



**HAL**  
open science

# A note on welfare measurement with income-dependent discrete choice

P Delle Site, Karim Kilani

► **To cite this version:**

P Delle Site, Karim Kilani. A note on welfare measurement with income-dependent discrete choice. 2024. hal-04527839v2

**HAL Id: hal-04527839**

**<https://hal.science/hal-04527839v2>**

Preprint submitted on 19 Oct 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# **Welfare measurement with income-dependent discrete choice**

P. Delle Site<sup>a,1</sup>, K. Kilani<sup>b</sup>

<sup>a</sup> *Niccolò Cusano University, Rome, Italy*

<sup>b</sup> *Conservatoire National des Arts et Métiers, Paris, France*

## **Abstract**

The expectation of the random compensating variation is the welfare change measure that is used in discrete choice models. The expectation of the equivalent variation is equally founded theoretically. When choices are income independent and income enters utilities linearly, the two measures are identical. The case of income-dependent choices remains an area for exploration. The paper provides the equivalent variation counterparts of the formulas that are available for the expectation of the compensating variation.

## **Keywords**

compensating variation, discrete choice, equivalent variation, random utility, welfare

## **JEL classification**

D11, D60

---

(<sup>1</sup>) Corresponding author: [paolo.dellesite@unicusano.it](mailto:paolo.dellesite@unicusano.it)

## 1. Introduction

There is a long record of contributions to welfare measurement when choices are discrete. This is of use in cost-benefit analysis in the transport sector, one where discrete travel choices (if to travel or otherwise, destination, mode of transport, route) are of relevance.

Since the introduction of discrete choice models, where utilities are modelled as random, in the seventies, researchers addressed the measurement of welfare change, adapting the measures that are used for classical divisible goods demand models.

The first case was the one of income-independent choices, where income enters the systematic utilities linearly with a constant and alternative-independent coefficient which is easily interpreted as the marginal utility of income. Williams (1977) proposed a representative consumer approach: the measure of welfare change is the representative consumer surplus difference (without and with the policy) which equals the difference in the expectation of the maximum utility divided by the marginal utility of income. This is the popular logsum formula in the case of the multinomial logit model. Small and Rosen (1981) obtained the same formula based on the aggregate compensating variation derived from the variation of the expenditure function.

For the case with income-dependent choices, one can apply the Hicksian concepts of compensating and equivalent variation of classical demand theory by using the expectation of the maximum utility as indirect utility. This approach is found in Hau (1985 for theory, and 1987 for an application in the transport sector).

Today, welfare measurement within discrete choice models is grounded in the theory of random Hicksian measures proposed by McFadden (1999). The expectation of the random compensating variation is commonly used. If random terms are interpreted as individual specific, then the measure is the average in a population of individuals who are identical in terms of systematic utilities.

When choices are income independent and income enters utilities linearly, this measure equals the difference in the expectations of the maximum utility divided by the marginal utility of income. Therefore, this represents a case where the representative consumer approach and the random utility approach coincide. By contrast, when choices are income dependent, the two approaches lead to different measures (see the numerical exploration by Tra, 2013).

If the random approach is followed, there are two ways to compute the expectation of the compensating variation. The first is simulation based on draws from the distribution of the random terms. Herriges and Kling (1999) used this method with a translog in residual income specification of the utilities, the residual income being the difference between disposable income and expenditure on the alternative. This specification is appealing, because it provides a decreasing marginal utility of income, i.e. one additional dollar has less value for the richer than for the poorer. The translog specification is studied in Delle Site (2014).

The second way to compute the expectation of the compensating variation is based on the solution to one-dimensional integrals. Karlström (2014) obtained a formula starting from the distribution of the random expenditure function. De Palma and Kilani (2011) first calculated the probabilities of observed choice transitions between alternatives. Then, they obtained a formula starting from the distributions of the expectations of the compensating variation that are conditional on the observed transition.

The random equivalent variation is equally valid theoretically. Bhattacharya (2015) provided a formula for the expectation of the equivalent variation which restricts to price changes. The computation of the expectation in the case of an unrestricted change in systematic utilities is to date unexplored. The present letter fills this gap and provides the counterparts of the formulas that were obtained by Karlström (2014) and de Palma and Kilani (2011) for the expectation of the compensating variation.

Additionally, the letter demonstrates how an income-dependent discrete choice model, with translog or power in residual income specification of the utilities, can be obtained from the consumer behaviour problem formulated in terms of direct utility. This result, inspired by Small and Rosen (1981), is not found in the literature.

The letter has the following organisation. The theoretical findings are in sections 2 and 3. Section 4 concludes with discussion and future research.

