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► **To cite this version:**

Fabienne Comte, Nicolas Marie. Nonparametric Estimation of the Transition Density Function for Diffusion Processes. 2024. hal-04527294v2

**HAL Id: hal-04527294**

**<https://hal.science/hal-04527294v2>**

Preprint submitted on 1 Nov 2024

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# NONPARAMETRIC ESTIMATION OF THE TRANSITION DENSITY FUNCTION FOR DIFFUSION PROCESSES

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ABSTRACT. We assume that we observe  $N$  independent copies of a diffusion process on a time-interval  $[0, 2T]$ . For a given time  $t$ , we estimate the transition density  $p_t(x, y)$ , namely the conditional density of  $X_{t+s}$  given  $X_s = x$ , under conditions on the diffusion coefficients ensuring that this quantity exists. We use a least squares projection method on a product of finite dimensional spaces, prove risk bounds for the estimator and propose an anisotropic model selection method, relying on several reference norms. A simulation study illustrates the theoretical part for Ornstein-Uhlenbeck or square-root (Cox-Ingersoll-Ross) processes.

## 1. INTRODUCTION

In this paper, we consider the stochastic differential equation (SDE)

$$(1) \quad X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s ; t \in [0, 2T],$$

where  $x_0 \in \mathbb{R}$ ,  $W = (W_t)_{t \in [0, 2T]}$  is a Brownian motion,  $b, \sigma \in C^1(\mathbb{R})$ , and  $b'$  and  $\sigma'$  are bounded. Under these conditions on  $b$  and  $\sigma$ , Equation (1) has a unique (strong) solution  $X = (X_t)_{t \in [0, 2T]}$ . Under additional conditions, the transition density  $p_t(x, \cdot)$  is well defined and can be understood as the conditional density of  $X_{s+t}$  given  $X_s = x$ . The two time indexes  $s + t$  and  $s$  explain why the process is considered on  $[0, 2T]$ , while each index evolves in  $[0, T]$ . Our aim is to provide and study a nonparametric estimator of the function  $(x, y) \mapsto p_t(x, y)$  ( $t \in [0, T]$ ), computed from  $N$  independent copies of  $X$  observed on the time-interval  $[0, 2T]$ .

Let us start with some comments about the observation scheme. It is related to functional data analysis, dealing with samples of infinite dimensional data (see Ramsay and Silverman [34] and Wang *et al.* [37]). Several examples of applications in biology and meteorology (resp. energy) are developed in Ramsay and Silverman [34] (resp. Candanedo *et al.* [8]). In Section 2, we also present an example of financial model leading to this observation setting.

Since few years, statistical inference from copies of diffusion processes, especially estimators of the drift function  $b$ , has been deeply investigated. In that way, tools related to statistical inference for stochastic processes have been used in the world of functional data analysis. From independent copies of the solution  $X$  of an SDE, projection least squares estimators of  $b$  have been studied in Comte and Genon-Catalot [14] for continuous time observations, in Denis *et al.* [19] for discrete time observations with a classification purpose in a parametric setting, and in Denis *et al.* [18] for the same question in the nonparametric framework. Marie and Rosier [31] propose a kernel based Nadaraya-Watson estimator of the drift function  $b$ , with bandwidth selection relying on the Penalized Comparison to Overfitting criterion recently introduced in Lacour *et al.* [26]. Still for independent copies of  $X$ , Halconruy and Marie [22] investigate the properties of the projection least squares estimator of  $b$  when the Brownian motion  $W$  is replaced by a Lévy process in Equation (1), and Comte and Genon-Catalot [15] and Marie [29] deal with estimators of the drift function in non-autonomous SDE. More recently, copies-based estimation with dependency has been investigated. For instance, one may refer to Della Maestra and Hoffmann [17] or Belomestny *et al.* [7], dealing with nonparametric estimators in interacting particle systems and McKean-Vlasov models. However, the question of transition density estimation has not been considered in this setting, while this

function is a key quantity of the process.

A transition density is a conditional density, but it is beyond our scope to describe all works dealing with the nonparametric estimation of the conditional density of  $Y$  given  $X = x$  when  $N$  i.i.d. observations of the couples  $(X_i, Y_i)$  are available. We refer to the recent paper by Comte and Lacour [16], and the references therein, on this topic. The question of estimating a transition density has already been studied, by several authors, for discrete samples of general Markov chains. The discrete time observation of one path of the solution  $X$  of Equation (1), say  $X_{k\Delta}$  with  $k = 1, \dots, n$  and  $\Delta > 0$ , fits to this framework. Nonparametric estimation strategies can be based on contrast minimization, as in Lacour [24], an approach we also follow and adapt to our setting. A transition can also be seen as a ratio of two densities, namely the density of  $(X_s, X_{s+t})$  divided by that of  $X_s$ , and then the ratio of the corresponding density estimators may be proposed and studied (see Lacour [25]). The paper by Sart [35] presents and studies estimators corresponding to the two strategies. Recently, in higher dimension, Löffler and Picard [28] estimate a general transition operator relying on singular value decomposition.

The framework considered in the present work is new. Indeed, the question of estimating the transition density  $p_t(x, y)$  ( $t \in (0, T]$ ) from the observation on  $[0, 2T]$  of  $N$  independent copies  $X^i := \mathcal{I}(x_0, W^i)$  ( $i \in \{1, \dots, N\}$ ) of the solution  $X$  of Equation (1), where  $\mathcal{I}(\cdot)$  is the Itô map for Equation (1) and  $W^1, \dots, W^N$  are  $N$  independent copies of the Brownian motion  $W$ , has not been yet investigated. Let us consider  $\mathcal{S}_{\mathbf{m}} := \mathcal{S}_{\varphi, m_1} \otimes \mathcal{S}_{\psi, m_2}$ , where  $\mathcal{S}_{\varphi, m_1} := \text{span}\{\varphi_1, \dots, \varphi_{m_1}\}$  (resp.  $\mathcal{S}_{\psi, m_2} := \text{span}\{\psi_1, \dots, \psi_{m_2}\}$ ),  $\varphi_1, \dots, \varphi_{N_T}$  (resp.  $\psi_1, \dots, \psi_{N_T}$ ) are continuous functions from  $I$  (resp.  $J$ ) into  $\mathbb{R}$  such that  $(\varphi_1, \dots, \varphi_{N_T})$  (resp.  $(\psi_1, \dots, \psi_{N_T})$ ) is an orthonormal family in  $\mathbb{L}^2(I, dx)$  (resp.  $\mathbb{L}^2(J, dx)$ ),  $N_T := [NT] + 1$  and  $I, J \subset \mathbb{R}$  are non-empty intervals. In this paper, we study the following projection least squares estimator of the transition density function  $p_t$ :

$$\widehat{p}_{\mathbf{m}, t}(x, y) := \sum_{j=1}^{m_1} \sum_{k=1}^{m_2} \widehat{\Theta}_{j, k} \varphi_j(x) \psi_k(y),$$

where  $\mathbf{m} = (m_1, m_2) \in \{1, \dots, N_T\}^2$ , and the coefficients  $\widehat{\Theta}_{j, k}$  are computed from the observed paths of the processes  $(X_t^i)_{0 \leq t \leq 2T}$  ( $i \in \{1, \dots, N\}$ ). This estimator is the minimizer over functions  $\tau \in \mathcal{S}_{\mathbf{m}}$  of an objective function defined as the empirical version of

$$\|\tau - p_t\|_f^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\tau(x, y) - p_t(x, y))^2 f(x) dx dy;$$

the  $f$ -weighted distance between  $\tau$  and  $p_t$  with

$$f(\cdot) = \frac{1}{T} \int_0^T p_s(x_0, \cdot) ds.$$

Roughly speaking,  $\widehat{p}_{\mathbf{m}, t}$  mimics the projection of  $p_t$  on  $\mathcal{S}_{\mathbf{m}}$  with respect to the  $f$ -weighted norm  $\|\cdot\|_f$ , and then our purpose is to prove that the empirical norm and the empirical criterion defining  $\widehat{p}_{\mathbf{m}, t}$  are near of their theoretical counterparts with high probability. In this way, we are able to evaluate the quadratic risk of the estimator.

A risk bound is established on  $\widehat{p}_{\mathbf{m}, t}$ , for a fixed  $\mathbf{m}$ . Under regularity assumptions and an adequate choice of  $\mathbf{m} = (m_1, m_2)$ , a rate of convergence is provided for the projection least squares estimator of  $p_t$ . Then, we focus on the example where  $p_t$  belongs to an anisotropic 2-dimensional Sobolev-Hermite space. Clearly, other regularity spaces may be considered, but since the Hermite basis is  $\mathbb{R}$ -supported, this example is very instructive and allows us to rely on some results already established in Comte and Lacour [16] for the estimation of the conditional density. The established rates are achieved when  $m_1$  and  $m_2$  - the coordinates of  $\mathbf{m}$  - are chosen as  $N$  to a power depending on the regularity orders of  $p_t$ . Since these regularity parameters are unknown, the previous choice of  $\mathbf{m}$  cannot be done in practice. This is why a data-driven selection procedure is provided and studied, allowing us to select a model  $\widehat{\mathbf{m}} = (\widehat{m}_1, \widehat{m}_2)$  directly from the data. The couple of dimensions  $(\widehat{m}_1, \widehat{m}_2)$  is selected in a random subset of  $\{1, \dots, N_T\}^2$  thanks to a penalized version of the estimation contrast, which corresponds to the idea of a data-driven squared-bias/variance compromise. Then, a risk bound is established on the adaptive estimator  $\widehat{p}_{\widehat{\mathbf{m}}, t}$ ,

which automatically achieves the squared-bias/variance tradeoff up to a  $\log(N)$  factor. We also prove that the bound can be improved, and the  $\log(N)$ -loss corrected, in two special cases: when  $t > t_0$  for a given  $t_0 > 0$ , and when  $t > 0$  with  $I$  - the support of the basis  $\varphi$  - compact.

To start with, in a short preliminary Section 2, we propose two application settings where our observation scheme is meaningful, and where our estimator of the transition density may be used. The projection least squares estimator is defined in Section 3, and non-adaptive risk bounds on a given model are established in Section 4. A model selection procedure and an adaptive risk bound are provided in Section 5. The whole procedure is illustrated through simulations in Section 6. Finally, Section 7 provides concluding remarks and proofs are gathered in Appendix A.

### Notation and basic definitions:

- The space of  $m_1 \times m_2$  matrices with coefficients in  $\mathbb{R}$  is denoted by  $\mathcal{M}_{m_1, m_2}(\mathbb{R})$ .
- For every  $M \in \mathcal{M}_{m_1, m_2}(\mathbb{R})$ , the transpose of  $M$  is denoted by  $M^*$ , and the largest eigenvalue of  $MM^*$  is denoted by  $\|M\|_{\text{op}}^2$ .
- The usual inner product (resp. norm) on  $\mathbb{L}^2(\mathbb{R}^2)$  is denoted by  $\langle \cdot, \cdot \rangle$  (resp.  $\|\cdot\|$ ). Recall that, for every  $\tau, \tau^* \in \mathbb{L}^2(\mathbb{R}^2)$ ,

$$\langle \tau, \tau^* \rangle = \int_{\mathbb{R}^2} \tau(x, y) \tau^*(x, y) dx dy.$$

For the sake of readability, the usual inner product and the associated norm on  $\mathbb{L}^2(\mathbb{R})$  are denoted in the same way.

## 2. TWO MOTIVATING EXAMPLES

In this section, we briefly present two possible applications of our estimation method of  $p_t$ .

- Assume that  $\sigma(\cdot)^2 > 0$ , and consider the parabolic partial differential equation defined by

$$(2) \quad \frac{\partial u}{\partial t}(t, x) + \frac{1}{2} \sigma(x)^2 \frac{\partial^2 u}{\partial x^2}(t, x) + b(x) \frac{\partial u}{\partial x}(t, x) = 0, \quad u(T, x) = v(x),$$

where  $v : \mathbb{R} \rightarrow \mathbb{R}$  is a known twice continuously differentiable function. Moreover, let  $X^{t,x}$  be the solution of Equation (1) starting from  $x \in \mathbb{R}$  at time  $t \in [0, T)$ . By Lamberton and Lapeyre [27], Theorem 5.1.7, the solution of Equation (2) is given by

$$F(t, x) := \mathbb{E}(\varphi(X_T^{t,x})) = \int_{-\infty}^{\infty} v(y) p_{T-t}(x, y) dy.$$

Thus, the solution of Equation (2) can be estimated by

$$(3) \quad \widehat{F}_{\mathbf{m}}(t, x) := \int_{-\infty}^{\infty} v(y) \widehat{p}_{\mathbf{m}, T-t}(x, y) dy.$$

In the same spirit as Milstein *et al.* [33], (3) provides a Monte-Carlo method which may replace the usual finite difference algorithm to solve Equation (2) numerically. Note that in this example,  $b$ ,  $\sigma$  and  $v$  are assumed to be known, and then one may simulate a large amount  $N$  of copies of  $X$  observed at high-frequency on  $[0, 2T]$  in order to compute  $\widehat{p}_{\mathbf{m}, T-t}$ , and then  $\widehat{F}_{\mathbf{m}}(t, x)$ . In other words, the copies-based observation scheme is appropriate in this context.

- The options pricing in finance is another possible application of our estimation method of  $p_t$ . Let  $X = (X_t)_{t \in \mathbb{R}}$  be the prices process of a risky asset, which risk-neutral dynamics are modeled by

$$(4) \quad \frac{dX_t}{X_t} = (r - \delta) dt + \sigma(X_t) dW_t,$$

where  $r > 0$  is the risk-free rate,  $\delta > 0$  the dividend rate,  $W = (W_t)_{t \in \mathbb{R}}$  is a two-sided Brownian motion, and  $X_0$  is a  $\sigma((W_t)_{t \in \mathbb{R}_-})$ -measurable square integrable random variable. Consider also the option with payoff  $v(X_T^{0,x})$ , which price  $P(x, T)$  satisfies

$$P(x, T) = e^{-rT} \mathbb{E}(v(X_T^{0,x})) = e^{-rT} \int_0^{\infty} v(y) p_T(x, y) dy.$$

Thus, when  $r < \delta$ , an estimator of  $P(x, T)$  is given by

$$\widehat{P}_{\mathbf{m}}(x, T) := e^{-rT} \int_0^\infty v(y) \widehat{p}_{\mathbf{m}, T}(x, y) dy,$$

where the copies  $X^1, \dots, X^N$  of  $(X_t^{0, x})_{t \in [0, 2T]}$  are here constructed from  $(X_t)_{t \in \mathbb{R}_-}$ , the past of the (recurrent Markov) prices process, by following the same line as in Marie [30], Remark 2.3. So, the copies-based observation scheme remains appropriate in this context, but requires high-frequency data on a long time-interval.

Note that risk bounds on truncated versions of  $\widehat{F}_{\mathbf{m}}(t, x)$  and  $\widehat{P}_{\mathbf{m}}(x, T)$  may be deduced from the results of Section 4.2, but are omitted since they may not be optimal.

### 3. A PROJECTION LEAST SQUARES ESTIMATOR OF THE TRANSITION DENSITY FUNCTION

**3.1. Assumptions on the coefficients of Equation (1).** Throughout the paper, we consider copies  $X^1, \dots, X^N$  of the solution  $X$  of Equation (1) under the condition:

$$(5) \quad x_0 \in \mathbb{R}, W = (W_t)_{t \in [0, 2T]} \text{ is a Brownian motion, } b, \sigma \in C^1(\mathbb{R}) \text{ and } b', \sigma' \text{ are bounded.}$$

We also assume that  $\sigma$  satisfies the following non-degeneracy condition:

$$(6) \quad \exists \alpha, A > 0 : \forall x \in \mathbb{R}, \alpha \leq |\sigma(x)| \leq \alpha + A.$$

As already mentioned, under condition (5), Equation (1) has a unique solution. Under the condition (6), the transition density function  $p_t$  is well-defined. Conditions (5) and (6) are assumed to be fulfilled throughout the paper. Some properties of the density  $p_t(x_0, \cdot)$  of  $X_t$  are provided in the following proposition.

**Proposition 3.1.** *There exist two positive constants  $\mathbf{c}_T$  and  $\mathbf{m}_T$ , depending on  $T$  but not on  $t$ , such that for every  $x, y \in \mathbb{R}$ ,*

$$(7) \quad p_t(x_0, x) \leq \mathbf{c}_T t^{-\frac{1}{2}} \exp\left(-\mathbf{m}_T \frac{(x - x_0)^2}{t}\right).$$

Then, the density function

$$(8) \quad f(\cdot) := \frac{1}{T} \int_0^T p_s(x_0, \cdot) ds$$

is well-defined and satisfies the following properties:

- (1)  $f$  is upper bounded on  $\mathbb{R}$ .
- (2) For every compact interval  $I \subset \mathbb{R}$ , there exists a positive constant  $\underline{\mathbf{m}} = \underline{\mathbf{m}}(I)$  such that  $f(\cdot) \geq \underline{\mathbf{m}}$  on  $I$ .
- (3) For every  $\kappa \in \mathbb{R}_+$ ,  $|b|^\kappa \in \mathbb{L}^2(\mathbb{R}, f(x) dx)$ .

*Proof.* First, under the conditions (5) and (6), by Menozzi *et al.* [32], Theorem 2.1, there exist two positive constants  $\bar{\mathbf{c}}_T$  and  $\bar{\mathbf{m}}_T$ , depending on  $T$  but not on  $t$ , such that for every  $x, y \in \mathbb{R}$ ,

$$(9) \quad p_t(x, y) \leq \bar{\mathbf{c}}_T t^{-\frac{1}{2}} \exp\left(-\bar{\mathbf{m}}_T \frac{(y - \theta_t(x))^2}{t}\right),$$

where  $s \mapsto \theta_s(x)$  is the solution of the differential equation

$$z_s = x + \int_0^s b(z_u) du ; s \in [0, T].$$

For any  $u \in (0, T]$  and  $x \in \mathbb{R}$ ,

$$|x - x_0| \leq |x - \theta_u(x_0)| + u S_T(x_0) \quad \text{with} \quad S_T(x_0) = \sup_{s \in [0, T]} |b(\theta_s(x_0))|,$$

leading to

$$-\frac{1}{2}(x - x_0)^2 + u^2 S_T(x_0)^2 \geq -(x - \theta_u(x_0))^2.$$

Thus, by Inequality (9),

$$p_t(x_0, x) \leq \mathbf{c}_T t^{-\frac{1}{2}} \exp\left(-\mathbf{m}_T \frac{(x - x_0)^2}{t}\right)$$

with  $\mathbf{c}_T = \bar{\mathbf{c}}_T e^{\bar{\mathbf{m}}_T T S_T(x_0)^2}$  and  $\mathbf{m}_T = \frac{\bar{\mathbf{m}}_T}{2}$ .

In particular,  $t \mapsto p_t(x_0, x)$  belongs to  $\mathbb{L}^1([0, T])$ , which legitimates to consider the density function  $f$  defined by (8). Still by Inequality (7):

- $f$  is upper bounded on  $\mathbb{R}$ . Indeed, for every  $x \in \mathbb{R}$ ,

$$f(x) \leq \frac{1}{T} \int_0^T \mathbf{c}_T s^{-\frac{1}{2}} ds = 2\mathbf{c}_T T^{-\frac{1}{2}}.$$

- Since  $b'$  is bounded (and then  $b$  has linear growth),

$$|b|^\kappa \in \mathbb{L}^2(\mathbb{R}, f(x)dx) ; \forall \kappa \in \mathbb{R}_+.$$

Indeed, for every  $v \in \mathbb{R}_+$  and  $t \in (0, T]$ ,

$$\begin{aligned} \mathbb{E}(|X_t|^v) &= \int_{-\infty}^{\infty} |x|^v p_t(x_0, x) dx \\ &\leq t^{-\frac{1}{2}} \mathbf{c}_T \underbrace{\int_{-\infty}^{\infty} |x|^v \exp\left(-\mathbf{m}_T \frac{(x - x_0)^2}{T}\right) dx}_{=: \mathbf{c}_{T,v} < \infty} \end{aligned}$$

and then, for every  $\kappa \in \mathbb{R}_+$ ,

$$\begin{aligned} \int_{-\infty}^{\infty} |b(x)|^{2\kappa} f(x) dx &= \frac{1}{T} \int_0^T \mathbb{E}(|b(X_s)|^{2\kappa}) ds \\ &\leq \mathbf{c}_1 \left(1 + \frac{1}{T} \int_0^T \mathbb{E}(|X_s|^{2\kappa}) ds\right) \leq \mathbf{c}_1 (1 + 2\mathbf{c}_{T,2\kappa} T^{-\frac{1}{2}}) < \infty, \end{aligned}$$

where  $\mathbf{c}_1$  is a positive constant depending only on  $b$  and  $\kappa$ .

