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Maximal Admissible State-Dependent Disturbances for a Candidate Invariant Set

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Abstract: There exist many state-dependent locally bounded additive disturbances for which a given set can be deemed as invariant with respect to a stable linear discrete-time dynamic system. We introduce the notion of maximal disturbance set which is unique for each system and candidate invariant set and shares many of the geometric properties of its associated invariant set. We demonstrate how maximal disturbance sets may be constructed for arbitrary invariant sets by constructing them from the union of maximal disturbance sets for convex minimal robust positively invariant sets.

Keywords: Linear Systems, Robustness analysis, Time-invariant systems

1. INTRODUCTION

The set-theoretic properties of dynamic systems subject to additive disturbances have been extensively studied due to their applications in robust and constrained control Aubin et al. (2011); Blanchini et al. (2008); Mayne et al. (2005). Traditionally, most research has focused on the invariant sets of autonomous stable discrete-time systems subject to a state-independent bounded disturbance. For example, it is known that the set of states reachable from the origin Hirata and Ohta (2003) – the minimal Robust Positively Invariant (mRPI) set – exists is bounded and unique Kolmanovsky et al. (1998); Rakovič et al. (2005); Kuntsevich and Pshenichnyi (1996); Ong and Gilbert (2006). A number of works have considered the case when the disturbance is state-dependent for example Oлару and Ito (2018); De Santis (1998); Schaich and Cannon (2015); Rakovič et al. (2006) which considered the ultimate bounds, limitations of constrained systems and reachability respectively. In the same vein and of particular relevance to the current work are Townsend et al. (2024) and Athanasopoulos et al. (2024) which characterised the existence and properties of mRPI sets using set-valued recursive maps and, alternatively, by parameterised polyhedra, respectively. Importantly, Townsend et al. (2024) proved that the existence of bounded invariant sets is not guaranteed in the case of state-dependent locally bounded disturbances and that many different disturbances may give rise to the same invariant sets.

Continuing these developments, here we construct the largest set of disturbances for a given invariant set. We achieve this by characterising the largest set of disturbances for each state of a system with a convex mRPI set and demonstrating that the union of these disturbance sets correspond to the union of the convex mRPI sets.

We prove that the maximal set of state-dependent disturbances is unique and shares many of the algebraic and geometric properties of the underlying invariant set.

2. PRELIMINARIES

Dynamics and disturbance characterisation We adopt the notational conventions and assumptions of Townsend et al. (2024). We recall here the essential ones for clarity. Consider a stable discrete-time linear system subject to an additive disturbance

$$x(k+1) = Ax(k) + w(k) \quad (1)$$

where state transition matrix $A \in \mathbb{R}^{n \times n}$ is Schur. The disturbances w are constrained to be locally bounded state-dependent set i.e. $w \in W(x)$ where W is the map

$$W : \mathbb{R}^n \rightarrow \mathcal{P}_{cl}(\mathbb{R}^n) \quad (2)$$

where $\mathcal{P}_{cl}(\mathbb{R}^n)$ is the set of all closed subsets of \mathbb{R}^n .

Assumption 1. For all $x \in \mathbb{R}^n$ that

$$\hat{w}(x) := \max \{\|w\| : w \in W(x)\} < \infty$$

and that the disturbance is locally bounded i.e. there is $M > 0$ such that $\hat{w}(x) \leq M \|x\|$ for all $x \in \mathbb{R}^n$.

To ensure (1) is well-defined in the case when $W(x) = \emptyset$ we consider the extension $\overline{W} : \mathbb{R}^n \rightarrow \mathcal{P}_{cl}(\mathbb{R}^n)$

$$\overline{W}(x) := \begin{cases} W(x), & x \in \text{supp}(W) \\ \{0\}, & x \notin \text{supp}(W) \end{cases} \quad (3)$$

where $\text{supp}(\cdot)$ is the support of the function W . This extension ensures that (1) is forward complete for all x without affecting the dynamics of the system.

Invariant Sets We are interested in the *Robust Positive Invariant* (RPI) sets of (1).

