dots,k\}} V(\Sigma_j) + S_j(y)\right)\\ &= \sup_{x\in\Sigma_i}\sup_{y\in\sigma^{-1}(x),y\notin\Sigma_i}\max_{j\in\{1,\dots,L\}} (A(y) + V(\Sigma_j) + S_j(y))\\ &= \max_{j\in\{1,\dots,L\}}\sup_{x\in\Sigma_i}\sup_{y\in\sigma^{-1}(x),y\notin\Sigma_i} (A(y) + V(\Sigma_j) + S_j(y))\\ &= \max_{j\in\{1,\dots,L\}} V(\Sigma_j) + \left(\sup_{x\in\Sigma_i}\sup_{y\in\sigma^{-1}(x),y\notin\Sigma_i} A(y) + S_j(y)\right).\end{aligned}$$

Then we have

$$\gamma + V(\Sigma_i) = \max_{j\in\{1,\dots,L\}} V(\Sigma_j) + a_{ij}.$$

The proof for the case $i \in \{k+1, ..., L\}$ is similar, except that γ disappears. Indeed, if Σ_i has entropy $h' < h$, then (6) can be rewritten as:

$$(e^{P(\beta A)-h'} - 1)e^{h'}\int H_\beta(x)\, d\nu_i(x) = \int \sum_{y\in\sigma^{-1}(x),\, y\notin\Sigma_i} e^{\beta A(y)} H_{\beta A}(y)\, d\nu_i(x).$$

The rest of the computations holds, except that γ has to be removed. Hence, we get:

$$V(\Sigma_i) = \max_{j\in\{1,\dots,L\}} V(\Sigma_j) + a_{ij}.$$

□

3.4 Component with small entropy do not play a role

We remember that $h_{top}(\Sigma_i) = h$ if $i \in \{1, ..., k\}$ and $h_{top}(\Sigma_i) < h$ if $i \in \{k+1, ..., L\}$.

Proposition 22. *If $i \in \{k+1, ..., L\}$ then for every $l \in \{1, ..., L\}$, there exists $j \in \{1, ..., k\}$ such that*

$$V(\Sigma_i) + a_{li} \le V(\Sigma_j) + a_{lj}.$$

Proof. From equation (5) and item 2. of Lemma 19, there exists $j_1 \neq i$ such that

$$V(\Sigma_i) = V(\Sigma_{j_1}) + a_{ij_1}. \tag{8}$$

Hence, item 3. of Lemma 19 yields

$$V(\Sigma_i) + a_{li} = V(\Sigma_{j_1}) + a_{ij_1} + a_{li} \le V(\Sigma_{j_1}) + a_{lj_1}.$$

If $j_1 \in \{1, ..., k\}$ then the proposition is proved. If not, then we apply the same reasoning with j_1 instead of i. This produces $j_2 \neq j_1$ such that

$$V(\Sigma_{j_1}) = V(\Sigma_{j_2}) + a_{j_1 j_2}. \tag{9}$$

Equations (8) and (9) along with Lemma 19 yields

$$V(\Sigma_i) = V(\Sigma_{j_2}) + a_{ij_1} + a_{j_1 j_2} \leq V(\Sigma_{j_2}) + a_{ij_2}. \tag{10}$$

Again, if $j_2 \in \{1, ..., k\}$, then we are done. Otherwise we continue the reasoning. Since there are finitely many components, either we eventually find some j_l in $\{1, ..., k\}$, or some j_r will be equal to j_l with $l < r$. This last case is impossible because we reach a contradiction. Indeed, by applying a similar equation to (9) recursively from j_l, we get

$$V(\Sigma_{j_l}) = V(\Sigma_{j_{l+1}}) + a_{j_l j_{l+1}} = ... = V(\Sigma_{j_r}) + a_{j_l j_{l+1}} + ... + a_{j_{r-1} j_r}$$

Using that $j_r = j_l$ and applying items 2. and 3. of Lemma 19, we obtain

$$V(\Sigma_{j_l}) = V(\Sigma_{j_l}) + a_{j_l j_{l+1}} + ... + a_{j_{r-1} j_l} \leq V(\Sigma_{j_l}) + a_{j_l j_l} < V(\Sigma_{j_l})$$

which is a contradiction.

□

3.5 End of the proof of Theorem A

Proposition 23. *The limit* $\gamma = \lim_{\beta \to \infty} \frac{1}{\beta} \log(P(\beta A) - h)$ *exists. Furthermore it is the unique eigenvalue for the* $k \times k$ *matrix in the Max-Plus formalism whose entries are the* a_{ij}*'s.*

Proof. Suppose that $\Omega = \Sigma_1 \cup ... \cup \Sigma_k \cup \Sigma_{k+1} \cup ... \cup \Sigma_L$ where $h_{top}(\Sigma_i) = h$ if $i \in \{1, ..., k\}$ and $h_{top}(\Sigma_i) < h$ if $i \in \{k+1, ..., L\}$. Let β_j be a sequence such that $\lim_{\beta_j \to \infty} \frac{1}{\beta_j} \log(P(\beta_j A) - h) = \gamma$ and take a subsequence of this one such that $\frac{1}{\beta_{j_l}} \log(H_{\beta_{j_l}})$ converges for a function V. From equation (4) γ is a real number and we have

$$\gamma + V(\Sigma_i) = \max_{j \in \{1, ..., L\}} V(\Sigma_j) + a_{ij}, \ \forall\, i \in \{1, ..., k\}.$$

Applying Proposition 22 we get that such maximum is reached at some $j \in \{1, ..., k\}$ and therefore

$$\gamma + V(\Sigma_i) = \max_{j \in \{1, ..., k\}} V(\Sigma_j) + a_{ij}, \ \forall\, i \in \{1, ..., k\},$$

which in Max-plus formalism can be rewritten as

$$\gamma \otimes \begin{bmatrix} V(\Sigma_1) \\ \vdots \\ V(\Sigma_k) \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{pmatrix} \otimes \begin{bmatrix} V(\Sigma_1) \\ \vdots \\ V(\Sigma_k) \end{bmatrix}. \tag{11}$$

Observe that the matrix $M = (a_{ij})$ does not depend of the subsequence which defines V and γ and all its entries are real numbers. Therefore, [7, Th 3.23, p111] yields that M has a unique eigenvalue. Hence, the function $\beta \to \frac{1}{\beta}\log(P(\beta A) - h)$ admits a unique accumulation point, thus converges. □

Remark 24. *In the case $L = k$,* i.e., *all the irreducible components have maximal entropy, setting $\vec{V} := (V(\Sigma_1), \dots, V(\Sigma_k))$, $\vec{V}$ is an eigenvector for the matrix M. Nevertheless, uniqueness does not necessarily holds. It is thus meaning full to ask for which eigenvectors can be selected.*
The question still have meaning if $k < L$, and in that case the role of components with small entropy is even more unpredictable.

