



HAL
open science

A note on welfare measurement with income-dependent discrete choice

P Delle Site, Karim Kilani

► **To cite this version:**

P Delle Site, Karim Kilani. A note on welfare measurement with income-dependent discrete choice. 2024. hal-04527839

HAL Id: hal-04527839

<https://hal.science/hal-04527839>

Preprint submitted on 31 Mar 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A note on welfare measurement with income-dependent discrete choice

P. Delle Site^{a,1}, K. Kilani^b

^a *Niccolò Cusano University, Rome, Italy*

^b *Conservatoire National des Arts et Métiers, Paris, France*

Abstract

The expectation of the random compensating variation is the welfare change measure that is used in discrete choice models. The expectation of the equivalent variation is equally founded theoretically. When choices are income independent, the two measures take identical values. The case of income-dependent choices remains an area for exploration. The paper provides a twofold theoretical contribution. First, it derives the discrete choice model with n alternatives and a translog and power specification of the residual income term of the indirect utilities from a quasilinear direct utility in the consumption of the discrete goods and a numéraire. Second, it provides the equivalent variation counterparts of the formulas that are available for the expectation of the compensating variation: the one based on the random expenditure function, and the one based on conditional on observed transition expectations of the compensating variation. An example related to multinomial logit and choice of the transport mode of urban travel illustrates the insights.

Keywords

compensating variation, discrete choice, equivalent variation, random utility, welfare

1. Introduction

There is a long record of contributions to welfare measurement when choices are discrete. This is of use in cost-benefit analysis in the transport sector, one where discrete travel choices (if to travel or otherwise, destination, mode of transport, route) are of relevance.

Since the introduction of discrete choice models, where utilities are modeled as random, in the seventies, researchers addressed the measurement of welfare change, adapting the measures that are used for classical divisible goods demand models.

The first case was the one of income-independent choices where income enters the systematic utilities linearly with a constant and alternative-independent coefficient which is easily interpreted as the marginal utility of income. Williams (1977) proposed a representative consumer approach: the measure of welfare change is the representative consumer surplus difference (without and with the policy) which equals the difference in the expectation of the maximum utility divided by the marginal utility of income. This is the popular logsum formula in the case of the multinomial logit model. Small and Rosen (1981) obtained the same formula based on the aggregate compensating variation derived from the variation of the expenditure function.

For the case with income-dependent choices, one can apply the Hicksian concepts of compensating and equivalent variation of classical demand theory by using the expectation of the maximum utility as indirect utility. This approach is found in Hau (1985 for theory, and 1987 for an application in the transport sector).

Today, welfare measurement within discrete choice models is grounded in the theory of random Hicksian measures proposed by McFadden (1999). The expectation of the random compensating variation is commonly used. If random terms are interpreted as individual specific, then the measure is the average in a population of individuals who are identical in terms of systematic utilities.

When choices are income independent, this measure equals the difference in the expectations of the maximum utility divided by the marginal utility of income. Therefore, this represents a case where the representative consumer approach and the random utility approach coincide. By contrast, when

(¹) Corresponding author: paolo.dellesite@unicusano.it

choices are income dependent, the two approaches lead to different measures (see the numerical exploration by Tra, 2013).

If the random approach is followed, there are two ways to compute the expectation of the compensating variation. The first is simulation based on draws from the distribution of the random terms. Herriges and Kling (1999) used this method with a translog in residual income specification of the utilities, the residual income being the difference between disposable income and expenditure on the alternative. This specification is appealing, because it provides a decreasing marginal utility of income, i.e. one additional dollar has less value for the richer than for the poorer. The translog specification is studied in Delle Site (2014).

The second way to compute the expectation of the compensating variation is based on the solution to one-dimensional integrals. Karlström (2014) obtained a formula starting from the distribution of the random expenditure function. De Palma and Kilani (2011) obtained a formula starting from the distribution of the conditional on the observed transition expectations.

The random equivalent variation is equally valid theoretically. Bhattacharya (2015) provided a formula for the expectation of the equivalent variation which restricts to price changes. The computation of the expectation in the case of an unrestricted change in systematic utilities is to date unexplored. The present note fills this gap and provides the counterparts of the formulas that were obtained by Karlström (2014) and de Palma and Kilani (2011) for the expectation of the compensating variation.

Additionally, the note demonstrates how an income-dependent discrete choice model, with translog or power in residual income specification of the utilities, can be obtained from the consumer behaviour problem formulated in terms of direct utility.