Definition 2. A non-empty set $\Omega \subset \mathbb{R}^n$ is RPI with respect to (1) if $Ax + w \in \Omega$ for all $x \in \Omega$ and all $w \in \overline{W}(x)$. Furthermore the *mRPI* set is the closed invariant set which is contained in all other closed invariant sets.

As demonstrated in Townsend et al. (2024) the existence of RPI sets in the case of state-dependent disturbances is not trivial. We do not directly consider this existence question

here, but we address the converse problem of finding the maximal set of disturbances for a given invariant set.

Extended State-Disturbance Space Before defining the set-recursive maps it is necessary to define the extended state-disturbance space

$$\overline{\mathcal{W}} := \left\{ \begin{pmatrix} x \\ w \end{pmatrix} \in \mathbb{R}^{2n} : w \in \overline{\mathcal{W}}(x) \right\} \quad (4)$$

which collects all the state-disturbance pairs. The set $\overline{\mathcal{W}}$ restricted to an invariant set is the main object studied in this paper. $\mathcal{W} = \overline{\mathcal{W}} \cap (X \times \mathbb{R}^n)$.

$$\mathcal{W} = \left\{ \begin{pmatrix} x \\ w \end{pmatrix} \in \mathbb{R}^{2n} : x \in X \right\} \quad (5)$$

This definition of \mathcal{W} differs from the one used in Townsend et al. (2024) as it restricts \mathcal{W} to an invariant set X rather than the $\text{supp}(\mathcal{W})$. An embedding of the dynamics of (1) in \mathcal{W} is given by

$$\begin{pmatrix} x(k+1) \\ w(k+1) \end{pmatrix} = \begin{pmatrix} A & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ w(k) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w(k+1) \quad (6)$$

We note the non-causal $w(k+1)$ term in (6) does not affect the dynamics of the system.

Set Recursive Maps The set-recursive maps introduced in Townsend et al. (2024) enable us to characterise, as fixed points, and converge to invariant sets from non-invariant subsets of the state-space.

$$F : \mathcal{P}_{cl}(\mathbb{R}^n) \times \mathcal{P}_{cl}(\mathbb{R}^{2n}) \rightarrow \mathcal{P}_{cl}(\mathbb{R}^n) \quad (7)$$

$$F(X, \overline{\mathcal{W}}) := cl(F'(X, \overline{\mathcal{W}}))$$

for which

$$F'(X, \overline{\mathcal{W}}) := [A \ I] (\overline{\mathcal{W}} \cap (X \times \mathbb{R}^n)) \quad (8)$$

and where $cl(\cdot)$ is the closure of a set, $X \subset \mathbb{R}^n$ is some set and $[A \ I] \in \mathbb{R}^{n \times 2n}$ is a non-square extension of A . The map, F' , is the set of all forward images of a set of states under (1) subject to the state-dependent disturbances $\mathcal{W}(x)$. Lemmas 3 and 4 are results from Townsend et al. (2024) given here for reference.

Lemma 3. (Converging from Outside). Let X_0 be a non-empty, closed and bounded RPI set and define the sequence of sets (X_k) by

$$X_{k+1} := F(X_k, \overline{\mathcal{W}}) \quad (9)$$

Then the limit

$$X_\infty := \lim_{k \rightarrow \infty} X_k$$

exists and is a fixed point of the map F .

Lemma 4. (Converging from Inside). Suppose X_0 is non-empty and contained in a bounded RPI set. Then,

$$X_{k+1} = F \left(\bigcup_{i \leq k} X_i, \overline{\mathcal{W}} \right) \cup X_0 \quad (10)$$

converges to the fixed point, $X_\infty \supseteq X_0$.

Fixed Sets To introduce maximal disturbance sets in Definition 6 the definition of fixed sets of (7) in terms of the map F is instrumental.

Definition 5. (Fixed sets). We say a set $X \subset \mathbb{R}^n$ is *fixed* if $F(X, \overline{\mathcal{W}}) = X$.

We note X may contain disjoint subsets and that if X is a fixed set. Then it is a fixed point of both (9) and (10).