4 Proof of Theorem B

From now on, we consider $X = X_2$ the full 2-shift. A is a non-positive potential of the form

$$A|_{[01]} = b,\ A|_{[10]} = d,\ A(0^\infty) = A(1^\infty) = 0,\ A|_{[0^n1]} = a_n,\ A|_{[1^n0]} = c_n.$$

It is Lipschitz continuous which yields that the series $\sum a_n$ and $\sum c_n$ converge. Furthermore, we remind that all numbers b, d a_n and c_n are negative, which yields that the Aubry set is $\{0^\infty, 1^\infty\}$.

In that case Theorem A holds, and we set

$$\gamma := \lim_{\beta\to+\infty} \frac{1}{\beta}\log P(\beta A),$$

as periodic orbits are subshift of finite type with zero entropy.

We consider a family (B_β) of potentials satisfying $|B_\beta|_\infty < e^{\beta\delta}$, $\delta < \gamma$. In the study of selection of subaction we consider also the hypothesis $Lip(B_\beta) \leq \beta c$ for some constant c and any β.

We also remind that H_ϕ stand for the eigenfunction for the potential ϕ. In the case $\phi = \beta A$ the normalization is

$$\boxed{H_{\beta A}(0^\infty) = 1.} \tag{12}$$

The section runs as follows: we first give some technical results. Then we prove the convergence for $\frac{1}{\beta}\log P(\beta A + B_\beta)$ and $\frac{1}{\beta}\log H_{\beta A+B_\beta}$. Finally we prove the convergence for the measure.

4.1 Technical results

4.1.1 Accumulation points for $\frac{1}{\beta}log(H_{\beta A+B_\beta})$

We remind the double-inequality (3)

$$P(\beta A) - |B_\beta|_\infty \leq P(\beta A + B_\beta) \leq P(\beta A) + |B_\beta|_\infty.$$

Hence, our assumption $|B_\beta|_\infty < e^{\beta\delta}$, $\delta < \gamma$ immediately yields

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log(P(\beta A + B_\beta)) = \gamma. \tag{13}$$

Next Lemma is a re-writing of the computation we did to get Equalities (6) and (7).

Lemma 25. *Suppose $Lip(B_\beta) \leq \beta c$ for some constant c and any β and that $|B_\beta|_\infty < e^{\beta\delta}$, $\delta < \gamma$. Any accumulation point $\widetilde{V}$ for $\frac{1}{\beta}\log(H_{\beta A+B_\beta})$ is a calibrated subaction of A satisfying*

$$\gamma = A(10^\infty) + \tilde{V}(10^\infty) - \tilde{V}(0^\infty)$$

and

$$\gamma = A(01^\infty) + \tilde{V}(01^\infty) - \tilde{V}(1^\infty).$$

Proof. From Lemma 11 there exist convergent sub-sequences and any limit is a calibrated subaction for A. For any x we have

$$e^{P(\beta A+B_\beta)} H_{\beta A+B_\beta}(x) = \sum_{y,\ \sigma(y)=x} e^{\beta A(y)+B_\beta(y)} H_{\beta A+B_\beta}(y).$$

In the special case $x = 0^\infty$, we get

$$(e^{P(\beta A+B_\beta)} - e^{B_\beta(0^\infty)}) H_{\beta A+B_\beta}(0^\infty) = e^{\beta A(10^\infty)+B_\beta(10^\infty)} H_{\beta A+B_\beta}(10^\infty).$$

From (13) and the hypothesis $|B_\beta|_\infty < e^{\beta\delta}$ we obtain

$$\lim_{\beta\to+\infty} \frac{1}{\beta} \log(e^{P(\beta A+B_\beta)} - e^{B_\beta(0^\infty)}) = \gamma.$$

Therefore

$$\gamma + \widetilde{V}(0^\infty) = A(10^\infty) + \widetilde{V}(10^\infty).$$

The proof of $\gamma = A(01^\infty) + \tilde{V}(01^\infty) - \tilde{V}(1^\infty)$ is similar. □

4.1.2 A property for one 2×2 Max-Plus Matrix

Proposition 26. *Let a, b, c, d be real numbers. Let M be the matrix*

$$M := \begin{pmatrix} a+b+d & c+d \\ a+b & b+c+d \end{pmatrix}.$$

Then its unique eigenvalue (for max-plus laws) is given by

$$\lambda = \max\left\{a+b+d, b+c+d, \frac{a+b+c+d}{2}\right\}.$$

Furthermore, its eigen-space has dimension 1.

Proof. Again, [7, Th 3.23, p111] yields that M has a unique eigen-value, say λ, which is equal to

$$\max\left\{a+b+d, b+c+d, \frac{a+b+c+d}{2}\right\}.$$

It thus remains to study the dimension of the eigenspace.

Equality

$$\begin{pmatrix} a+b+d & c+d \\ a+b & b+c+d \end{pmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \otimes \begin{bmatrix} x \\ y \end{bmatrix}$$

can be rewritten as

$$\begin{cases} \max\{a+b+d+x,\ c+d+y\} = \lambda + x & \text{(14a)} \\ \max\{a+b+x,\ b+c+d+y\} = \lambda + y & \text{(14b)} \end{cases}$$

To prove that the eigen-space has dimension 1 means to prove that $y-x$ is constant. We consider three cases.

• Case 1: If $\lambda = a+b+d$. Equation (14a) yields $c+d+y \leq a+b+d+x$. Hence $c+d+y < a+x$ and then $b+c+d+y < a+b+x$.

Using this in (14b) we get $a+b+x = \lambda + y$ and then, as $\lambda = a+b+d$, we get $\boxed{y = x-d}$.

• Case 2: If $\lambda = b+c+d$. The computation is the same than in the previous case, up to doing the exchanges

$$a \leftrightarrow c,\ b \leftrightarrow d,\ x \leftrightarrow y.$$

This yields $\boxed{x = y-b}$

• Case 3: If $\lambda = \dfrac{a+b+c+d}{2}$. Equation (14a) yields $c+d+y \leq \dfrac{a+b+c+d}{2} + x$ which means

$$y \leq \frac{a+b-c-d}{2} + x.$$

Conversely, Equation (14b) yields $a+b+x \leq \dfrac{a+b+c+d}{2} + y$ which means

$$y \geq \frac{a+b-c-d}{2} + x.$$

This finally yields $\boxed{y = x + \dfrac{a+b-c-d}{2}}$.

□

4.2 Proof of the convergence for $\frac{1}{\beta} \cdot log(H_{\beta A+B_\beta})$

Here we prove that perturbation does not affect the selection of the calibrated subaction. Note that Theorem A holds in our case and we shall re-write Equation (11). For that purpose we have to compute the Mañé potential. For simplicity we set $S_i(\cdot) := S(i^\infty, \cdot)$ with $i = 0, 1$.