Now, Menozzi *et al.* [32], Theorem 1.2 also says that there exist two positive constants  $\underline{\mathbf{c}}_T$  and  $\underline{\mathbf{m}}_T$ , depending on  $T$  but not on  $t$ , such that for every  $x, y \in \mathbb{R}$ ,

$$(10) \quad p_t(x, y) \geq \underline{\mathbf{c}}_T t^{-\frac{1}{2}} \exp\left(-\underline{\mathbf{m}}_T \frac{(y - \theta_t(x))^2}{t}\right).$$

For any  $u \in (0, T]$  and  $x \in \mathbb{R}$ ,

$$|x - \theta_u(x_0)| \leq |x - x_0| + u S_T(x_0),$$

leading to

$$-(x - \theta_u(x_0))^2 \geq -2(x - x_0)^2 - 2u^2 S_T(x_0)^2.$$

So, by Inequality (10),

$$p_t(x_0, x) \geq \mathbf{c}_T^* t^{-\frac{1}{2}} \exp\left(-\mathbf{m}_T^* \frac{(x - x_0)^2}{t}\right)$$

with  $\mathbf{c}_T^* = \underline{\mathbf{c}}_T e^{-2\underline{\mathbf{m}}_T T S_T(x_0)^2}$  and  $\mathbf{m}_T^* = 2\underline{\mathbf{m}}_T$

and then, for every compact interval  $I \subset \mathbb{R}$  and  $x \in I$ ,

$$\begin{aligned} f(x) &\geq \frac{\mathbf{c}_T^*}{T} \int_{T/2}^T s^{-\frac{1}{2}} \exp\left(-\frac{\mathbf{m}_T^*}{s} (x - x_0)^2\right) ds \\ &\geq \underline{\mathbf{m}}(I) := \frac{\mathbf{c}_T^*}{T} \exp\left(-\frac{2\mathbf{m}_T^*}{T} \max\{(\max(I) - x_0)^2, (\min(I) - x_0)^2\}\right) \int_{T/2}^T s^{-\frac{1}{2}} ds > 0. \end{aligned}$$

This concludes the proof. □

**3.2. The projection least squares estimator and some related definitions.** The objective function  $\gamma_N$  is defined by

$$\gamma_N(\tau) := \frac{1}{NT} \sum_{i=1}^N \left( \int_0^T \int_{-\infty}^{\infty} \tau(X_s^i, y)^2 dy ds - 2 \int_0^T \tau(X_s^i, X_{s+t}^i) ds \right); \tau \in \mathcal{S}_{\mathbf{m}}.$$

In order to explain the relevance of the criterion, let us compute its expectation:

$$\mathbb{E}(\gamma_N(\tau)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\tau(x, y) - p_t(x, y))^2 f(x) dx dy - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_t(x, y)^2 f(x) dx dy.$$

This shows that, the closer  $\tau$  is to  $p_t$ , the smaller  $\mathbb{E}(\gamma_N(\tau))$ , and that  $\gamma_N(\tau)$  is the empirical version of the  $f$ -weighted  $\mathbb{L}^2$ -distance between  $\tau$  and  $p_t$ . This is the reason why we define the estimator of  $p_t$  as the minimizer of  $\gamma_N$  in  $\mathcal{S}_{\mathbf{m}}$ .

**Remark.** Note that any  $\tau \in \mathcal{S}_{\mathbf{m}}$  can be written  $\tau = \sum_{j,\ell} \Theta_{j,\ell} (\varphi_j \otimes \psi_\ell)$  with  $\Theta \in \mathcal{M}_{m_1, m_2}(\mathbb{R})$ . Then,  $\tau$  is characterized by the matrix  $\Theta$ .

First, let us study the conditions ensuring that  $\gamma_N$  has a unique minimizer in  $\mathcal{S}_{\mathbf{m}}$ . For  $\tau = \sum_{j,\ell} \Theta_{j,\ell} (\varphi_j \otimes \psi_\ell)$  with  $\Theta \in \mathcal{M}_{m_1, m_2}(\mathbb{R})$ ,

$$\nabla_{\tau} \gamma_N(\tau) = 2(\widehat{\Psi}_{m_1} \Theta - \widehat{Z}_{\mathbf{m},t}),$$

where

$$(11) \quad \widehat{\Psi}_{m_1} := \left( \frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X_s^i) \varphi_{j'}(X_s^i) ds \right)_{j, j' \in \{1, \dots, m_1\}}$$

and

$$(12) \quad \widehat{Z}_{\mathbf{m},t} := \left( \frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X_s^i) \psi_\ell(X_{s+t}^i) ds \right)_{(j,\ell) \in \{1, \dots, m_1\} \times \{1, \dots, m_2\}}.$$

The symmetric matrix  $\widehat{\Psi}_{m_1}$  is positive semidefinite because for any  $y \in \mathbb{R}^{m_1}$ ,

$$y^* \widehat{\Psi}_{m_1} y = \frac{1}{NT} \sum_{i=1}^N \int_0^T \left( \sum_{j=1}^{m_1} y_j \varphi_j(X_s^i) \right)^2 ds \geq 0.$$

If in addition  $\widehat{\Psi}_{m_1}$  is invertible, it is positive definite, and then

$$(13) \quad \widehat{p}_{\mathbf{m},t} = \sum_{j=1}^{m_1} \sum_{\ell=1}^{m_2} [\widehat{\Theta}_{\mathbf{m},t}]_{j,\ell} (\varphi_j \otimes \psi_\ell) \quad \text{with} \quad \widehat{\Theta}_{\mathbf{m},t} = \widehat{\Psi}_{m_1}^{-1} \widehat{Z}_{\mathbf{m},t}$$

is the only minimizer of  $\gamma_N$  in  $\mathcal{S}_{\mathbf{m}}$  called the projection least squares estimator of  $p_t$ .

Now, let us consider the theoretical counterpart  $\Psi_{m_1} := \mathbb{E}(\widehat{\Psi}_{m_1})$  of  $\widehat{\Psi}_{m_1}$  (see Equality (11)), the empirical inner product  $\langle \cdot, \cdot \rangle_N$  such that

$$\langle \tau, \tau^* \rangle_N := \frac{1}{NT} \sum_{i=1}^N \int_0^T \int_{-\infty}^{\infty} \tau(X_s^i, y) \tau^*(X_s^i, y) dy ds; \forall \tau, \tau^* \in \mathcal{S}_{\mathbf{m}},$$

and the  $f$ -weighted inner product  $\langle \cdot, \cdot \rangle_f$  such that

$$\langle \tau, \tau^* \rangle_f := \mathbb{E}(\langle \tau, \tau^* \rangle_N) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tau(x, y) \tau^*(x, y) f(x) dx dy; \forall \tau, \tau^* \in \mathcal{S}_{\mathbf{m}}.$$

The empirical norm associated to  $\langle \cdot, \cdot \rangle_N$  (resp.  $\langle \cdot, \cdot \rangle_f$ ) is denoted by  $\|\cdot\|_N$  (resp.  $\|\cdot\|_f$ ). For any  $\tau \in \mathcal{S}_{\mathbf{m}}$ ,

$$\gamma_N(\tau) = \|\tau\|_N^2 - \frac{2}{NT} \sum_{i=1}^N \int_0^T \tau(X_s^i, X_{s+t}^i) ds$$

and then, interestingly,

$$\mathbb{E}(\gamma_N(\tau)) = \|\tau\|_f^2 - 2\langle \tau, p_t \rangle_f,$$

justifying again the choice of the objective function  $\gamma_N$  to define our estimator  $\widehat{p}_{\mathbf{m},t}$  of  $p_t$ . It is standard in nonparametric regression-type settings to have to handle a random norm depending on the observations, and to control the proximity with its expectation thanks to deviation inequalities.

**3.3. Examples of bases.** In addition to the empirical norm, related to the data, and its theoretical counterpart - the  $f$ -weighted  $\mathbb{L}^2$ -norm  $\|\cdot\|_f$  -, the usual  $\mathbb{L}^2$ -norm  $\|\cdot\|$  defined at the end of the introduction section appears through the bases and their characteristics.

First, let  $u = (u_j)_{j \geq 1}$  be an orthonormal  $I$ -supported family of  $\mathbb{L}^2(\mathbb{R}, dx)$  (equipped with  $\langle \cdot, \cdot \rangle$ ), and consider  $\mathcal{S}_{u,m} := \text{span}\{u_1, \dots, u_m\}$ ; the linear space generated by the  $m \in \mathbb{N}^*$  first functions of the family  $u$ . Consider also

$$\mathfrak{L}_u(m) := \sup_{x \in I} \sum_{j=1}^m u_j(x)^2.$$

One may easily check that

$$\mathfrak{L}_u(m) = \sup_{\tau \in \mathcal{S}_{u,m}: \|\tau\|=1} \|\tau\|_\infty \quad (\text{see Lemma 1 in Bige and Massart [4], p. 337}).$$

So,  $\mathfrak{L}_u(m)$  measures the link between the  $\mathbb{L}^2$  and  $\mathbb{L}^\infty$  norms in the function space  $\mathcal{S}_{u,m}$ : when  $I = [0, 1]$ ,

$$\|\tau\|^2 \leq \|\tau\|_\infty^2 \leq \mathfrak{L}_u(m) \|\tau\|^2; \quad \forall \tau \in \mathcal{S}_{u,m}.$$

Now, let us provide examples of usual bases  $u$ , and controls of  $\mathfrak{L}_u(m)$  for each of them:

- The trigonometric basis is defined on  $I = [0, 1]$  by

$$\text{trig}_1(x) := 1, \quad \text{trig}_{2j}(x) := \sqrt{2} \cos(2\pi j x), \quad \text{trig}_{2j+1}(x) := \sin(2\pi j x)$$

for  $j = 1, \dots, (m-1)/2$  with  $m \in \mathbb{N} \setminus (2\mathbb{N})$ . Since  $\cos(\cdot)^2 + \sin(\cdot)^2 = 1$ ,  $\mathfrak{L}_{\text{trig}}(m) = m$ . In the same way,  $\mathfrak{L}_u(m)$  is of order  $m$  for splines, wavelets, or any orthonormal bounded bases.

- For the Legendre basis  $(\ell_j)_{j \geq 1}$ , defined via normalized Legendre's polynomials on  $I = [-1, 1]$  (see Tsybakov [36], Section 1.2.2),  $\mathfrak{L}_\ell(m) \lesssim m^2$  as established in Cohen *et al.* [10] (p. 831).
- Finally, let us focus on the Hermite basis  $(h_j)_{j \in \mathbb{N}}$ , defined on  $I = \mathbb{R}$  by

$$(14) \quad h_j(x) := \mathfrak{c}_j H_j(x) e^{-\frac{x^2}{2}} \quad \text{with} \quad \mathfrak{c}_j = (2^j j! \sqrt{\pi})^{-\frac{1}{2}} \quad \text{and} \quad H_j(x) = (-1)^j e^{x^2} \frac{d^j}{dx^j} e^{-x^2}$$

for every  $x \in \mathbb{R}$  and  $j \in \mathbb{N}$ . The sequence  $(h_j)_{j \in \mathbb{N}}$  is an orthonormal family of  $\mathbb{L}^2(\mathbb{R}, dx)$ . Moreover, the  $h_j$ 's are bounded by  $\pi^{-1/4}$ , and by Lemma 1 in Comte and Lacour [16], there exists a constant  $\mathfrak{c}_h > 0$  such that

$$(15) \quad \mathfrak{L}_h(m) \leq \mathfrak{c}_h \sqrt{m}.$$

Without loss of generality, in the sequel, we assume that  $\mathfrak{L}_u(m) \geq 1$ . Otherwise,  $\mathfrak{L}_u(m)$  can be replaced by  $1 \vee \mathfrak{L}_u(m)$ .

#### 4. RISK BOUNDS ON THE PROJECTION LEAST SQUARES ESTIMATOR

**4.1. Risk bound with respect to the empirical norm.** This subsection deals with a nonadaptive risk bound, with respect to the empirical norm  $\|\cdot\|_N$ , on our estimator  $\widehat{p}_{\mathbf{m},t}$  of  $p_t$ , built in  $\mathcal{S}_{\mathbf{m}} = \mathcal{S}_{\varphi, m_1} \otimes \mathcal{S}_{\psi, m_2}$ . In the sequel,  $\mathbf{m} = (m_1, m_2)$  fulfills the following assumption.

**Assumption 4.1.** *There exist two constants  $\mathfrak{c}_{4.1} > 0$  and  $q \in \mathbb{N}^*$ , not depending on  $\mathbf{m}$  and  $N$ , such that*

$$(16) \quad \mathfrak{L}_\psi(m_2) \leq \mathfrak{c}_{4.1} N^q.$$

Moreover,

$$(17) \quad \mathfrak{L}_\varphi(m_1) (\|\Psi_{m_1}^{-1}\|_{\text{op}} \vee 1) \leq \frac{\mathfrak{c}_\Lambda}{2} \cdot \frac{NT}{\log(NT)}$$



with

$$\mathbf{c}_\Lambda = \frac{1 - \log(2)}{(1+p)T} \quad \text{and} \quad p = 2(q+4) + 1.$$

The first part (16) of Assumption 4.1 is a weak limit on the maximal dimension  $m_2$  that can be considered (see Section 3.3). The second part (17) of Assumption 4.1 is a generalization of the so-called *stability condition* introduced in the nonparametric regression framework in Cohen *et al.* [10], and already extended to the context of independent copies of continuous diffusion processes framework in Comte and Genon-Catalot [14]. In fact, (17) is in turn a limit on the maximal dimension  $m_1$  that can be considered. It can be stronger than  $m_1 \leq NT$ , but to what extent? First, let us answer this question when  $I$  is a compact interval. By Proposition 3.1.(2), there exists  $\underline{\mathbf{m}} > 0$  such that  $f(\cdot) \geq \underline{\mathbf{m}}$  on  $I$ , and then

$$\begin{aligned} \|\Psi_{m_1}^{-1}\|_{\text{op}} &= \frac{1}{\lambda_{\min}(\Psi_{m_1})} = \left( \inf_{\theta: \|\theta\|_{2, m_1} = 1} \sum_{j, j'=1}^{m_1} \theta_j \theta_{j'} [\Psi_{m_1}]_{j, j'} \right)^{-1} \\ &= \left[ \inf_{\theta: \|\theta\|_{2, m_1} = 1} \int_I \left( \sum_{j=1}^{m_1} \theta_j \varphi_j(x) \right)^2 f(x) dx \right]^{-1} \leq \frac{1}{\underline{\mathbf{m}}} \quad \text{for every } m_1 \in \{1, \dots, NT\}. \end{aligned}$$

Moreover, assume that there exists a constant  $\mathbf{c}_\varphi > 0$ , not depending on  $m_1$  and  $N$ , such that  $\mathfrak{L}_\varphi(m_1) \leq \mathbf{c}_\varphi^2 m_1$ . For instance, all bounded bases satisfy this condition. Then, (17) is fulfilled by any

$$m_1 \leq \frac{\mathbf{c}_\Lambda}{2\mathbf{c}_\varphi^2 (\underline{\mathbf{m}}^{-1} \vee 1)} \cdot \frac{NT}{\log(NT)}.$$

In other words, when  $I$  is compact and the  $\varphi_j$ 's are bounded (i.e.  $\max_j \|\varphi_j\|_\infty \leq \Phi$  with  $\Phi > 0$ ),  $m_1$  needs to be of order  $NT/\log(NT)$ , which remains a weak condition. Now, when  $I$  is a non-compact interval, the set of possible values for  $m_1$  becomes smaller. For instance, for Hermite's basis, it holds that  $\|\Psi_{m_1}^{-1}\|_{\text{op}} \gtrsim \sqrt{m_1}$  by Comte and Genon-Catalot [13], Proposition 8, which necessarily reduces the collection of authorized values for  $m_1$ .

Under Assumption 4.1, we can prove the following risk bound.

**Theorem 4.2.** *Consider  $p_{I \times J, t} := (p_t)_{|I \times J}$ . Under Assumption 4.1, there exists a constant  $\mathbf{c}_{4.2} > 0$ , not depending on  $\mathbf{m}$  and  $N$ , such that for every  $t \in [0, T]$ ,*

$$(18) \quad \mathbb{E}(\|\widehat{p}_{\mathbf{m}, t} - p_{I \times J, t}\|_N^2) \leq \min_{\tau \in \mathcal{S}_{\mathbf{m}}} \|\tau - p_{I \times J, t}\|_f^2 + \frac{2m_1 \mathfrak{L}_\psi(m_2)}{N} + \frac{\mathbf{c}_{4.2}}{N}.$$

We emphasize that the risk bound in Theorem 4.2 is sharp since the constant in front of the bias term  $\min_{\tau \in \mathcal{S}_{\mathbf{m}}} \|\tau - p_{I \times J, t}\|_f^2$  in Inequality (18) is 1, and the constant 2 in the variance term  $2m_1 \mathfrak{L}_\psi(m_2)/N$  may be  $1 + \varepsilon$  for  $\varepsilon > 0$ , up to additional technicalities.

**Sketch of proof.** Let us provide a sketch of the proof of Theorem 4.2, which is detailed in Subsection A.1. For every function  $h$  from  $\mathbb{R}^2$  into  $\mathbb{R}$ , let

$$\widehat{\Pi}_{\mathbf{m}}(h) := \arg \min_{\tau \in \mathcal{S}_{\mathbf{m}}} \|\tau - h\|_N^2 \quad \text{be its empirical projection on } \mathcal{S}_{\mathbf{m}}.$$

Then,

$$\begin{aligned} \mathbb{E}(\|\widehat{p}_{\mathbf{m}, t} - p_{I \times J, t}\|_N^2) &= \mathbb{E} \left( \min_{\tau \in \mathcal{S}_{\mathbf{m}}} \|\tau - p_{I \times J, t}\|_N^2 \right) + \mathbb{E}(\|\widehat{p}_{\mathbf{m}, t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2) \\ &\leq \min_{\tau \in \mathcal{S}_{\mathbf{m}}} \|\tau - p_{I \times J, t}\|_f^2 + \mathbb{E}(\|\widehat{p}_{\mathbf{m}, t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2). \end{aligned}$$

The first term is the expected squared bias term, and

$$\begin{aligned} \mathbb{E}(\|\widehat{p}_{\mathbf{m}, t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2) &= \mathbb{E}(\|\widehat{p}_{\mathbf{m}, t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2 \mathbf{1}_{\Omega_{m_1}}) \\ &\quad + \mathbb{E}(\|\widehat{p}_{\mathbf{m}, t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2 \mathbf{1}_{\Omega_{m_1}^c}) =: A + B, \end{aligned}$$

where  $\Omega_{m_1}$  is the event on which, for every  $\mathbf{x} \in \mathbb{R}^{m_1}$ ,  $\mathbf{x}^* \widehat{\Psi}_{m_1} \mathbf{x} \leq 2\mathbf{x}^* \Psi_{m_1} \mathbf{x}$ . This means that on  $\Omega_{m_1}$ ,  $\widehat{\Psi}_{m_1}$  can be replaced by its expectation up to a factor two in controls. In a first step,  $A$  is controlled by  $m_1 \mathfrak{L}_\psi(m_2)/N$ , the main variance contribution, by using that

$$(19) \quad \|\widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2 = \sup_{\tau \in \mathcal{S}_{\mathbf{m}}: \|\tau\|_N=1} \left[ \frac{1}{NT} \sum_{i=1}^N \int_0^T \left( \tau(X_s^i, X_{s+t}^i) - \int_{-\infty}^{\infty} \tau(X_s^i, y) p_t(X_s^i, y) dy \right) ds \right]^2,$$

and by bounding its expectation thanks to the definition of  $\Omega_{m_1}$ . Indeed, on  $\Omega_{m_1}$ ,  $\|\tau\|_N^2 \leq 1$  implies that  $\|\tau\|_f^2 \leq 2$ , and then the random unit ball on which the supremum is taken can be replaced by a deterministic one. In a second step, we prove that  $B = O(1/N)$  thanks to Comte and Genon-Catalot [14], Lemma 6.1, which says that  $\mathbb{P}(\Omega_{m_1}^c) \lesssim 1/N^p$ .

**Remark.** Let us discuss the order of the variance term in Inequality (18) for the bases presented in Section 3.3. Obviously, the choice of  $\varphi$  does not matter as it leads in any case to the factor  $m_1$ . So, we discuss the choice of  $\psi$ , using the results of Section 3.3. First, for  $\psi$  chosen as splines, wavelets or trigonometric bases,  $\mathfrak{L}_\psi(m_2) \lesssim m_2$ , and then the variance term in the risk bound on  $\widehat{p}_{\mathbf{m},t}$  is of order  $m_1 m_2/N$  as for the usual projection estimator of a 2-dimensional density function. Now, for Legendre's basis,  $\mathfrak{L}_\psi(m_2) \lesssim m_2^2$ , leading to a variance term of order  $m_1 m_2^2/N$  in Inequality (18). Finally, for the Hermite basis, it is worth noting that, by (15), the variance term in Inequality (18) is of order  $m_1 \sqrt{m_2}/N$ .