3. MAXIMAL DISTURBANCE SET

The maximal disturbance set, Definition 6, is the largest collection of disturbances which ensure (1) admits a given fixed set, X .

Definition 6. (Maximal Disturbance Set). Given $X \subset \mathbb{R}^n$, we say \mathcal{W} is the *maximal disturbance set* over X for (1) if X is the fixed set for (7) and $\mathcal{W}' \subseteq \mathcal{W}$ for all disturbance sets \mathcal{W}' for which X is the fixed set. The *maximal disturbance set* is described explicitly as

$$\mathcal{W} := \left\{ \begin{pmatrix} x \\ w \end{pmatrix} : x \in X \text{ and } Ax + w \in X \right\}.$$

3.1 Fixed points

A subset of the set of fixed points of (1) is always contained in and intersects the boundary of the maximal disturbance set. This is proven in Theorem 8.

Definition 7. (Fixed point). A point $z = (x, w_x) \in \mathcal{W}$ is a *fixed point* if $x = Ax + w_x$.

Throughout we denote

$$\mathcal{Z} := \left\{ \begin{pmatrix} x \\ (I - A)x \end{pmatrix} : x \in \mathbb{R}^n \right\}$$

which is the collection of all fixed points of (1). As

$$x = Ax + (I - A)x$$

Whilst non-essential due to the importance of convexity to subsequent results we note that \mathcal{Z} is convex.

Theorem 8. The set $X \subset \mathbb{R}^n$ is fixed if and only if

$$\mathcal{Z}|_X := \left\{ \begin{pmatrix} x \\ (I - A)x \end{pmatrix} : x \in X \right\} \subset \mathcal{W}$$

where \mathcal{W} is the maximal disturbance set.

Proof. This follows by observing that $\mathcal{Z}|_X$ is the collection of fixed points for all $x \in X$. Thus if X is fixed it will preserve this property for all w_x such that $x = Ax + w_x$. Furthermore if \mathcal{W} is maximal but it does not contain a pair (x, w_x) such that $x = Ax + w_x$ then such $x \notin X$.

We will later prove that fixed points $(x, (I - A)x) \in \partial\mathcal{W}$ whenever $x \in \partial X$.

3.2 Open Fixed Sets

We now give a first application of maximal disturbance sets. As mentioned above, the closure, (7), of the map (8) was necessary in Townsend et al. (2024) as the set recursions (9) and (10) could converge to open sets whose closure was not invariant. Control over the disturbance set allows us to address this generation of open sets.

Lemma 9. Suppose X is not closed and is a fixed point of F' with maximal disturbance set \mathcal{W} . Then \mathcal{W} is not closed and $cl(X)$ is a fixed point of F' with maximal disturbance $cl(\mathcal{W})$.

Proof. Let $(x_i) \subset X$ be a sequence converging to x . As X is a fixed point of F' for each x_i there must be w_i such that

$$x_{i+1} = Ax_i + w_i$$

4. CONSTRUCTION

As \overline{W} is maximal we have that $(x_i, w_i) \in \overline{W}$. As (x_i) converges to x the sequence is Cauchy. So for any $\varepsilon > 0$ there is I such that

$$\|x_i - x_{i+1}\| < \frac{\varepsilon}{1 + \|A\|}$$

for all $i \geq I$. Whilst not essential we note $\|A\| < 1$ and is constant. We consider the sequence (w_i) .

$$\begin{aligned} \|w_i - w_{i+1}\| &= \|Ax_i + w_i - Ax_{i+1} - w_{i+1} + Ax_{i+1} - Ax_i\| \\ &\leq \|Ax_i + w_i - Ax_{i+1} - w_{i+1}\| + \dots \\ &\quad \dots + \|A\| \|x_{i+1} - x_i\| \\ &\leq \|x_{i+1} - x_{i+2}\| + \|A\| \|x_{i+1} - x_i\| \\ &< \varepsilon \end{aligned}$$

i.e. the sequence (w_i) is Cauchy and as the space is complete it converges to a limit w . We have that