The note has the following organisation. The theoretical findings are in section 2. Section 3 provides an illustrative example. Section 4 concludes with discussion and future research.

2. Theoretical findings

2.1 Derivation of income-dependent discrete choice random utility model

Consider n discrete goods indexed by i and the numéraire indexed by 0. Let the respective quantities be $X_i, i = 1, \dots, n, X_0$. Let the prices of the n goods be $p_i, i = 1, \dots, n$. The numéraire has unitary price. Let the disposable income of the consumer be y .

Consider the consumer's utility maximisation problem:

$$\max_{X_0, X_1, \dots, X_n} U(X_0, X_1, \dots, X_n) \quad (1)$$

$$U = \alpha f(X_0) + \sum_{i=1}^n (\bar{v}_i + \epsilon_i) X_i, \quad \alpha > 0 \quad (2)$$

$$X_0 + \sum_{i=1}^n p_i X_i = y \quad (3)$$

$$\sum_{i=1}^n X_i = 1 \quad (4)$$

$$X_0 \geq 0; X_i \geq 0, \quad i = 1, \dots, n \quad (5)$$

where U is the direct utility, \bar{v}_i and ϵ_i are a quality index and a random term associated with good i , respectively.

Based on Eq. (2), the direct utility is quasilinear: linear in the quantities $X_i, i = 1, \dots, n$, of the discrete goods, and nonlinear in the numéraire. Eq. (3) is the budget constraint. Based on Eq. (4), only a unitary quantity of the discrete goods is consumed. Eqs (5) impose the non-negativity of the quantities consumed. Notice that the integrality constraint on the quantities of the discrete goods is not imposed.

We assume that the function $f(X_0)$ takes one of the following functional forms:

$$f(X_0) = X_0^\beta, 0 < \beta < 1 \quad (6)$$

$$f(X_0) = \ln(X_0) \quad (7)$$

The functional form of Eq. (6) is power, the one of Eq. (7) the logarithmic. Both imply a decreasing marginal direct utility with the numéraire.

The following proposition states that, under the assumptions made, a unitary quantity of one discrete good only plus the numéraire are consumed.

Proposition 1. The solution to the consumer's utility maximisation problem in Eqs (1)-(7), in terms of demand functions, is the discrete choice random utility model with conditional indirect utilities:

$$u_i = \alpha f(y - p_i) + \bar{v}_i + \epsilon_i, \quad i = 1, \dots, n \quad (8)$$

Proof. Since the constraints of the direct utility maximization problem are linear, regularity conditions are satisfied. Therefore, the following Karush-Kuhn-Tucker conditions hold. With a locally optimal point $X_i, i = 1, \dots, n, X_0$, there exist multipliers $\lambda_1, \lambda_2, \rho_i, i = 1, \dots, n$ such that, in addition to constraints (3)-(5), the following are satisfied:

$$\begin{aligned} -(\bar{v}_i + \epsilon_i) + \lambda_1 p_i + \lambda_2 - \rho_i &= 0, \quad i = 1, \dots, n \\ -\frac{\partial f}{\partial X_0} + \lambda_1 - \rho_0 &= 0 \\ \rho_i X_i &= 0, \quad i = 0, 1, \dots, n \\ \rho_i &\geq 0, \quad i = 0, 1, \dots, n \end{aligned}$$

We obtain:

$$\begin{aligned} -(\bar{v}_i + \epsilon_i) + \lambda_1 p_i + \lambda_2 = \rho_i &\geq 0, \quad i = 1, \dots, n \\ -\frac{\partial f}{\partial X_0} + \lambda_1 = \rho_0 &\geq 0 \end{aligned} \quad (9)$$

The first case is the one where the numéraire is consumed: $X_0 > 0$. Then, we have $\rho_0 = 0$ and $\lambda_1 = \frac{\partial f}{\partial X_0}$. We can have only one good with $X_i > 0$ because this inequality implies $\rho_i = 0$ and

$$-(\bar{v}_i + \epsilon_i) + \lambda_1 p_i + \lambda_2 = 0$$

which can be satisfied for one good i only. The discrete choice random utility model where only one alternative is consumed, plus the numéraire, follows. The conditional indirect utility of alternative i takes the functional form in Eq. (8).

The second case is the one where the numéraire is not consumed: $X_0 = 0$. Then, we have $\rho_0 \geq 0$. Due to Eqs (6) and (7), we have $\frac{\partial f}{\partial X_0} = \infty$, which gives, by Eq. (9), $\rho_0 = -\infty$ for a contradiction. Therefore, the case where the numéraire is not consumed is impossible.