**4.2. Risk bound with respect to the  $f$ -weighted norm, on a truncated version of the projection least squares estimator.** In Theorem 4.2, the risk of  $\widehat{p}_{\mathbf{m},t}$  is related to the empirical norm  $\|\cdot\|_N$ . Then, a result for a risk related to the theoretical counterpart  $\|\cdot\|_f$  of  $\|\cdot\|_N$  is of interest, especially because the bias term in Inequality (18) involves  $\|\cdot\|_f$ . In order to establish such a result from the "discrete" (empirical) one, an empirical version of the constraint (17) is required to ensure that the eigenvalues of  $\widehat{\Psi}_{m_1}$  are not too close to 0. Thus, this subsection deals with a risk bound, with respect to the  $f$ -weighted norm  $\|\cdot\|_f$ , on the following truncated version of our estimator  $\widehat{p}_{\mathbf{m},t}$ :

$$\widetilde{p}_{\mathbf{m},t} := \widehat{p}_{\mathbf{m},t} \mathbf{1}_{\Lambda_{m_1}},$$

where

$$(20) \quad \Lambda_{m_1} := \left\{ \mathfrak{L}_\varphi(m_1) (\|\widehat{\Psi}_{m_1}^{-1}\|_{\text{op}} \vee 1) \leq \mathfrak{c}_\Lambda \frac{NT}{\log(NT)} \right\}.$$

Under (17), the probability of  $\Lambda_{m_1}$  is near of one. Moreover, as expected, on the event  $\Lambda_{m_1}$ ,  $\widehat{\Psi}_{m_1}$  is invertible because

$$\inf\{\text{sp}(\widehat{\Psi}_{m_1})\} \geq \frac{\mathfrak{L}_\varphi(m_1)}{\mathfrak{c}_\Lambda} \cdot \frac{\log(NT)}{NT},$$

where  $\text{sp}(\widehat{\Psi}_{m_1})$  is the spectrum of the random matrix  $\widehat{\Psi}_{m_1}$ , and then  $\widetilde{p}_{\mathbf{m},t}$  is well-defined.

**Theorem 4.3.** *Under Assumption 4.1,*

(1) *There exists a constant  $\mathfrak{c}_{4.3,1} > 0$ , not depending on  $\mathbf{m}$  and  $N$ , such that for every  $t \in (0, T]$ ,*

$$(21) \quad \mathbb{E}(\|\widetilde{p}_{\mathbf{m},t} - p_{I \times J, t}\|_f^2) \leq 9 \min_{\tau \in \mathcal{S}_{\mathbf{m}}} \|\tau - p_{I \times J, t}\|_f^2 + \frac{8m_1 \mathfrak{L}_\psi(m_2)}{N} + \frac{\mathfrak{c}_{4.3,1}(1 + R_f(t))}{N}$$

with  $R_f(t) = R(t) + \|p_{I \times J, t}\|_f^2$  and

$$R(t) = \frac{1}{T} \mathbb{E} \left[ \left( \int_0^T \int_{-\infty}^{\infty} p_t(X_s, y)^2 dy ds \right)^2 \right]^{\frac{1}{2}}.$$

(2) *There exists a constant  $\mathfrak{c}_{4.3,2} > 0$ , not depending on  $\mathbf{m}$  and  $N$ , such that*

$$(22) \quad \frac{1}{T} \int_0^T \mathbb{E}(\|\widetilde{p}_{\mathbf{m},t} - p_{I \times J, t}\|_f^2) dt \leq 9 \min_{\tau \in \mathcal{S}_{\mathbf{m}}} \left\{ \frac{1}{T} \int_0^T \|\tau - p_{I \times J, t}\|_f^2 dt \right\} + \frac{8m_1 \mathfrak{L}_\psi(m_2)}{N} + \frac{\mathfrak{c}_{4.3,2}}{N}.$$

**Sketch of proof.** Let us provide a sketch of the proof of Theorem 4.3, which is detailed in Subsection A.2. In a first step, thanks to the definitions of both  $\Omega_{m_1}$  and the orthogonal projection of  $p_{I \times J, t}$  onto  $\mathcal{S}_{\mathbf{m}}$  for the inner product  $\langle \cdot, \cdot \rangle_f$ , we show that

$$(23) \quad \mathbb{E}(\|\tilde{p}_{\mathbf{m}, t} - p_{I \times J, t}\|_f^2) \lesssim \min_{\tau \in \mathcal{S}_{\mathbf{m}}} \|\tau - p_{I \times J, t}\|_f^2 + A + B + C,$$

where

$$A := \mathbb{E}(\|\widehat{p}_{\mathbf{m}, t} - p_{I \times J, t}\|_N^2 \mathbf{1}_{\Omega_{m_1} \cap \Lambda_{m_1}}),$$

$$B := \mathbb{E}(\|p_{I \times J, t}\|_N^2 \mathbf{1}_{\Omega_{m_1} \cap \Lambda_{m_1}^c}) \quad \text{and} \quad C := \mathbb{E}(\|\tilde{p}_{\mathbf{m}, t} - p_{I \times J, t}\|_f^2 \mathbf{1}_{\Omega_{m_1}^c}).$$

The (first) bias term in the right-hand side of Inequality (23) remains the same as in Inequality (18). In the same way,  $A$  is of order  $m_1 \mathfrak{L}_\psi(m_2)/N + 1/N$  as in Theorem 4.2. Since  $\mathbb{P}(\Lambda_{m_1}^c) \lesssim 1/N^p$  by Comte and Genon-Catalot [14], Lemma 6.1, we get that  $B \lesssim R(t)/N$ . Lastly,  $C$  is controlled by  $(1 + \|p_{I \times J, t}\|_f^2)/N$  by using that

$$\|\widehat{p}_{\mathbf{m}, t}\|_f^2 = \text{trace}(\Psi_{m_1} \widehat{\Theta}_{\mathbf{m}, t} \widehat{\Theta}_{\mathbf{m}, t}^*) \quad \text{and} \quad \mathbb{P}(\Omega_{m_1}^c) \lesssim \frac{1}{N^p}.$$

Inequality (22) is deduced from Inequalities (9) and (21).

#### Remarks:

- (1) When  $I$  is a compact interval, the norms  $\|\cdot\|$  and  $\|\cdot\|_f$  are equivalent by Proposition 3.1.(1,2). In that case, one can deduce a risk bound on  $\tilde{p}_{\mathbf{m}, t}$ , involving  $\|\cdot\|$  and not  $\|\cdot\|_f$ , from Theorem 4.3. This is no longer true when  $I$  is a non-compact interval, and the risk needs to be evaluated via the  $f$ -weighted norm  $\|\cdot\|_f$ .
- (2) By Jensen's inequality, Inequality (10) and the change of variable formula,

$$\begin{aligned} R(t) &\geq \frac{1}{T} \int_0^T \int_{-\infty}^{\infty} \mathbb{E}(p_t(X_s, y)^2) dy ds = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_t(x, y)^2 f(x) dx dy \\ &\geq \frac{\underline{c}_T^2}{t} \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} \exp\left(-2\underline{m}_T \frac{(y - \theta_t(x))^2}{t}\right) dy dx \\ &= \frac{\underline{c}_T^2}{t} \int_{-\infty}^{\infty} f(x) dx \int_{-\infty}^{\infty} \exp\left[-2\underline{m}_T \left(\frac{y}{t^{1/2}}\right)^2\right] dy \\ &= \frac{\underline{c}_T^2}{t^{1/2}} \int_{-\infty}^{\infty} e^{-2\underline{m}_T y^2} dy = \frac{2\underline{c}_T^2 \sqrt{\pi}}{\sqrt{2\underline{m}_T t}} \xrightarrow{t \rightarrow 0^+} +\infty. \end{aligned}$$

Therefore,

$$\sup_{t \in (0, T]} R(t) = +\infty,$$

and this is the reason why Inequality (22) is also provided. Otherwise, only times  $t \in [t_0, T]$ , for some fixed  $t_0 \in (0, T)$ , must be considered.

**4.3. Rates in the anisotropic Sobolev-Hermite spaces.** In order to control the bias term in Theorems 4.2 and 4.3, we assume that  $p_t$  belongs to a Sobolev-Hermite space. In dimension  $d = 1$ , these function spaces have been introduced by Bongioanni and Torrea [5]. The connection between Hermite's coefficients and functional regularity was established later (see Belomestny *et al.* [6]) and are summarized in Comte and Genon-Catalot [12], Appendix A. The definition of these spaces can be extended on  $A := \mathbb{R}^d$  for any  $d \in \mathbb{N}^*$  (see Comte and Lacour [16], Section 2.3).

#### Notations:

- For every  $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$  and  $\mathbf{s} = (s_1, \dots, s_d) \in (0, \infty)^d$ ,  $\mathbf{k}^{\mathbf{s}} := k_1^{s_1} \times \dots \times k_d^{s_d}$ .
- For every  $g \in \mathbb{L}^2(A)$  and  $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$ ,  $a_{\mathbf{k}}(g) := \langle g, h_{k_1} \otimes \dots \otimes h_{k_d} \rangle$ .

Throughout this subsection, both  $\mathcal{S}_{\varphi, m_1}$  and  $\mathcal{S}_{\psi, m_2}$  are generated by the Hermite basis, and then  $I = J = \mathbb{R}$ . First, let us recall the definition of the Sobolev-Hermite ellipsoid on  $A := \mathbb{R}^d$ , of order  $\mathbf{s} = (s_1, \dots, s_d) \in (0, \infty)^d$  and of radius  $L > 0$ .

**Definition 4.4.** (Sobolev-Hermite ellipsoids) The Sobolev-Hermite ellipsoid  $W_{\mathbf{s}}^{(d)}(A, L)$  of order  $\mathbf{s}$  and of radius  $L$  is defined by

$$W_{\mathbf{s}}^{(d)}(A, L) := \left\{ g \in \mathbb{L}^2(A) : \sum_{\mathbf{k} \in \mathbb{N}^d} a_{\mathbf{k}}(g)^2 \mathbf{k}^{\mathbf{s}} \leq L \right\}.$$

Now, by assuming that  $p_t$  belongs to  $W_{\mathbf{s}}^{(2)}(A, L)$ , the bias term in Theorems 4.2 and 4.3 decreases to 0 with polynomial rate. Indeed, noting  $p_{\mathbf{m},t}$  the orthogonal projection of  $p_t$  on  $\mathcal{S}_{\mathbf{m}}$ ,

$$\begin{aligned} \|p_t - p_{\mathbf{m},t}\|^2 &= \sum_{\mathbf{k} \in \mathbb{N}^2: \exists q \in \{1,2\}, k_q \geq m_q} a_{\mathbf{k}}(p_t)^2 \\ &\leq \sum_{q=1}^2 \sum_{\mathbf{k} \in \mathbb{N}^2: k_q \geq m_q} a_{\mathbf{k}}(p_t)^2 k_q^{s_q} k_q^{-s_q} \leq L(m_1^{-s_1} + m_2^{-s_2}) \end{aligned}$$

and then, since  $p_{I \times J, t} = p_t$  and  $f(\cdot) \leq 2\mathbf{c}_T T^{-1/2}$ ,

$$\min_{\tau \in \mathcal{S}_{\mathbf{m}}} \|\tau - p_{I \times J, t}\|_f^2 \leq \frac{2\mathbf{c}_T L}{\sqrt{T}} \|p_t - p_{\mathbf{m},t}\|^2 \leq \frac{2\mathbf{c}_T L}{\sqrt{T}} (m_1^{-s_1} + m_2^{-s_2}).$$

Thanks to this control of the bias term, and since the variance term in Theorem 4.2 is of order  $m_1 \sqrt{m_2}/N$  when  $\mathcal{S}_{\psi, m_2}$  is generated by the Hermite basis (see Subsection 4.1), one can establish the following proposition.

**Proposition 4.5.** Assume that  $p_t$  belongs to  $W_{\mathbf{s}}^{(2)}(A, L)$ , and consider  $\mathbf{m}^* = (m_1^*, m_2^*)$  with

$$m_1^* \propto N^{\frac{s_2}{s_1 s_2 + s_1/2 + s_2}} \quad \text{and} \quad m_2^* \propto N^{\frac{s_1}{s_1 s_2 + s_1/2 + s_2}}.$$

Then,

$$\mathbb{E}(\|\widehat{p}_{\mathbf{m}^*, t} - p_t\|_N^2) = O\left(N^{-\frac{1}{1 + \frac{1}{s_1} + \frac{1}{2s_2}}}\right),$$

provided that  $m_1^*$  satisfies the stability condition (17) in Assumption 4.1.

The proof of Proposition 4.5 follows the same line as Comte and Lacour [16], Proposition 2, and is omitted here. We mention that, for conditional density estimation, Comte and Lacour [16] also prove a lower bound for this rate (see their Theorem 2 in Section 3.5), under some additional conditions. Note that, for instance, other rates may be obtained on Sobolev-Fourier spaces associated with the trigonometric basis.

## 5. MODEL SELECTION

Clearly the values of  $m_1^*$  and  $m_2^*$  depend on  $s_1$  and  $s_2$ , the regularity indexes of the unknown function to estimate. Therefore, these results have theoretical interest to show the consistency and evaluate the rate of the estimators, but cannot be used in practice. In this section, we study a model selection method leading to a data-driven choice of  $m_1$  and  $m_2$ .

**5.1. General case.** In order to introduce an appropriate model selection criterion, throughout this section, the  $\varphi_j$ 's and the  $\psi_\ell$ 's fulfill the following additional assumption.

**Assumption 5.1.** The  $\varphi_j$ 's and the  $\psi_\ell$ 's fulfill the three following conditions:

- (1) For every  $m_1, M_1 \in \{1, \dots, N_T\}$ , if  $M_1 > m_1$ , then  $\mathcal{S}_{\varphi, m_1} \subset \mathcal{S}_{\varphi, M_1}$ .
- (2) For every  $m_2, M_2 \in \{1, \dots, N_T\}$ , if  $M_2 > m_2$ , then  $\mathcal{S}_{\psi, m_2} \subset \mathcal{S}_{\psi, M_2}$ .
- (3) There exists a constant  $\mathbf{c}_\varphi \geq 1$ , not depending on  $N$ , such that

$$\mathfrak{L}_\varphi(m_1) \leq \mathbf{c}_\varphi^2 m_1 ; \forall m_1 \in \{1, \dots, N_T\}.$$

Note that the first two conditions above mean that the two univariate collections of models are nested, which of course does not imply that the product spaces are. For instance, the compactly supported trigonometric basis, and both the non-compactly supported Laguerre and Hermite bases, fulfill Assumption 5.1.

Now, we define a data-driven criterion allowing to select a model  $\widehat{\mathbf{m}}$  making a squared-bias/variance compromise. Such a criterion is a function, to minimize with respect to  $\mathbf{m}$  taken in a finite collection, estimating the  $\mathbb{L}^2$ -risk as the sum of the estimated squared bias of  $\widehat{p}_{\mathbf{m},t}$  penalized by a term having the order of its variance term. On the one hand, the squared bias of  $\widehat{p}_{\mathbf{m},t}$  is estimated by  $\gamma_N(\widehat{p}_{\mathbf{m},t}) = -\|\widehat{p}_{\mathbf{m},t}\|_N^2$ . On the other hand, the order of the variance should be estimated by its bound  $m_1 \mathfrak{L}_\psi(m_2)/N$ , up to a multiplicative constant to calibrate. But in the general case, we need to take a penalty of order  $m_1 \mathfrak{L}_\psi(m_2) \log(N)/N$ , implying a slightly degraded rate of the adaptive estimator. Precisely, we consider

$$(24) \quad \widehat{\mathbf{m}} = \widehat{\mathbf{m}}(t) := \arg \min_{\mathbf{m} \in \widehat{\mathcal{M}}_N} \{-\|\widehat{p}_{\mathbf{m},t}\|_N^2 + 2\kappa \text{pen}(\mathbf{m})\},$$

where  $\kappa \geq \kappa_0$ ,  $\kappa_0 > 0$  is defined later,

$$(25) \quad \text{pen}(\mathbf{m}) := (1 + \log(N)) \frac{m_1 \mathfrak{L}_\psi(m_2)}{N}; \quad \forall \mathbf{m} = (m_1, m_2) \in \{1, \dots, N_T\}^2,$$

and  $\widehat{\mathcal{M}}_N := \mathcal{U}_N \cap (\widehat{\mathcal{V}}_N \times \mathcal{N})$  with  $\mathcal{N} = \{1, \dots, N \wedge N_T\}$ ,

$$\begin{aligned} \mathcal{U}_N &= \{(m_1, m_2) \in \mathcal{N}^2 : m_1 \mathfrak{L}_\psi(m_2) \leq N\}, \\ \widehat{\mathcal{V}}_N &= \left\{ m_1 \in \mathcal{N} : \mathfrak{c}_\varphi^2 m_1 (\|\widehat{\Psi}_{m_1}^{-1}\|_{\text{op}}^2 \vee 1) \leq \mathfrak{d} \frac{NT}{\log(NT)} \right\} \end{aligned}$$

and

$$\mathfrak{d} = \min \left( \frac{1}{8\mathfrak{c}_\varphi^2 T (\|f\|_\infty + (3\mathfrak{c}_\varphi)^{-1} \sqrt{\mathfrak{c}_\Lambda/8})(1+p)}, \frac{\mathfrak{c}_\Lambda}{8} \right) \quad (\mathfrak{c}_\Lambda = \frac{1 - \log(2)}{(1+p)T}).$$

Note that since  $\mathfrak{c}_\varphi$ ,  $\mathfrak{c}_\Lambda$ ,  $p$ ,  $T$  and  $\|f\|_\infty$  are all positive, then  $\mathfrak{d}$  is well-defined. Note also that for every  $\mathbf{m} = (m_1, m_2) \in \mathcal{U}_N$ ,  $m_2$  fulfills the first part (16) of Assumption 4.1 with  $q = 1$  (or larger), and then we take  $p = 11$  (recall that  $p = 2(q+4) + 1$ ) in this section. Consider also the theoretical counterpart  $\mathcal{M}_N := \mathcal{U}_N \cap (\mathcal{V}_N \times \mathcal{N})$  of  $\widehat{\mathcal{M}}_N$ , where

$$\mathcal{V}_N := \left\{ m_1 \in \mathcal{N} : \mathfrak{c}_\varphi^2 m_1 (\|\Psi_{m_1}^{-1}\|_{\text{op}}^2 \vee 1) \leq \frac{\mathfrak{d}}{4} \cdot \frac{NT}{\log(NT)} \right\}.$$

The following theorem provides a risk bound on the adaptive estimator  $\widehat{p}_{\widehat{\mathbf{m}},t}$ .

**Theorem 5.2.** *Under Assumption 5.1, for  $\kappa_0 = 44a$  and  $a \geq (2 \cdot 84 \sqrt{\mathfrak{d}T})^2/2$ ,*

(1) *There exists a constant  $\mathfrak{c}_{5.2,1} > 0$ , not depending on  $N$ , such that for every  $t \in (0, T]$ ,*

$$\mathbb{E}(\|\widehat{p}_{\widehat{\mathbf{m}},t} - p_{I \times J,t}\|_N^2) \leq 6 \min_{\mathbf{m} \in \widehat{\mathcal{M}}_N} \{\mathbb{E}(\|\widehat{p}_{\mathbf{m},t} - p_{I \times J,t}\|_N^2) + \kappa \text{pen}(\mathbf{m})\} + \frac{\mathfrak{c}_{5.2,1}(1 + R(t))}{N}.$$

(2) *There exists a constant  $\mathfrak{c}_{5.2,2} > 0$ , not depending on  $N$ , such that*

$$\frac{1}{T} \int_0^T \mathbb{E}(\|\widehat{p}_{\widehat{\mathbf{m}}(t),t} - p_{I \times J,t}\|_N^2) dt \leq 6 \min_{\mathbf{m} \in \widehat{\mathcal{M}}_N} \left\{ \frac{1}{T} \int_0^T \mathbb{E}(\|\widehat{p}_{\mathbf{m},t} - p_{I \times J,t}\|_N^2) dt + \kappa \text{pen}(\mathbf{m}) \right\} + \frac{\mathfrak{c}_{5.2,2}}{N}.$$

The first result in Theorem 5.2 means that the final estimator  $\widehat{p}_{\widehat{\mathbf{m}},t}$  makes automatically (up to a multiplicative constant which may be taken equal to  $6 + 2\kappa$ ) the bias-variance tradeoff by keeping in mind Inequality (18) which provides a risk bound on  $\mathbb{E}(\|\widehat{p}_{\mathbf{m},t} - p_{I \times J,t}\|_N^2)$  for every  $\mathbf{m} \in \mathcal{M}_N$ .