$$x = Ax + w$$

If X is not closed \mathcal{W} is not closed. Taking the closure of X , we have that all limit points of sequences in X are now added to X . These limit points are fixed points contained in \mathcal{W} . It remains to prove that $cl(\mathcal{W})$ is maximal. Suppose not. Suppose X is not closed and the limit point $x \notin X$. We have that $x \in cl(X)$ and the sequence $((x_i, w_i)) \in \mathcal{W}$ with the limit point $(x, w) \in cl(\mathcal{W})$. Let $\mathcal{W}' \supset cl(\mathcal{W})$ be the maximal disturbance set for $cl(X)$. Take $(x, \hat{w}) \in \mathcal{W}' \setminus \mathcal{W}$. As $cl(X)$ is fixed under \mathcal{W}' we have

$$Ax + \hat{w} = y \in cl(X)$$

We observe there must be a sequence $(y_i) \subset X$ which converges to $y \in cl(X)$. By maximality of \mathcal{W} for each i there is $(x_i, \hat{w}_i) \in \mathcal{W}$ such that

$$Ax_i + \hat{w}_i = y_i$$

The sequence (\hat{w}_i) limits to \hat{w} i.e. $(x, \hat{w}) \in cl(\mathcal{W})$. Contradicting the assumption that $\mathcal{W}' \supset cl(\mathcal{W})$.

Lemma 10 revisits the classical case of state-independent bounded disturbance $w \in V$. For such disturbances, the mRPI exists and is given by

$$X := cl \left(\bigoplus_{i=0}^{\infty} A^i V \right) \quad (11)$$

We note without taking the closure that (11) may result in a non-closed set and in the state-independent case X is invariant if and only if $\bigoplus_{i=0}^{\infty} A^i V$ is invariant.

Lemma 10 discuss the maximal disturbance set starting from the classical construction

$$\mathcal{V} := X \times V$$

where X is as in (11).

Lemma 10. Suppose (1) is subject to the state-independent bounded disturbance \mathcal{V} . Then \mathcal{V} is not maximal.

Proof. In this case the mRPI set exists and is

$$X := cl \left(\bigoplus_{i=0}^{\infty} A^i V \right) \supset V$$

The maximal allowable disturbance at 0 is

$$W(0) = X$$

As the maximal disturbance set

$$\mathcal{W} \supseteq W(0) \cup \mathcal{V}$$

We thus conclude that \mathcal{V} is not maximal.

We first consider disturbances which generate unique convex fixed sets i.e. disturbances such that (1) admits a *convex mRPI*. In which case we can explicitly construct the maximal disturbance set. We then demonstrate how to combine such disturbance sets so we may obtain the maximal disturbance set for any fixed set.

We restrict our analysis to closed invariant sets. However we may consider open or semi-open invariant sets by removing extremal points from the invariant set X and the associated maximal disturbance set \mathcal{W} .

For notational convenience we introduce the map $f(z) = f(x, w) := Ax + w$.

4.1 Convex mRPI

Theorem 11. The mRPI set X is convex if and only if the maximal disturbance set \mathcal{W} is convex.

Proof. Suppose X is non-convex. As \mathcal{W} is maximal we have that $W(0) = X$. Thus, by the non-convexity of X , there exist $(x_1, 0), (x_2, 0) \in \mathcal{W}$ and $\lambda \in (0, 1)$ such that

$$\lambda(x_1, 0) + (1 - \lambda)(x_2, 0) \notin \mathcal{W}$$

Suppose X is convex but \mathcal{W} is non-convex and maximal. There exist $(x, w_x), (y, w_y) \in \mathcal{W}$ and $\lambda \in (0, 1)$ such that

$$\lambda(x, w_x) + (1 - \lambda)(y, w_y) \notin \mathcal{W}$$

However

$$\lambda f(x, w_x) + (1 - \lambda)f(y, w_y) \in X$$

by the convexity of X , where

$$f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^n : (x, w) \mapsto Ax + w$$

Furthermore as f is linear

$$\lambda f(x, w_x) + (1 - \lambda)f(y, w_y)$$

equals

$$f(\lambda x + (1 - \lambda)y, \lambda w_x + (1 - \lambda)w_y)$$

i.e. the pair

$$z := (\lambda x + (1 - \lambda)y, \lambda w_x + (1 - \lambda)w_y)$$

maps to a point in the mRPI X and therefore there exists $z \in \mathcal{W}'$. Hence $z \notin \mathcal{W}$ contradicts the maximality of \mathcal{W} .