We let the reader check that the functions S_i are constant on the 2-cylinders [01] and [10] with values:

$$S_0(10) = b + \sum_{n\geq 2} a_n,\ S_0(01) = \sum_{n\geq 2} a_n,$$
$$S_1(10) = \sum_{n\geq 2} c_n,\ S_1(01) = d + \sum_{n\geq 2} c_n.$$

Equation (11) yields that any accumulation point V for $\frac{1}{\beta}\log H_{\beta A}$ satisfies the equation

$$\gamma \otimes \begin{bmatrix} V(0^\infty) \\ V(1^\infty) \end{bmatrix} = \begin{pmatrix} A(10)+S_0(10) & A(10)+S_1(10) \\ A(01)+S_0(01) & A(01)+S_1(01) \end{pmatrix} \otimes \begin{bmatrix} V(0^\infty) \\ V(1^\infty) \end{bmatrix},$$

which can be rewritten as

$$\gamma \otimes \begin{bmatrix} V(0^\infty) \\ V(1^\infty) \end{bmatrix} = \begin{pmatrix} d+b+\sum_{n\geq 2} a_n & d+\sum_{n\geq 2} c_n \\ b+\sum_{n\geq 2} a_n & b+d+\sum_{n\geq 2} c_n \end{pmatrix} \otimes \begin{bmatrix} V(0^\infty) \\ V(1^\infty) \end{bmatrix}. \tag{15}$$

Equation (15) can be rewritten as

$$\gamma \otimes \begin{bmatrix} V(0^\infty) \\ V(1^\infty) \end{bmatrix} = M \otimes \begin{bmatrix} V(0^\infty) \\ V(1^\infty) \end{bmatrix},$$

where M satisfies Proposition 26, with $a := \sum_{n\geq 2} a_n$ and $c := \sum_{n\geq 2} c_n$. This yields

$$\gamma = \max\left\{ \sum_{n\geq 2} a_n + b + d,\ \sum_{n\geq 2} c_n + b + d,\ \frac{\sum_{n\geq 2} a_n + \sum_{n\geq 2} c_n + b + d}{2} \right\}. \tag{16}$$

Our condition $H_{\beta A}(0^\infty) = 1$ yields $V(0^\infty) = 0$ and, as the eigenspace has dimension 1 (see Proposition 26), this fixes the value for $V(1^\infty)$. As $\Omega = \{0^\infty, 1^\infty\}$, applying Lemma 14, the calibrated subaction V is determined for any point $x \in \{0,1\}^{\mathbb{N}}$.

Now we consider the potential $\beta A + B_\beta$. Lemma 25 shows that any accumulation point $\widetilde{V}$ for $\frac{1}{\beta}\log(H_{\beta A+B_\beta})$ is a calibrated subaction for A which satisfies

$$\gamma + \tilde{V}(0^\infty) = A(10^\infty) + \tilde{V}(10^\infty)$$

and

$$\gamma + \tilde{V}(1^\infty) = A(01^\infty) + \tilde{V}(01^\infty).$$

By Lemma 14 this can be rewritten as

$$\gamma + \tilde{V}(0^\infty) = A(10^\infty) + \max_{i\in\{1,2\}} S_{i^\infty}(10) + \tilde{V}(i^\infty)$$

and

$$\gamma + \tilde{V}(1^\infty) = A(01^\infty) + \max_{i\in\{1,2\}} S_{i^\infty}(01) + \tilde{V}(i^\infty).$$

Then, as in (15), we have

$$\gamma \otimes \begin{pmatrix} \widetilde{V}(0^\infty) \\ \widetilde{V}(1^\infty) \end{pmatrix} = \begin{pmatrix} d+b+\sum_{n\geq 2} a_n & d+\sum_{n\geq 2} c_n \\ b+\sum_{n\geq 2} a_n & b+d+\sum_{n\geq 2} c_n \end{pmatrix} \otimes \begin{pmatrix} \widetilde{V}(0^\infty) \\ \widetilde{V}(1^\infty) \end{pmatrix}.$$

Using the condition $H_{\beta A+B_\beta}(0^\infty) = 1$ we have $\widetilde{V}(0^\infty) = 0 = V(0^\infty)$, therefore Prop. 26 shows that $\widetilde{V}(1^\infty) = V(1^\infty)$. Applying Lemma 14 we conclude that $\tilde{V} = V$.

4.3 Proof of the convergence for $\mu_{\beta A+B_\beta}$

4.3.1 The key condition to get convergence

Now we present some auxiliary results for the study of selection of measure. For any cylinder $[\omega] \subset X$ let us denote by $\mathbb{1}_{[\omega]} : X \to \{0,1\}$ the function satisfying

$$\mathbb{1}_{[\omega]}(x) = \begin{cases} 1 & \text{if } x \in [\omega] \\ 0 & \text{if } x \notin [\omega] \end{cases}.$$

Lemma 27. *Let $A : X \to \mathbb{R}$ be any Lipschitz function such that $\mu_{\beta A}$ converges to a probability μ_∞. Let $[\omega]$ be a fixed cylinder and assume that for any $\widetilde{\delta} < \gamma$ the one parameter family of functions $C_\beta := e^{\beta\widetilde{\delta}}\mathbb{1}_{[\omega]}$ satisfies*

$$\limsup_{\beta\to+\infty} \mu_{\beta A+C_\beta}([\omega]) \leq \mu_\infty([\omega]) \text{ and } \liminf_{\beta\to+\infty} \mu_{\beta A-C_\beta}([\omega]) \geq \mu_\infty([\omega]).$$

Then for any $\widetilde{\delta}' < \gamma$ and any Lipschitz continuous functions B_β satisfying $|B_\beta|_\infty < e^{\beta\widetilde{\delta}'}$, $\lim_{\beta\to+\infty} \mu_{\beta A+B_\beta}([\omega]) = \mu_\infty([\omega])$ holds.

Proof. Let us assume $|B_\beta|_\infty < e^{\beta\widetilde{\delta}'}$ for some $\widetilde{\delta}' < \gamma$ holds. Let $\widetilde{\delta}$ be a real number in $(\widetilde{\delta}', \gamma)$, let us fix some cylinder $[\omega]$. We set $C_\beta := e^{\beta\widetilde{\delta}}\mathbb{1}_{[\omega]}$. By definition of being equilibrium state, the next inequalities hold:

$$\int \beta A + C_\beta \, d\mu_{\beta A+C_\beta} + h(\mu_{\beta A+C_\beta}) \geq \int \beta A + C_\beta \, d\mu_{\beta A+B_\beta} + h(\mu_{\beta A+B_\beta})$$

and

$$\int \beta A + B_\beta \, d\mu_{\beta A+C_\beta} + h(\mu_{\beta A+C_\beta}) \leq \int \beta A + B_\beta \, d\mu_{\beta A+B_\beta} + h(\mu_{\beta A+B_\beta}).$$

This yields

$$\int C_\beta - B_\beta \, d\mu_{\beta A+C_\beta} \geq \int C_\beta - B_\beta \, d\mu_{\beta A+B_\beta}. \tag{17}$$