**Sketch of proof.** Let us provide a sketch of the proof of Theorem 5.2, which is detailed in Subsection A.3. In a first step, by using that  $\widehat{\Pi}_{\mathbf{m}_1}(\widehat{p}_{\mathbf{m}_2,t}) = \widehat{p}_{\mathbf{m}_1,t}$  for every  $\mathbf{m}_1, \mathbf{m}_2 \in \mathcal{N}^2$  satisfying  $\mathcal{S}_{\mathbf{m}_1} \subset \mathcal{S}_{\mathbf{m}_2}$ ,

we show that for every  $\mathbf{m} \in \widehat{\mathcal{M}}_N$ ,

$$(26) \quad \|\widehat{p}_{\widehat{\mathbf{m}},t} - p_{I \times J,t}\|_N^2 \leq 6\|\widehat{p}_{\mathbf{m},t} - p_{I \times J,t}\|_N^2 + 4\kappa \text{pen}(\mathbf{m}) + 11 \left( \|\widehat{p}_{\widehat{\mathbf{m}},t} - \widehat{\Pi}_{\widehat{\mathbf{m}}}(p_t)\|_N^2 - \frac{2}{11} \kappa \text{pen}(\widehat{\mathbf{m}}) \right)_+.$$

Now, consider the event  $\Xi_N := \{\mathcal{M}_N \subset \widehat{\mathcal{M}}_N \subset \mathfrak{M}_N\}$  with  $\mathfrak{M}_N = \mathcal{U}_N \cap (\mathfrak{V}_N \times \mathcal{N})$  and

$$\mathfrak{V}_N = \left\{ m_1 \in \mathcal{N} : \mathfrak{c}_\varphi^2 m_1 (\|\Psi_{m_1}^{-1}\|_{\text{op}}^2 \vee 1) \leq 4\mathfrak{d} \frac{NT}{\log(NT)} \right\},$$

and recall that  $\Omega_{m_1}$  ( $m_1 \in \mathcal{N}$ ) is an event on which the empirical norm  $\|\cdot\|_{N,1}$  and its theoretical counterpart  $\|\cdot\|_{f,1}$  are equivalent for functions in  $\mathcal{S}_{\varphi,m_1}$ . By Comte and Genon-Catalot [14], Lemma 6.1 and Inequality (6.17), the events  $\Xi_N$  and  $\Omega_N := \cap_{m_1 \in \mathfrak{V}_N} \Omega_{m_1}$  have probability near of 1:

$$(27) \quad \mathbb{P}(\Xi_N^c) \lesssim \frac{1}{N^{p-1}} \quad \text{and} \quad \mathbb{P}(\Omega_N^c) \lesssim \frac{1}{N^{p-1}}.$$

Consider also the empirical process  $\nu_N$  defined by

$$\nu_N(\tau) := \frac{1}{NT} \sum_{i=1}^N \int_0^T \left( \tau(X_s^i, X_{s+t}^i) - \int_{-\infty}^{\infty} \tau(X_s^i, y) p_t(X_s^i, y) dy \right) ds; \quad \forall \tau \in \mathcal{S}_{\mathbf{m}}.$$

In a second step, thanks to Equality (19), we deduce from (27) and Inequality (26) that

$$\begin{aligned} \mathbb{E}(\|\widehat{p}_{\widehat{\mathbf{m}},t} - p_{I \times J,t}\|_N^2) &\lesssim \min_{\mathbf{m} \in \mathcal{M}_N} \{ \mathbb{E}(\|\widehat{p}_{\mathbf{m},t} - p_{I \times J,t}\|_N^2) + \kappa \text{pen}(\mathbf{m}) \} \\ &\quad + \sum_{\mathbf{m} \in \mathfrak{M}_N} \mathbb{E} \left[ \left( \sup_{\tau \in \mathcal{S}_{\mathbf{m}}: \|\tau\|_f \leq 2} \nu_N(\tau)^2 - \frac{2}{11} \kappa \text{pen}(\mathbf{m}) \right)_+ \mathbf{1}_{\Xi_N \cap \Omega_N} \right] + \frac{1+R(t)}{N}. \end{aligned}$$

In a third step, we prove that

$$\sum_{\mathbf{m} \in \mathfrak{M}_N} \mathbb{E} \left[ \left( \sup_{\tau \in \mathcal{S}_{\mathbf{m}}: \|\tau\|_f \leq 2} \nu_N(\tau)^2 - \frac{2}{11} \kappa \text{pen}(\mathbf{m}) \right)_+ \mathbf{1}_{\Xi_N \cap \Omega_N} \right] \lesssim \frac{1}{N}$$

thanks to Klein and Rio's extension of Talagrand's inequality (see [23], Theorem 1.2). In a fourth step, we deduce Theorem 5.2.(2) from Theorem 5.2.(1) and Inequality (9).

**5.2. Two special cases:  $t > t_0 > 0$  and compactly supported bases.** This subsection deals with two interesting special cases, where the extra  $\log(N)$  term in the penalty given by (25), can be avoided. First, let us consider  $t \in [t_0, T]$  with a fixed  $t_0 > 0$ . This condition on  $t$  leads to

$$\sup_{(x,y) \in I \times J} p_t(x, y) \leq p_0 := \bar{c}_T t_0^{-\frac{1}{2}} \quad \text{by Inequality (9).}$$

The following model selection criterion, simpler than (24), may be considered:

$$(28) \quad \widetilde{\mathbf{m}} = \widetilde{\mathbf{m}}(t) := \arg \min_{\mathbf{m} \in \widehat{\mathcal{M}}_N} \{ -\|\widehat{p}_{\mathbf{m},t}\|_N^2 + 2\kappa_b \text{pen}_b(\mathbf{m}) \},$$

where  $\kappa_b \geq \kappa_{b,0}$ ,  $\kappa_{b,0} > 0$  is defined later, and

$$\text{pen}_b(\mathbf{m}) := \frac{m_1 \mathfrak{L}_\psi(m_2)}{N}; \quad \forall \mathbf{m} = (m_1, m_2) \in \{1, \dots, N_T\}^2.$$

In the sequel, the map  $(m_1, m_2) \mapsto m_1 \mathfrak{L}_\psi(m_2)$  fulfills the following additional but usual condition.

**Assumption 5.3.** For every  $\xi > 0$ , there exists  $S(\xi) > 0$ , not depending on  $N$ , such that

$$\sum_{1 \leq m_1, m_2 \leq N} \exp(-\xi m_1 \mathfrak{L}_\psi(m_2)) \leq S(\xi) < \infty.$$

Assumption 5.3 means that the complexity of the models collection is not too high. Note that if  $\mathfrak{L}_\psi(m_2) = 1 \vee (\mathbf{c}_\psi m_2)$ , or even if  $\mathfrak{L}_\psi(m_2) = 1 \vee (\mathbf{c}_\psi \sqrt{m_2})$ , then  $(m_1, m_2) \mapsto m_1 \mathfrak{L}_\psi(m_2)$  fulfills Assumption 5.3 because

$$\sum_{m_1=1}^N \exp(-\xi \mathfrak{L}_\psi(m_2))^{m_1} \leq \frac{e^{-\xi \mathfrak{L}_\psi(m_2)}}{1 - e^{-\xi}}; \quad \forall m_2 \in \mathbb{N}^*.$$

So, for instance,  $(m_1, m_2) \mapsto m_1 \mathfrak{L}_\psi(m_2)$  fulfills Assumption 5.3 when  $\mathcal{S}_{\psi, m_2}$  is generated by the trigonometric or the Hermite basis.

**Proposition 5.4.** *Under Assumptions 5.1 and 5.3, for  $\kappa_{b,0} = 16.5$ , there exists a constant  $\mathbf{c}_{5.4} > 0$ , not depending on  $N$ , such that for every  $t \in [t_0, T]$  with  $t_0 > 0$ ,*

$$\mathbb{E}(\|\widehat{p}_{\mathbf{m},t} - p_{I \times J,t}\|_N^2) \leq 6 \min_{\mathbf{m} \in \mathcal{M}_N} \{\mathbb{E}(\|\widehat{p}_{\mathbf{m},t} - p_{I \times J,t}\|_N^2) + \kappa_b \text{pen}_b(\mathbf{m})\} + \frac{\mathbf{c}_{5.4}(1 + t_0^{-1/2})}{N}.$$

Now, let us briefly present the second interesting special case by assuming that  $I$  is a compact interval. As already established in Section 3.1, by Proposition 3.1.(2), there exists  $\underline{\mathbf{m}} > 0$  such that  $f(\cdot) \geq \underline{\mathbf{m}}$  on  $I$ . This implies that  $\|\Psi_{m_1}^{-1}\|_{\text{op}} \leq 1/\underline{\mathbf{m}}$ . So, for  $t \in (0, T]$  (resp.  $t \in [t_0, T]$  with a fixed  $t_0 > 0$ ), the model selection criterion (24) (resp. (28)) may be simplified in another way:

$$\begin{aligned} \widehat{\mathbf{m}}^* &= \arg \min_{\mathbf{m} \in \mathcal{M}_N^*} \{-\|\widehat{p}_{\mathbf{m},t}\|_N^2 + 2\kappa \text{pen}(\mathbf{m})\} \\ &\text{(resp. } \widetilde{\mathbf{m}}^* = \arg \min_{\mathbf{m} \in \mathcal{M}_N^*} \{-\|\widehat{p}_{\mathbf{m},t}\|_N^2 + 2\kappa_b \text{pen}_b(\mathbf{m})\}), \end{aligned}$$

where  $\mathcal{M}_N^* := \mathcal{U}_N \cap (\mathcal{V}_N^* \times \mathcal{N})$  and

$$\mathcal{V}_N^* := \left\{ m_1 \in \mathcal{N} : m_1 \leq \frac{\mathbf{c}_\Lambda}{2\mathbf{c}_\varphi^2(\underline{\mathbf{m}}^{-1} \vee 1)} \cdot \frac{NT}{\log(NT)} \right\}.$$

A result similar to Theorem 5.2 (resp. Proposition 5.4) may be established on the adaptive estimator  $\widehat{p}_{\widehat{\mathbf{m}}^*,t}$  (resp.  $\widehat{p}_{\widetilde{\mathbf{m}}^*,t}$ ) by taking  $\widehat{\mathcal{M}}_N = \mathcal{M}_N = \mathfrak{M}_N := \mathcal{M}_N^*$  in the proof (of Theorem 5.2 (resp. Proposition 5.4)). This means that the collection in which  $\widehat{\mathbf{m}}^*$  or  $\widetilde{\mathbf{m}}^*$  are selected is larger and not random.

## 6. NUMERICAL EXPERIMENTS

We propose a brief simulation study to illustrate our estimation method. The implementation is done using the Hermite basis defined by (14) ( $I = J = \mathbb{R}$ ). The Hermite polynomials are computed thanks to the recursion formula  $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$  with  $H_0(x) = 1$  and  $H_1(x) = x$  (see Abramowitz and Stegun [1], (22.7)).

We fix  $t = 1$ , so we can take the penalty involved in Proposition 5.4:  $\text{pen}_b(\mathbf{m}) = m_1 \sqrt{m_2}/N$ . Moreover, we choose  $\kappa_b = 2$ ; a value obtained from preliminary calibration experiments. A cutoff test excludes the dimensions  $m_1$  such that the largest eigenvalue of  $\widehat{\Psi}_{m_1}^{-1}$  is too large (see Comte and Genon-Catalot [14]).

We simulate discrete samples in three models, obtained from an exact discretization of  $d$ -dimensional Ornstein-Uhlenbeck processes  $U_1, \dots, U_N$ :

$$(29) \quad dU_i(t) = -\frac{r}{2}U_i(t)dt + \frac{\gamma}{2}dW_{i,d}(t), \quad U_i(0) \sim \mathcal{N}_d\left(0, \frac{\gamma^2}{4r}I_d\right),$$

where  $W_{i,d}$  is a  $d$ -dimensional standard Brownian motion. An exact simulation is generated with step  $\Delta > 0$  by computing

$$U_i((k+1)\Delta) = e^{-\frac{r\Delta}{2}}U_i(k\Delta) + \varepsilon_i((k+1)\Delta), \quad \varepsilon_i(k\Delta) \sim_{\text{iid}} \mathcal{N}_d\left(0, \frac{\gamma^2(1 - e^{-r\Delta})}{4r}I_d\right).$$

In all cases, we take  $k \in \{0, \dots, n\}$  with  $n = 1000$ ,  $\Delta = 0.01$ , and as already mentioned, we fix  $t = 1$  in the function  $(x, y) \mapsto p_t(x, y)$  to estimate.

**Example 1.**  $X_i(t) = U_i(t)$ , where  $U_i(t)$  is defined by (29) with  $d = 1$ . The  $X_i$ 's are independent copies of the solution of Equation (1) with

$$b(x) = -\frac{rx}{2}, \quad \sigma(x) = \frac{\gamma}{2}, \quad r = 2 \quad \text{and} \quad \gamma = 2.$$

Here, the transition density function is given by

$$p_t^{(1)}(x, y) = \sqrt{\frac{2r}{\pi\gamma^2(1 - e^{-rt})}} \exp\left(-\frac{2r}{\gamma^2(1 - e^{-rt})}\left(y - xe^{-\frac{rt}{2}}\right)^2\right).$$

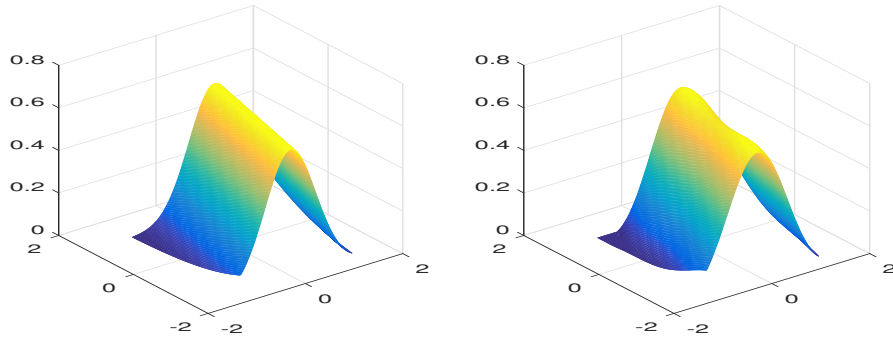


FIGURE 1. Example 1. Transition density (left) and the estimation (right). Selected dimensions (4,5),  $100 \cdot \text{MISE} = 0.22$ .  $N = 200$ ,  $T = 10$ ,  $\Delta = 0.01$ ,  $t = 1$ .

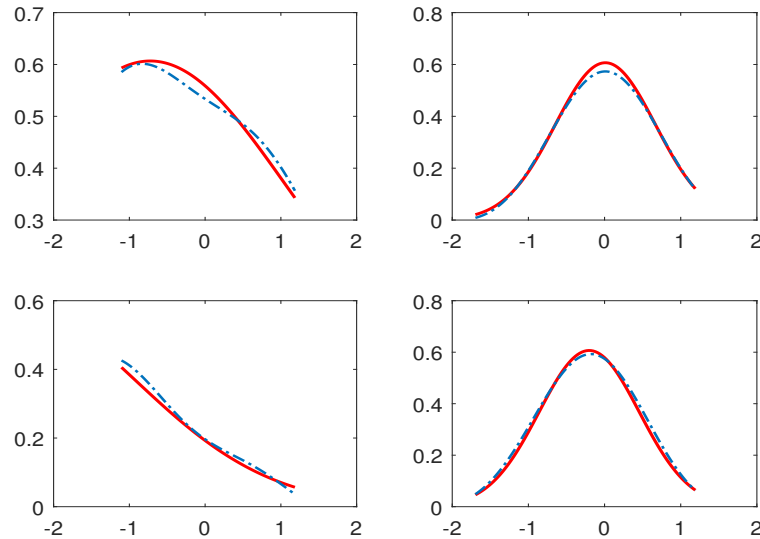


FIGURE 2. Example 1. Full red line, the true and the estimation in dotted blue. Left:  $x \mapsto p_t(x, y)$  for a fixed value of  $y$  ( $y = -0.27$  top and  $y = -1$  bottom). Right:  $y \mapsto p_t(x, y)$  for a fixed value of  $x$  ( $x = 0.03$  top and  $x = -0.55$  bottom).  $N = 200$ ,  $T = 10$ ,  $\Delta = 0.01$ ,  $t = 1$ .

**Example 2.**  $X_i(t) = \tanh(U_i(t))$ , where  $U_i(t)$  is defined by (29) with  $d = 1$ . The  $X_i$ 's are independent



copies of the solution of Equation (1) with

$$b(x) = (1 - x^2) \left( -\frac{r}{2} \operatorname{atanh}(x) - \frac{\gamma^2}{4} x \right), \quad \sigma(x) = \frac{\gamma}{2} (1 - x^2), \quad r = 4 \quad \text{and} \quad \gamma = 1.$$

Here, the transition density function is given by

$$p_t^{(2)}(x, y) = \frac{p_t^{(1)}(\operatorname{atanh}(x), \operatorname{atanh}(y))}{1 - y^2}.$$

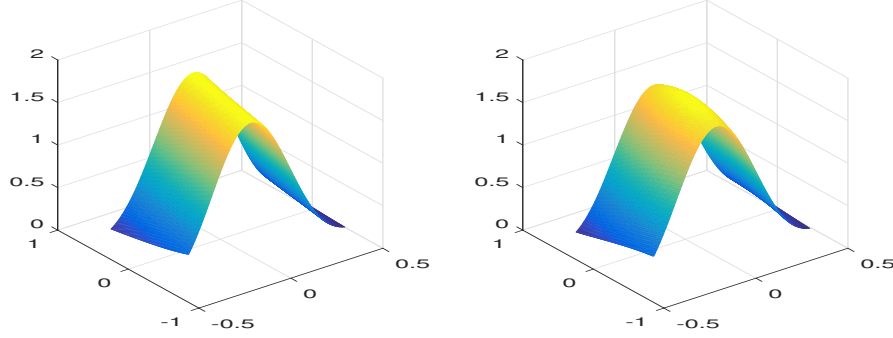


FIGURE 3. Example 2. Transition density (left) and the estimation (right). Selected dimensions (2,41),  $100 \cdot \text{MISE} = 0.16$ .  $N = 200$ ,  $T = 10$ ,  $\Delta = 0.01$ ,  $t = 1$ .

**Example 3.** (Cox-Ingersoll-Ross or square-root process)  $X_i(t) = \|U_i(t)\|_{2,d}^2$ , where  $U_i(t)$  is defined by (29) with  $d = 6$ . The  $X_i$ 's are independent copies of the solution of Equation (1) with

$$b(x) = \frac{d\gamma^2}{4} - rx, \quad \sigma(x) = \gamma\sqrt{x}, \quad r = 1 \quad \text{and} \quad \gamma = 1.$$

Here, the transition density function is given by

$$p_t^{(3)}(x, y) = c_t \exp(-c_t(xe^{-rt} + y)) \times \left( \frac{y}{xe^{-rt}} \right)^{\frac{d}{4} - \frac{1}{2}} \mathcal{I} \left( \frac{d}{2} - 1, 2c_t \sqrt{xye^{-rt}} \right), \quad \text{where} \quad c_t := \frac{2r}{\gamma^2(1 - e^{-rt})}$$

and  $\mathcal{I}(p, x)$  is the modified Bessel function of the first kind of order  $p$  at point  $x$  (see the formula (20) in Aït-Sahalia [2]).

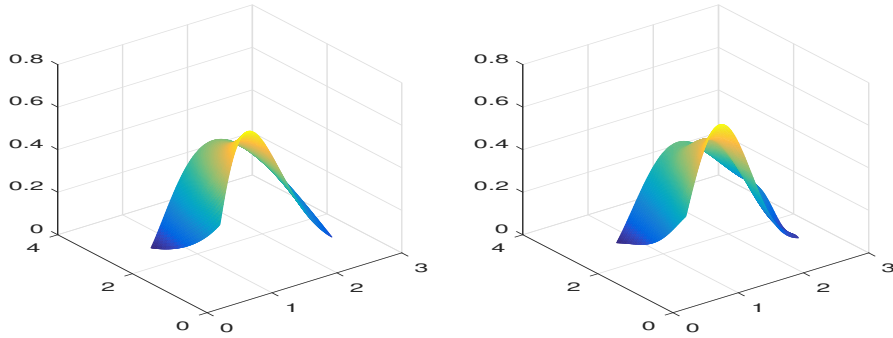


FIGURE 4. Example 3. Transition density (left) and the estimation (right). Selected dimensions (6,9),  $100 \cdot \text{MISE} = 0.29$ .  $N = 200$ ,  $T = 10$ ,  $\Delta = 0.01$ ,  $t = 1$ .