Since the objective function is strictly concave and the feasible set is convex (because constraints are linear) the local optimum is a global maximum. \square

Based on the expressions of the conditional indirect utilities of Eqs (8) and the assumption of the functional forms of Eqs (6) and (7), the marginal utility of income is decreasing with income. When Eq. (7) is used, then we have the translog in residual income functional form.

2.2 Random compensating and equivalent variations

Consider two states of the world: the state without the policy, denoted by the superscript 0, and the state with the policy, denoted by the superscript 1. Assume the random term ϵ_i of each alternative i is unchanged when transitioning between these states.

Based on McFadden (1999), the random compensating variation cv is defined as the solution to the following equation:

$$\max_{i=1,\dots,n} [\alpha f(y - p_i^0) + \bar{v}_i^0 + \epsilon_i] = \max_{i=1,\dots,n} [\alpha f(y - cv - p_i^1) + \bar{v}_i^1 + \epsilon_i]$$

A random equivalent variation ev can be defined as the solution to the equation:

$$\max_{i=1,\dots,n} [\alpha f(y + ev - p_i^0) + \bar{v}_i^0 + \epsilon_i] = \max_{i=1,\dots,n} [\alpha f(y - p_i^1) + \bar{v}_i^1 + \epsilon_i]$$

While these equations bear a striking resemblance to each other, they subtly differ in that the compensating variation cv is subtracted in the with-policy utility, whereas the equivalent variation ev is added in the without-policy utility, reflecting their distinct conceptual implications.

Figure 1 and 2 provide a geometric representation in a case of two alternatives where the utilities take the following values without and with the policy:

$$u_1^0 = \ln(20 - 19.9) + 13.8; \quad u_1^1 = \ln(20 - 12) + 14$$

$$u_2^0 = \ln(20 - 19.9) + 14.302; \quad u_2^1 = \ln(20 - 9) + 12$$

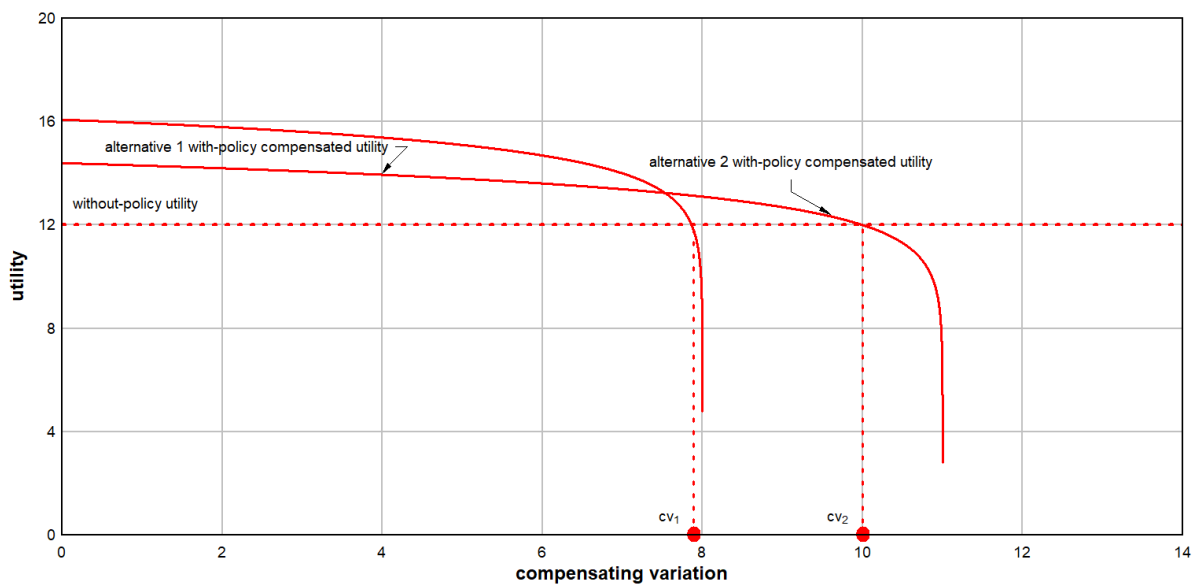


Figure 1. Random compensating variation

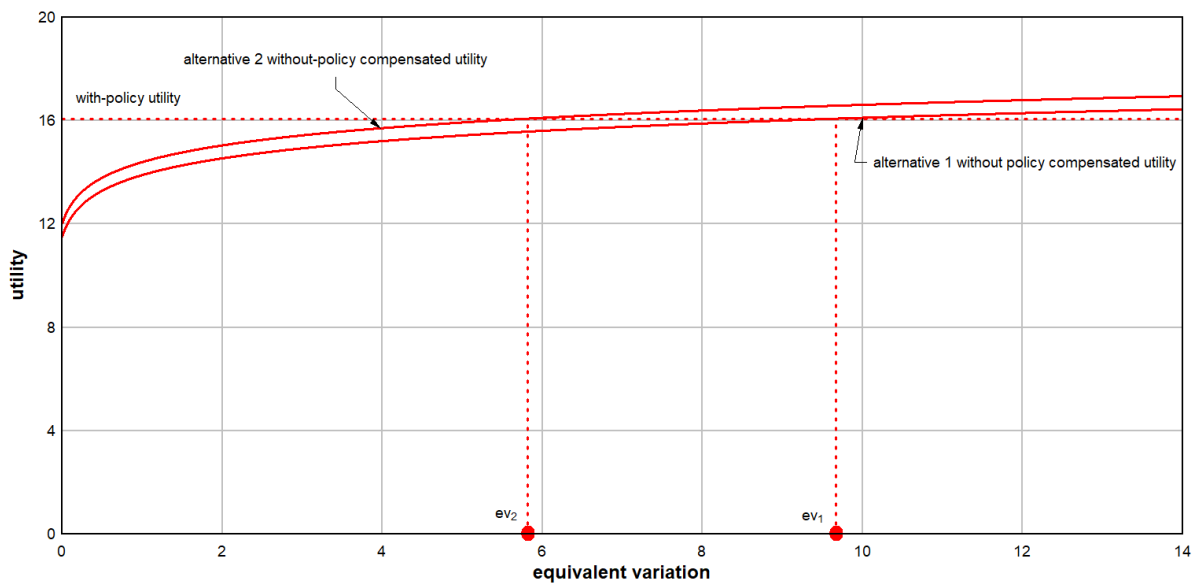


Figure 2. Random equivalent variation

With reference to the notation in the charts, the compensating variation is $cv_2 = 10$, the equivalent variation is $ev_2 = 5.812$.

Due to the increasing monotonicity of utility with residual income, the random compensating variation cv is the maximum among the values obtained by the intersection of the with-policy compensated utility and the horizontal without-policy utility line. In the case here: $cv = \max(cv_1, cv_2)$.

Symmetrically, the random equivalent variation is the minimum among the values obtained by the intersection of the without-policy compensated utility and the horizontal with-policy utility line. In the case here: $ev = \min(ev_1, ev_2)$.

The measurement of welfare change by the expectation of the compensating variation $\mathbb{E}[cv]$ is today well established. The expectation of the equivalent variation $\mathbb{E}[ev]$ is also theoretically justified.

The two expectations can be obtained by simulation, i.e. drawing from the distribution of the random terms:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{r=1}^N cv(\epsilon_{1r}, \dots, \epsilon_{nr}) = \mathbb{E}[cv]$$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{r=1}^N ev(\epsilon_{1r}, \dots, \epsilon_{nr}) = \mathbb{E}[ev]$$

where $r = 1, \dots, N$ is the index of draw.

Formulas based on one-dimensional integrals are also available. For the expectation of the compensating variation $\mathbb{E}[cv]$ there are the following two. One is obtained by Karlström (2014), who uses an argument based on the distribution of the random expenditure function. One is obtained by de Palma and Kilani (2011), who use an argument based on the distribution of the conditional on the observed transition compensating variation. Both formulas require numerical integration.

The expectation of the compensating variation $\mathbb{E}[cv]$ is, based on Karlström (2014):

$$\mathbb{E}[cv] = y - \sum_{i=1}^n \int_0^{\mu_{ii}} P_i[g_1(m), \dots, g_n(m)] dm \quad (10)$$

where μ_{ii} is the expenditure needed with the policy to restore the without-policy level of utility if alternative i is chosen without and with the policy:

$$\alpha f(y - p_i^0) + \bar{v}_i^0 = \alpha f(\mu_{ii} - p_i^1) + \bar{v}_i^1$$

P_i is the probability of choice of alternative i having as arguments the systematic utilities of the alternatives, and

$$g_i(m) = \max[\alpha f(y - p_i^0) + \bar{v}_i^0, \alpha f(m - p_i^1) + \bar{v}_i^1], i = 1, \dots, n$$

For use of the formula with the translog in residual income specification of Eq. (7), there is a need to set at minus infinity the logarithm function of negative values of the argument, since m can be lower than p_i^1 .