## 2. Derivation of income-dependent discrete choice random utility model

Consider  $n$  discrete goods indexed by  $i$  and the numéraire indexed by 0. Let the respective quantities be  $X_i, i = 1, \dots, n, X_0$ . Let the prices of the  $n$  goods be  $p_i, i = 1, \dots, n$ . The numéraire has unitary price. Let the disposable income of the consumer be  $y$ .

Consider the consumer's utility maximisation problem:

$$\max_{X_0, X_1, \dots, X_n} U(X_0, X_1, \dots, X_n) \quad (1)$$

$$U = \alpha f(X_0) + \sum_{i=1}^n (\bar{v}_i + \epsilon_i) X_i, \quad \alpha > 0 \quad (2)$$

$$X_0 + \sum_{i=1}^n p_i X_i = y \quad (3)$$

$$\sum_{i=1}^n X_i = 1 \quad (4)$$

$$X_i X_j = 0, \quad i \neq j, \quad i, j = 1, \dots, n \quad (5)$$

$$X_0 \geq 0; X_i \geq 0, \quad i = 1, \dots, n \quad (6)$$

where  $U$  is the direct utility,  $\bar{v}_i$  and  $\epsilon_i$  are a quality index and a random term associated with good  $i$ , respectively.

Based on Eq. (2), the direct utility is quasilinear: linear in the quantities  $X_i, i = 1, \dots, n$ , of the discrete goods, and nonlinear in the numéraire. Eq. (3) is the budget constraint. Eqs (6) impose the non-negativity of the quantities consumed. Eq (4) together with Eqs (5) and (6) imposes that one discrete good only in unitary quantity is consumed.

We assume that the function  $f(X_0)$  takes one of the following functional forms:

$$f(X_0) = X_0^\beta, \quad 0 < \beta < 1 \quad (7)$$

$$f(X_0) = \ln(X_0) \quad (8)$$

The functional form of Eq. (6) is power, the one of Eq. (7) the logarithmic. Both imply a decreasing marginal direct utility with the numéraire.

**Lemma.** The solution to the consumer's direct utility maximisation problem in Eqs (1)-(8), in terms of demand functions, is the discrete choice random utility model with conditional indirect utilities:

$$u_i = \alpha f(y - p_i) + \bar{v}_i + \epsilon_i, \quad i = 1 \dots, n \quad (9)$$

*Proof.* The discrete choice random utility model where only one alternative is consumed, plus the numéraire, follows directly from the formulation. The conditional indirect utility of alternative  $i$  takes the functional form in Eq. (9). Since the objective function is strictly concave and the feasible set is convex (because it is the intersection of convex sets), the local optimum is a global maximum.  $\square$

Based on the expressions of the conditional indirect utilities of Eqs (9) and the assumption of the functional forms of Eqs (7) and (8), the marginal utility of income is decreasing with income. When Eq. (8) is used, then we have the translog in residual income functional form. Notice that the residual income equals the consumption of the numéraire.

## 3. Random compensating and equivalent variations

Consider two states of the world: the state without the policy, denoted by the superscript 0, and the state with the policy, denoted by the superscript 1. Assume the random term  $\epsilon_i$  of each alternative  $i$  is unchanged when transitioning between these states.

Based on McFadden (1999), the random compensating variation  $cv$  is defined as the solution to the following equation:

$$\max_{i=1,\dots,n} [\alpha f(y - p_i^0) + \bar{v}_i^0 + \epsilon_i] = \max_{i=1,\dots,n} [\alpha f(y - cv - p_i^1) + \bar{v}_i^1 + \epsilon_i]$$

A random equivalent variation  $ev$  can be defined as the solution to the equation:

$$\max_{i=1,\dots,n} [\alpha f(y + ev - p_i^0) + \bar{v}_i^0 + \epsilon_i] = \max_{i=1,\dots,n} [\alpha f(y - p_i^1) + \bar{v}_i^1 + \epsilon_i]$$

The measurement of welfare change by the expectation of the compensating variation  $\mathbb{E}[cv]$  is today well established. The expectation of the equivalent variation  $\mathbb{E}[ev]$  is also theoretically justified.

The two expectations can be obtained by Monte Carlo simulation, i.e. by drawing from the distribution of the random terms. The Monte Carlo estimators  $s_1$  and  $s_2$  of, respectively, the expectation of the compensating variation and the expectation of the equivalent variation are:

$$s_1 = \frac{1}{N} \sum_{r=1}^N cv(\epsilon_{1r}, \dots, \epsilon_{nr})$$

$$s_2 = \frac{1}{N} \sum_{r=1}^N ev(\epsilon_{1r}, \dots, \epsilon_{nr})$$

where  $r = 1, \dots, N$  is the index of draw. These two estimators are unbiased and have variance equal to  $1/N$  times the variance of the respective welfare change random measure.