Similarly, inequalities

$$\int \beta A + B_\beta \, d\mu_{\beta A+B_\beta} + h(\mu_{\beta A+B_\beta}) \geq \int \beta A + B_\beta \, d\mu_{\beta A-C_\beta} + h(\mu_{\beta A-C_\beta})$$

and

$$\int \beta A - C_\beta \, d\mu_{\beta A+B_\beta} + h(\mu_{\beta A+B_\beta}) \leq \int \beta A - C_\beta \, d\mu_{\beta A-C_\beta} + h(\mu_{\beta A-C_\beta})$$

yield

$$\int B_\beta + C_\beta \, d\mu_{\beta A+B_\beta} \geq \int B_\beta + C_\beta \, d\mu_{\beta A-C_\beta}. \tag{18}$$

Then we have

$$\begin{aligned}
\mu_{\beta A+C_\beta}([\omega]) &= \int \mathbb{1}_{[\omega]} \, d\mu_{\beta A+C_\beta} \\
&= e^{-\beta\widetilde{\delta}} \cdot \int C_\beta \, d\mu_{\beta A+C_\beta} \\
&= e^{-\beta\widetilde{\delta}} \cdot \int C_\beta - B_\beta + B_\beta \, d\mu_{\beta A+C_\beta} \\
\text{by(17)} \quad &\geq e^{-\beta\widetilde{\delta}} \cdot \int C_\beta - B_\beta \, d\mu_{\beta A+B_\beta} + e^{-\beta\widetilde{\delta}} \cdot \int B_\beta \, d\mu_{\beta A+C_\beta} \\
&\geq \int \mathbb{1}_{[\omega]} \, d\mu_{\beta A+B_\beta} - e^{-\beta\widetilde{\delta}} \cdot \int B_\beta \, d\mu_{\beta A+B_\beta} + e^{-\beta\widetilde{\delta}} \cdot \int B_\beta \, d\mu_{\beta A+C_\beta} \\
&\geq \mu_{\beta A+B_\beta}([\omega]) - 2e^{\beta(\widetilde{\delta}'-\widetilde{\delta})},
\end{aligned}$$

where we use in the last inequality that $\mu_{\beta A+B_\beta}$ and $\mu_{\beta A+C_\beta}$ are probability measures. Since $\widetilde{\delta}' < \widetilde{\delta}$, letting $\beta \to +\infty$ yields

$$\mu_\infty([\omega]) \geq \limsup_{\beta\to+\infty} \mu_{\beta A+C_\beta}([\omega]) \geq \limsup_{\beta\to+\infty} \mu_{\beta A+B_\beta}([\omega]).$$

Similarly, using (18), we also get:

$$\mu_{\beta A-C_\beta}([\omega]) \leq \mu_{\beta A+B_\beta}([\omega]) + 2e^{\beta(\widetilde{\delta}'-\widetilde{\delta})}$$

and so

$$\mu_\infty([\omega]) \leq \liminf_{\beta\to+\infty} \mu_{\beta A-C_\beta}([\omega]) \leq \liminf_{\beta\to+\infty} \mu_{\beta A+B_\beta}([\omega]).$$

□

Corollary 28. *Let $A : X \to \mathbb{R}$ be any Lipschitz function. Assume that $\mu_{\beta A}$ converges to μ_∞ and that for any fixed $n \in \mathbb{N}$, $\widetilde{\delta} < \gamma$ and any family of functions C_β depending on n coordinates and satisfying $|C_\beta|_\infty \leq e^{\beta\widetilde{\delta}}$, $\mu_{\beta A+C_\beta} \to \mu_\infty$ holds. Then, for any $\widetilde{\delta}' < \gamma$ and any Lipschitz functions B_β satisfying $|B_\beta|_\infty < e^{\beta\widetilde{\delta}'}$ we have $\mu_{\beta A+B_\beta} \to \mu_\infty$.*

Proof. The selection of measure is determined by the limits

$$\lim_{\beta\to\infty} \mu_{\beta A+B_\beta}([\omega]),\ \ [\omega]\ \text{ is cylinder.}$$

For each fixed cylinder $[\omega]$, we can construct a function $C_\beta := e^{\beta\widetilde{\delta}}\mathbb{1}_{[\omega]}$, which depends of finite coordinates. By hipothesis,

$$\lim_{\beta\to\infty} \mu_{\beta A+C_\beta}([\omega]) = \lim_{\beta\to\infty} \mu_{\beta A-C_\beta}([\omega]) = \mu_\infty([\omega]).$$

Then, from Lemma 27 we have

$$\lim_{\beta\to\infty} \mu_{\beta A+B_\beta}([\omega]) = \mu_\infty([\omega]).$$

□

4.3.2 Selection of maximizing measure

Now we prove the convergence of the measures $\mu_{\beta A+B_\beta}$ and $\mu_{\beta A}$ as stated in Theorem B. We will present explicitly the limit measure.

By Lemma 6, any accumulation point for $\mu_{\beta A+B_\beta}$ is A-maximizing, hence of the form $s\delta_{0^\infty} + (1-s)\delta_{1^\infty}$, with $s \in [0,1]$. Note that it is sufficient to get that $\mu_{\beta A+B_\beta}([0])$ and $\mu_{\beta A+B_\beta}([1])$ converge to determine s. As $\mu_{\beta A+B_\beta}([1]) = 1 - \mu_{\beta A+B_\beta}([0])$, it thus sufficient to get that $\mu_{\beta A+B_\beta}([0])$ converges. Furthermore, Lemma 27 yields that the convergence for $\mu_{\beta A+B_\beta}([0])$ follows from the convergence of $\mu_{\beta A+C_\beta\mathbb{1}_{[0]}}([0])$ with $|C_\beta|$ going faster to 0 than $e^{\beta\gamma}$. Therefore we can suppose that B_β only depends on the first coordinate and satisfies

$$B_\beta(0) = a_\beta \ \text{ and } \ B_\beta(1) = 0,$$

with $|a_\beta| \le e^{\beta\delta}, \delta < \gamma$.

Let us set

$$S_0(\beta) = \frac{1 + \sum_{j=1}^{\infty}(j+1)e^{\beta(a_2+\ldots+a_{1+j})+ja_\beta - jP(\beta A+B_\beta)}}{1 + \sum_{j=1}^{\infty} e^{\beta(a_2+\ldots+a_{1+j})+ja_\beta - jP(\beta A+B_\beta)}},$$

$$S_1(\beta) = \frac{1 + \sum_{j=1}^{\infty}(j+1)e^{\beta(c_2+\ldots+c_{1+j})- jP(\beta A+B_\beta)}}{1 + \sum_{j=1}^{\infty} e^{\beta(c_2+\ldots+c_{1+j})- jP(\beta A+B_\beta)}}.$$

The potential $\beta A + B_\beta$ does not satisfy (2) because it can be positive in a neighborhood of 0^∞ while it is equal to zero in 1^∞, but the same proof of Proposition 6 in [3] can be applied in order to conclude that

$$\mu_{\beta A+B_\beta}([0]) = \frac{S_0(\beta)}{S_0(\beta)+S_1(\beta)} \qquad \text{and} \qquad \mu_{\beta A+B_\beta}([1]) = \frac{S_1(\beta)}{S_0(\beta)+S_1(\beta)}.$$

In a first step we prove that the limit for $\frac{S_0(\beta)}{S_1(\beta)} = \frac{\mu_{\beta A+B_\beta}([0])}{\mu_{\beta A+B_\beta}([1])}$ does not depend on a_β. In a second moment we prove that such limit exists. For that purpose, we consider a series

of claims. From now on in this section the notation $f(\beta) \sim g(\beta)$ means $\lim_{\beta\to+\infty} \frac{f(\beta)}{g(\beta)} = 1$. Clearly if $S_0(\beta) \sim F(\beta)$ and $S_1(\beta) \sim G(\beta)$ then $\lim_{\beta\to+\infty} \frac{S_0(\beta)}{S_1(\beta)} = \lim_{\beta\to+\infty} \frac{F(\beta)}{G(\beta)}$. Next lemma present basic properties of $\sim$ which will be used.