The expectation of the equivalent variation is not provided by Karlström (2014). The following proposition fills the gap.

Proposition 2. The expectation of the equivalent variation is:

$$\mathbb{E}[ev] = \sum_{i=1}^n \int_0^{\mu'_{ii}} P_i[g'_1(m), \dots, g'_n(m)] dm - y \quad (11)$$

where μ'_{ii} is the expenditure needed without the policy to obtain the with-policy level of utility if alternative i is chosen without and with the policy:

$$\alpha f(\mu'_{ii} - p_i^0) + \bar{v}_i^0 = \alpha f(y - p_i^1) + \bar{v}_i^1$$

and

$$g'_i(m) = \max[\alpha f(m - p_i^0) + \bar{v}_i^0, \alpha f(y - p_i^1) + \bar{v}_i^1], i = 1, \dots, n$$

Proof. The random expenditure in Karlström (2014) $m = y - cv$ satisfies:

$$\max_{i=1, \dots, n} \alpha f(y - p_i^0) + \bar{v}_i^0 = \max_{i=1, \dots, n} \alpha f(m - p_i^1) + \bar{v}_i^1 \quad (12)$$

Taking expectations we obtain $\mathbb{E}[cv] = y - \mathbb{E}[m]$, where $\mathbb{E}[m]$ equals the sum of the integrals in the right-hand side of Eq. (10).

Now, define $m' = y + ev$ which satisfies:

$$\max_{i=1, \dots, n} \alpha f(m' - p_i^0) + \bar{v}_i^0 = \max_{i=1, \dots, n} \alpha f(y - p_i^1) + \bar{v}_i^1 \quad (13)$$

Eq. (13) can be re-written as:

$$\max_{i=1, \dots, n} \alpha f(y - p_i^1) + \bar{v}_i^1 = \max_{i=1, \dots, n} \alpha f(m' - p_i^0) + \bar{v}_i^0 \quad (14)$$

which is Eq. (12) with the superscripts 0 and 1 swapped.

Taking expectations we obtain $\mathbb{E}[ev] = \mathbb{E}[m'] - y$. The expression of $\mathbb{E}[m']$ is simply obtained, in the light of Eq. (14), from the expression of $\mathbb{E}[m]$ by swapping the superscripts 0 and 1. \square

The expectation of the compensating variation $\mathbb{E}[cv]$ is, based on de Palma and Kilani (2011):

$$\mathbb{E}[cv] = \bar{\psi}_n - \sum_{i=1}^n \int_{\psi_{ii}}^{\bar{\psi}_n} P_i[\alpha f(y - p_i^0) + \bar{v}_i^0 + \delta_i^+(c), \dots, \alpha f(y - p_n^0) + \bar{v}_n^0 + \delta_n^+(c)] dc$$

where:

$$\bar{\psi}_n = \max_{i=1, \dots, n} \psi_{ni},$$

$$\psi_{ij} \text{ satisfies } \delta_j(\psi_{ij}) = (\delta_j - \delta_i)^+,$$

$$\delta_j(c) = \alpha f(y - c - p_j^1) + \bar{v}_j^1 - \alpha f(y - p_j^0) - \bar{v}_j^0,$$

alternatives are in ascending order of $\delta_i = \alpha f(y - p_i^1) + \bar{v}_i^1 - \alpha f(y - p_i^0) - \bar{v}_i^0$, and

$$x^+ = (x, 0).$$

The expectation of the equivalent variation is not provided by de Palma and Kilani (2011). The following proposition fills the gap.

Proposition 3. The expectation of the equivalent variation is:

$$\mathbb{E}[ev] = \sum_{i=1}^n \int_{\psi'_{ii}}^{\bar{\psi}'_n} P_i[\alpha f(y - p_i^1) + \bar{v}_i^1 + \delta_i'^+(c), \dots, \alpha f(y - p_n^1) + \bar{v}_n^1 + \delta_n'^+(c)] dc - \bar{\psi}'_n$$

where:

$$\bar{\psi}'_n = \max_{i=1, \dots, n} \psi'_{in},$$

$$\psi'_{ij} \text{ satisfies } \delta_i'(\psi'_{ij}) = (\delta_i' - \delta_j')^+,$$

$$\delta_i'(c) = \alpha f(y - c - p_i^0) + \bar{v}_i^0 - \alpha f(y - p_i^1) - \bar{v}_i^1,$$

alternatives are in ascending order of $\delta_i' = \alpha f(y - p_i^0) + \bar{v}_i^0 - \alpha f(y - p_i^1) - \bar{v}_i^1$.