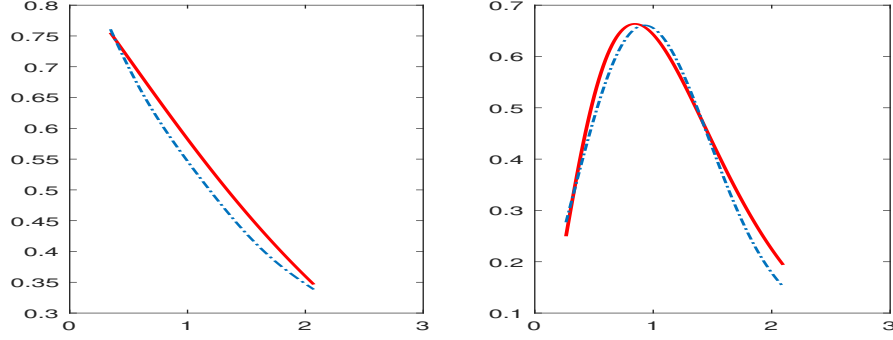


FIGURE 5. Example 3. Full red line, the true and the estimation in dotted blue. Left:  $x \mapsto p_t(x, y)$  for a fixed value of  $y$  ( $y = 0.71$ ). Right:  $y \mapsto p_t(x, y)$  for a fixed value of  $x$  ( $x = 0.76$ ).  $N = 200$ ,  $T = 10$ ,  $\Delta = 0.01$ ,  $t = 1$ .

Model		$N = 100$	$N = 400$	$N = 1000$
1	MISE	1.62 (2.64)	0.48 (0.99)	0.18 (0.40)
	Medians	0.76	0.21	0.10
	Dim (10,12)	(3.05,4.91)	(4.23,5.92)	(5.00,7.00)
3	MISE	6.80 (10.5)	2.14 (5.62)	1.27 (5.62)
	Medians	1.65	0.34	0.19
	dim (12,15)	(5.01, 6.68)	(7.07, 9.07)	(8.94, 12.2)

TABLE 1. Line MISE:  $100 \cdot \text{MISE}$  (with  $100 \cdot$ standard deviation) computed over 200 repetitions, by (30). Line Medians: Median values of  $100 \cdot \text{MISE}$ . Line Dim with maximal proposals for  $(m_1, m_2)$ : means of the selected couples  $(\hat{m}_1, \hat{m}_2)$ .

Figures 1, 3 and 4 show the true surface and the estimated one for Examples 1, 2 and 3 respectively. We see that the *shape* of the estimated transition density is very similar to the true one, and the selected dimensions can be of any orders, especially in the  $y$ -direction (see Figure 3, where the selected couple of dimensions is (2,41)). Figures 2 and 5 represent sections of the curve for fixed values of  $y$  or  $x$  for the same simulated paths as in Figure 1 and 4 respectively. One may notice that the estimation in the  $y$ -direction is better than in the  $x$ -direction.

In Table 1, we compute normalized squared errors for Models 1 and 3, which are defined by

$$(30) \quad \text{MISE} = \frac{\frac{1}{K} \sum_{k=1}^K \frac{DXY^{(k)}}{N_I N_J} \sum_{i=1}^{N_I} \sum_{j=1}^{N_J} (p_t(x_i^{(k)}, y_j^{(k)}) - \hat{p}_t^{(k)}(x_i^{(k)}, y_j^{(k)}))^2}{\frac{DXY^{(K)}}{N_I N_J} \sum_{i=1}^{N_I} \sum_{j=1}^{N_J} p_t^2(x_i^{(K)}, y_j^{(K)})},$$

where

$$DXY^{(k)} := (bX^{(k)} - aX^{(k)})(bY^{(k)} - aY^{(k)}),$$

$bX^{(k)}$  and  $aX^{(k)}$  are the 98% and 2% quantiles of the  $X_t^{(k)}$ 's, and  $bY^{(k)}$  and  $aY^{(k)}$  are the 99% and 1% quantiles of the  $X_{t+1}^{(k)}$ 's. The super-index  $k$  denotes the repetition number, the points  $(x_i^{(k)}, y_j^{(k)})$  are equispaced in the range of the observations of the path  $k$ , and we take  $N_I = N_J = 100$  and  $K = 200$ . We also give the associated standard deviation, together with the mean of the selected dimensions in each direction. To spare time of computation, we've adjusted maximal proposals for  $m_1$  and  $m_2$  to the choices corresponding to each example in such a way that the largest proposal is never selected (i.e. is always

too large).

The results in Table 1 show that, as could be expected, the error is getting smaller when  $N$  increases, and in the same time, the selected dimensions are increasing. This is expected from Proposition 4.5. We also notice that medians are much smaller in all cases than means: this indicates that the performance of the estimation is most of the time much better than what the mean indicates. Probably few bad results deteriorate the mean. The variability can be checked in the results to be unrelated to the selected dimensions, which are quite stable.

We have also implemented the (half)-trigonometric basis, or a product basis with  $\varphi = t$  and  $\psi = h$ . They work well, with some questions on the nature of the theoretical bias term and underlying regularity spaces in the mixed ( $t$  and  $h$ ) case.

## 7. CONCLUDING REMARKS

In this paper, for a fixed  $t \in (0, T]$ , we have proposed a least squares contrast estimator  $\widehat{p}_{\mathbf{m},t}(x, y)$  of the transition density  $p_t(x, y)$  of the solution  $X$  to Equation (1). The estimator is defined through the estimators of the coefficients of its development on a basis of a finite dimensional space  $\mathcal{S}_{\mathbf{m}} = \mathcal{S}_{\varphi, m_1} \otimes \mathcal{S}_{\psi, m_2}$ ,  $\mathbf{m} = (m_1, m_2)$ . In our observation setting,  $N$  independent copies  $X^1, \dots, X^N$  of  $X$  are available. We provide an upper bound on a risk defined as the expectation of the empirical or integrated distance between  $\widehat{p}_{\mathbf{m},t}$  and  $p_t$ , exhibiting a squared-bias/variance decomposition up to negligible terms. An adaptive procedure is then tailored to automatically select the couple  $\mathbf{m} = (m_1, m_2)$ , and it is proved to give good results, both in theory and in practice.

Of course, the simulation part immediately faces the question of handling high frequency but discrete samples and the topic may be developed. However, similarly to what happens in functional data analysis (FDA), small step samples often require a specific study from the beginning and can not be directly deduced from continuous-time constructions. In general, the continuous time results are extended to the high frequency one, up to some constraints linking the sample step and the number of observations. This would be worth being investigated. Note that FDA often considers both the influence of noise and discretization, which would mean here observations, for  $i = 1, \dots, N$ , of  $X^i(t_j) + \eta_{i,j}$  with  $t_j = j\Delta/n$ ,  $j = 1, \dots, n$  and  $\eta_{i,j}$  independent and identically distributed noises with common variance (see Chagny *et al.* [9]). It is not clear if both questions may be simultaneously solved in our setting.

Other questions may be: 1) To study functions related to the conditional density, the main example being the conditional cumulative distribution function; 2) A recent study by Amorino *et al.* [3] defines a concept of local differential privacy in the context of i.i.d. diffusion processes, and it may be worth studying the estimation of  $f$  or  $p_t$  under such confidentiality condition; (3) To extend the results of our paper - Theorems 4.2, 4.3 and 5.2 - to an estimator minimizing a generalized objective function  $\gamma_{d,N}$  for  $d$ -dimensional diffusion processes. As a matter of fact, if  $X$  is a  $d$ -dimensional diffusion process with  $d \in \mathbb{N}^*$ , the objective function  $\gamma_N$  is naturally extended by

$$(31) \quad \gamma_{d,N}(\tau) := \frac{1}{NT} \sum_{i=1}^N \left( \int_0^T \int_{\mathbb{R}^d} \tau(X_s^i, y)^2 dy ds - 2 \int_0^T \tau(X_s^i, X_{s+t}^i) ds \right); \tau \in \mathcal{S}_{\mathbf{m}_1, \dots, \mathbf{m}_d},$$

where  $\mathcal{S}_{\mathbf{m}_1, \dots, \mathbf{m}_d} := \mathcal{S}_{\mathbf{m}_1} \otimes \dots \otimes \mathcal{S}_{\mathbf{m}_d}$  and  $\mathbf{m}_1, \dots, \mathbf{m}_d \in \{1, \dots, N_T\}^2$ . This is out of the scope of our paper, but note that one could get an expression of the solution of the minimization problem  $\min_{\mathcal{S}_{\mathbf{m}_1, \dots, \mathbf{m}_d}} \gamma_{d,N}$  - similar to (13) - involving hyper-matrices in the spirit of Dussap [21]. However, this requires the definition of additional tools, and if the theory can be generalized, the implementation becomes algorithmically difficult and a computer specialist matter.

## APPENDIX A. PROOFS

**A.1. Proof of Theorem 4.2.** The proof of Theorem 4.2 relies on two technical lemmas stated first.

**Lemma A.1.** *Consider the event*

$$\Omega_{m_1} := \left\{ \sup_{\tau \in \mathcal{S}_{\varphi, m_1}} \left| \frac{\|\tau\|_{N,1}^2}{\|\tau\|_{f,1}^2} - 1 \right| \leq \frac{1}{2} \right\}$$

where, for every  $h \in \mathbb{L}^2(\mathbb{R}, f(x)dx)$ ,

$$\|h\|_{N,1}^2 := \frac{1}{NT} \sum_{i=1}^N \int_0^T h(X_s^i)^2 ds \quad \text{and} \quad \|h\|_{f,1}^2 := \int_{-\infty}^{\infty} h(x)^2 f(x) dx.$$

Recall also that  $\Lambda_{m_1}$  is defined by (20). Under Assumption 4.1, there exists a constant  $\mathbf{c}_{A.1} > 0$ , not depending on  $m_1$  and  $N$ , such that

$$\mathbb{P}(\Omega_{m_1}^c) \leq \frac{\mathbf{c}_{A.1}}{N^p} \quad \text{and} \quad \mathbb{P}(\Lambda_{m_1}^c) \leq \frac{\mathbf{c}_{A.1}}{N^p} \quad \text{with} \quad p = 2(q+4) + 1.$$

See Comte and Genon-Catalot [14], Lemma 6.1 for a proof. Now, let us introduce two additional empirical maps:

- The empirical process  $\nu_N$ , defined by

$$\nu_N(\tau) := \frac{1}{NT} \sum_{i=1}^N \int_0^T \left( \tau(X_s^i, X_{s+t}^i) - \int_{-\infty}^{\infty} \tau(X_s^i, y) p_t(X_s^i, y) dy \right) ds$$

for every  $\tau \in \mathcal{S}_{\mathbf{m}}$ . Note that for  $\widehat{Z}_{\mathbf{m},t}$  defined by (12), it holds

$$(32) \quad [\widehat{Z}_{\mathbf{m},t}]_{j,\ell} = \frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X_s^i) \psi_\ell(X_{s+t}^i) ds = \langle p_t, \varphi_j \otimes \psi_\ell \rangle_N + \nu_N(\varphi_j \otimes \psi_\ell)$$

for every  $j \in \{1, \dots, m_1\}$  and  $\ell \in \{1, \dots, m_2\}$ .

- The empirical orthogonal projection  $\widehat{\Pi}_{\mathbf{m}}$ , defined by

$$(33) \quad \widehat{\Pi}_{\mathbf{m}}(\cdot) \in \arg \min_{\tau \in \mathcal{S}_{\mathbf{m}}} \|\tau - \cdot\|_N^2.$$

For any function  $h$  from  $\mathbb{R}^2$  into  $\mathbb{R}$ ,

$$\widehat{\Pi}_{\mathbf{m}}(h) = \sum_{j=1}^{m_1} \sum_{\ell=1}^{m_2} [\widehat{\Psi}_{m_1}^{-1} \widehat{P}_{\mathbf{m}}(h)]_{j,\ell} (\varphi_j \otimes \psi_\ell)$$

with  $\widehat{P}_{\mathbf{m}}(h) = (\langle h, \varphi_j \otimes \psi_\ell \rangle_N)_{(j,\ell) \in \{1, \dots, m_1\} \times \{1, \dots, m_2\}}$ .

**Lemma A.2.** For every  $t \in [0, T]$ ,

$$\|\widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2 = \sup_{\tau \in \mathcal{S}_{\mathbf{m}}: \|\tau\|_N = 1} \nu_N(\tau)^2.$$

The proof of Lemma A.2 is postponed to Subsubsection A.1.2.

A.1.1. *Steps of the proof.* First of all, by the definition of  $\widehat{\Pi}_{\mathbf{m}}$  (see (33)),

$$\|\widehat{p}_{\mathbf{m},t} - p_{I \times J,t}\|_N^2 = \min_{\tau \in \mathcal{S}_{\mathbf{m}}} \|\tau - p_{I \times J,t}\|_N^2 + \|\widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2,$$

and by Lemma A.2,

$$\begin{aligned} \mathbb{E}(\|\widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2) &= \mathbb{E} \left( \mathbf{1}_{\Omega_{m_1}} \sup_{\tau \in \mathcal{S}_{\mathbf{m}}: \|\tau\|_N = 1} \nu_N(\tau)^2 \right) + \mathbb{E} \left( \mathbf{1}_{\Omega_{m_1}^c} \sup_{\tau \in \mathcal{S}_{\mathbf{m}}: \|\tau\|_N = 1} \nu_N(\tau)^2 \right) \\ &=: A + B. \end{aligned}$$

Let us find suitable bounds on  $A$  and  $B$ .

**Step 1 (bound on  $A$ ).** First, since

$$\|\cdot\|_{f,1}^2 \mathbf{1}_{\Omega_{m_1}} \leq 2 \|\cdot\|_{N,1}^2 \mathbf{1}_{\Omega_{m_1}} \quad \text{on} \quad \mathcal{S}_{\varphi, m_1}$$

by the definition of  $\Omega_{m_1}$ , for every  $\tau \in \mathcal{S}_{\mathbf{m}}$ ,

$$\begin{aligned} \|\tau\|_f^2 \mathbf{1}_{\Omega_{m_1}} &= \int_{-\infty}^{\infty} \|\tau(\cdot, y)\|_{f,1}^2 \mathbf{1}_{\Omega_{m_1}} dy \\ &\leq 2 \int_{-\infty}^{\infty} \|\tau(\cdot, y)\|_{N,1}^2 \mathbf{1}_{\Omega_{m_1}} dy = 2\|\tau\|_N^2 \mathbf{1}_{\Omega_{m_1}}. \end{aligned}$$

Then,

$$\{\tau \in \mathcal{S}_{\mathbf{m}} : \|\tau\|_N = 1\} \subset \{\tau \in \mathcal{S}_{\mathbf{m}} : \|\tau\|_f^2 \leq 2\} \quad \text{on } \Omega_{m_1},$$

leading to

$$A \leq \mathbb{E} \left( \sup_{\tau \in \mathcal{S}_{\mathbf{m}} : \|\tau\|_f^2 \leq 2} \nu_N(\tau)^2 \right).$$

Since  $(\varphi_1, \dots, \varphi_{m_1})$  is an orthonormal family of  $\mathbb{L}^2(\mathbb{R}, dx)$ ,  $\varphi_1, \dots, \varphi_{m_1}$  are linearly independent, and one may consider the basis  $(\varphi_1^f, \dots, \varphi_{m_1}^f)$  of  $\mathcal{S}_{\varphi, m_1}$ , orthonormal in  $\mathbb{L}^2(\mathbb{R}, f(x)dx)$ , obtained from  $(\varphi_1, \dots, \varphi_{m_1})$  via the Gram-Schmidt process. Thus,  $(\varphi_j^f \otimes \psi_\ell)_{j,\ell}$  is an orthonormal basis of  $\mathcal{S}_{\mathbf{m}}$  equipped with  $\langle \cdot, \cdot \rangle_f$ , and denoting by  $\|\cdot\|_{2,\mathbf{m}}$  the Fröbenius norm on  $\mathcal{M}_{m_1, m_2}(\mathbb{R})$ ,

$$A \leq \mathbb{E} \left[ \sup_{\Theta : \|\Theta\|_{2,\mathbf{m}}^2 \leq 2} \left( \sum_{j=1}^{m_1} \sum_{\ell=1}^{m_2} \Theta_{j,\ell} \nu_N(\varphi_j^f \otimes \psi_\ell) \right)^2 \right] \leq 2\mathbb{E} \left( \sum_{j=1}^{m_1} \sum_{\ell=1}^{m_2} \nu_N(\varphi_j^f \otimes \psi_\ell)^2 \right).$$

Now, note that for every  $j \in \{1, \dots, m_1\}$  and  $\ell \in \{1, \dots, m_2\}$ ,

$$\begin{aligned} \mathbb{E}(\nu_N(\varphi_j^f \otimes \psi_\ell)) &= \frac{1}{NT} \mathbb{E} \left( \sum_{i=1}^N \int_0^T \varphi_j^f(X_s^i) (\psi_\ell(X_{s+t}^i) - \mathbb{E}(\psi_\ell(X_{s+t}^i) | X_s^i)) ds \right) \\ &= \frac{1}{T} \int_0^T \mathbb{E}(\varphi_j^f(X_s) \psi_\ell(X_{s+t}) - \mathbb{E}(\varphi_j^f(X_s) \psi_\ell(X_{s+t}) | X_s)) ds = 0. \end{aligned}$$

Therefore, since  $\|\varphi_j^f\|_f = 1$  for every  $j \in \{1, \dots, m_1\}$ ,

$$\begin{aligned} A &\leq 2 \sum_{j=1}^{m_1} \sum_{\ell=1}^{m_2} \text{var}(\nu_N(\varphi_j^f \otimes \psi_\ell)) \\ &\leq \frac{2}{N} \sum_{j=1}^{m_1} \sum_{\ell=1}^{m_2} \mathbb{E} \left[ \left( \frac{1}{T} \int_0^T \varphi_j^f(X_s) (\psi_\ell(X_{s+t}) - \mathbb{E}(\psi_\ell(X_{s+t}) | X_s)) ds \right)^2 \right] \\ &\leq \frac{2}{NT} \sum_{j=1}^{m_1} \sum_{\ell=1}^{m_2} \int_0^T (\mathbb{E}(\varphi_j^f(X_s)^2 \psi_\ell(X_{s+t})^2) + \mathbb{E}(\varphi_j^f(X_s)^2 \mathbb{E}(\psi_\ell(X_{s+t}) | X_s)^2) \\ &\quad - 2\mathbb{E}(\varphi_j^f(X_s) \psi_\ell(X_{s+t}) \mathbb{E}(\varphi_j^f(X_s) \psi_\ell(X_{s+t}) | X_s))) ds \\ &\leq \frac{2}{NT} \sum_{j=1}^{m_1} \sum_{\ell=1}^{m_2} \int_0^T \mathbb{E}(\varphi_j^f(X_s)^2 \psi_\ell(X_{s+t})^2) ds \leq \frac{2m_1 \mathfrak{L}_\psi(m_2)}{N}. \end{aligned}$$

**Step 2 (bound on B).** Since  $\varphi_1, \dots, \varphi_{m_1}$  are linearly independent as mentioned in Step 1, one may consider the basis  $(\varphi_1^N, \dots, \varphi_{m_1}^N)$  of  $\mathcal{S}_{\varphi, m_1}$ , orthonormal for the empirical inner product  $\langle \cdot, \cdot \rangle_{N,1}$ , obtained from  $(\varphi_1, \dots, \varphi_{m_1})$  via the Gram-Schmidt process. Then,  $(\varphi_j^N \otimes \psi_\ell)_{j,\ell}$  is an orthonormal basis of  $\mathcal{S}_{\mathbf{m}}$

equipped with  $\langle \cdot, \cdot \rangle_N$ , and by Cauchy-Schwarz's and Jensen's inequalities,

$$\begin{aligned}
\sup_{\tau \in \mathcal{S}_{\mathbf{m}}: \|\tau\|_N=1} \nu_N(\tau)^2 &= \sup_{\Theta: \|\Theta\|_{2,\mathbf{m}}^2=1} \left( \sum_{j=1}^{m_1} \sum_{\ell=1}^{m_2} \Theta_{j,\ell} \nu_N(\varphi_j^N \otimes \psi_\ell) \right)^2 \\
&\leq \sum_{j=1}^{m_1} \sum_{\ell=1}^{m_2} \left( \frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j^N(X_s^i) (\psi_\ell(X_{s+t}^i) - \mathbb{E}(\psi_\ell(X_{s+t}^i) | X_s^i)) ds \right)^2 \\
(34) \quad &\leq 4\mathfrak{L}_\psi(m_2) \underbrace{\sum_{j=1}^{m_1} \left( \frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j^N(X_s^i)^2 ds \right)}_{=1} = 4m_1 \mathfrak{L}_\psi(m_2).
\end{aligned}$$

Therefore, by Assumption 4.1 and Lemma A.1, there exists a constant  $\mathbf{c}_1 > 0$ , not depending on  $\mathbf{m}$ ,  $N$  and  $t$ , such that

$$B \leq 4m_1 \mathfrak{L}_\psi(m_2) \mathbb{P}(\Omega_{m_1}^c) \leq \mathbf{c}_1 N^{-1}.$$