For convex X we may characterise the maximal disturbance set \mathcal{W} as the convex hull of some set of extremal points. In Lemma 12 we relate the extrema of \mathcal{W} to those of X .

Lemma 12. Suppose X is the convex mRPI set with respect to the maximal disturbance set \mathcal{V} . Then $ex(X)$ and $ex(\mathcal{W})$ are the extremal points of X and \mathcal{W} if and only if

$$ex(\mathcal{W}) = \{(x_j, x_i - Ax_j) : x_i, x_j \in ex(X)\}$$

Proof. Take

$$z \in \{(x_j, x_i - Ax_j) : x_i, x_j \in ex(X)\}$$

We have

$$f(z) = f(x_j, x_i - Ax_j) = Ax_j + x_i - Ax_j = x_i \in ex(X)$$

If z is not extremal either $z \in int(\mathcal{W})$ or there exist $z_1, z_2 \in ex(\mathcal{W})$ and $\lambda \in (0, 1)$ such that

$$z = \lambda z_1 + (1 - \lambda) z_2$$

In the first case there is $\varepsilon > 0$ such that $B_\varepsilon(z) \in \mathcal{W}$. Hence the point

$$(1 + \varepsilon)z \in \mathcal{W}$$

We have

$$f((1 + \varepsilon)z) = (1 + \varepsilon)f(x_j, x_i - Ax_j) = (1 + \varepsilon)x_i$$

Contradicting the assumption that $x_i \in \text{ex}(X)$. In the second case, using the linearity of f

$$f(z) = \lambda f(z_1) + (1 - \lambda)f(z_2) = \lambda x_k + (1 - \lambda)x_l$$

which implies

$$x_i = \lambda x_k + (1 - \lambda)x_l$$

contradicting $x_i \in \text{ex}(X)$. Hence

$$\{(x_j, x_i - Ax_j) : x_i, x_j \in \text{ex}(X)\} \subseteq \text{ex}(\mathcal{W})$$

Furthermore as \mathcal{W} is convex

$$\text{co}(\{(x_j, x_i - Ax_j) : x_i, x_j \in \text{ex}(X)\}) \subseteq \mathcal{W}$$

Take

$$z \in \text{ex}(\mathcal{W}) \setminus \{(x_j, x_i - Ax_j) : x_i, x_j \in \text{ex}(X)\}$$

As \mathcal{W} is restricted to X we know for $z = (x, w) \in \mathcal{W}$ that $x \in X$. By convexity of X we have

$$x = \lambda x_i + (1 - \lambda)x_j$$

For some $x_i, x_j \in \text{ex}(X)$ and $\lambda \in [0, 1]$. We have $f(z) = \gamma x_k + (1 - \gamma)x_l \in X$ for some $x_k, x_l \in \text{ex}(X)$ and $\gamma \in [0, 1]$. Thus

$$f(z) = Ax + w = A(\lambda x_i + (1 - \lambda)x_j) + w = \gamma x_l + (1 - \gamma)x_k$$

Re-arranging for the disturbance

$$w = \gamma x_l + (1 - \gamma)x_k - \lambda Ax_i - (1 - \lambda)Ax_j$$

We have the points

$$z_1 := (x_i, x_l - Ax_i), z_2 := (x_i, x_k - Ax_i),$$

$$z_3 := (x_j, x_l - Ax_j), z_4 := (x_j, x_k - Ax_j)$$

are in $\text{ex}(\mathcal{W})$ and the convex combination

$$\lambda(\gamma z_1 + (1 - \gamma)z_2) + (1 - \lambda)(\gamma z_3 + (1 - \gamma)z_4) = (x, w)$$

i.e. $(x, w) \in \text{co}(\{(x_j, x_i - Ax_j) : x_i, x_j \in \text{ex}(X)\})$ Contradicting (x, w) being an extremal point.