Proof. Along the same lines of the proof of proposition 2. It is left to the reader. \square

When income is heterogenous and the probability density function of income $h(y)$ is available, the expectation of the compensating variation $\mathbb{E}[cv]$ and of the equivalent variation $\mathbb{E}[ev]$ are obtained by the law of total expectation:

$$\mathbb{E}[cv] = \int_0^{\infty} \mathbb{E}[cv|y]h(y)dy \quad (15)$$

$$\mathbb{E}[ev] = \int_0^{\infty} \mathbb{E}[ev|y]h(y)dy \quad (16)$$

where $\mathbb{E}[cv|y]$ and $\mathbb{E}[ev|y]$ are the expectations of the compensating and the equivalent variation conditional on income y .

3. Illustrative example

The example relates to the choice of the transport mode by travelers in urban areas. There are three alternatives: metro, bus and car. A multinomial logit model is estimated on simulated data mimicking conditions found in the Rome urban area in the morning peak. Details on how data are generated are found in Delle Site and Salucci (2013). Let M denote metro, B bus and C car. The systematic utilities are translog in residual income:

$$\begin{aligned} v_M &= \alpha \ln(y - p_{MB}) + \beta_1 t_{1,M} + \beta_2 t_{2,M} + \beta_3 \\ v_B &= \alpha \ln(y - p_{MB}) + \beta_1 t_{1,B} + \beta_2 t_{2,B} + \beta_4 \\ v_C &= \alpha \ln(y - p_C) + \beta_2 t_{2,C} \end{aligned}$$

where t_1 is access travel time, t_2 is on-board travel time, and $\alpha, \beta_1, \beta_2, \beta_3, \beta_4$ are estimation coefficients. Their values are in Table 1. The price attribute is the monthly ticket for metro and bus, and the monthly parking charge at destination for car. All coefficients have the expected sign and are statistically significant at 5% level. Access time is valued relatively more than on-board time.

Welfare analysis is conducted for a bus prioritization policy. This may consist in measures including priority lanes and switching of services to bus rapid transit. The on-board travel time of bus is assumed to decrease because of the policy. This implies demand is shifted from car and metro to bus. The travel time of car is assumed to be unchanged, which is realistic insofar as the lower road space available offsets the lower number of cars.

Figure 3 shows the expectations of the compensating variation and the equivalent variation (monthly values) when the travel time of bus decreases by a percent value in the range between 5% and 40% of the value in the without policy conditions.

The chart considers a single origin-destination pair with the following values of the attributes: (pre-government) monthly income 2387 EUR, metro and bus monthly ticket 30 EUR, parking at destination monthly ticket 70 EUR, metro access time 13.5 minutes, bus access time 8.1 minutes, metro on-board time 10.8 minutes, bus on-board time 18.2 minutes, car travel time 22.8 minutes. The associated shares are 48.3% for metro, 24.1% for bus and 27.6% for car.

Income data are from the Ministry of Economy and Finance and relate to year 2021. For residents of Rome municipality, the average pre-government monthly income is 2387 EUR and the standard deviation is 1122 EUR. The empirical income distribution is approximated by the lognormal distribution in Figure 4. Figure 5 shows the expectations of the compensating variation and the equivalent variation (monthly values), for the same range of variation of bus travel time of Figure 3, for a population of income-heterogenous travelers of the origin-destination pair.

In both cases of income set at the average value and of heterogenous income, the expectation of the equivalent variation and the expectation of the compensating variation are positive. They increase and diverge as the percent decrease of bus travel time gets larger.

For computation, a python code is used. The expectations of the compensating variation and the equivalent variation with average income are obtained from the one-dimensional integral formulas of Eqs (10) and (11). Those with heterogenous income from Eqs (15) and (16). Computation by simulation is also performed for comparison. Simulation requires high computation time to achieve comparable accuracy. Computation that uses the formulas is more efficient. With a personal computer having 2.70 GHz CPU and 16.00 MB RAM, computation time is in the order of fractions of second using the formulas, while several minutes are needed by simulation.