Formulas based on one-dimensional integrals are also available. For the expectation of the compensating variation  $\mathbb{E}[cv]$  there are the following two. One is obtained by Karlström (2014), who uses an argument based on the distribution of the random expenditure function. One is obtained by de Palma and Kilani (2011), who use an argument based of the distributions of the compensating variation that are conditional on the observed transition. Both formulas require numerical integration.

The expectation of the compensating variation  $\mathbb{E}[cv]$  is, based on Karlström (2014):

$$\mathbb{E}[cv] = y - \sum_{i=1}^n \int_0^{\mu_{ii}} P_i[g_1(m), \dots, g_n(m)] dm \quad (10)$$

where  $\mu_{ii}$  is the expenditure needed with the policy to restore the without-policy level of utility if alternative  $i$  is chosen without and with the policy:

$$\alpha f(y - p_i^0) + \bar{v}_i^0 = \alpha f(\mu_{ii} - p_i^1) + \bar{v}_i^1$$

$P_i$  is the probability of choice of alternative  $i$  having as arguments the systematic utilities of the alternatives, and

$$g_i(m) = \max[\alpha f(y - p_i^0) + \bar{v}_i^0, \alpha f(m - p_i^1) + \bar{v}_i^1], \quad i = 1, \dots, n$$

For use of the formula with the translog in residual income specification of Eq. (7), there is a need to set at minus infinity the logarithm function of negative values of the argument, since  $m$  can be lower than  $p_i^1$ .

The expectation of the equivalent variation is not provided by Karlström (2014). The following proposition fills the gap.

**Proposition 1.** The expectation of the equivalent variation is:

$$\mathbb{E}[ev] = \sum_{i=1}^n \int_0^{\mu'_{ii}} P_i[g'_1(m), \dots, g'_n(m)] dm - y$$

where  $\mu'_{ii}$  is the expenditure needed without the policy to obtain the with-policy level of utility if alternative  $i$  is chosen without and with the policy:

$$\alpha f(\mu'_{ii} - p_i^0) + \bar{v}_i^0 = \alpha f(y - p_i^1) + \bar{v}_i^1$$

and

$$g'_i(m) = \max[\alpha f(m - p_i^0) + \bar{v}_i^0, \alpha f(y - p_i^1) + \bar{v}_i^1], \quad i = 1, \dots, n$$

*Proof.* The random expenditure in Karlström (2014)  $m = y - cv$  satisfies:

$$\max_{i=1,\dots,n} [\alpha f(y - p_i^0) + \bar{v}_i^0 + \epsilon_i] = \max_{i=1,\dots,n} [\alpha f(m - p_i^1) + \bar{v}_i^1 + \epsilon_i] \quad (11)$$

Taking expectations we obtain  $\mathbb{E}[cv] = y - \mathbb{E}[m]$ , where  $\mathbb{E}[m]$  equals the sum of the integrals in the right-hand side of Eq. (10).

Now, define  $m' = y + ev$  which satisfies:

$$\max_{i=1,\dots,n} [\alpha f(m' - p_i^0) + \bar{v}_i^0 + \epsilon_i] = \max_{i=1,\dots,n} [\alpha f(y - p_i^1) + \bar{v}_i^1 + \epsilon_i] \quad (12)$$

Eq. (12) can be re-written as:

$$\max_{i=1,\dots,n} [\alpha f(y - p_i^1) + \bar{v}_i^1 + \epsilon_i] = \max_{i=1,\dots,n} [\alpha f(m' - p_i^0) + \bar{v}_i^0 + \epsilon_i] \quad (13)$$

which is Eq. (11) with the superscripts 0 and 1 swapped.

Taking expectations we obtain  $\mathbb{E}[ev] = \mathbb{E}[m'] - y$ . The expression of  $\mathbb{E}[m']$  is simply obtained, in the light of Eq. (13), from the expression of  $\mathbb{E}[m]$  in Eq. (10) by swapping the superscripts 0 and 1.  $\square$

The expectation of the compensating variation  $\mathbb{E}[cv]$  is, based on de Palma and Kilani (2011):

$$\mathbb{E}[cv] = \bar{\psi}_n - \sum_{i=1}^n \int_{\psi_{ii}}^{\bar{\psi}_n} P_i [\alpha f(y - p_i^0) + \bar{v}_i^0 + \delta_i^+(c), \dots, \alpha f(y - p_n^0) + \bar{v}_n^0 + \delta_n^+(c)] dc$$

where:

$$\delta_j(c) = \alpha f(y - c - p_j^1) + \bar{v}_j^1 - \alpha f(y - p_j^0) - \bar{v}_j^0$$

$$x^+ = \max(x, 0)$$

$$\psi_{ij} \text{ satisfies } \delta_j(\psi_{ij}) = (\delta_j - \delta_i)^+,$$

$$\bar{\psi}_n = \max_{i=1,\dots,n} \psi_{ni}, \text{ and}$$

alternatives are in ascending order of  $\delta_i = \alpha f(y - p_i^1) + \bar{v}_i^1 - \alpha f(y - p_i^0) - \bar{v}_i^0$ .