**Step 3 (conclusion).** By the two previous steps,

$$\mathbb{E}(\|\widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2) \leq \frac{2m_1 \mathfrak{L}_\psi(m_2)}{N} + \frac{\mathbf{c}_1}{N}.$$

Thus,

$$\begin{aligned}
\mathbb{E}(\|\widehat{p}_{\mathbf{m},t} - p_{I \times J, t}\|_N^2) &\leq \min_{\tau \in \mathcal{S}_{\mathbf{m}}} \mathbb{E}(\|\tau - p_{I \times J, t}\|_N^2) + \mathbb{E}(\|\widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2) \\
&\leq \min_{\tau \in \mathcal{S}_{\mathbf{m}}} \|\tau - p_{I \times J, t}\|_f^2 + \frac{2m_1 \mathfrak{L}_\psi(m_2)}{N} + \frac{\mathbf{c}_1}{N}.
\end{aligned}$$

□

A.1.2. *Proof of Lemma A.2.* First, by Cauchy-Schwarz's inequality,

$$\sup_{\tau \in \mathcal{S}_{\mathbf{m}}: \|\tau\|_N=1} \langle \widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t), \tau \rangle_N^2 \leq \|\widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2,$$

and since

$$\tau^* := \frac{\widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t)}{\|\widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N} \in \mathcal{S}_{\mathbf{m}}$$

satisfies both  $\|\tau^*\|_N = 1$  and

$$\langle \widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t), \tau^* \rangle_N^2 = \|\widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2,$$

then

$$(35) \quad \|\widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2 = \sup_{\tau \in \mathcal{S}_{\mathbf{m}}: \|\tau\|_N=1} \langle \widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t), \tau \rangle_N^2.$$

Now, let us show that

$$(36) \quad \langle \widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t), \tau \rangle_N = \nu_N(\tau) ; \forall \tau \in \mathcal{S}_{\mathbf{m}}.$$

By the decompositions of  $\widehat{p}_{\mathbf{m},t}$  and  $\widehat{\Pi}_{\mathbf{m}}(p_t)$  in the basis  $(\varphi_j \otimes \psi_\ell)_{j,\ell}$  of  $\mathcal{S}_{\mathbf{m}}$ , and by (32),

$$\widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t) = \sum_{j=1}^{m_1} \sum_{\ell=1}^{m_2} [\widehat{\Psi}_{m_1}^{-1}(\widehat{Z}_{\mathbf{m},t} - \widehat{P}_{\mathbf{m}}(p_t))]_{j,\ell} (\varphi_j \otimes \psi_\ell) = \sum_{j=1}^{m_1} \sum_{\ell=1}^{m_2} [\widehat{\Psi}_{m_1}^{-1} \widehat{\Delta}_{\mathbf{m}}]_{j,\ell} (\varphi_j \otimes \psi_\ell),$$

where

$$\begin{aligned}
\widehat{\Delta}_{\mathbf{m}} &:= \left( \frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X_s^i) \psi_\ell(X_{s+t}^i) ds \right. \\
&\quad \left. - \frac{1}{NT} \sum_{i=1}^N \int_0^T \int_{-\infty}^{\infty} \varphi_j(X_s^i) \psi_\ell(y) p_t(X_s^i, y) dy ds \right)_{j,\ell} = (\nu_N(\varphi_j \otimes \psi_\ell))_{j,\ell}.
\end{aligned}$$

Then, for any  $\tau = \sum_{j,\ell} \Theta_{j,\ell}(\varphi_j \otimes \psi_\ell)$  with  $\Theta \in \mathcal{M}_{m_1, m_2}(\mathbb{R})$ ,

$$\langle \widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t), \tau \rangle_N = \sum_{j,j'=1}^{m_1} \sum_{\ell,\ell'=1}^{m_2} [\widehat{\Psi}_{m_1}^{-1} \widehat{\Delta}_{\mathbf{m}}]_{j,\ell} \Theta_{j',\ell'} \langle \varphi_j \otimes \psi_\ell, \varphi_{j'} \otimes \psi_{\ell'} \rangle_N$$

As

$$\langle \varphi_j \otimes \psi_\ell, \varphi_{j'} \otimes \psi_{\ell'} \rangle_N = \frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X_s^i) \varphi_{j'}(X_s^i) ds \int_{-\infty}^{\infty} \psi_\ell(y) \psi_{\ell'}(y) dy = [\widehat{\Psi}_{m_1}]_{j,j'} \delta_{\ell,\ell'}$$

we get

$$\begin{aligned} \langle \widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t), \tau \rangle_N &= \sum_{j,j'=1}^{m_1} \sum_{\ell,\ell'=1}^{m_2} [\widehat{\Psi}_{m_1}^{-1} \widehat{\Delta}_{\mathbf{m}}]_{j,\ell} \Theta_{j',\ell'} [\widehat{\Psi}_{m_1}]_{j,j'} \delta_{\ell,\ell'} \\ &= \sum_{j'=1}^{m_1} \sum_{\ell=1}^{m_2} \Theta_{j',\ell} \underbrace{\sum_{j=1}^{m_1} [\widehat{\Psi}_{m_1}]_{j,j'} [\widehat{\Psi}_{m_1}^{-1} \widehat{\Delta}_{\mathbf{m}}]_{j,\ell}}_{=[\widehat{\Delta}_{\mathbf{m}}]_{j',\ell}} \\ &= \sum_{j'=1}^{m_1} \sum_{\ell=1}^{m_2} \Theta_{j',\ell} \nu_N(\varphi_{j'} \otimes \psi_\ell) = \nu_N(\tau). \end{aligned}$$

Therefore, by Equalities (35) and (36) together,

$$\|\widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2 = \sup_{\tau \in \mathcal{S}_{\mathbf{m}}: \|\tau\|_N=1} \nu_N(\tau)^2.$$

□

**A.2. Proof of Theorem 4.3.** First of all,

$$\begin{aligned} \mathbb{E}(\|\widetilde{p}_{\mathbf{m},t} - p_{I \times J, t}\|_f^2) &= \mathbb{E}(\|\widetilde{p}_{\mathbf{m},t} - p_{I \times J, t}\|_f^2 \mathbf{1}_{\Omega_{m_1}}) + \mathbb{E}(\|\widetilde{p}_{\mathbf{m},t} - p_{I \times J, t}\|_f^2 \mathbf{1}_{\Omega_{m_1}^c}) \\ &=: A + B. \end{aligned}$$

Let us find suitable bounds on  $A$  and  $B$ .

**Step 1 (bound on  $A$ ).** As already mentioned, since

$$\|\cdot\|_{f,1}^2 \mathbf{1}_{\Omega_{m_1}} \leq 2 \|\cdot\|_{N,1}^2 \mathbf{1}_{\Omega_{m_1}} \quad \text{on } \mathcal{S}_{\varphi, m_1}$$

by the definition of  $\Omega_{m_1}$ , for every  $\tau \in \mathcal{S}_{\mathbf{m}}$ ,

$$\begin{aligned} \|\tau\|_f^2 \mathbf{1}_{\Omega_{m_1}} &= \int_{-\infty}^{\infty} \|\tau(\cdot, y)\|_{f,1}^2 \mathbf{1}_{\Omega_{m_1}} dy \\ &\leq 2 \int_{-\infty}^{\infty} \|\tau(\cdot, y)\|_{N,1}^2 \mathbf{1}_{\Omega_{m_1}} dy = 2 \|\tau\|_N^2 \mathbf{1}_{\Omega_{m_1}}. \end{aligned}$$

Then, noting  $p_{I \times J, t}^f$  as the orthogonal projection of  $p_{I \times J, t}$  onto  $\mathcal{S}_{\mathbf{m}}$  for the inner product  $\langle \cdot, \cdot \rangle_f$ ,

$$\begin{aligned} \|\widetilde{p}_{\mathbf{m},t} - p_{I \times J, t}\|_f^2 \mathbf{1}_{\Omega_{m_1}} &= (\|\widetilde{p}_{\mathbf{m},t} - p_{I \times J, t}^f\|_f^2 + \|p_{I \times J, t}^f - p_{I \times J, t}\|_f^2) \mathbf{1}_{\Omega_{m_1}} \\ &\leq \|p_{I \times J, t}^f - p_{I \times J, t}\|_f^2 + 2 \|\widetilde{p}_{\mathbf{m},t} - p_{I \times J, t}^f\|_N^2 \mathbf{1}_{\Omega_{m_1}} \\ &\leq \min_{\tau \in \mathcal{S}_{\mathbf{m}}} \|\tau - p_{I \times J, t}\|_f^2 + 4 \|\widetilde{p}_{\mathbf{m},t} - p_{I \times J, t}\|_N^2 + 4 \|p_{I \times J, t} - p_{I \times J, t}^f\|_N^2. \end{aligned}$$

Moreover,

$$\|\widetilde{p}_{\mathbf{m},t} - p_{I \times J, t}\|_N^2 \leq \|\widehat{p}_{\mathbf{m},t} - p_{I \times J, t}\|_N^2 + \|p_{I \times J, t}\|_N^2 \mathbf{1}_{\Lambda_{m_1}^c}$$

and, by Lemma A.1,

$$\begin{aligned} \mathbb{E}(\|p_{I \times J, t}\|_N^2 \mathbf{1}_{\Lambda_{m_1}^c}) &\leq \mathbb{E} \left[ \left( \frac{1}{NT} \sum_{i=1}^N \int_0^T \int_{-\infty}^{\infty} p_t(X_s^i, y)^2 dy ds \right)^2 \right]^{\frac{1}{2}} \mathbb{P}(\Lambda_{m_1}^c)^{\frac{1}{2}} \\ &\leq \mathbf{c}_{A.1}^{\frac{1}{2}} \frac{R(t)}{N^{p/2}} \quad \text{with} \quad R(t) = \frac{1}{T} \mathbb{E} \left[ \left( \int_0^T \int_{-\infty}^{\infty} p_t(X_s, y)^2 dy ds \right)^2 \right]^{\frac{1}{2}}. \end{aligned}$$

Therefore, by Theorem 4.2, there exists a constant  $\mathbf{c}_1 > 0$ , not depending on  $\mathbf{m}$ ,  $N$  and  $t$ , such that

$$\begin{aligned} A &\leq 5 \min_{\tau \in \mathcal{S}_{\mathbf{m}}} \|\tau - p_{I \times J, t}\|_f^2 + 4\mathbb{E}(\|\tilde{p}_{\mathbf{m}, t} - p_{I \times J, t}\|_N^2) \\ &\leq 9 \min_{\tau \in \mathcal{S}_{\mathbf{m}}} \|\tau - p_{I \times J, t}\|_f^2 + \frac{8m_1 \mathfrak{L}_{\psi}(m_2)}{N} + \frac{\mathbf{c}_1(1 + R(t))}{N}. \end{aligned}$$

**Step 2 (bound on  $B$ ).** Since  $\hat{\Theta}_{\mathbf{m}, t} \hat{\Theta}_{\mathbf{m}, t}^*$  is a positive semidefinite (symmetric) matrix,

$$\begin{aligned} \|\hat{p}_{\mathbf{m}, t}\|_f^2 &= \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} \left( \sum_{j=1}^{m_1} \sum_{\ell=1}^{m_2} [\hat{\Theta}_{\mathbf{m}, t}]_{j, \ell} \varphi_j(x) \psi_{\ell}(y) \right)^2 dy dx \\ &= \int_{-\infty}^{\infty} f(x) \sum_{j, j'=1}^{m_1} \sum_{\ell, \ell'=1}^{m_2} [\hat{\Theta}_{\mathbf{m}, t}]_{j, \ell} [\hat{\Theta}_{\mathbf{m}, t}]_{j', \ell'} \varphi_j(x) \varphi_{j'}(x) \int_{-\infty}^{\infty} \psi_{\ell}(y) \psi_{\ell'}(y) dy dx \end{aligned}$$

Now using that  $\int_{-\infty}^{\infty} \psi_{\ell}(y) \psi_{\ell'}(y) dy = \delta_{\ell, \ell'}$ , we get

$$\begin{aligned} \|\hat{p}_{\mathbf{m}, t}\|_f^2 &= \sum_{j, j'=1}^{m_1} \sum_{\ell=1}^{m_2} [\hat{\Theta}_{\mathbf{m}, t}]_{j, \ell} [\hat{\Theta}_{\mathbf{m}, t}]_{j', \ell} \int_{-\infty}^{\infty} \varphi_j(x) \varphi_{j'}(x) f(x) dx \\ &= \sum_{j=1}^{m_1} \sum_{\ell=1}^{m_2} [\hat{\Theta}_{\mathbf{m}, t}]_{\ell, j} \underbrace{\sum_{j'=1}^{m_1} [\Psi_{m_1}]_{j, j'} [\hat{\Theta}_{\mathbf{m}, t}]_{j', \ell}}_{=[\Psi_{m_1} \hat{\Theta}_{\mathbf{m}, t}]_{j, \ell}} \text{ as } \int_{-\infty}^{\infty} \varphi_j(x) \varphi_{j'}(x) f(x) dx = [\Psi_{m_1}]_{j, j'} \\ &= \sum_{j=1}^{m_1} [\Psi_{m_1} \hat{\Theta}_{\mathbf{m}, t} \hat{\Theta}_{\mathbf{m}, t}^*]_{j, j} = \text{trace}(\Psi_{m_1} \hat{\Theta}_{\mathbf{m}, t} \hat{\Theta}_{\mathbf{m}, t}^*) \leq \|\Psi_{m_1}\|_{\text{op}} \text{trace}(\hat{\Theta}_{\mathbf{m}, t} \hat{\Theta}_{\mathbf{m}, t}^*). \end{aligned}$$

Let us find suitable controls on  $\|\Psi_{m_1}\|_{\text{op}}^2$  and  $\mathbb{E}(\text{trace}(\hat{\Theta}_{\mathbf{m}, t} \hat{\Theta}_{\mathbf{m}, t}^*)^2)$ . On the one hand, by Cauchy-Schwarz's and Jensen's inequalities,

$$\begin{aligned} \|\Psi_{m_1}\|_{\text{op}}^2 &= \sup_{y: \|y\|_{2, m_1} = 1} \sum_{j=1}^{m_1} \left( \sum_{j'=1}^{m_1} y_{j'} \int_{-\infty}^{\infty} \varphi_j(x) \varphi_{j'}(x) f(x) dx \right)^2 \\ &\leq \sum_{j, j'=1}^{m_1} \left( \int_{-\infty}^{\infty} \varphi_j(x) \varphi_{j'}(x) f(x) dx \right)^2 \\ &\leq \sum_{j, j'=1}^{m_1} \int_{-\infty}^{\infty} \varphi_j(x)^2 \varphi_{j'}(x)^2 f(x) dx \leq \mathfrak{L}_{\varphi}(m_1)^2 \underbrace{\int_{-\infty}^{\infty} f(x) dx}_{=1}. \end{aligned}$$

On the other hand, since  $\hat{Z}_{\mathbf{m}, t} \hat{Z}_{\mathbf{m}, t}^*$  is a positive semidefinite (symmetric) matrix,

$$\text{trace}(\hat{\Theta}_{\mathbf{m}, t} \hat{\Theta}_{\mathbf{m}, t}^*) = \text{trace}((\hat{\Psi}_{m_1}^{-1})^2 \hat{Z}_{\mathbf{m}, t} \hat{Z}_{\mathbf{m}, t}^*) \leq \|\hat{\Psi}_{m_1}^{-1}\|_{\text{op}}^2 \text{trace}(\hat{Z}_{\mathbf{m}, t} \hat{Z}_{\mathbf{m}, t}^*).$$



Moreover,

$$\begin{aligned} \mathbb{E}(\text{trace}(\widehat{Z}_{\mathbf{m},t}\widehat{Z}_{\mathbf{m},t}^*)) &= \mathbb{E}\left[\left(\sum_{j=1}^{m_1}\sum_{\ell=1}^{m_2}[\widehat{Z}_{\mathbf{m},t}]_{j,\ell}^2\right)^2\right] \\ &\leq \frac{m_1m_2}{N^4T^4}\sum_{j=1}^{m_1}\sum_{\ell=1}^{m_2}\mathbb{E}\left[\left(\sum_{i=1}^N\int_0^T\varphi_j(X_s^i)\psi_\ell(X_{s+t}^i)ds\right)^4\right] \leq m_1m_2\mathfrak{L}_\varphi(m_1)^2\mathfrak{L}_\psi(m_2)^2 \end{aligned}$$

and, on the event  $\Lambda_{m_1}$ ,

$$\|\widehat{\Psi}_{m_1}^{-1}\|_{\text{op}}^4 \leq \mathfrak{c}_\Lambda^4 \frac{N^4T^4}{\log(NT)^4\mathfrak{L}_\varphi(m_1)^4}.$$

Then, there exists a constant  $\mathfrak{c}_2 > 0$ , not depending on  $\mathbf{m}$ ,  $N$  and  $t$ , such that

$$\mathbb{E}(\|\widetilde{p}_{\mathbf{m},t}\|_f^4) \leq \mathfrak{c}_2N^4m_1m_2\mathfrak{L}_\psi(m_2)^2.$$

So, by Lemma A.1, there exists a constant  $\mathfrak{c}_3 > 0$ , not depending on  $\mathbf{m}$ ,  $N$  and  $t$ , such that

$$\begin{aligned} B &\leq \mathbb{E}(\|\widetilde{p}_{\mathbf{m},t} - p_{I \times J,t}\|_f^4)^{\frac{1}{2}}\mathbb{P}(\Omega_{m_1}^c)^{\frac{1}{2}} \\ &\leq \mathfrak{c}_3(1 + \|p_{I \times J,t}\|_f^2)N^{3+q-\frac{p}{2}} \leq \frac{\mathfrak{c}_3(1 + \|p_{I \times J,t}\|_f^2)}{N} \quad \text{because } p = 2(q+4) + 1 \text{ in Lemma A.1.} \end{aligned}$$

**Step 3 (conclusion).** By the two previous steps, there exists a constant  $\mathfrak{c}_4 > 0$ , not depending on  $\mathbf{m}$ ,  $N$  and  $t$ , such that

$$\mathbb{E}(\|\widetilde{p}_{\mathbf{m},t} - p_{I \times J,t}\|_f^2) \leq 9 \min_{\tau \in S_{\mathbf{m}}} \|\tau - p_{I \times J,t}\|_f^2 + \frac{8m_1\mathfrak{L}_\psi(m_2)}{N} + \frac{\mathfrak{c}_4(1 + R_f(t))}{N}$$

with  $R_f(t) = R(t) + \|p_{I \times J,t}\|_f^2$ . Moreover, by Inequality (9), and since  $y \mapsto p_t(x, y)$  is a density function for every  $x \in \mathbb{R}$ ,

$$\begin{aligned} R(t) &= \frac{1}{T}\mathbb{E}\left[\left(\int_0^T\int_{-\infty}^{\infty}p_t(X_s, y)^2dyds\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{T}\mathbb{E}\left[\left(\sup_{(u,v) \in \mathbb{R}^2}p_t(u, v)\int_0^T\int_{-\infty}^{\infty}p_t(X_s, y)dyds\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{\bar{\mathfrak{c}}_T}{Tt^{1/2}}\mathbb{E}\left[\left(\int_0^T\int_{-\infty}^{\infty}p_t(X_s, y)dyds\right)^2\right]^{\frac{1}{2}} = \bar{\mathfrak{c}}_Tt^{-\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} \|p_{I \times J,t}\|_f^2 &= \int_I\int_Jp_t(x, y)^2f(x)dx dy \\ &\leq \sup_{(u,v) \in \mathbb{R}^2}p_t(u, v)\int_{-\infty}^{\infty}f(x)\int_{-\infty}^{\infty}p_t(x, y)dy dx \\ &\leq \bar{\mathfrak{c}}_Tt^{-\frac{1}{2}}\int_{-\infty}^{\infty}f(x)\int_{-\infty}^{\infty}p_t(x, y)dy dx = \bar{\mathfrak{c}}_Tt^{-\frac{1}{2}}, \end{aligned}$$

leading to

$$\frac{1}{T}\int_0^TR_f(t)dt \leq 4\bar{\mathfrak{c}}_TT^{-\frac{1}{2}}.$$

In conclusion,

$$\frac{1}{T}\int_0^T\mathbb{E}(\|\widetilde{p}_{\mathbf{m},t} - p_{I \times J,t}\|_f^2)dt \leq 9 \min_{\tau \in S_{\mathbf{m}}}\left\{\frac{1}{T}\int_0^T\|\tau - p_{I \times J,t}\|_f^2dt\right\} + \frac{8m_1\mathfrak{L}_\psi(m_2)}{N} + \frac{\mathfrak{c}_4(1 + 4\bar{\mathfrak{c}}_TT^{-1/2})}{N}.$$

□

**A.3. Proof of Theorem 5.2.** The proof of Theorem 5.2 relies on the two following technical lemmas.

**Lemma A.3.** *Under Assumption 5.1, for any  $\mathbf{m} = (m_1, m_2)$  and  $\mathbf{M} = (M_1, M_2)$  belonging to  $\mathcal{N}^2$ , if  $\mathcal{S}_{\mathbf{m}} \subset \mathcal{S}_{\mathbf{M}}$ , then*

$$\widehat{\Pi}_{\mathbf{m}}(\widehat{p}_{\mathbf{M},t}) = \widehat{p}_{\mathbf{m},t}.$$

**Lemma A.4.** *Consider the event*

$$\Xi_N := \{\mathcal{M}_N \subset \widehat{\mathcal{M}}_N \subset \mathfrak{M}_N\},$$

where  $\mathfrak{M}_N := \mathcal{U}_N \cap (\mathfrak{V}_N \times \mathcal{N})$  and

$$\mathfrak{V}_N := \left\{ m_1 \in \mathcal{N} : \mathfrak{c}_{\varphi}^2 m_1 (\|\Psi_{m_1}^{-1}\|_{\text{op}}^2 \vee 1) \leq 4\mathfrak{d} \frac{NT}{\log(NT)} \right\}.$$

Under Assumption 5.1, there exists a constant  $\mathfrak{c}_{A.4} > 0$ , not depending on  $N$ , such that

$$\mathbb{P}(\Xi_N^c) \leq \frac{\mathfrak{c}_{A.4}}{N^{p-1}}.$$

The proof of Lemma A.3 is postponed to Subsubsection A.3.2, and Lemma A.4 is a straightforward consequence of Comte and Genon-Catalot [14], Inequality (6.17), because

$$\{\mathcal{V}_N \subset \widehat{\mathcal{V}}_N \subset \mathfrak{V}_N\} \subset \Xi_N.$$

Note that  $\widehat{\mathbf{m}}$  needs to be selected in the empirical set  $\widehat{\mathcal{M}}_N$ , instead of  $\mathcal{M}_N$ , because  $\mathcal{M}_N$  depends on  $m_1 \mapsto \|\Psi_{m_1}^{-1}\|_{\text{op}}$  which has no explicit control in general. However, and this is the purpose of Lemma A.4, we need that with high probability:

- All the possible couples of dimensions in  $\mathcal{M}_N$  may be selected in  $\widehat{\mathcal{M}}_N$ .
- The couple of dimensions  $\widehat{\mathbf{m}}(\omega)$  selected in  $\widehat{\mathcal{M}}_N(\omega)$  belongs to  $\mathfrak{M}_N$  in order that  $\Psi_{\widehat{m}_1(\omega)}$  satisfies Assumption 4.1, and then to apply Lemma A.1 and Theorem 4.2 in the (following) proof of Theorem 5.2.