Table 1. Estimation results

coefficient	attribute	estimate (t-statistic)
α	residual income (EUR/month)	4.10986 (3.868)
β_1	access time (minutes)	-0.19967 (-7.012)
β_2	on-board time (minutes)	-0.09829 (-5.767)
β_3	alternative specific of metro	2.00681 (4.151)
β_4	alternative specific of bus	0.96316 (3.447)

2916 simulated observations

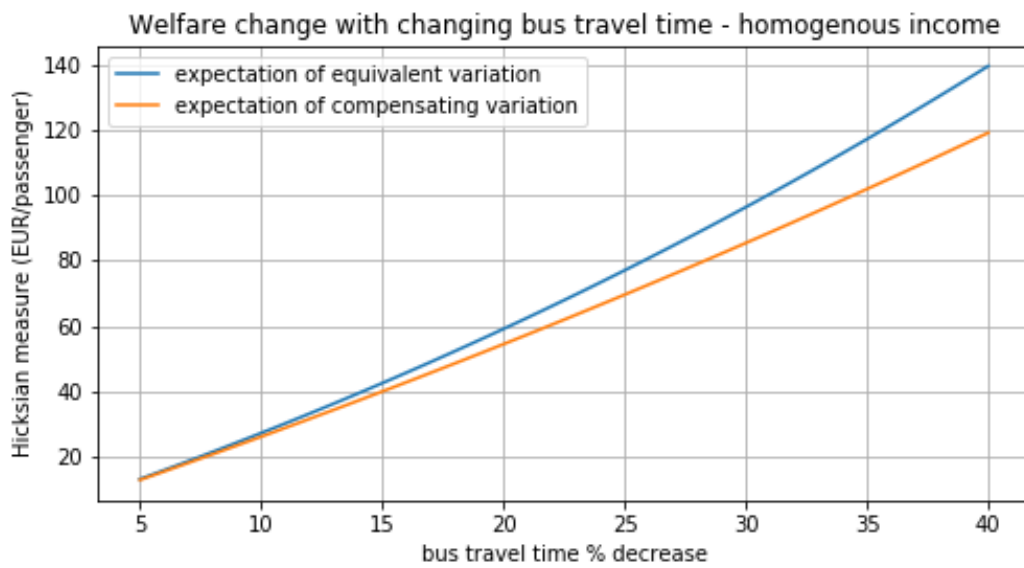


Figure 3. Hicksian measures for varying bus travel time, homogenous income

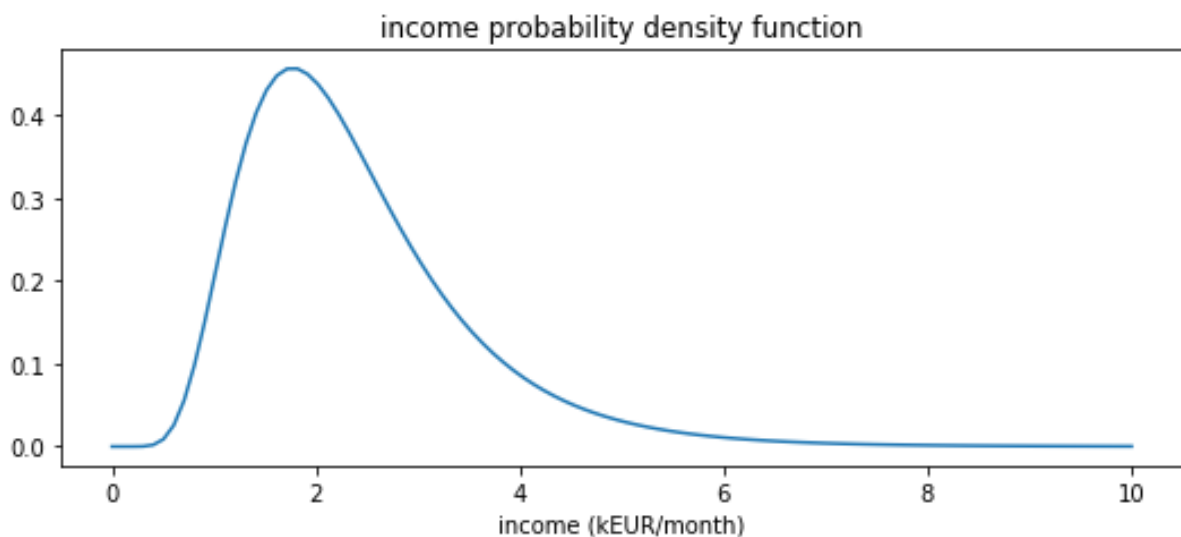


Figure 4. Probability density function of income based on lognormal approximation