Lemma 29. *For any positive functions f, g we have:*
1. If $\lim_{\beta\to+\infty} f(\beta) = 0$ then $(1 + f(\beta) + g(\beta)) \sim (1 + g(\beta))$;
2. If $\lim_{\beta\to+\infty} f(\beta) = 1$ then $(1 + f(\beta) \cdot g(\beta)) \sim (1 + g(\beta))$;
3. If $\lim_{\beta\to+\infty} f(\beta) = 1$ and h is a function satisfying $(1+g(\beta)) \leq 1+h(\beta) \leq 1+f(\beta)\cdot g(\beta)$ then $(1 + h(\beta)) \sim (1 + g(\beta))$;
4. If $f(\beta) \sim f_0(\beta)$ then $(1 + f(\beta)g(\beta)) \sim (1 + f_0(\beta)g(\beta))$.

Proof. The proof of item 1. follows from

$$\lim_{\beta\to+\infty} \frac{1 + f(\beta) + g(\beta)}{1 + g(\beta)} = 1 + \lim_{\beta\to+\infty} \frac{f(\beta)}{1 + g(\beta)} = 1,$$

because $f(\beta) \to 0$ and g is positive.

The proof of item 2. follows from

$$\lim_{\beta\to+\infty} \frac{1 + f(\beta) \cdot g(\beta)}{1 + g(\beta)} = 1 + \lim_{\beta\to+\infty} \frac{(f(\beta) - 1) \cdot g(\beta)}{1 + g(\beta)} = 1,$$

because $(f(\beta) - 1) \to 0$ and $0 < \frac{g(\beta)}{1+g(\beta)} < 1$.

The proof of item 3. is a consequence of 2.. Indeed, we have

$$1 = \frac{1 + g(\beta)}{1 + g(\beta)} \leq \frac{1 + h(\beta)}{1 + g(\beta)} \leq \frac{1 + f(\beta)g(\beta)}{1 + g(\beta)}$$

and the right hand side function converges to 1 (because item 2.).

The proof of item 4. is again a consequence of item 2.. We have

$$1 + f(\beta)g(\beta) = 1 + \frac{f(\beta)}{f_0(\beta)} f_0(\beta)g(\beta) \sim 1 + f_0(\beta)g(\beta)$$

because, by hypothesis, $\frac{f(\beta)}{f_0(\beta)} \to 1$.

□

Claim 1.

$$\lim_{\beta\to+\infty} \sum_{j<\beta} (j + 1)e^{\beta(a_2+\ldots+a_{1+j})+ja_\beta-jP(\beta A+B_\beta)} = 0. \tag{19}$$

Proof of the Claim. Note that $P(\beta A + B_\beta) \geq P(\beta A) - e^{\beta\delta} > 0$ holds for β large enough. Hence,

$$0 \leq \sum_{j<\beta} (j + 1)e^{\beta(a_2+\ldots+a_{1+j})+ja_\beta-jP(\beta A+B_\beta)} \leq (\beta + 1)^2 e^{\beta\cdot a_2+\beta\cdot|a_\beta|}.$$

As $a_\beta \to 0$ and $a_2 < 0$ we conclude (19). □

Remark 30. *We emphasize that this is the unique place where we assume $a_2 < 0$ and $c_2 < 0$. Except here, assumptions $\sum a_n \leq 0$, $\sum c_n \leq 0$, $b + d < 0$ + the fact that the Aubry set is $\{0^\infty, 1^\infty\}$ are sufficient.* ■

It follows from Claim 1 and item 1. of Lemma 29 that

$$S_0(\beta) \sim \frac{1 + \sum_{j\geq\beta}(j+1)e^{\beta(a_2+\ldots+a_{1+j})+ja_\beta - jP(\beta A+B_\beta)}}{1 + \sum_{j\geq\beta} e^{\beta(a_2+\ldots+a_{1+j})+ja_\beta - jP(\beta A+B_\beta)}}. \tag{20}$$

In the following a stands for $\sum a_n$ and c stands for $\sum c_n$.

Claim 2. *There exists $C > 0$ such that for every $j \geq 1$,*

$$c \leq c_2 + \ldots + c_{1+j} \leq c + C\theta^j \text{ and } a \leq a_2 + \ldots + a_{1+j} \leq a + C\theta^j$$

Proof of the Claim. Indeed, as A is Lipschitz, there exists $C_0 > 0$ such that, for $n \geq 2$,

$$-c_n = -A(1^n0^\infty) \leq -A(1^\infty) + C_0 \cdot d(1^n0^\infty, 1^\infty) = C_0\theta^n.$$

Then, for each $j \geq \beta$, we have

$$c_2 + \ldots + c_{1+j} = \sum_{i=2}^{\infty} c_i - \sum_{i=j+2}^{\infty} c_i \leq \sum_{i=2}^{\infty} c_i + \sum_{i=j+2}^{\infty} C_0\theta^i \leq \sum_{i=2}^{\infty} c_i + \frac{C_0\theta^{j+2}}{1-\theta}.$$

A similar computation can be used for a's. □

It follows from Claim 2 that

$$1 + e^{\beta a}\sum_{j\geq\beta} e^{-jP(\beta A+B_\beta)} \leq 1 + \sum_{j\geq\beta} e^{\beta(a_2+\ldots+a_{1+j})-jP(\beta A+B_\beta)} \leq 1 + e^{\beta C\theta^\beta}\cdot e^{\beta a}\sum_{j\geq\beta} e^{-jP(\beta A+B_\beta)}.$$

As $\lim_{\beta\to+\infty} e^{\beta C\theta^\beta} = 1$, applying item 3. of Lemma 29, we get

$$1 + \sum_{j\geq\beta} e^{\beta(a_2+\ldots+a_{1+j})-jP(\beta A+B_\beta)} \sim 1 + e^{\beta a}\sum_{j\geq\beta} e^{-jP(\beta A+B_\beta)}.$$

With a similar computation we also obtain

$$1 + \sum_{j\geq\beta}(j+1)e^{\beta(a_2+\ldots+a_{1+j})+ja_\beta-jP(\beta A+B_\beta)} \sim 1 + e^{\beta a}\sum_{j\geq\beta}(j+1)e^{ja_\beta-jP(\beta A+B_\beta)}.$$