In the case when X is convex the extrema of X correspond with the extrema of \mathcal{W} . This is proven in Corollary 13 which also establishes that no extrema of X map to the same extrema of \mathcal{W} .

Corollary 13. Let X and \mathcal{W} be as in Lemma 12. Then

$$|\text{ex}(\mathcal{W})| = |\text{ex}(X)|^2$$

Proof. We note that each extremal point of \mathcal{W} is unique. As if

$$(x_1, x_2 - Ax_1) = (x_3, x_4 - Ax_3)$$

Then we have $x_1 = x_3$ which implies that $x_2 = x_4$. For each x_i there are therefore $|\text{ex}(X)|$ distinct points of the form

$$(x_i, x_j - Ax_i)$$

4.2 Union of Convex mRPI

We now construct maximal disturbance sets for non-convex fixed sets from the maximal disturbance sets for convex fixed sets.

We may express a general fixed set as an arbitrary union of convex sets. This union will be fixed under the union of the individual maximal disturbance sets. However the union

of the maximal disturbance sets is not itself maximal. To address this we define

$$U(i, j) := \bigcup_{x \in X_i} \{(x, w) : Ax + w \in X_j, j \neq i\}$$

which is the set of state-disturbance pairs which map between fixed sets X_i and X_j and denote by

$$U(i) := \bigcup_{j \neq i} U(i, j)$$

Lemma 14 gives the maximal disturbance set for the union of arbitrarily many fixed sets in terms of $U(i)$ and the individual maximal disturbance sets.

Lemma 14. Suppose $\{X_1, \dots, X_n, \dots\}$ are fixed sets with maximal disturbance sets $\{\mathcal{W}_1, \dots, \mathcal{W}_n, \dots\}$. Then

$$\mathcal{W} := \bigcup_i (\mathcal{W}_i \cup U(i)) \quad (12)$$

is the maximal disturbance set such that $\bigcup_i X_i$ is fixed.

Proof. Suppose

$$X := \bigcup_i X_i$$

is a fixed set. Take $x_1 \in X_1$ and w such that

$$y := Ax_1 + w \in X$$

if $y \in X_1$ then $w \in \mathcal{W}_1$ as \mathcal{W}_1 is maximal. Otherwise if $x_1 \in X_j \neq X_1$ we have

$$w \in \{(x, w) : Ax + w \in X_j \text{ and } x_1 \in X_1\}$$

$$\subseteq \left(\bigcup_i \left(\bigcup_{x \in X_i} \{(x, w) : Ax + w \in X_j, j \neq i\} \right) \right)$$

Let \mathcal{W} be a disturbance set. Then for any $x, y \in X$ there exists w such that $Ax + w = y$ and $(x, w) \in \mathcal{W}$ i.e. X is a fixed set with respect to \mathcal{W} .

We observe that because \mathcal{W} is a maximal disturbance set it is convex if and only if $\bigcup_i X_i$ is convex.

Convexity and extrema of U We now consider the geometry of the components of the maximal disturbance set. In Lemma 15 we characterise the convexity of the sets $U(i, j)$ in terms of the convexity of the mRPI sets X_i and X_j . The convexity of the sets $U(i, j)$ much like \mathcal{W}_i depend solely on the convexity of each mRPI set and not the union.

Lemma 15. The set $U(i, j)$ is convex if and only if X_i and X_j are convex.

Proof. Take $(x, w), (y, v) \in U(i, j)$. Consider

$$f\left(\lambda \begin{pmatrix} x \\ w \end{pmatrix} + (1 - \lambda) \begin{pmatrix} y \\ v \end{pmatrix}\right) = \dots$$

$$\dots = A(\lambda x + (1 - \lambda)y) + \lambda w + (1 - \lambda)v$$

As X_i is convex

$$p := \lambda x + (1 - \lambda)y \in X_i$$

Defining $h := f(x, w)$ and $g := f(y, v)$, convexity of X_j implies

$$\lambda h + (1 - \lambda)g \in X_j$$

Thus, for the disturbance u defined as

$$u := \lambda w + (1 - \lambda)v$$

We have

$$Ap + u \in X_j$$

Now, suppose X_i is not convex i.e. there are $x, y \in X_i$ and $\lambda' \in (0, 1)$ such that