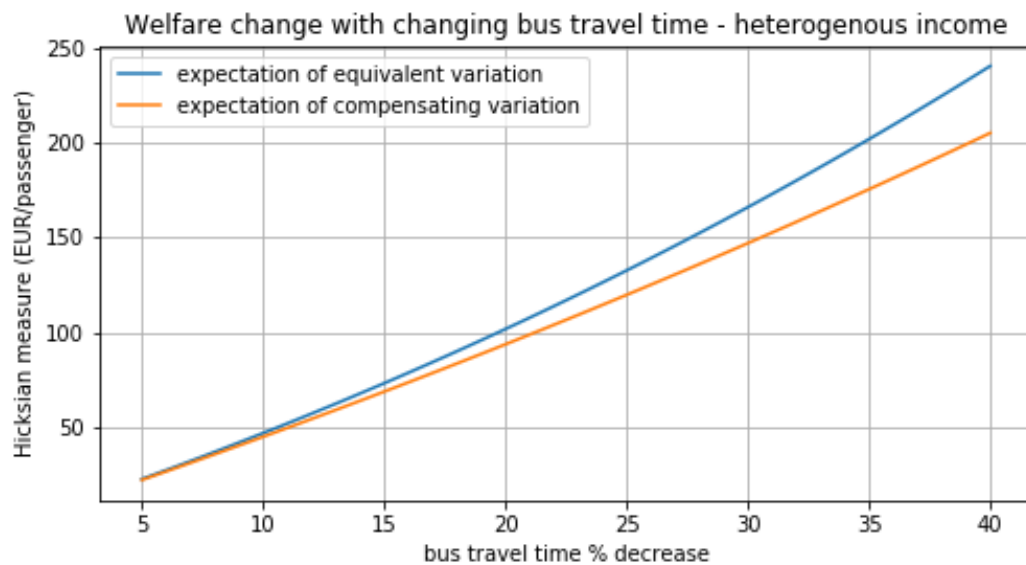


Figure 5. Hicksian measures for varying bus travel time, heterogenous income

4. Conclusions

With income-dependent choices, the expectations of the compensating variation and of the equivalent variation are different. Therefore, in policy evaluation, both should be considered. The computation of the two expectations by one-dimensional integral formulas is definitely more efficient than simulation, when accuracy and computation time are considered. This is true even with heterogenous income, a case where an additional level of integration is needed.

De Palma and Kilani (2011), provide expressions of the conditional on the observed transitions expectations of the compensating variation. Future research might provide expressions of the conditional on the observed transitions expectations of the equivalent variation. This is of interest to applied practice, because estimation of the welfare change accruing to the sub-populations of shifters and non-shifters is key to impact distribution analysis.

References

- Bhattacharya, D. (2015). Nonparametric welfare analysis for discrete choice. *Econometrica*, 83(2), 617-649.
- Delle Site, P. (2014). On price and income effects in discrete choice models. *Theoretical Economics Letters*, 4(6), 497-505.
- Delle Site, P., & Salucci, M. V. (2013). Transition choice probabilities and welfare analysis in random utility models with imperfect before-after correlation. *Transportation Research Part B: Methodological*, 58, 215-242.
- de Palma, A., & Kilani, K. (2011). Transition choice probabilities and welfare analysis in additive random utility models. *Economic Theory*, 46, 427-454.
- Hau, T.D. (1985). A Hicksian approach to cost-benefit analysis with discrete-choice models. *Economica*, 52(208).
- Hau, T. D. (1987). Using a Hicksian approach to cost-benefit analysis in discrete choice: An empirical analysis of a transportation corridor simulation model. *Transportation Research Part B: Methodological*, 21(5), 339-357.
- Herriges, J. A., & Kling, C. L. (1999). Nonlinear income effects in random utility models. *Review of Economics and Statistics*, 81(1), 62-72.

Karlström, A. (2014). Appraisal. In: Hess, S., and Daly, A. (eds) Handbook of Choice Modelling, Edward Elgar Publishing, 601-626.

McFadden, D. (1999). Computing willingness-to-pay in random utility models. In: Melvin, J.R., Moore, J.C., and Riezman, R.G. (eds) Trade, Theory and Econometrics, Routledge, 275-296.

Small, K. A., & Rosen, H. S. (1981). Applied welfare economics with discrete choice models. *Econometrica: Journal of the Econometric Society*, 49(1), 105-130.

Tra, C. I. (2013). Nonlinear income effects in random utility models: revisiting the accuracy of the representative consumer approximation. *Applied Economics*, 45(1), 55-63.

Williams, H. C. (1977). On the formation of travel demand models and economic evaluation measures of user benefit. *Environment and Planning A*, 9(3), 285-344.