Applying this results in equation (20) we get

$$S_0(\beta) \sim \frac{1 + e^{\beta a}\sum_{j\geq\beta}(j+1)e^{ja_\beta-jP(\beta A+B_\beta)}}{1 + e^{\beta a}\sum_{j\geq\beta} e^{ja_\beta-jP(\beta A+B_\beta)}}. \tag{21}$$

Let us consider now the next step. From simple computations we get, for any $z > 0$ and $k \in \mathbb{N}$,

$$\sum_{j=k}^{\infty} e^{-jz} = \frac{e^{-kz}}{1-e^{-z}} \quad \text{and} \quad \sum_{j=k}^{\infty}(j+1)e^{-jz} = \frac{e^{-kz}}{1-e^{-z}}\left(k + \frac{1}{1-e^{-z}}\right).$$

Claim 3.

$$\sum_{j\geq\beta} e^{ja_\beta - jP(\beta A+B_\beta)} \sim \frac{1}{P(\beta A)} \text{ and } \sum_{j\geq\beta}(j+1)e^{ja_\beta - jP(\beta A+B_\beta)} \sim \frac{1}{(P(\beta A))^2}.$$

Proof of the Claim. As $\lim_{x\to+\infty}\frac{x}{1-e^{-x}} = 1$, if z is a function such that $z(\beta)\downarrow 0$ and $\beta\cdot z(\beta)\to 0$ we get

$$\sum_{j\geq\beta} e^{-jz(\beta)} = \frac{e^{-\beta z(\beta)}}{1-e^{-z(\beta)}} \sim \frac{1}{z(\beta)}.$$

Furthermore,

$$\sum_{j\geq\beta}(j+1)e^{-jz(\beta)} = \frac{e^{-\beta z(\beta)}}{1-e^{-z(\beta)}}\left(\beta + \frac{1}{1-e^{-z(\beta)}}\right) \sim \frac{\beta}{z(\beta)} + \frac{1}{(z(\beta))^2} \sim \frac{\beta z(\beta)+1}{(z(\beta))^2} \sim \frac{1}{(z(\beta))^2}.$$

Finally we take $z(\beta) = -a_\beta + P(\beta A + B_\beta)$. Observe that $z(\beta)\sim P(\beta A)$ and $\beta\cdot P(\beta A)\to 0$ because $P(\beta A)$ and a_β converge exponentially fast to zero, respectively at rate $e^{\beta.\gamma}$ and $e^{\beta.\delta}$ with γ as given in (16) and $\delta<\gamma$. □

Claim 3, item 4. of Lemma 29 and equation (21) together yield

$$S_0(\beta) \sim \frac{1+e^{\beta a}\frac{1}{(P(\beta A)^2}}{1+e^{\beta a}\frac{1}{P(\beta A)}}.$$

Similarly we get

$$S_1(\beta) \sim \frac{1+e^{\beta c}\frac{1}{(P(\beta A))^2}}{1+e^{\beta c}\frac{1}{P(\beta A)}}$$

and therefore

$$\frac{S_0(\beta)}{S_1(\beta)} \sim \frac{(P(\beta A))^2+e^{\beta a}}{P(\beta A)+e^{\beta a}}\frac{P(\beta A)+e^{\beta c}}{(P(\beta A))^2+e^{\beta c}}. \tag{22}$$

Observe that the right hand side does not depend of B_β. Then the limit of the ratio $\frac{\mu_{\beta A+B_\beta}([0])}{\mu_{\beta A+B_\beta}([1])} = \frac{S_0(\beta)}{S_1(\beta)}$ does not depend of B_β. This finally shows that $\mu_{\beta A+B_\beta}$ converges to the same limit than $\mu_{\beta A}$, if it converges.

Next proposition shows that $\mu_{\beta A+B_\beta}$ actually converges and explicitly gives the limit.

Proposition 31. *Under above hypothesis concerning B_β and A:*
1. If $a=c$ then $\mu_{\beta A+B_\beta}\to\frac{\delta_{0^\infty}+\delta_{1^\infty}}{2}$.
2. If $a>c$ and $a+b+d<c$ then $\mu_{\beta A+B_\beta}\to\frac{\delta_{0^\infty}+\delta_{1^\infty}}{2}$.
3. If $a>c$ and $a+b+d>c$ then $\mu_{\beta A+B_\beta}\to\delta_{0^\infty}$
4. If $a>c$ and $a+b+d=c$ then $\mu_{\beta A+B_\beta}\to\frac{10+2\sqrt{5}}{20}\delta_{0^\infty}+\frac{10-2\sqrt{5}}{20}\delta_{1^\infty}$.
We get symmetric results in the case $c>a$.

Proof. We will consider (22). Furthermore, from (16) we have

$$\gamma = \max\left\{a+b+d,\ c+b+d,\ \frac{a+c+b+d}{2}\right\}.$$

The proof of item 1. is consequence of (22). In the case $a = c$, the right-hand side term in (22) equals 1, which yields that $\frac{S_0(\beta)}{S_1(\beta)}$ goes to 1 as $\beta \to +\infty$.

For the proof of other cases we consider $l(\beta) := P(\beta A)/e^{\beta\gamma}$. Then $P(\beta A) = l(\beta)e^{\beta\gamma}$ and $\lim_{\beta\to+\infty} \frac{1}{\beta}\log(l(\beta)) = 0$. It follows that for any constant $\epsilon > 0$ we have $e^{-\beta\epsilon} < l(\beta) < e^{\beta\epsilon}$ for sufficiently large β. With such notation, from (22) we have

$$S_0(\beta)/S_1(\beta) \sim \frac{l^2(\beta)e^{2\beta\gamma} + e^{\beta a}}{l(\beta)e^{\beta\gamma} + e^{\beta a}} \frac{l(\beta)e^{\beta\gamma} + e^{\beta c}}{l^2(\beta)e^{2\beta\gamma} + e^{\beta c}}.$$

Proof of item 2: If $a > c$ and $a+b+d < c$ then $\frac{a+b+d+c}{2} > a+b+d$ and so $\gamma = \frac{a+b+d+c}{2}$. We have also

$$a > c \Rightarrow a > c+b+d \Rightarrow \frac{a}{2} > \frac{c+b+d}{2} \Rightarrow a > \frac{a}{2} + \frac{c+b+d}{2} \Rightarrow a > \gamma$$

and similarly

$$c > a+b+d \Rightarrow c > \gamma.$$

As $a > \gamma$ and $c > \gamma$ it follows that

$$S_0(\beta)/S_1(\beta) \sim \frac{l^2(\beta)e^{2\beta\gamma} + e^{\beta a}}{l(\beta)e^{\beta\gamma} + e^{\beta a}} \frac{l(\beta)e^{\beta\gamma} + e^{\beta c}}{l^2(\beta)e^{2\beta\gamma} + e^{\beta c}} \to 1$$

and then $\mu_{\beta A+B_\beta} \to \frac{\delta_{0\infty}+\delta_{1\infty}}{2}$.