**A.3.1. Steps of the proof.** The proof of Theorem 5.2 is dissected in four steps.

**Step 1.** Let us prove that for any  $\mathbf{m} = (m_1, m_2) \in \widehat{\mathcal{M}}_N$ ,

$$(37) \quad \|\widehat{p}_{\widehat{\mathbf{m}},t} - p_{I \times J,t}\|_N^2 \leq 6\|\widehat{p}_{\mathbf{m},t} - p_{I \times J,t}\|_N^2 + 4\kappa \text{pen}(\mathbf{m}) + 11 \left( \|\widehat{p}_{\widehat{\mathbf{m}},t} - \widehat{\Pi}_{\widehat{\mathbf{m}}}(p_t)\|_N^2 - \frac{2}{11} \kappa \text{pen}(\widehat{\mathbf{m}}) \right)_+.$$

Consider  $\mathbf{M} = (\max(\mathcal{N}), \max(\mathcal{N}))$  and  $m \in \{\mathbf{m}, \widehat{\mathbf{m}}\} \subset \widehat{\mathcal{M}}_N \subset \mathcal{N}^2$ . First, let us show that

$$(38) \quad \|\widehat{p}_{\widehat{\mathbf{m}},t} - p_{I \times J,t}\|_N^2 \leq \|\widehat{p}_{\mathbf{m},t} - p_{I \times J,t}\|_N^2 - 2(\|\widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2 - \kappa \text{pen}(\mathbf{m})) \\ + 2(\|\widehat{p}_{\widehat{\mathbf{m}},t} - \widehat{\Pi}_{\widehat{\mathbf{m}}}(p_t)\|_N^2 - \kappa \text{pen}(\widehat{\mathbf{m}})) + R_{\mathbf{M}}$$

with

$$R_{\mathbf{M}} = 2\langle \widehat{\Pi}_{\widehat{\mathbf{m}}}(p_t) - \widehat{\Pi}_{\mathbf{m}}(p_t), \widehat{p}_{\mathbf{M},t} - \widehat{\Pi}_{\mathbf{M}}(p_t) \rangle_N.$$

Since  $\mathcal{S}_{\mathbf{m}} \subset \mathcal{S}_{\mathbf{M}}$  by Assumption 5.1.(1,2),

$$\widehat{\Pi}_{\mathbf{m}}(\widehat{p}_{\mathbf{M},t}) = \widehat{p}_{\mathbf{m},t} \quad \text{by Lemma A.3.}$$

Then,

$$\begin{aligned} \|\widehat{p}_{\mathbf{M},t} - \widehat{p}_{\mathbf{m},t}\|_N^2 &= \|\widehat{p}_{\mathbf{M},t}\|_N^2 + \|\widehat{p}_{\mathbf{m},t}\|_N^2 - 2\langle \widehat{p}_{\mathbf{M},t}, \widehat{p}_{\mathbf{m},t} \rangle_N \\ &= \|\widehat{p}_{\mathbf{M},t}\|_N^2 + \|\widehat{\Pi}_{\mathbf{m}}(\widehat{p}_{\mathbf{M},t})\|_N^2 \\ &\quad - 2 \underbrace{\langle \widehat{p}_{\mathbf{M},t} - \widehat{\Pi}_{\mathbf{m}}(\widehat{p}_{\mathbf{M},t}), \widehat{\Pi}_{\mathbf{m}}(\widehat{p}_{\mathbf{M},t}) \rangle_N}_{=0} - 2\langle \widehat{\Pi}_{\mathbf{m}}(\widehat{p}_{\mathbf{M},t}), \widehat{\Pi}_{\mathbf{m}}(\widehat{p}_{\mathbf{M},t}) \rangle_N \\ &= \|\widehat{p}_{\mathbf{M},t}\|_N^2 - \|\widehat{p}_{\mathbf{m},t}\|_N^2. \end{aligned}$$

So, by the definition of  $\widehat{\mathbf{m}}$ ,

$$\begin{aligned} \|\widehat{p}_{\mathbf{M},t} - \widehat{p}_{\widehat{\mathbf{m}},t}\|_N^2 + 2\kappa\text{pen}(\widehat{\mathbf{m}}) &= \|\widehat{p}_{\mathbf{M},t}\|_N^2 - \|\widehat{p}_{\widehat{\mathbf{m}},t}\|_N^2 + 2\kappa\text{pen}(\widehat{\mathbf{m}}) \\ &\leq \|\widehat{p}_{\mathbf{M},t}\|_N^2 - \|\widehat{p}_{\mathbf{m},t}\|_N^2 + 2\kappa\text{pen}(\mathbf{m}) = \|\widehat{p}_{\mathbf{M},t} - \widehat{p}_{\mathbf{m},t}\|_N^2 + 2\kappa\text{pen}(\mathbf{m}), \end{aligned}$$

and thus

$$\begin{aligned} \|\widehat{p}_{\widehat{\mathbf{m}},t} - p_{I \times J,t}\|_N^2 - \|\widehat{p}_{\mathbf{m},t} - p_{I \times J,t}\|_N^2 \\ &= \|\widehat{p}_{\widehat{\mathbf{m}},t} - \widehat{p}_{\mathbf{M},t}\|_N^2 - \|\widehat{p}_{\mathbf{m},t} - \widehat{p}_{\mathbf{M},t}\|_N^2 + 2\langle \widehat{p}_{\widehat{\mathbf{m}},t} - \widehat{p}_{\mathbf{m},t}, \widehat{p}_{\mathbf{M},t} - p_{I \times J,t} \rangle_N \\ &\leq 2\kappa\text{pen}(\mathbf{m}) - 2\kappa\text{pen}(\widehat{\mathbf{m}}) + 2\langle \widehat{p}_{\widehat{\mathbf{m}},t} - \widehat{p}_{\mathbf{m},t}, \widehat{p}_{\mathbf{M},t} - p_{I \times J,t} \rangle_N. \end{aligned}$$

Moreover, since  $\widehat{\Pi}_m \circ \widehat{\Pi}_{\mathbf{M}} = \widehat{\Pi}_m$  and  $\widehat{\Pi}_m(\widehat{p}_{\mathbf{M},t}) = \widehat{p}_{\mathbf{m},t}$ ,

$$\begin{aligned} \langle \widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_m(p_t), \widehat{p}_{\mathbf{M},t} - \widehat{\Pi}_{\mathbf{M}}(p_t) \rangle_N \\ &= \underbrace{\langle \widehat{\Pi}_m(\widehat{p}_{\mathbf{M},t} - \widehat{\Pi}_{\mathbf{M}}(p_t)), \widehat{p}_{\mathbf{M},t} - \widehat{\Pi}_{\mathbf{M}}(p_t) - \widehat{\Pi}_m(\widehat{p}_{\mathbf{M},t} - \widehat{\Pi}_{\mathbf{M}}(p_t)) \rangle_N}_{=0} + \|\widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_m(p_t)\|_N^2, \end{aligned}$$

leading to

$$\begin{aligned} \langle \widehat{p}_{\widehat{\mathbf{m}},t} - \widehat{p}_{\mathbf{m},t}, \widehat{p}_{\mathbf{M},t} - p_{I \times J,t} \rangle_N &= \langle \widehat{p}_{\widehat{\mathbf{m}},t} - \widehat{p}_{\mathbf{m},t}, \widehat{p}_{\mathbf{M},t} - \widehat{\Pi}_{\mathbf{M}}(p_t) \rangle_N + \underbrace{\langle \widehat{p}_{\widehat{\mathbf{m}},t} - \widehat{p}_{\mathbf{m},t}, \widehat{\Pi}_{\mathbf{M}}(p_t) - p_{I \times J,t} \rangle_N}_{=0} \\ &= \langle \widehat{p}_{\widehat{\mathbf{m}},t} - \widehat{\Pi}_{\widehat{\mathbf{m}}}(p_t), \widehat{p}_{\mathbf{M},t} - \widehat{\Pi}_{\mathbf{M}}(p_t) \rangle_N \\ &\quad + \langle \widehat{\Pi}_{\widehat{\mathbf{m}}}(p_t) - \widehat{\Pi}_{\mathbf{m}}(p_t), \widehat{p}_{\mathbf{M},t} - \widehat{\Pi}_{\mathbf{M}}(p_t) \rangle_N \\ &\quad + \langle \widehat{\Pi}_{\mathbf{m}}(p_t) - \widehat{p}_{\mathbf{m},t}, \widehat{p}_{\mathbf{M},t} - \widehat{\Pi}_{\mathbf{M}}(p_t) \rangle_N \\ &= \|\widehat{p}_{\widehat{\mathbf{m}},t} - \widehat{\Pi}_{\widehat{\mathbf{m}}}(p_t)\|_N^2 - \|\widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2 \\ &\quad + \langle \widehat{\Pi}_{\widehat{\mathbf{m}}}(p_t) - \widehat{\Pi}_{\mathbf{m}}(p_t), \widehat{p}_{\mathbf{M},t} - \widehat{\Pi}_{\mathbf{M}}(p_t) \rangle_N. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\widehat{p}_{\widehat{\mathbf{m}},t} - p_{I \times J,t}\|_N^2 &\leq \|\widehat{p}_{\mathbf{m},t} - p_{I \times J,t}\|_N^2 + 2\kappa\text{pen}(\mathbf{m}) - 2\kappa\text{pen}(\widehat{\mathbf{m}}) + 2\langle \widehat{p}_{\widehat{\mathbf{m}},t} - \widehat{p}_{\mathbf{m},t}, \widehat{p}_{\mathbf{M},t} - p_{I \times J,t} \rangle_N \\ &\leq \|\widehat{p}_{\mathbf{m},t} - p_{I \times J,t}\|_N^2 - 2(\|\widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2 - \kappa\text{pen}(\mathbf{m})) \\ &\quad + 2(\|\widehat{p}_{\widehat{\mathbf{m}},t} - \widehat{\Pi}_{\widehat{\mathbf{m}}}(p_t)\|_N^2 - \kappa\text{pen}(\widehat{\mathbf{m}})) + R_{\mathbf{M}}. \end{aligned}$$

Now, let us find a suitable bound on  $R_{\mathbf{M}}$ . Since  $\varphi_1, \dots, \varphi_{\max(\mathcal{N})}$  are linearly independent, one may consider the basis  $(\varphi_1^N, \dots, \varphi_{\max(\mathcal{N})}^N)$  of  $\mathcal{S}_{\varphi, \max(\mathcal{N})}$ , orthonormal for the empirical inner product  $\langle \cdot, \cdot \rangle_{N,1}$ , obtained from  $(\varphi_1, \dots, \varphi_{\max(\mathcal{N})})$  via the Gram-Schmidt process. Then,  $(\varphi_j^N \otimes \psi_\ell)_{j,\ell}$  is an orthonormal basis of  $\mathcal{S}_{\mathbf{M}}$  equipped with  $\langle \cdot, \cdot \rangle_N$ , so that

$$\begin{aligned} \widehat{\Pi}_{\mathbf{M}}(p_t) &= \sum_{j=1}^{\max(\mathcal{N})} \sum_{\ell=1}^{\max(\mathcal{N})} \langle p_t, \varphi_j^N \otimes \psi_\ell \rangle_N (\varphi_j^N \otimes \psi_\ell) \\ \text{and } \widehat{p}_{\mathbf{M},t} = \widehat{\Pi}_{\mathbf{M}}(\widehat{p}_{\mathbf{M},t}) &= \sum_{j=1}^{\max(\mathcal{N})} \sum_{\ell=1}^{\max(\mathcal{N})} \langle \widehat{p}_{\mathbf{M},t}, \varphi_j^N \otimes \psi_\ell \rangle_N (\varphi_j^N \otimes \psi_\ell). \end{aligned}$$

Thus,

$$\begin{aligned}
|R_{\mathbf{M}}| &= 2 \left| \left\langle \widehat{\Pi}_{\widehat{\mathbf{m}}}(p_t) - \widehat{\Pi}_{\mathbf{m}}(p_t), \sum_{j=1}^{m_1 \vee \widehat{m}_1} \sum_{\ell=1}^{m_2 \vee \widehat{m}_2} \langle \widehat{p}_{\mathbf{M},t} - p_t, \varphi_j^N \otimes \psi_\ell \rangle_N (\varphi_j^N \otimes \psi_\ell) \right\rangle_N \right| \\
&\leq \frac{1}{4} \|\widehat{\Pi}_{\widehat{\mathbf{m}}}(p_t) - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2 + 4 \left\| \sum_{j=1}^{m_1 \vee \widehat{m}_1} \sum_{\ell=1}^{m_2 \vee \widehat{m}_2} \langle \widehat{p}_{\mathbf{M},t} - p_t, \varphi_j^N \otimes \psi_\ell \rangle_N (\varphi_j^N \otimes \psi_\ell) \right\|_N^2 \\
&\leq \frac{1}{2} \|\widehat{\Pi}_{\widehat{\mathbf{m}}}(p_t) - p_{I \times J, t}\|_N^2 + \frac{1}{2} \|\widehat{\Pi}_{\mathbf{m}}(p_t) - p_{I \times J, t}\|_N^2 + 4 \sum_{j=1}^{\widehat{m}_1} \sum_{\ell=1}^{\widehat{m}_2} \langle \widehat{p}_{\mathbf{M},t} - p_t, \varphi_j^N \otimes \psi_\ell \rangle_N^2 \\
&\quad + 4 \sum_{j=1}^{m_1} \sum_{\ell=1}^{m_2} \langle \widehat{p}_{\mathbf{M},t} - p_t, \varphi_j^N \otimes \psi_\ell \rangle_N^2 \\
&= \frac{1}{2} \|\widehat{\Pi}_{\widehat{\mathbf{m}}}(p_t) - p_{I \times J, t}\|_N^2 + \frac{1}{2} \|\widehat{\Pi}_{\mathbf{m}}(p_t) - p_{I \times J, t}\|_N^2 \\
&\quad + 4 \|\widehat{\Pi}_{\widehat{\mathbf{m}}}(\widehat{p}_{\mathbf{M},t} - p_t)\|_N^2 + 4 \|\widehat{\Pi}_{\mathbf{m}}(\widehat{p}_{\mathbf{M},t} - p_t)\|_N^2 \\
(39) \quad &= \frac{1}{2} \|\widehat{p}_{\widehat{\mathbf{m}},t} - p_{I \times J, t}\|_N^2 + \frac{1}{2} \|\widehat{p}_{\mathbf{m},t} - p_{I \times J, t}\|_N^2 + \frac{7}{2} \|\widehat{p}_{\widehat{\mathbf{m}},t} - \widehat{\Pi}_{\widehat{\mathbf{m}}}(p_t)\|_N^2 + \frac{7}{2} \|\widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2.
\end{aligned}$$

Therefore, by Inequalities (38) and (39),

$$\begin{aligned}
\|\widehat{p}_{\widehat{\mathbf{m}},t} - p_{I \times J, t}\|_N^2 &\leq 3 \|\widehat{p}_{\mathbf{m},t} - p_{I \times J, t}\|_N^2 + 3 \|\widehat{p}_{\mathbf{m},t} - \widehat{\Pi}_{\mathbf{m}}(p_t)\|_N^2 \\
&\quad + 4\kappa \text{pen}(\mathbf{m}) + 11 \left( \|\widehat{p}_{\widehat{\mathbf{m}},t} - \widehat{\Pi}_{\widehat{\mathbf{m}}}(p_t)\|_N^2 - \frac{2}{11} \kappa \text{pen}(\widehat{\mathbf{m}}) \right) \\
&\leq 6 \|\widehat{p}_{\mathbf{m},t} - p_{I \times J, t}\|_N^2 + 4\kappa \text{pen}(\mathbf{m}) + 11 \left( \|\widehat{p}_{\widehat{\mathbf{m}},t} - \widehat{\Pi}_{\widehat{\mathbf{m}}}(p_t)\|_N^2 - \frac{2}{11} \kappa \text{pen}(\widehat{\mathbf{m}}) \right)_+.
\end{aligned}$$

**Step 2.** First of all, consider

$$\Omega_N := \bigcap_{m_1 \in \mathfrak{M}_N} \Omega_{m_1}.$$

For every  $m_1 \in \mathfrak{M}_N$ ,

$$\mathfrak{L}_\varphi(m_1) (\|\Psi_{m_1}\|_{\text{op}} \vee 1) \leq \mathfrak{c}_\varphi^2 m_1 (\|\Psi_{m_1}\|_{\text{op}}^2 \vee 1) \leq 4\mathfrak{d} \frac{NT}{\log(NT)} \leq \frac{\mathfrak{c}_\Lambda}{2} \cdot \frac{NT}{\log(NT)},$$

leading to

$$(40) \quad \mathbb{P}(\Omega_N^c) \leq \sum_{m_1 \in \mathfrak{M}_N} \mathbb{P}(\Omega_{m_1}^c) \leq \frac{\mathfrak{c}_{A.1}}{N^{p-1}} \quad \text{by Lemma A.1 (requiring Assumption 4.1).}$$