The expectation of the equivalent variation is not provided by de Palma and Kilani (2011). The following proposition fills the gap.

**Proposition 2.** The expectation of the equivalent variation is:

$$\mathbb{E}[ev] = \sum_{i=1}^n \int_{\psi'_{ii}}^{\bar{\psi}'_n} P_i [\alpha f(y - p_i^1) + \bar{v}_i^1 + \delta_i'^+(c), \dots, \alpha f(y - p_n^1) + \bar{v}_n^1 + \delta_n'^+(c)] dc - \bar{\psi}'_n$$

where:

$$\delta'_j(c) = \alpha f(y - c - p_j^0) + \bar{v}_j^0 - \alpha f(y - p_j^1) - \bar{v}_j^1,$$

$$\psi'_{ij} \text{ satisfies } \delta'_i(\psi'_{ij}) = (\delta'_i - \delta'_j)^+,$$

$$\bar{\psi}'_n = \max_{i=1,\dots,n} \psi'_{in} \text{ and}$$

alternatives are in ascending order of  $\delta'_i = \alpha f(y - p_i^0) + \bar{v}_i^0 - \alpha f(y - p_i^1) - \bar{v}_i^1$ .

*Proof.* Along the same lines of the proof of proposition 1. It is left to the reader.  $\square$

When income is heterogenous and the probability density function of income  $h(y)$  is available, the expectation of the compensating variation  $\mathbb{E}[cv]$  and of the equivalent variation  $\mathbb{E}[ev]$  are obtained by the law of total expectation:

$$\mathbb{E}[cv] = \int_0^\infty \mathbb{E}[cv|y] h(y) dy$$

$$\mathbb{E}[ev] = \int_0^\infty \mathbb{E}[ev|y] h(y) dy$$

where  $\mathbb{E}[cv|y]$  and  $\mathbb{E}[ev|y]$  are the expectations of the compensating and the equivalent variation conditional on income  $y$ .

#### 4. Conclusions

With income-dependent choices, the expectations of the compensating variation and of the equivalent variation are different. Therefore, in policy evaluation, both should be considered. The computation of the two expectations by one-dimensional integral formulas is more efficient than Monte Carlo simulation, when accuracy and computation time are considered. This is true even with heterogeneous income, a case where an additional level of integration is needed.

De Palma and Kilani (2011) provided expressions of the expectations of the distributions of compensating variation that are conditional on the observed transition. Future research might provide expressions of the expectations of the distributions of equivalent variation that are conditional on the observed transition. This is of interest to applied practice, because estimation of the welfare change accruing to the sub-populations of shifters and non-shifters is key to impact distribution analysis.

#### References

- Bhattacharya, D. (2015). Nonparametric welfare analysis for discrete choice. *Econometrica*, 83(2), 617-649.
- Delle Site, P. (2014). On price and income effects in discrete choice models. *Theoretical Economics Letters*, 4(6), 497-505.
- de Palma, A., & Kilani, K. (2011). Transition choice probabilities and welfare analysis in additive random utility models. *Economic Theory*, 46, 427-454.
- Hau, T.D. (1985). A Hicksian approach to cost-benefit analysis with discrete-choice models. *Economica*, 52(208).
- Hau, T. D. (1987). Using a Hicksian approach to cost-benefit analysis in discrete choice: An empirical analysis of a transportation corridor simulation model. *Transportation Research Part B: Methodological*, 21(5), 339-357.
- Herriges, J. A., & Kling, C. L. (1999). Nonlinear income effects in random utility models. *Review of Economics and Statistics*, 81(1), 62-72.
- Karlström, A. (2014). Appraisal. In: Hess, S., and Daly, A. (eds) *Handbook of Choice Modelling*, Edward Elgar Publishing, 601-626.
- McFadden, D. (1999). Computing willingness-to-pay in random utility models. In: Melvin, J.R., Moore, J.C., and Riezman, R.G. (eds) *Trade, Theory and Econometrics*, Routledge, 275-296.
- Small, K. A., & Rosen, H. S. (1981). Applied welfare economics with discrete choice models. *Econometrica: Journal of the Econometric Society*, 49(1), 105-130.
- Tra, C. I. (2013). Nonlinear income effects in random utility models: revisiting the accuracy of the representative consumer approximation. *Applied Economics*, 45(1), 55-63.
- Williams, H. C. (1977). On the formation of travel demand models and economic evaluation measures of user benefit. *Environment and Planning A*, 9(3), 285-344.