Proof of item 3: If $a+b+d > c$ then $\frac{a+b+d+c}{2} < a+b+d$ and then $\gamma = a+b+d$. It follows that $c < \gamma < a$. Then, using any auxiliary number δ such that $c < \delta < \gamma$ and $2\gamma < \delta$ we have

$$S_0(\beta)/S_1(\beta) \sim \frac{l^2(\beta)e^{2\beta\gamma} + e^{\beta a}}{l(\beta)e^{\beta\gamma} + e^{\beta a}} \frac{l(\beta)e^{\beta\gamma} + e^{\beta c}}{l^2(\beta)e^{2\beta\gamma} + e^{\beta c}} \sim 1 \cdot \frac{l(\beta)e^{\beta(\gamma-\delta)} + e^{\beta(c-\delta)}}{l^2(\beta)e^{\beta(2\gamma-\delta)} + e^{\beta(c-\delta)}} \to +\infty.$$

Proof of item 4: As $a+b+d = c$ we have $a+b+d = \frac{a+b+d+c}{2}$ and then $\gamma = a+b+d = \frac{a+b+d+c}{2}$. In this case $\gamma < a$ and $\gamma = c$. Then we have

$$S_0(\beta)/S_1(\beta) \sim 1 \cdot \frac{l(\beta)e^{\beta c} + e^{\beta c}}{l^2(\beta)e^{2\beta c} + e^{\beta c}} \sim \frac{l(\beta)+1}{l^2(\beta)e^{\beta c}+1} \sim l(\beta) + 1.$$

It remains to study the limit for $l(\beta)$. From [23] corollary 3.5 we have (here in the present paper $b_j = b$ and $d_j = d$ for any j)

$$e^{2P(\beta A)} = e^{\beta(b+d)}\left(1 + \sum_{j\geq 1} e^{\beta(a_2+\ldots+a_{1+j})-jP(\beta A)}\right)\left(1 + \sum_{j\geq 1} e^{\beta(c_2+\ldots+c_{1+j})-jP(\beta A)}\right).$$

Applying similar computations as above (in order to get (22)) we obtain

$$e^{2P(\beta A)} \sim e^{\beta(b+d)} \left(1 + \frac{e^{\beta a}}{P(\beta A)}\right)\left(1 + \frac{e^{\beta c}}{P(\beta A)}\right).$$

Then

$$e^{2P(\beta A)} \sim e^{\beta(b+d)} + \frac{e^{\beta(b+d+c)}}{P(\beta A)} + \frac{e^{\beta(b+d+a)}}{P(\beta A)} + \frac{e^{\beta(b+d+a+c)}}{(P(\beta A))^2}$$

and

$$e^{2P(\beta A)} \sim e^{\beta(b+d)} + \frac{e^{\beta(b+d+c)}}{l(\beta)e^{\beta\gamma}} + \frac{e^{\beta(b+d+a)}}{l(\beta)e^{\beta\gamma}} + \frac{e^{\beta(b+d+a+c)}}{l^2(\beta)e^{2\beta\gamma}}.$$

As $e^{2P(\beta A)} \to 1$ and from the hyphothesis of item 4. we have $\gamma = c = a+b+d = \frac{a+b+d+c}{2}$, we get

$$1 \sim \frac{1}{l(\beta)} + \frac{1}{l^2(\beta)}.$$

As $l(\beta)$ is positive we obtain $l(\beta) \to \frac{1+\sqrt{5}}{2}$. Finally we conclude that $S_0(\beta)/S_1(\beta) \to \frac{3+\sqrt{5}}{2}$. Consequently $\mu_{\beta A}([0]) \to \frac{3+\sqrt{5}}{5+\sqrt{5}} = \frac{10+2\sqrt{5}}{20}$ and therefore $\mu_{\beta A}([1]) \to \frac{2}{5+\sqrt{5}} = \frac{10-2\sqrt{5}}{20}$. □

Remark 32. *It is noteworthy that knowing A close to 0^∞ and 1^∞ is not sufficient to determine the selection. This has been already pointed out in [3]. Indeed, $b+d$ plays a role to determine the limit.*

However, we point ou that in cases 1 and 2 in Proposition , γ is given by the length-2 cycle $0^\infty \to 1^\infty \to 0^\infty$ in the matrix M, and then μ_∞ is $\frac{1}{2}(\delta_{0^\infty} + \delta_{1^\infty})$. In case 3, γ is given by the length-1 cycle $0^\infty \to 0^\infty$ and the limit measure is δ_{0^∞}. Finally in case 4, γ is given both by length-1 cycle $0^\infty \to 0^\infty$ and length-2 cycle $0^\infty \to 1^\infty \to 0^\infty$, and the limit measure is an asymmetric combination of δ_{0^∞} and δ_{1^∞}.

In all these cases, $a \geq c$ and μ_∞ contains some positive part of δ_{0^∞}. This confirm that flatness may also be a criteria for selection (see [18]).

A About the hypothesis in Theorem B

Consider a Lipschitz function $A : X \to \mathbb{R}$ satisfying $m(A) = 0$ and suppose there exists the limit $\gamma = \lim_{\beta\to+\infty} \frac{1}{\beta}\log(P(\beta A) - h)$, where $h = \sup_{\mu\in\mathcal{M}_{max}(A)} h_\mu(\sigma)$. Considering the hypothesis in Theorem B concerning the perturbation, it is natural to ask what happens if B_β converges to zero slower than $e^{\beta\gamma}$. Next example present a discussion in this way and we will show a change in the selection of subaction and measure. We assume, in this example, that the eigenfuntions H_β are normalized by $H_\beta(0^\infty) = 1$.

Example 33. *In this example we suppose $X = \{0,1\}^{\mathbb{N}}$ and consider two numbers γ and η satisfying $\gamma < \eta < 0$. Let $A, B_\beta : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ be functions which are constant in each 2-cylinder and defined by*

$$A(0,1) = A(1,0) = \gamma, \;\; A(0,0) = A(1,1) = 0,$$

$$B_\beta(0,0) = B_\beta(1,0) = B_\beta(0,1) = 0 \ \ and \ \ B_\beta(1,1) = \log(1+e^{\beta\eta}).$$

We claim that, as β goes to $+\infty$:
1. $\frac{1}{\beta}\log(P(\beta A)) \to \gamma$*;*
2. $\frac{1}{\beta}\log|B_\beta|_\infty = \eta$ *(then $|B_\beta|_\infty$ seems $e^{\beta\eta}$ with $\eta > \gamma$);*
3. $\frac{1}{\beta}\log(H_{\beta A})(1^\infty) \to 0$*;*
4. $\frac{1}{\beta}\log(H_{\beta A+B_\beta})(1^\infty) \to \eta - \gamma$*;*
5. $\mu_{\beta A} \to \frac{1}{2}(\delta_{0^\infty} + \delta_{1^\infty})$*;*
6. $\mu_{\beta A+B_\beta} \to \delta_{1^\infty}$*.*
From items 1-6 clearly the perturbation B_β converges to zero exponentially fast, but slower than $e^{\beta\gamma}$, and it changes the results concerning selections of subaction and measure.