Now, since  $\Xi_N = \{\mathcal{M}_N \subset \widehat{\mathcal{M}}_N \subset \mathfrak{M}_N\}$ , on the event  $\Xi_N \cap \Omega_N$ , Inequality (37) remains true for every  $\mathbf{m} \in \mathcal{M}_N$ . Then,

$$\begin{aligned}
\|\widehat{p}_{\widehat{\mathbf{m}},t} - p_{I \times J, t}\|_N^2 &= \|\widehat{p}_{\widehat{\mathbf{m}},t} - p_{I \times J, t}\|_N^2 \mathbf{1}_{\Xi_N \cap \Omega_N} + \|\widehat{p}_{\widehat{\mathbf{m}},t} - p_{I \times J, t}\|_N^2 \mathbf{1}_{\Xi_N^c \cup \Omega_N^c} \\
&\leq \min_{\mathbf{m} \in \mathcal{M}_N} \{6 \|\widehat{p}_{\mathbf{m},t} - p_{I \times J, t}\|_N^2 + 4\kappa \text{pen}(\mathbf{m})\} \\
&\quad + 11 \left( \|\widehat{p}_{\widehat{\mathbf{m}},t} - \widehat{\Pi}_{\widehat{\mathbf{m}}}(p_t)\|_N^2 - \frac{2}{11} \kappa \text{pen}(\widehat{\mathbf{m}}) \right)_+ \mathbf{1}_{\Xi_N \cap \Omega_N} \\
&\quad + \left( \|\widehat{p}_{\widehat{\mathbf{m}},t} - \widehat{\Pi}_{\widehat{\mathbf{m}}}(p_t)\|_N^2 + \min_{\tau \in \mathcal{S}_{\widehat{\mathbf{m}}}} \|\tau - p_{I \times J, t}\|_N^2 \right) (\mathbf{1}_{\Xi_N^c} + \mathbf{1}_{\Omega_N^c}) \\
(41) \quad &\leq \min_{\mathbf{m} \in \mathcal{M}_N} \{6 \|\widehat{p}_{\mathbf{m},t} - p_{I \times J, t}\|_N^2 + 4\kappa \text{pen}(\mathbf{m})\} + A + B
\end{aligned}$$

where, by Lemma A.2,

$$A := \left( \sup_{\tau \in \mathcal{S}_{\widehat{\mathbf{m}}}: \|\tau\|_N = 1} \nu_N(\tau)^2 + \|p_{I \times J, t}\|_N^2 \right) (\mathbf{1}_{\Xi_N^c} + \mathbf{1}_{\Omega_N^c})$$

$$\text{and } B := 11 \left( \sup_{\tau \in \mathcal{S}_{\widehat{\mathbf{m}}}: \|\tau\|_N = 1} \nu_N(\tau)^2 - \frac{2}{11} \kappa \text{pen}(\widehat{\mathbf{m}}) \right)_+ \mathbf{1}_{\Xi_N \cap \Omega_N}.$$

Let us find suitable bounds on  $\mathbb{E}(A)$  and  $\mathbb{E}(B)$ . On the one hand, since

$$\sup_{\tau \in \mathcal{S}_{\mathbf{m}}: \|\tau\|_N = 1} \nu_N(\tau)^2 \leq 4m_1 \mathfrak{L}_\psi(m_2); \quad \forall \mathbf{m} = (m_1, m_2) \in \mathcal{N}^2$$

as established in the proof of Theorem 4.2 (see Inequality (34)), since  $\mathbb{E}(\|p_{I \times J, t}\|_N^4)^{1/2} \leq R(t)$  as established in the proof of Theorem 4.3 (see Step 1), and by Inequality (40) and Lemma A.4,

$$\mathbb{E}(A) \leq \mathbf{c}_1 \left( R(t) + \sum_{m_1, m_2=1}^{\max(\mathcal{N})} m_1 \mathfrak{L}_\psi(m_2) \right) (\mathbb{P}(\Omega_N^c)^{\frac{1}{2}} + \mathbb{P}(\Xi_N^c)^{\frac{1}{2}}) \leq \frac{\mathbf{c}_2(1 + R(t))}{N},$$

where  $\mathbf{c}_1$  and  $\mathbf{c}_2$  are positive constants not depending on  $N$  and  $t$ . On the other hand, since  $\widehat{\mathbf{m}} \in \mathfrak{M}_N$  on the event  $\Xi_N \cap \Omega_N$ ,

$$\mathbb{E}(B) \leq 11 \sum_{\mathbf{m} \in \mathfrak{M}_N} \mathbb{E} \left[ \left( \sup_{\tau \in \mathcal{S}_{\mathbf{m}}: \|\tau\|_N = 1} \nu_N(\tau)^2 - \frac{2}{11} \kappa \text{pen}(\mathbf{m}) \right)_+ \mathbf{1}_{\Xi_N \cap \Omega_N} \right].$$

Moreover, for every  $\mathbf{m} = (m_1, m_2) \in \mathfrak{M}_N$ ,

$$\{\tau \in \mathcal{S}_{\mathbf{m}} : \|\tau\|_N = 1\} \subset \{\tau \in \mathcal{S}_{\mathbf{m}} : \|\tau\|_f^2 \leq 2\} \quad \text{on } \Omega_{m_1} \supset \Omega_N.$$

Then,

$$\mathbb{E}(B) \leq 11 \sum_{\mathbf{m} \in \mathfrak{M}_N} \mathbb{E} \left[ \underbrace{\left( \sup_{\tau \in \mathcal{S}_{\mathbf{m}}: \|\tau\|_f \leq 2} \nu_N(\tau)^2 - \frac{2}{11} \kappa \text{pen}(\mathbf{m}) \right)_+}_{=: b_{\mathbf{m}}} \mathbf{1}_{\Xi_N \cap \Omega_N} \right].$$

**Step 3.** To prepare for the application of the integrated version (see Comte [11], Theorem A.1) of the Klein and Rio extension of Talagrand's inequality (see Klein and Rio [23], Theorem 1.2), this step deals with a suitable control of  $b_{\mathbf{m}}$ ,  $\mathbf{m} = (m_1, m_2) \in \mathfrak{M}_N$ . First, for every  $\tau = \sum_{j, \ell} T_{j, \ell}(\varphi_j \otimes \psi_\ell)$  belonging to  $\mathcal{F} := \{\tau \in \mathcal{S}_{\mathbf{m}} : \|\tau\|_f^2 \leq 2\}$ ,

$$\|\tau\|^2 = \text{trace}(T T^* \Psi_{m_1} \Psi_{m_1}^{-1}) \leq \|\Psi_{m_1}^{-1}\|_{\text{op}} \underbrace{\text{trace}(T^* \Psi_{m_1} T)}_{= \|\tau\|_f^2} \leq 2 \|\Psi_{m_1}^{-1}\|_{\text{op}},$$

and then

$$\|\tau\|_\infty = \sup_{(x, y) \in I \times J} \left\{ \sum_{j=1}^{m_1} \sum_{\ell=1}^{m_2} T_{j, \ell} \varphi_j(x) \psi_\ell(y) \right\}$$

$$\leq \mathfrak{L}_\psi(m_2)^{\frac{1}{2}} \sup_{x \in I} \left\{ \sum_{j=1}^{m_1} |\varphi_j(x)| \sum_{\ell=1}^{m_2} |T_{j, \ell}| \right\} \leq \sqrt{2} \mathfrak{L}_\varphi(m_1)^{\frac{1}{2}} \mathfrak{L}_\psi(m_2)^{\frac{1}{2}} \|\Psi_{m_1}^{-1}\|_{\text{op}}^{\frac{1}{2}}.$$

Since  $m_1 \in \mathfrak{V}_N$ , and since  $\mathfrak{L}_\psi(m_2) \leq m_1 \mathfrak{L}_\psi(m_2) \leq N$  by the definition of  $\mathcal{U}_N$ ,

$$\sup_{\tau \in \mathcal{F}} \|\tau\|_\infty \leq M \quad \text{with } M = \frac{\mathbf{c}_3 N}{\log(N)^{1/2}} \quad \text{and } \mathbf{c}_3 = 2\sqrt{2\delta T}.$$

Now, since

$$\mathbb{E} \left( \sup_{\tau \in \mathcal{F}} \nu_N(\tau)^2 \right) \leq \frac{2m_1 \mathfrak{L}_\psi(m_2)}{N}$$

as established in the proof of Theorem 4.2 (see Step 1),

$$\mathbb{E} \left( \sup_{\tau \in \mathcal{F}} |\nu_N(\tau)| \right) \leq H \quad \text{with} \quad H^2 = \frac{2m_1 \mathfrak{L}_\psi(m_2)}{N},$$

and then

$$N \sup_{\tau \in \mathcal{F}} \mathbb{E}(\nu_N(\tau)^2) \leq v \quad \text{with} \quad v = NH^2.$$

By the aforementioned Talagrand's inequality, for  $\alpha := a \log(N)$ ,

$$\mathbb{E} \left[ \left( \sup_{\tau \in \mathcal{F}} \nu_N(\tau)^2 - 2(1 + 2\alpha)H^2 \right)_+ \right] \leq A(N, \alpha, H, v) + B(N, \alpha, M, H),$$

where

$$\begin{aligned} A(N, \alpha, H, v) &:= \frac{24v}{N} \exp \left( -\frac{\alpha NH^2}{6v} \right) \\ &= \frac{48m_1 \mathfrak{L}_\psi(m_2)}{N} e^{-\frac{a}{6} \log(N)} \leq 48N^{-\frac{a}{6}} \quad \text{because } \mathbf{m} \in \mathcal{U}_N, \end{aligned}$$

and

$$\begin{aligned} B(N, \alpha, M, H) &:= \frac{\mathfrak{c}_4 M^2}{N^2} \exp \left( -\frac{\sqrt{2}}{42} \cdot \frac{NH\sqrt{\alpha}}{M} \right) \quad \text{with } \mathfrak{c}_4 = 7056 \\ &= \frac{\mathfrak{c}_3^2 \mathfrak{c}_4}{\log(N)} \exp \left( -\frac{\sqrt{2}}{42} \cdot \frac{\sqrt{2a} \log(N)}{\mathfrak{c}_3} \right) \leq \mathfrak{c}_3^2 \mathfrak{c}_4 N^{-\frac{\sqrt{2a}}{84\sqrt{\mathfrak{c}_3 T}}}. \end{aligned}$$

Therefore, since  $a \geq (2 \cdot 84\sqrt{\mathfrak{c}_3 T})^2/2$ , and since

$$\kappa \geq \kappa_0 = 44a \geq \frac{22(1 + 2a \log(N))}{1 + \log(N)} = 11(1 + 2a \log(N)) \frac{H^2}{\text{pen}(\mathbf{m})},$$

there exists a constant  $\mathfrak{c}_5 > 0$ , not depending on  $N$ ,  $m_1$ ,  $m_2$  and  $t$ , such that

$$\begin{aligned} b_{\mathbf{m}} &\leq \mathbb{E} \left[ \left( \sup_{\tau \in \mathcal{S}_{\mathbf{m}} : \|\tau\|_f \leq 2} \nu_N(\tau)^2 - \frac{2}{11} \kappa \text{pen}(\mathbf{m}) \right)_+ \right] \\ &\leq \mathbb{E} \left[ \left( \sup_{\tau \in \mathcal{F}} \nu_N(\tau)^2 - 2(1 + 2\alpha)H^2 \right)_+ \right] \leq \frac{\mathfrak{c}_5}{N^2}. \end{aligned}$$

**Step 4 (conclusion).** By Steps 2 and 3,

$$\mathbb{E}(A) \leq \frac{\mathfrak{c}_2(1 + R(t))}{N} \quad \text{and} \quad \mathbb{E}(B) \leq 11 \sum_{\mathbf{m} \in \mathfrak{M}_N} b_{\mathbf{m}} \leq \frac{11\mathfrak{c}_5}{N}.$$

Thus, by Inequality (41), there exists a constant  $\mathfrak{c}_6 > 0$ , not depending on  $N$  and  $t$ , such that

$$\mathbb{E}(\|\widehat{p}_{\widehat{\mathbf{m}}, t} - p_{I \times J, t}\|_N^2) \leq 6 \min_{\mathbf{m} \in \mathfrak{M}_N} \{ \mathbb{E}(\|\widehat{p}_{\mathbf{m}, t} - p_{I \times J, t}\|_N^2) + \kappa \text{pen}(\mathbf{m}) \} + \frac{\mathfrak{c}_6(1 + R(t))}{N}.$$

Moreover, Theorem 5.2.(2) is a consequence of Theorem 5.2.(1) thanks to Inequality (9) as in the last step of the proof of Theorem A.2.

A.3.2. *Proof of Lemma A.3.* By the definition of  $\widehat{\Pi}_{\mathbf{m}}$  (see (33)),

$$\widehat{\Pi}_{\mathbf{m}}(\widehat{p}_{\mathbf{M},t}) = \sum_{j=1}^{m_1} \sum_{\ell=1}^{m_2} [\widehat{\Psi}_{m_1}^{-1} \widehat{P}_{\mathbf{m}}(\widehat{p}_{\mathbf{M},t})]_{j,\ell} (\varphi_j \otimes \psi_\ell)$$

with, for every  $j \in \{1, \dots, m_1\}$  and  $\ell \in \{1, \dots, m_2\}$ ,

$$\begin{aligned} \widehat{P}_{\mathbf{m}}(\widehat{p}_{\mathbf{M},t}) &= \frac{1}{NT} \sum_{i=1}^N \int_0^T \int_{-\infty}^{\infty} \widehat{p}_{\mathbf{M},t}(X_s^i, y) \varphi_j(X_s^i) \psi_\ell(y) dy ds \\ &= \sum_{j'=1}^{M_1} \sum_{\ell'=1}^{M_2} [\widehat{\Psi}_{M_1}^{-1} \widehat{Z}_{\mathbf{M},t}]_{j',\ell'} \\ &\quad \times \frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X_s^i) \varphi_{j'}(X_s^i) \underbrace{\int_{-\infty}^{\infty} \psi_\ell(y) \psi_{\ell'}(y) dy}_{=\delta_{\ell,\ell'}} ds \\ &= \sum_{j'=1}^{M_1} [\widehat{\Psi}_{M_1}^{-1} \widehat{Z}_{\mathbf{M},t}]_{j',\ell} \underbrace{\frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X_s^i) \varphi_{j'}(X_s^i) ds}_{=[\widehat{\Psi}_{M_1}]_{j,j'}} \\ &= [\widehat{Z}_{\mathbf{M},t}]_{j,\ell} = [\widehat{Z}_{\mathbf{m},t}]_{j,\ell} \quad \text{because } \mathcal{S}_{\mathbf{m}} \subset \mathcal{S}_{\mathbf{M}}. \end{aligned}$$

Therefore,

$$\widehat{\Pi}_{\mathbf{m}}(\widehat{p}_{\mathbf{M},t}) = \sum_{j=1}^{m_1} \sum_{\ell=1}^{m_2} [\widehat{\Psi}_{m_1}^{-1} \widehat{Z}_{\mathbf{m},t}]_{j,\ell} (\varphi_j \otimes \psi_\ell) = \widehat{p}_{\mathbf{m},t}.$$

A.4. **Proof of Proposition 5.4.** The proof is mainly the same as that of Theorem 5.2, except for the control  $M$  of  $\sup_{\tau \in \mathcal{F}} \|\tau\|_\infty$  and the control  $\nu$  of  $N \sup_{\tau \in \mathcal{F}} \mathbb{E}(\nu_N(\tau)^2)$  involved in Talagrand's inequality (see Step 3 in Subsubsection A.3.1).

**Step 3 (bis).** Thanks to the Klein and Rio version of Talagrand's inequality (see Klein and Rio [23]), this step deals with a suitable control of  $b_{\mathbf{m}}$ ,  $\mathbf{m} = (m_1, m_2) \in \mathfrak{M}_N$ . First, for every  $\tau = \sum_{j,\ell} T_{j,\ell}(\varphi_j \otimes \psi_\ell)$  belonging to  $\mathcal{F} = \{\tau \in \mathcal{S}_{\mathbf{m}} : \|\tau\|_{\mathcal{F}}^2 \leq 2\}$ , we still get

$$\|\tau\|_\infty \leq \sqrt{2} \mathfrak{L}_\varphi(m_1)^{\frac{1}{2}} \mathfrak{L}_\psi(m_2)^{\frac{1}{2}} \|\Psi_{m_1}^{-1}\|_{\text{op}}^{\frac{1}{2}}.$$

By Assumption 5.1.(3), and since  $\mathbf{m} = (m_1, m_2)$  belongs to  $\mathfrak{M}_N$ ,

$$\mathfrak{L}_\varphi(m_1)^{\frac{1}{2}} \|\Psi_{m_1}^{-1}\|_{\text{op}}^{\frac{1}{2}} \leq \mathfrak{c}_\varphi^{\frac{1}{2}} m_1^{\frac{1}{4}} (\mathfrak{c}_\varphi^2 m_1 \|\Psi_{m_1}^{-1}\|_{\text{op}}^2)^{\frac{1}{4}} \leq \mathfrak{c}_\varphi^{\frac{1}{2}} m_1^{\frac{1}{4}} \left(4\mathfrak{d} \frac{NT}{\log(NT)}\right)^{\frac{1}{4}}.$$

Thus, for  $NT \geq e$ ,

$$\sup_{\tau \in \mathcal{F}} \|\tau\|_\infty \leq M \quad \text{with} \quad M = \mathfrak{c}_{3,b} m_1^{\frac{1}{4}} \mathfrak{L}_\psi(m_2)^{\frac{1}{2}} N^{\frac{1}{4}} \quad \text{and} \quad \mathfrak{c}_{3,b} = 2\sqrt{\mathfrak{c}_\varphi} (\mathfrak{d}T)^{\frac{1}{4}}.$$

Now, since

$$\mathbb{E} \left( \sup_{\tau \in \mathcal{F}} \nu_N(\tau)^2 \right) \leq \frac{2m_1 \mathfrak{L}_\psi(m_2)}{N}$$

as established in the proof of Theorem 4.2 (see Step 1),

$$\mathbb{E} \left( \sup_{\tau \in \mathcal{F}} |\nu_N(\tau)| \right) \leq H \quad \text{with} \quad H^2 = \frac{2m_1 \mathfrak{L}_\psi(m_2)}{N}.$$

Lastly,

$$\begin{aligned} \sup_{\tau \in \mathcal{F}} \left\{ \text{var} \left( \frac{1}{T} \int_0^T (\tau(X_s, X_{s+t}) - \mathbb{E}(\tau(X_s, X_{s+t}) | X_s)) ds \right) \right\} \\ \leq \sup_{\tau \in \mathcal{F}} \mathbb{E} \left[ \left( \frac{1}{T} \int_0^T \tau(X_s, X_{s+t}) ds \right)^2 \right] \leq p_0 \sup_{\tau \in \mathcal{F}} \|\tau\|_f^2 \leq v \quad \text{with} \quad v = 2p_0. \end{aligned}$$

By the aforementioned Talagrand's inequality, for  $\alpha = 1/4$ ,

$$\mathbb{E} \left[ \left( \sup_{\tau \in \mathcal{F}} \nu_N(\tau)^2 - 3H^2 \right)_+ \right] \leq A(N, H, v) + B(N, M, H),$$

where

$$A(N, H, v) := \frac{24v}{N} \exp\left(-\frac{NH^2}{24v}\right) = \frac{48p_0}{N} \exp\left(-\frac{m_1 \mathfrak{L}_\psi(m_2)}{24p_0}\right),$$

and

$$\begin{aligned} B(N, M, H) &:= \frac{\mathfrak{c}_{4,b} M^2}{N^2} \exp\left(-\frac{\sqrt{2}}{84} \cdot \frac{NH}{M}\right) \quad \text{with} \quad \mathfrak{c}_{4,b} = 7056 \\ &\leq \frac{\mathfrak{c}_{4,b} \mathfrak{c}_{3,b}^2}{\sqrt{N}} \exp\left(-\frac{m_1^{1/4} N^{1/4}}{42 \mathfrak{c}_{3,b}}\right) \quad \text{because} \quad \mathbf{m} \in \mathcal{U}_N. \end{aligned}$$

Since  $m_1 \geq 1$ ,

$$B(N, M, H) \leq \frac{\mathfrak{c}_{5,b}}{\sqrt{N}} \exp(-\mathfrak{c}_{6,b} N^{1/4}) \quad \text{with} \quad \mathfrak{c}_{5,b} = \mathfrak{c}_{4,b} \mathfrak{c}_{3,b}^2 \quad \text{and} \quad \mathfrak{c}_{6,b} = \frac{1}{42 \mathfrak{c}_{3,b}}.$$

So, for  $\kappa_b \geq \kappa_{b,0} = 33/2$  and  $t \in [t_0, T]$ ,

$$\begin{aligned} b_{\mathbf{m}} &\leq \mathbb{E} \left[ \left( \sup_{\tau \in \mathcal{S}_{\mathbf{m}}: \|\tau\|_f \leq 2} \nu_N(\tau)^2 - \frac{2}{11} \kappa \text{pen}(\mathbf{m}) \right)_+ \right] \leq \mathbb{E} \left[ \left( \sup_{\tau \in \mathcal{F}} \nu_N(\tau)^2 - 3H^2 \right)_+ \right] \\ &\leq \frac{48p_0}{N} \exp\left(-\frac{m_1 \mathfrak{L}_\psi(m_2)}{24p_0}\right) + \frac{\mathfrak{c}_{5,b}}{\sqrt{N}} \exp(-\mathfrak{c}_{6,b} N^{1/4}). \end{aligned}$$

Therefore, by Assumption 5.3, there exists a constant  $\mathfrak{c}_{7,b} > 0$ , not depending on  $N$ , such that

$$\mathbb{E}(B) \leq 11 \sum_{\mathbf{m} \in \mathfrak{M}_N} b_{\mathbf{m}} \leq \frac{\mathfrak{c}_{7,b}}{N}.$$

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