$$p := \lambda'x + (1 - \lambda')y \notin X_i$$

As there exists a pair $(x, \cdot), (y, \cdot) \in U(i, j)$ but no pair $(z, \cdot) \in U(i, j)$ we have that $U(i, j)$ is non-convex. Suppose instead that X_j is non-convex i.e. there exist $h, g \in X_j$ and $\lambda' \in (0, 1)$ such that

$$\lambda'g + (1 - \lambda')h \notin X_j$$

Let $(x, w), (y, v) \in U(i, j)$ such that

$$f(x, w) = g \text{ and } f(y, v) = h$$

We see that

$$\lambda'(x, w) + (1 - \lambda')(y, v) \notin U(i, j)$$

Lemma 16 characterises the extrema of $U(i, j)$ in terms of the extrema of X_i and X_j .

Lemma 16. (Extrema of U). Suppose X_i and X_j are convex. Then a point $(x, w) \in ex(U(i, j))$ if and only if $x = x_i \in ex(X_i)$ and $w = x_j - Ax_i$ where $x_j \in ex(X_j)$.

Proof. We know $U(i, j)$ is convex by Lemma 15

Suppose $(x_i, w) \in ex(U(i, j))$ but $x_i \notin ex(X_i)$ i.e. there exists λ and $x_1, x_2 \in X_i$ such that

$$x_i = \lambda x_1 + (1 - \lambda)x_2$$

In this case

$$\begin{pmatrix} x_i \\ w \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ w \end{pmatrix} + (1 - \lambda) \begin{pmatrix} x_2 \\ w \end{pmatrix}$$

the points $(x_1, w), (x_2, w) \in U(i, j)$ by convexity of $U(i, j)$ i.e. (x_i, w) is not extremal. Similarly if $x_j \notin ex(X_j)$. Suppose for $x_i \in ex(X)$ there exists $(x, w) \in ex(U(i, j))$ such that $w \neq x_k - Ax_i$ for $x_k \in ex(X_j)$. We have

$$f(x_i, w) = Ax_i + w = \sum_j \alpha_j x_j \in X_j$$

where the final equality follows by the convexity of X_j . Thus

$$w = \left(\sum_j \alpha_j x_j \right) - Ax_i = \sum_j \alpha_j (x_j - Ax_i)$$

Contradicting $(x_i, w) \in ex(U(i, j))$.

The converse follows by linearity.

4.3 Convex Hull of Fixed Sets

We conclude this section with Theorem 17 which proves that the maximal disturbance set for the convex hull of the union of mRPI sets is the convex hull of \mathcal{W} .

Theorem 17. Suppose $\{X_1, \dots, X_n, \dots\}$ are mRPI sets with maximal disturbance sets $\{\mathcal{W}_1, \dots, \mathcal{W}_n, \dots\}$. Then

$$co(X) = co \left(\bigcup_i X_i \right)$$

has maximal disturbance set $co(\mathcal{W})$.

Proof. We show the extrema of the maximal disturbance set for $co(X)$ are identically the extrema of $co(\mathcal{W})$.

The extrema

$$ex(co(X)) \subseteq \bigcup_i ex(X_i) \quad (13)$$

Let $\hat{\mathcal{W}}$ be the maximal disturbance set for $co(X)$. By Lemma 12 we have

$$ex(\hat{\mathcal{W}}) = \{(x_1, x_2 - Ax_1) : x_1, x_2 \in ex(co(X))\} \quad (14)$$

We have $x_1 \in X_i$ and $x_2 \in X_j$ for some – potentially equal – X_i and X_j .

Now consider the extrema of $co(\mathcal{W})$.

$$ex(co(\mathcal{W})) \subseteq \bigcup_i (ex(\mathcal{W}_i) \cup ex(U(i))) \quad (15)$$

The extrema of $U(i)$ cover the case when x_1, x_2 belong to distinct X_i and X_j and the extrema of \mathcal{W}_i cover the case when x_1, x_2 belong to the same X_i – where x_1, x_2 are as in (14).