Proof of 1.: In this case note that Theorem A holds. The matrix M in the Prop. 26 is then equal to

$$M = \begin{pmatrix} 2\gamma & \gamma \\ \gamma & 2\gamma \end{pmatrix},$$

and its unique eigenvalue for the Max-Plus formalism is γ. As entropy for 0^∞ and 1^∞ is zero we get 1.

Proof of 3.: the eigenfunction associated to $\lambda_{\beta A}$ depends just of one coordinate and satisfies

$$\lambda_\beta H_{\beta A}(0) = L_{\beta A}(H_{\beta A})(0) = 1 \cdot H_{\beta A}(0) + e^{\beta\gamma} H_{\beta A}(1).$$

Hence, as by convention $H_{\beta A}(0) = 1$, we get

$$H_{\beta A}(1) = \frac{\lambda_\beta - 1}{e^{\beta\gamma}},$$

with $\lambda_\beta = e^{P(\beta)}$. Doing $\frac{1}{\beta}\log$ in this last equality, and $\beta \to +\infty$ we get $V(1^\infty) = \lim_{\beta\to\infty}\frac{1}{\beta}\log(H_{\beta A}(1)) = 0$.

Proof of 5.: as A is locally constant we know that $\mu_{\beta A}$ converges (see [8, 17]). Furthermore, symmetry yields $\mu_{\beta A} \to \frac{\delta_{0^\infty}+\delta_{1^\infty}}{2}$.

Proof of 2.: as $\log(1+x) = x[1+o(x)]$, for x near to 0, we get

$$\lim_{\beta\to\infty}\frac{1}{\beta}\log(|B_\beta|_\infty) = \eta.$$

Proof of 4. the associated matrix for the Ruelle operator $L_{\beta A+B_\beta}$ is given by

$$\begin{pmatrix} 1 & e^{\beta\gamma} \\ e^{\beta\gamma} & 1+e^{\beta\eta} \end{pmatrix}$$

with dominating eigenvalue

$$\tilde{\lambda}_\beta = 1 + \frac{e^{\beta\eta} + \sqrt{e^{2\beta\eta} + 4e^{2\beta\gamma}}}{2}. \tag{23}$$

The associated main eigenfunction depends just of one coordinate and satisfies

$$\tilde{\lambda}_\beta H_{\beta A+B_\beta}(0) = H_{\beta A+B_\beta}(0) + e^{\beta\gamma} H_{\beta A+B_\beta}(1).$$

Hence, using again that $H_{\beta A+B_\beta}(0) = 1$ *by convention, we get*

$$H_{\beta A+B_\beta}(1) = \frac{\tilde{\lambda}_\beta - 1}{e^{\beta\gamma}} = \frac{e^{\beta\eta} + \sqrt{e^{2\beta\eta} + 4e^{2\beta\gamma}}}{2e^{\beta\gamma}}. \tag{24}$$

It follows that

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log(H_{\beta A+B_\beta}(1)) = \lim_{\beta\to\infty} \frac{1}{\beta} \log\left(\frac{e^{\beta\eta} + \sqrt{e^{2\beta\eta} + 4e^{2\beta\gamma}}}{2e^{\beta\gamma}}\right) = \eta - \gamma > 0.$$

Proof of 6.: consider the normalized function

$$\overline{\beta A + B_\beta} := \beta A + B_\beta + \log(H_{\beta A+B_\beta}) - \log(H_{\beta A+B_\beta} \circ \sigma) - \log(\tilde{\lambda}_\beta).$$

The associated Ruelle operator is identified with the matrix

$$\begin{pmatrix} \frac{1}{\tilde{\lambda}_\beta} & \frac{e^{\beta\gamma}}{H_{\beta A+B_\beta}(1)\tilde{\lambda}_\beta} \\ \frac{e^{\beta\gamma} H_{\beta A+B_\beta}(1)}{\tilde{\lambda}_\beta} & \frac{1+e^{\beta\eta}}{\tilde{\lambda}_\beta} \end{pmatrix}.$$

In order to study the selection of $\mu_\beta = \mu_{\beta A+B_\beta}$ *it is sufficiently to study* $p_0^\beta = \mu_\beta([0])$ *and* $p_1^\beta = \mu_\beta([1])$*. As* p_0^β *and* p_1^β *satisfy*

$$\begin{pmatrix} \frac{1}{\tilde{\lambda}_\beta} & \frac{e^{\beta\gamma}}{H_{\beta A+B_\beta}(1)\tilde{\lambda}_\beta} \\ \frac{e^{\beta\gamma} H_{\beta A+B_\beta}(1)}{\tilde{\lambda}_\beta} & \frac{1+e^{\beta\eta}}{\tilde{\lambda}_\beta} \end{pmatrix} \begin{bmatrix} p_0^\beta \\ p_1^\beta \end{bmatrix} = \begin{bmatrix} p_0^\beta \\ p_1^\beta \end{bmatrix}$$

we have

$$p_0^\beta e^{\beta\gamma} H_{\beta A+B_\beta}(1) + p_1^\beta (1 + e^{\beta\eta}) = \tilde{\lambda}_\beta p_1^\beta$$

and so, using that $p_1^\beta = 1 - p_0^\beta$*, we get*

$$p_0^\beta = \frac{\tilde{\lambda}_\beta - 1 - e^{\beta\eta}}{e^{\beta\gamma} H_{\beta A+B_\beta}(1) + \tilde{\lambda}_\beta - 1 - e^{\beta\eta}} \overset{(24)}{=} \frac{\tilde{\lambda}_\beta - 1 - e^{\beta\eta}}{2(\tilde{\lambda}_\beta - 1) - e^{\beta\eta}}$$

$$\overset{(23)}{=} \frac{\frac{-e^{\beta\eta} + \sqrt{e^{2\beta\eta} + 4e^{2\beta\gamma}}}{2}}{\sqrt{e^{2\beta\eta} + 4e^{2\beta\gamma}}} = \frac{1}{2} - \frac{e^{\beta\eta}}{2\sqrt{e^{2\beta\eta} + 4e^{2\beta\gamma}}}.$$

As $\eta > \gamma$ *we conclude that* $\lim_{\beta\to+\infty} p_0^\beta = \frac{1}{2} - \frac{1}{2} = 0$ *and so*

$$\lim_{\beta\to\infty} \mu_{\beta A+B_\beta} = \delta_{1^\infty}.$$

Remark 34. *It is possible to get similar conclusions as in above example in the case $B_\beta = B$ does not depend of β and satisfies*

$$B(0,0) = B(1,0) = B(0,1) = 0 \text{ and } B(1,1) > 0.$$

Indeed, just write $B(1,1) = \log(1+e^\eta)$ where $\eta \in \mathbb{R}$ and replace $e^{\beta\eta}$ by e^η in above computations in order to get $\lim_{\beta\to\infty} \frac{1}{\beta}\log(H_{\beta A+B_\beta}(1)) = -\gamma > 0$ and $\lim_{\beta\to\infty} \mu_{\beta A+B} = \delta_{1^\infty}$.

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