It now remains to show that if there is an extremal point in RHS of (13) which is not in LHS of (13). Then the corresponding extremal point is not in the LHS of (15) and the converse.

Suppose x_1 is in the RHS of (13) but is not in the LHS of (13) i.e. $x_1 \in ex(X_i)$ but $x_1 \in int(X_j)$ for some $j \neq i$.

$$x_1 = \lambda x_j + (1 - \lambda)\hat{x}_j$$

where $x_j, \hat{x}_j \in ex(X_j)$. Consider the point

$$\begin{pmatrix} x_1 \\ y - Ax_1 \end{pmatrix} \in U(i, j)$$

We may rewrite this as

$$\lambda \begin{pmatrix} x_j \\ y - Ax_j \end{pmatrix} + (1 - \lambda) \begin{pmatrix} \hat{x}_j \\ y - A\hat{x}_j \end{pmatrix} \in ex(\mathcal{W}_j)$$

Thus for each extremal point missing in the LHS of (13) there is an extremal point missing in (15).

5. EXAMPLE

Consider the system

$$x(k+1) = 0.5x(k) + w(k) \quad (16)$$

We give the maximal disturbance set for the union of two convex mRPI sets

$$X_1 := [-1, 1] \text{ and } X_2 := [3, 5]$$

These sets are shown by the solid red lines in Figure 1. Their respective maximal disturbance sets are given by the shaded blue and green regions. The set

$$U(1, 2) = co(\{(-1, 5.5), (-1, 3.5), (1, 4.5), (1, 2.5)\})$$

is shown by the orange region and the set

$$U(2, 1) = co(\{(5, -3.5), (5, -1.5), (3, -2.5), (3, -0.5)\})$$

is shown by the purple region.

We also give the convex hull $co(X_1 \cup X_2) = [-1, 5]$ which is the solid black line. The maximal disturbance set $co(\mathcal{W})$ is shown as the black shaded region.

6. CONCLUSIONS

We developed results which describe the invariant sets of discrete linear dynamic systems affected by additive disturbances. By imposing the requirement that a certain set be invariant, we addressed what were the maximal state-dependent bounds on the additive disturbances that ensure the given set was positively robust invariant.

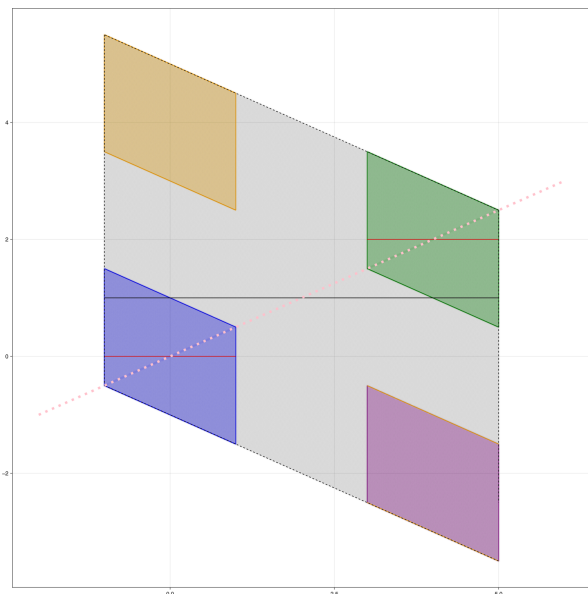


Fig. 1. The maximal disturbance set for the union of two convex mRPI sets $X_1 \cup X_2$ for the system (16). We also show the maximal disturbance set for the convex hull $co(X_1 \cup X_2)$. The fixed points of (1) are shown by the dotted pink line.

A series of geometric and topological properties have been established between the fixed set of the set mappings and the maximal disturbance set.

Further studies will allow a characterisation of how the maximal disturbance sets evolve by scaling or other operations on the invariant sets.

Of most interest is to extend the maximal disturbance sets beyond the fixed set X . Considering both the convex and non-convex extensions. Knowing the maximal disturbance set for the entire state-space would enable us to determine if a given general disturbance admits a desired invariant set.

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